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Performance study of the multiwavelet discontinuous Galerkin approach for solving the Green-Naghdi equations

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Abstract

This paper presents a multiresolution discontinuous Galerkin scheme for the adaptive solution of Boussinesq-type equations. The model combines multiwavelet-based grid adaptation with a discontinuous Galerkin (DG) solver based on the system of fully nonlinear and weakly dispersive Green-Naghdi (GN) equations. The key feature of the adaptation procedure is to conduct a multiresolution analysis using multiwavelets on a hierarchy of nested grids to improve the efficiency of the reference DG scheme on a uniform grid by computing on a locally refined adapted grid. This way the local resolution level will be determined by manipulating multiwavelet coefficients controlled by a single user-defined threshold value. The proposed adaptive multiwavelet discontinuous Galerkin solver for GN equations (MWDG-GN) is assessed using several benchmark problems related to wave propagation and transformation in nearshore areas. The numerical results demonstrate that the proposed scheme retains the accuracy of the reference scheme, while significantly reducing the computational cost.

Keywords: Multiwavelets; Discontinuous Galerkin; Boussinesq-type equations; Green-Naghdi equations; Multiresolution analysis; Nearshore wave processes

1- Introduction

The Boussinesq-type (BT) equations have been used as an alternative to the free-surface Euler equations for modelling of propagation and transformations of waves in nearshore areas. These types

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of applications usually correspond to the shallow water regime, where the horizontal length scale λ is large compared to the water depth scale h_0, so that the shallowness parameter is $\mu = h_0^2/\lambda^2 \ll 1$. Neglecting all the terms of order $O(\mu)$ from the Euler equations leads to the so-called Nonlinear Shallow Water (NSW) equations, whereas keeping them results in the simplest form of BT equations[1]. While this simple BT model is, in essence, weakly dispersive and only valid for long waves with $kh_0 < 0.75$ (k being the wavenumber), better dispersive behaviour and more accurate BT models can be achieved by incorporation of more terms and related manipulations[2]. The nonlinearity parameter is another related identifier, which is defined as the ratio of the wave amplitude scale to the water depth scale, $\epsilon = a/h_0$. Most of the BT equations impose a smallness amplitude assumption as $\epsilon = O(\mu^2)$, which is too restrictive for many applications in nearshore areas. Removing this assumption (i.e. let $\epsilon = O(1)$) while keeping all the $O(\mu)$ terms, gives the so-called Green-Naghdi (GN) equations[3–5]. The GN equations share the same characteristics of other BT models. However, they allow relative ease in computational implementation, which makes them very favourable in coastal engineering applications[6,7].

To numerically solve various BT wave models, different approaches have been used based on Finite Difference (FD), Finite Volume (FV), Finite Element (FE) and spectral element[2,8]. The Discontinuous Galerkin (DG) method is a more modern alternative for these approaches, which exploits the properties of the FV and FE methods. The DG method thereby provides faster convergence rates and better quality predictions on coarse meshes as compared to an equally accurate FV approach[9–11]. DG methods are becoming increasingly popular in solving BT equations[7,11–19]. However, the runtime cost of DG methods is high, given their demands for storage and evolution of local degrees of freedom within each computational cell and their restrictive CFL condition when applied with explicit Runge-Kutta (RK) time stepping. These costs would even be higher when modelling wave propagation and transformation in coastal areas, where the multitude of spatial and temporal scales further increase the wave feature and complexity.

Classical Adaptive Mesh Refinement (AMR) techniques were initially used in an attempt to reduce fine resolution costs by adapting the mesh resolution[20–22]. However, it turned out that classical AMR approaches bring about new issues owing to the inherently decoupled nature between the mesh and the

numerical solution. In order to control grid refinement/coarsening, AMR methods usually either use Richardson extrapolation[23] or heuristic criteria[24], which gives no information about the errors related to the adaptation process, making the effectiveness of an AMR approach subject to a-posteriori error estimates[25]. Moreover, most of the available AMR developments lack a general adaptivity sensor, so that they either need separate criteria for refinement/coarsening[26,27] or problem specific criteria[28,29] or are reported to be highly dependent on the type of refinement criteria[30]. Also, deploying a classical AMR method dictates extra corrections in the numerical scheme to address the loss of well-balancedness property for the case of the NSW equations[24,31–33].

Multiscale methods based on the Multiresolution Analysis (MRA) of wavelets provide an alternative that can preserve the quality of numerical methods on adaptive meshes[34–38]. Theoretical analyses show that only an error threshold value is needed with this category of adaptive solvers in order to bound the accumulated errors and preserve the accuracy of the reference uniform solver at the finest resolution grid[39–41]. Initially, this concept has been particularly verified with FV solvers, which later appeared to give marginal computational savings and introduce unacceptably large errors for low-order schemes. Therefore, the combination of DG methods with Multiwavelets[42] (MWs) has emerged recently. MWs preserve locality in line with the local and accurate structure of the DG method, which enables greater compression rates alongside small computational stencil compared to wavelets. Compared to the classic AMR methods, multiscale-based methods have been shown to exhibit larger compression rates and more gains in CPU time[43,44].

The MW-based DG solvers have been successfully used for adaptive modelling of Euler[41,45,46] and NSW equations[47–49], suggesting that just by the use of a single threshold value, the adaptive MWDG solver keeps the accuracy of the adaptive solution in the same order as the accuracy of the uniform solution, while reducing the computational cost.

Among the few existing works on wavelet-based grid adaptation for solving BT models, Smith et al.[50] extended the Haar Wavelet-Finite Volume (HWFV) model of Müller[51] to the case of weakly nonlinear, weakly dispersive model of Madsen and Sørensen[52]. Their analysis reported good

performance of the wavelet adaptation process, but also reported on instabilities in areas with fine resolutions, linking them to the treatment of the third spatial derivative in the BT equations.

This work, therefore, presents a first exploration of MW-based grid adaptation combined with DG discretization for modelling the GN equations (denoted hereafter by MWDG-GN). An existing uniform mesh DG solver for the GN equations (DG-GN)[11] is extended to adaptive form, following the MWDG method introduced in Kesserwani et al.[47] applied to the NSW equations (MWDG-NSW). The behaviour of the adaptive MWDG-GN solver in modelling different levels of nonlinearity and dispersion related to wave propagation is studied from both accuracy and efficiency point of view. The suitable range for the threshold parameter to reach the same quality of the solutions as the uniform DG-GN solver is also identified. The rest of the paper is organized as follows: In Section 2 we briefly recall the governing GN equations and in Section 3 the main ingredients of the uniform DG-GN solver are introduced. Section 4 describes the main ideas behind the MRA, and Section 5 explains the details of the MWDG-GN solver. Section 6 presents a series of numerical experiments that demonstrate the efficiency of the MWDG-GN solver. A summary of conclusions is presented in Section 7.

2- The Green-Naghdi (GN) equations

The one-dimensional (1D) GN system can be cast as the conventional NSW equations combined with source terms accounting for the dispersive effects, in the following conservative form[53]:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, z) = \mathbf{S}_\mathrm{b}(\mathbf{U}, z) - \mathbf{D}(\mathbf{U}, z) \tag{1}$$

$$\mathbf{U} = \begin{bmatrix} h \\ q \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}, z) = \begin{bmatrix} q \\ \frac{q^2}{h} + \frac{1}{2} g h^2 \end{bmatrix}, \quad \mathbf{S}_\mathrm{b}(\mathbf{U}, z) = \begin{bmatrix} 0 \\ -gh\partial_x z \end{bmatrix}, \quad \mathbf{D}(\mathbf{U}, z) = \begin{bmatrix} 0 \\ \mathcal{D}_c \end{bmatrix} \tag{2}$$

where $\mathbf{U}$ is the vector of flow variables i.e. water depth h and discharge $q = hu$, $\mathbf{F}$ represents the fluxes, z is the topography, g refers to the gravitational constant and $\mathbf{S}_\mathrm{b}$ is the topography source term. In this formulation, $\mathbf{D}$ denotes the dispersive source term, with $\mathcal{D}_c$ defined as

$$\mathcal{D}_c = -\frac{1}{\alpha} gh\partial_x\zeta + \left[1+\alpha\mathbb{T}[h_b]\right]^{-1}\left[\frac{1}{\alpha} gh\partial_x\zeta + h\left(\mathcal{Q}_1(u)+g\mathcal{Q}_2(\zeta)\right) + g\mathcal{Q}_3\left(\left[1+\alpha\mathbb{T}[h_b]\right]^{-1}(gh\partial_x\zeta)\right)\right] \tag{3}$$

where $u(x,t)$ is the horizontal velocity, h_b corresponds to the undisturbed state, $h(x,t)=\zeta(x,t)+h_b$ is the water height, $\zeta(x,t)$ stands for the free-surface elevation and $z(x)$ is the variation of the bottom with respect to the rest state (Fig. 1), and α is an optimization parameter[53]. The differential operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are expressed as follows:

$$\mathcal{Q}_1(u) = 2h\partial_x h(\partial_x u)^2 + \frac{4}{3}h^2\partial_x u(\partial_x^2 u) + h\partial_x z(\partial_x u)^2 + uh\partial_x u(\partial_x^2 z) + u^2\partial_x\zeta(\partial_x^2 z) + \frac{h}{2}u^2(\partial_x^3 z) \tag{4}$$

$$\mathcal{Q}_2(\zeta) = -\left(\partial_x\zeta\partial_x z + \frac{h}{2}\partial_x^2 z\right)\partial_x\zeta \tag{5}$$

For a given scalar function w, the second-order differential operator $\mathbb{T}$ is defined as:

$$\mathbb{T}[h_b](w) = -\frac{h_b^3}{3}\partial_x^2\left(\frac{w}{h_b}\right) - h_b^2\partial_x h_b\partial_x\left(\frac{w}{h_b}\right) \tag{6}$$

and $\mathcal{Q}_3$ admits the simplified notation:

$$\mathcal{Q}_3(w) = \frac{1}{6}\partial_x\left(h^2-h_b^2\right)\partial_x w + \frac{h^2-h_b^2}{3}\partial_x^2 w - \frac{1}{6}\partial_x^2\left(h^2-h_b^2\right)w \tag{7}$$

3- The uniform DG-GN model

The 1D computational domain $\Omega=[x_{\min},x_{\max}]$ is divided into N uniform and non-overlapping cells $\{I_i\}_{i=1,\dots,N}$ with cell $I_i=\left[x_{i-1/2},x_{i+1/2}\right]$ having size $\Delta x = x_{i+1/2}-x_{i-1/2}$ and centre $\Delta x = \left(x_{i+1/2}+x_{i-1/2}\right)/2$. Eq. (1) is approximated with a modal DG discretization with polynomials of degree p. Let V_p be

$$V_p = \left\{v\in L^2(\Omega): v|_{I_i}\in\Pi_p(I_i), i=1,\dots,N\right\} \tag{8}$$

where $\Pi_p(I_i)$ is the space of polynomials of degree at most p on I_i. Here, Legendre polynomials will be used, define as (e.g. for $0 \leq l \leq 3$):

$$P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{1}{2}(3\xi^2 - 1), \quad P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi) \quad (\xi \in [-1,1]) \tag{9}$$

which are compactly-supported on $[-1,1]$, inherently discontinuous, and orthogonal for the L^2-norm based on the following inner product:

$$\langle f, g \rangle_\Omega = \int_\Omega f(\xi) g(\xi) d\xi \tag{10}$$

The L^2-orthonormal basis $\varphi_l(\xi)$ can be defined by normalizing $P_l(\xi)$ for the L^2-norm such that $\langle \varphi_l, \varphi_{l'} \rangle_{L^2(\Omega)} = \delta_{ll'}$, where δ is the Kronecker delta. Since the reference domain spans $[-1, 1]$, the orthonormal basis is[54]:

$$\varphi_l(\xi) = \sqrt{\frac{2l+1}{2}} P_l(\xi) \quad (\xi \in [-1,1]) \tag{11}$$

Accordingly, two sets of basis functions will be defined over I_i: the primal basis $\Phi_i = \{\varphi_{i,0}, \varphi_{i,1}, \dots, \varphi_{i,p}\}$, and the dual basis $\tilde{\Phi}_i = \{\tilde{\varphi}_{i,0}, \tilde{\varphi}_{i,1}, \dots, \tilde{\varphi}_{i,p}\}$ [41]:

$$\varphi_{i,l}(x) = \sqrt{2}\varphi_l(\xi) \quad \text{and} \quad \tilde{\varphi}_{i,l}(x) = \frac{\varphi_{i,l}(x)}{\Delta x} \tag{12}$$

The primal and dual basis are chosen so that they are biorthogonal

$$\langle \varphi_{i,l}, \tilde{\varphi}_{i',l'} \rangle_{L^2(\Omega)} = \delta_{i,l}\delta_{i',l'} \tag{13}$$

To get an FE local weak formulation, Eq. (1) is multiplied by a test function selected as the dual basis $\tilde{\varphi}_{i,l}$, then integrated by parts over the control volume I_i to give:

$$\int_{I_i} \partial_t \mathbf{U}_\mathrm{h}(x,t)\tilde{\varphi}_{i,l}(x)dx - \int_{I_i} \mathbf{F}\big(\mathbf{U}_\mathrm{h}(x,t)\big)\partial_x\tilde{\varphi}_{i,l}(x)dx$$
$$+ \left[\tilde{\mathbf{F}}\left(\mathbf{U}_\mathrm{h}\big(x_{i+1/2},t\big)\right)\tilde{\varphi}_{i,l}\big(x_{i+1/2}\big) - \tilde{\mathbf{F}}\left(\mathbf{U}_\mathrm{h}\big(x_{i-1/2},t\big)\right)\tilde{\varphi}_{i,l}\big(x_{i-1/2}\big)\right] \tag{14}$$
$$= \int_{I_i} \mathbf{S}_\mathrm{b}(\mathbf{U}_\mathrm{h}(x,t),z_\mathrm{h})\tilde{\varphi}_{i,l}(x)dx - \int_{I_i} \mathbf{D}_\mathrm{h}(\mathbf{U}_\mathrm{h}(x,t),z_\mathrm{h})\,\tilde{\varphi}_{i,l}(x)dx$$

in which, $\mathbf{U}_\mathrm{h}$, $\mathbf{D}_\mathrm{h}$ and z_h are local approximations of $\mathbf{U}$, $\mathbf{D}$ and z, which are also spanned by FE expansion coefficients, and $\tilde{\mathbf{F}}$ is a nonlinear numerical flux function based on an HLL approximate Riemann solver[55]. On I_i the local solution can be expanded using the primal basis $\varphi_{i,l}$ as:

$$\mathbf{U}_\mathrm{h}(x,t)|_{I_i} = \sum_{l=0}^{p} \mathbf{U}_{i,l}(t)\varphi_{i,l}(x) \qquad (x \in I_i) \tag{15}$$

$$\mathbf{D}_\mathrm{h}(x,t)|_{I_i} = \sum_{l=0}^{p} \mathbf{D}_{i,l}(t)\varphi_{i,l}(x) \qquad (x \in I_i) \tag{16}$$

$$z_\mathrm{h}(x,t)|_{I_i} = \sum_{l=0}^{p} z_{i,l}(t)\varphi_{i,l}(x) \qquad (x \in I_i) \tag{17}$$

where $\mathbf{U}_{i,l}$, $\mathbf{D}_{i,l}$ and $z_{i,l}$ are time-dependent expansion coefficients. These initial states are obtained by projecting a given initial condition onto the dual basis. The local semi-discrete DG formulation for each l-th coefficient of polynomial accuracy over a cell I_i reads:

$$\partial_t\left(\mathbf{U}_{i,l}(t)\right) = -\frac{\sqrt{2l+1}}{\Delta x}\left\{\left[\tilde{\mathbf{F}}_{i+\frac{1}{2}} - (-1)^l\tilde{\mathbf{F}}_{i-\frac{1}{2}}\right] - \int_{-1}^{+1} \mathbf{F}\left(\mathbf{U}_h\left(x_i + \xi\frac{\Delta x}{2},t\right)\right)\left(\frac{\partial[P_l(\xi)]}{\partial\xi}\right)d\xi\right.$$
$$\left. - \int_{-1}^{+1} \mathbf{S_b}\left(\mathbf{U}_h\left(x_i + \xi\frac{\Delta x}{2},t\right),z_h\right)P_l(\xi)d\xi\right\} - \mathbf{D}_{i,l}(t) \tag{18}$$

Here, piecewise linear polynomial basis (i.e. $l = 0, 1$) are chosen, resulting in a second order DG scheme, hereafter called DG2. The local integral terms are computed by the two-point Gauss-Legendre rule and time integration is achieved by locally applying a 2-stage explicit RK time stepping scheme to solve the ODEs in Eq. (18) with a CFL number less than 1/3 for stability. In order to consistently discretize the higher order derivatives in dispersive terms, the so-called Local Discontinuous Galerkin (LDG)

approach[56] is used. The complete explanations regarding the DG solving procedure e.g. slope limiting, wetting/drying and solving the dispersive source terms can be found in Sharifian et al.[11].

4- Multi-resolution analysis

Considering the reference interval [-1,1], a hierarchy of nested grids, $\{I_j^n\}_{j=0,1,\dots,2^n-1}$ with increasing resolution $n = 0, 1, 2, \dots$ is defined by midpoint sub-division of the reference interval, i.e. $I_j^n = [-1 + 2^{-n+1}j, -1 + 2^{-n+1}(j+1)]$. On each sub-interval I_j^n at resolution n, any continuous function is approximated as a vector space V_p^n denoting the space of piecewise polynomial functions of degree at most p. The spaces V_p^n have degrees of freedom $2^n(p+1)$ and form a nested structure of closed subspaces (Fig. 2)

$$V_p^0 \subset V_p^1 \subset \cdots \subset V_p^n \subset \cdots \tag{19}$$

For the Legendre polynomials used in the DG method (Eqs. 15-17) $\Phi = \{\varphi_0, \varphi_1, \dots, \varphi_p\}$ consisting of $p+1$ functions spanning the space V_p^0 on [-1,1], it is possible to obtain the basis $\Phi_j^n = \{\varphi_{j,0}^n, \varphi_{j,1}^n, \dots, \varphi_{j,p}^n\}$ containing $2^n(p+1)$ functions, spanned over a sub-space V_p^n supported on I_j^n, by translation and dilation of Φ [42]:

$$\varphi_{jl}^n(x) = 2^{n/2}\varphi_l(2^n(x+1) - 2j - 1), \quad l = 0, \dots, p, \;\; j = 0, \dots, 2^n - 1, \;\; x \in I_j^n \tag{20}$$

in which j denotes the translation or shifting factor over sub-intervals $\{I_j^n\}_{j=0,1,\dots,2^n-1}$ and 2^n is the dilatation factor. Functions φ_l are called *scaling functions*. By considering the nested property (Eq. 19), the multiwavelet sub-space W_p^n can be defined as the orthogonal complement of V_p^n inside V_p^{n+1}, i.e.

$$V_p^n \oplus W_p^n = V_p^{n+1} \tag{21}$$

such that $V_p^n \perp W_p^n$ and $W_p^n \subset V_p^{n+1}$. The orthonormal basis W_p^0 comprises $p+1$ polynomials $\Psi = \{\psi_0, \psi_1, \dots, \psi_p\}$ defined on [-1,1], also known as *multiwavelet* Legendre polynomials (Fig. 3)[41,54,57]. Similarly, space W_p^n is spanned by functions $\Psi_j^n = \{\psi_{j,0}^n, \psi_{j,1}^n, \dots, \psi_{j,p}^n\}$, obtained by translation and dilation as

$$\psi_{jl}^{n}(x) = 2^{n/2}\psi_{l}(2^{n}(x+1) - 2j - 1), \quad l = 0, \dots, p, \quad j = 0, \dots, 2^{n} - 1, \quad x \in I_{j}^{n} \tag{22}$$

Using functions φ_{jl}^{n}, any arbitrary function $f \in L^{2}(-1, +1)$ can be reconstructed or decomposed across multiple scales of resolution. This is because by recursively applying Eq. (21), V_{p}^{n} can be decomposed into a single V_{p}^{0} space along with a sequence of W_{p}:

$$V_{p}^{n} = V_{p}^{0} \oplus W_{p}^{0} \oplus W_{p}^{1} \oplus \cdots \oplus W_{p}^{n-1} \tag{23}$$

The orthogonal projection of $f(x)$ onto V_{p}^{n} takes the following form:

$$P_{p}^{n} f(x) = \sum_{j=0}^{2^{n}-1} \sum_{l=0}^{p} s_{j,l}^{n} \, \varphi_{j,l}^{n}(x) \tag{24}$$

where P_{p}^{n} is the projection operator. Eq. (24) gives the so-called *single-scale decomposition* of the approximate solution on level *n*. The *single-scale* coefficients, $s_{j,l}^{n}$, can be derived from a L^{2} projection onto an orthonormal basis:

$$s_{j,l}^{n} = \langle f, \varphi_{j,l}^{n} \rangle = \int_{-1+2^{-n+1}(j)}^{-1+2^{-n+1}(j+1)} f(x) \varphi_{j,l}^{n} dx \tag{25}$$

Note that, for any $f \in V_{p}^{n}$, the following relation holds[42]:

$$P_{p}^{n} f = f \tag{26}$$

It is also possible to expand f by deploying multiwavelets as

$$Q_{p}^{n} f(x) = P_{p}^{n+1} f(x) - P_{p}^{n} f(x) = \sum_{j=0}^{2^{n}-1} \sum_{l=0}^{p} d_{j,l}^{n} \, \psi_{j,l}^{n}(x) \tag{27}$$

where the detail coefficients are obtained from

$$d_{j,l}^{n} = \langle f, \psi_{j,l}^{n} \rangle = \int_{-1+2^{-n+1}(j)}^{-1+2^{-n+1}(j+1)} f(x) \psi_{j,l}^{n} dx \tag{28}$$

Recursive use of Eq. (27), leads to *multi-scale decomposition* of *f* on level *n*

$$P_p^n f(x) = P_p^{n-1} f(x) + Q_p^{n-1} f(x) = P_p^{n-2} f(x) + Q_p^{n-2} f(x) + Q_p^{n-1} f(x) = \cdots$$
$$= P_p^0 f(x) + \sum_{m=0}^{n-1} Q_p^m f(x) = \sum_{l=0}^{p} s_{0,l}^0 \varphi_l(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{l=0}^{p} d_{j,l}^m \psi_{j,l}^m(x) \quad (29)$$

The single scale coefficients $\{s_{0,l}^0\}_{l=0}^p$ represent the information on the coarsest level $m = 0$, while detail coefficients $\{d_{j,l}^m\}$ carry multi-scale information, or fluctuations of the solution which, if added to the lowest-resolution information, enrich it up to level n of resolution[48].

4-1- Two-scale transformation for down- and up-scaling local information

In order to reconstruct or decompose the local solution expansion between two successive resolution levels, a two-scale transformation can be derived. Without loss of generality, the two-scale transformation is considered between levels $m = 0$ and $m = 1$. The so-called Quadrature Mirror Filter (QMF) coefficients will be used in decomposition and reconstruction steps, which are of two types[47]: low-pass filter coefficients (derived from scaling functions), and high-pass filter coefficients (derived from multiwavelet functions). The low-pass filter coefficients are defined as $h_{l,r}^j = \langle \varphi_l, \varphi_{j,r}^1 \rangle$ ($j = 0, 1$; $r = 0, \dots, p$). Considering $h_{l,r}^0$ we will have:

$$h_{l,r}^0 = \langle \varphi_l, \varphi_{0,r}^1 \rangle = \int_{-1}^{+1} \varphi_l(x) \varphi_{0,r}^1(x) dx = \sqrt{2} \int_{-1}^{0} \varphi_l(x) \varphi_r(2x+1) dx \quad (30)$$

in which $x \in [-1,0]$ comes from the fact that $\varphi_r(2x+1)$ is nonzero only if $(2x+1) \in [-1,+1]$. Accordingly, by changing the variables the following holds:

$$h_{l,r}^0 = \langle \varphi_l, \varphi_{0,r}^1 \rangle = \sqrt{2} \int_{-1}^{0} \varphi_l(x) \varphi_r(2x+1) dx$$
$$= \frac{1}{\sqrt{2}} \int_{-1}^{+1} \varphi_l\left(\frac{y-1}{2}\right) \varphi_r(y) dy \approx \frac{1}{\sqrt{2}} \sum_{G=0}^{p} w_G \varphi_l\left(\frac{\hat{x}_G - 1}{2}\right) \varphi_r(\hat{x}_G) \quad (31)$$

where the Gauss-Legendre quadrature rules are deployed to compute the integral. Similarly, for $h_{l,r}^1$

$$h_{l,r}^1 = \langle \varphi_l, \varphi_{0,r}^1 \rangle = \sqrt{2} \int_{0}^{1} \varphi_l(x) \varphi_r(2x-1) dx \quad (32)$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^{+1} \varphi_l\left(\frac{y+1}{2}\right) \varphi_r(y) dy \approx \frac{1}{\sqrt{2}} \sum_{G=0}^{p} w_G \varphi_l\left(\frac{\hat{x}_G + 1}{2}\right) \varphi_r(\hat{x}_G)$$

Based on multiwavelet functions $\psi_l \in W_p^0,\ l = 0, \dots, p$, the relation $W_p^0 \subset W_p^1$ leads to $\varphi_l \in W_p^1$.

Therefore, following the same procedure, the *high-pass filter coefficients* will be derived as:

$$g_{l,r}^0 \approx \frac{1}{\sqrt{2}} \sum_{G=0}^{p} w_G \psi_l\left(\frac{\hat{x}_G - 1}{2}\right) \varphi_r(\hat{x}_G) \tag{33}$$

$$g_{l,r}^1 \approx \frac{1}{\sqrt{2}} \sum_{G=0}^{p} w_G \psi_l\left(\frac{\hat{x}_G + 1}{2}\right) \varphi_r(\hat{x}_G) \tag{34}$$

Now, in order to define the multiwavelet decomposition, Eqs. (20) and (25) will result in

$$s_{j,l}^{n-1} = \langle f, \varphi_{j,l}^{n-1} \rangle = \sum_{r=0}^{p} \left(h_{l,r}^0 s_{2j,r}^n + h_{l,r}^1 s_{2j+1,r}^n \right) \tag{35}$$

and in the same manner,

$$d_{j,l}^{n-1} = \sum_{r=0}^{p} \left(g_{l,r}^0 s_{2j,r}^n + g_{l,r}^1 s_{2j+1,r}^n \right) \tag{36}$$

in which $l = 0, \dots, p$, $j = 0, \dots, 2^{n-1} - 1$. By forming matrices

$$\mathbf{H}_b = \left\{ h_{i,l}^b \right\}, \qquad \mathbf{G}_b = \left\{ g_{i,l}^b \right\}, \qquad b = 0, 1; \quad i, l = 0, \dots, p \tag{37}$$

and introducing the following vectors (for $j = 0, \dots, 2^n - 1$)

$$\begin{aligned} \mathbf{s}_j^n &= \left(s_{j,0}^n \quad \dots \quad s_{j,p}^n \right)^{\mathrm{T}} \\ \mathbf{d}_j^n &= \left(d_{j,0}^n \quad \dots \quad d_{j,p}^n \right)^{\mathrm{T}} \end{aligned} \tag{38}$$

the decomposition relations in Eqs. (35-36) can be reformulated as[54]:

$$\mathbf{s}_j^{n-1} = \mathbf{H}_0 \mathbf{s}_{2j}^n + \mathbf{H}_1 \mathbf{s}_{2j+1}^n \tag{39}$$

$$\mathbf{d}_j^{n-1} = \mathbf{G}_0 \mathbf{s}_{2j}^n + \mathbf{G}_1 \mathbf{s}_{2j+1}^n \tag{40}$$

Now, left-multiplying Eq. (39) by $\mathbf{H}_0^{\mathrm{T}}$ and Eq. (40) by $\mathbf{G}_0^{\mathrm{T}}$, then summing them would result in

$$\mathbf{H}_0^{\mathrm{T}}\mathbf{s}_j^{n-1} + \mathbf{G}_0^{\mathrm{T}}\mathbf{d}_j^{n-1} = \mathbf{s}_{2j}^{n} \tag{41}$$

and in the same way, multiplication by $\mathbf{H}_1^{\mathrm{T}}$ and $\mathbf{G}_1^{\mathrm{T}}$ leads to

$$\mathbf{H}_1^{\mathrm{T}}\mathbf{s}_j^{n-1} + \mathbf{G}_1^{\mathrm{T}}\mathbf{d}_j^{n-1} = \mathbf{s}_{2j+1}^{n} \tag{42}$$

In summary, Eqs. (39-40) and (41-42) define decomposition (also called multi-scale transformation) and reconstruction (also called inverse multi-scale transformation) formulas, respectively.

5- The Adaptive MWDG-GN model

In order to combine the DG-GN solver with the MW-based grid adaptation, the multi-resolution analysis introduced in Section 4 is applied to each cell I_i of the baseline grid. The DG formulation of multi-resolution scheme follows the same procedure as the non-adaptive case (Section 3), however, in the adaptive framework the computational domain would be a heterogeneous grid comprised of selectively chosen resolution levels of the grid hierarchy (see Section 5-2), on which the time evolution is actually performed.

5-1- Local multi-scale DG formulation

Therefore, each cell $I_i = \left[x_{i-1/2}, x_{\mathrm{i}+1/2}\right]$ is recursively subdivided into 2^n sub-intervals $\left\{I_{i,j}^n\right\}_{j=0,1,\dots,2^n-1}$ in a way that each cell $I_{i,j}^n = \left[x_{i-1/2} + \Delta x^{(n)} j, x_{i-1/2} + \Delta x^{(n)}(j+1)\right]$ would have the local resolution-dependent size of $\Delta x^{(n)} = 2^{-n}\Delta x$, centred by $x_{i,j}^n = x_{i-1/2} + \Delta x^{(n)}(j+1/2)$. In this notation, the sub-index i is introduced for referring to the respective baseline cell. Also, since DG approximations and scaling function expansions are composed of the same basis functions, there is a direct relation between them. By considering the baseline cell $I_i = [a, b]$ and denoting $\Delta x^{(n)} = (b-a)/2^n$ as the mesh size on level n, and $x_j^n = a + (j + \frac{1}{2})\Delta x^{(n)}$ as the centre of cell I_j^n, using Eq. (20) the global DG approximation of the solution on the domain can be expanded as:

$$\mathbf{U}_{\mathrm{h}}(x,t)=\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}\mathbf{U}_{j,l}^{(n)}(t)\varphi_l(\xi)=\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}\mathbf{U}_{j,l}^{(n)}(t)\varphi_l\left(\frac{2}{\Delta x^{(n)}}\left(x-x_j^n\right)\right)$$

$$=\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}\mathbf{U}_{j,l}^{(n)}(t)\varphi_l\left(\frac{2^{n+1}}{b-a}(x-a)-2j-1\right) \tag{43}$$

$$=\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}\mathbf{U}_{j,l}^{(n)}(t)\varphi_l(2^n(y+1)-2j-1)=2^{-\frac{n}{2}}\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}\mathbf{U}_{j,l}^{(n)}(t)\varphi_{j,l}^n(y)$$

in which $y=-1+2(x-a)/(b-a)$. On the other hand, over the reference domain [-1,1], based on properties of Eqs. (24) and (26), it holds that

$$\mathbf{U}_{\mathrm{h}}(x,t)=P_p^n\mathbf{U}_{\mathrm{h}}(x,t)=\sum_{j=0}^{2^n-1}\sum_{l=0}^{p}s_{j,l}^n\,\varphi_{j,l}^n(y) \tag{44}$$

Therefore, Eqs. (43) and (44) will lead to

$$2^{-\frac{n}{2}}\mathbf{U}_{j,l}^{(n)}=s_{j,l}^n \tag{45}$$

which gives the relation between DG and single-scale coefficients.

5-2- Resolution adaptivity

In order to select the appropriate resolution levels to form the adaptive grid, a selection process is applied on the multiwavelet coefficients resulting in a set of the significant details denoted by $SD\in\{(i,s,m),0\le i\le N,0\le s\le 2^m-1\}$, which will be used to determine those sub-cells across the different resolution levels that need to be active within the adaptive grid. In other words, the set of active cells $\left\{I_{i,s}^m\right\}_{(i,s,m)\in SD}$ will merge as $\cup_{(i,s,m)\in SD}\left\{I_{i,s}^m\right\}=[x_{\min},x_{\max}]$. To clarify this concept, if we express the local polynomial approximation of the flow vector $\mathbf{U}_{\mathrm{h}}$, over a cell $I_{i,s}$ in the form of the multi-scale decomposition up to a highest resolution n (Eq. 29) with coefficients $s_{i,0,l}^0$ and $d_{i,s,l}^m$ (index i refers to the baseline cell), respective detail coefficients at a resolution $m\le n$, i.e. $d_{i,s,l}^m$, would become increasingly smaller with increasing level of spatial resolutions, when the underlying function is smooth. In the opposite way, if discontinuities are present, the details usually stay significant for

increasing refinement level[41,46]. This property enables us to select the active cells by comparing the magnitudes of these details with a user-specified threshold value.

5-2-1- Thresholding and prediction

To apply the thresholding, a prescribed value ε will be defined by the user, based on which the level-dependent threshold value $\varepsilon_m = 2^{m-n}\varepsilon$ is introduced. The detail coefficients $d_{i,s,l}^m$ whose absolute values, scaled with the maximum value of the solution, fall below ε_m will be discarded from selection, i.e.

$$\hat{d}_{i,s,l}^m = \begin{cases} d_{i,s,l}^m & \text{if} \quad \max\limits_{\substack{l=0,\dots,p \\ i,s,m\in SD}} \left(\dfrac{\left|\left(d_{i,s,l}^m\right)_r\right|}{\max\left\{\max\limits_{i,s,m\in SD}\left|\left(\mathbf{U}_{i,s,0}^m\right)_r\right|, 1\right\}} \right) > \varepsilon_m \\ 0 & \text{else} \end{cases} \tag{46}$$

in which $\left(\mathbf{U}_{i,s,0}^m\right)_r$ is the average coefficient of the respective conserved quantity r, in cell $I_{i,s}^m$. This procedure is called *hard thresholding*. Also, since the flow field evolves in time, after each evolution step, adaptivity is performed to update the grid at the new time level. To guarantee that no significant future of the solution is lost at the new time level, a prediction step will be further applied on a selected set of significant details, in which the following constraints are considered[46,49]:

1- On account of the finite speed of propagation, the details in a local neighbourhood (in the same level) of a cell with significant detail may also become significant within one time step, and will be refined subsequently.

2- Formation of shocks may steepen the gradients, resulting in significant details on higher levels. Therefore, another constraint with the criterion $2^{\bar{M}+1}\varepsilon_m$ ($\bar{M} = p$ denotes the number of vanishing moments of the multiwavelets) is introduced, according to which the details at the higher level $m + 1$ will be set as significant.

3- The set of cells characterized with significant details possess a tree-like structure; i.e. if a cell in level m is detected significant, all its substructure cells on lower resolution levels are set as significant, regardless of the thresholding based on their respective details.

5-2-2- Adaptive MWDG-GN algorithm

In order to apply the adaptivity procedure, first the initialization is performed in the following steps:

1- The initial grid is formed by projection of the initial data on a fully refined grid at the finest level n (Fig. 4(a)).

2- The multi-scale transformations (Eqs. (39-40)) are applied to determine the detail coefficients on levels $m = 0, \dots, n-1$ (Fig. 4(b)).

3- Initial hard thresholding is applied to obtain the initial set of significant details (Fig. 4(c)).

After the initial significant details are determined, the main steps of the computations are performed in the following steps:

4- The prediction step is performed based on the available set of significant details.

5- The inverse multi-scale transformations (Eqs. (41-42)), will be recursively applied; proceeding level-wise from coarse to fine, in order to refine those cells flagged as significant from the previous steps, and also determines the respective DG coefficients (Fig. 4(d)). At the end of this step, the active cells which form the appropriate multi-scale adaptive grid are detected. The RKDG calculations will be performed over this grid.

6- The RKDG evolution is performed, following the same procedure as the non-adaptive RKDG. The only difference is the slope limiting. As previously stated, by choosing a suitable threshold value the grid is refined up to the finest level near discontinuities and if the solution is locally smooth, we expect the grid not to be refined up to the finest level in this region. This property has been used as an additional indicator for the limiting process such that the limiting process is only applied in cells on the finest level n.

7- The multi-scale transformations (Eqs. (39-40)) will be applied again for decomposition of the RK-updated solutions.

8- The new set of significant details is computed by hard thresholding.

Steps 4 to 8 will be performed in the main time loop of the computations.

5-2-3- Considerations regarding well-balancing

To justify the well-balancing property, some considerations are applied in the adaptivity process[48,49]. Since the topography, as opposed to the flow variables, does not evolve in time, a static (but not necessarily uniform) grid is considered for it. The corresponding set of significant details of the topography is then added in each time step to significant details of the flow variables as an additional constraint to the coarsening and the refinement procedure. Moreover, since depth is a poor indicator of regularity/complexity of the solution, in all the steps involved in the adaptivity process (except for RKDG evolution) the conserved variables vector must be rearranged as $\mathbf{U} = [h + z, q]^{\mathrm{T}}$, so that water surface elevation is analyzed by the multiresolution transformations, instead of flow depth.

6- Numerical results

The main idea behind the adaptive MWDG2 approach is to increase the computational efficiency of the reference DG2 scheme without losing accuracy. To do this, a choice of the threshold value ε is needed: a too large threshold would spoil the accuracy of the solution as a result of dominating additional error, while a too small threshold leads to over-refinement and inefficiency[41]. With NSW equations, a threshold value ranging between $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ is found to be enough for the adaptive MWDG2 solver to yield an appropriate balance between accuracy and efficiency[47,49]. For a wave with dispersive behaviour, the use of BT equations generally dictates finer resolutions to resolve more complex physical features of interest, and also higher computational costs compared to NSW equations. To find out the effects of these characteristics, the choice for a suitable range for the threshold value is here re-investigated for the GN solver for tests considering the propagation and transformation of solitary waves. The solitary waves can be considered as a balance between the nonlinearity and the frequency dispersion that maintains the permanent waveform. These waves have been generally used to model certain behaviours of nonlinear long waves, such as the leading wave of tsunamis and storm surges. Moreover, due to their locality, they are more likely to benefit from grid adaptation, compared to periodic waves. The first test provides an analytical investigation of the propagation of a solitary wave over a flat bottom, where convergence and thresholding criteria are analysed. The other test cases show the capability of the proposed MWDG-GN model compared to experimental data with wave transformation. The second test depicts the interaction of a solitary wave with a mild-slope beach and

297 the third test deals with the interaction of two solitary waves. In all the tests, the boundary conditions
298 are imposed based on solid wall, inflow and outflow boundary and the optimization parameter of the
299 GN equations is set to $\alpha = 1$. Regarding computational efficiency measurements, it should be noted
300 that all the simulations are performed on a 3.6 GHz Intel i7 quad-core processor.

301 **6-1- Propagation of a solitary wave**

302 To identify and analyse the properties of the MWDG-GN solver, the propagation of a solitary wave
303 over a flat bed is considered. The wave has a finite amplitude and permanent form resulting from the
304 balance between nonlinear and dispersive effects, and has an exact solution given by[3]

$$h(x,t) = h_0 + a\,\mathrm{sech}^2\left(\frac{\sqrt{3a}}{2h_0\sqrt{h_0+a}}(x-ct)\right)$$
$$u(x,t) = c\left(1-\frac{h_0}{h(x,t)}\right) \tag{47}$$

305 where h_0 is the initial local water depth, a the wave amplitude and $c = \sqrt{g(h_0+a)}$ the wave speed.
306 The solitary wave propagates in a 200 m long domain over a constant water depth of $h_0 = 1$ m, and its
307 wave crest is initially centred at $x_0 = 50$ m. This wave is moderately nonlinear with a relative amplitude
308 of $a/h_0 = 0.2$.

309 The adaptive MWDG-GN simulations are performed up to $t = 30$ s for a range of resolution
310 settings, where each setting is defined by the pair $\{N_b, L\}$ with N_b and L indicating the number of cells
311 of the baseline grid and the maximum refinement level, respectively. The settings are taken by fixing L
312 = 7 and considering different size for the baseline grid N_b = 1, 2, 3, 4 and 5, yielding grids with a
313 maximum of 128, 256, 384, 512 and 640 cells, respectively. On these fine uniform grids, simulations
314 using the DG-GN solver are also carried out to enable a relative comparison of accuracy and efficiency
315 for adaptive MWDG-GN solver over a range of threshold values between $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-2}$.

316 6.1.1 Choice of the threshold value with the adaptive MWDG-GN solver

317 The accuracy of the adaptive MWDG-GN scheme is measured using the normalized L^2 errors of
318 water height h and discharge hu based on the following formula

$$L^2{}_{error} = \sqrt{\frac{(U-U_T)^2 \Delta x_L}{{U_T}^2 \Delta x_L}} \tag{48}$$

where U_T is the analytical solution based on Eq. (47) and Δx_L denotes the grid size on the finest level of resolution. Fig. 5 shows the water depth (Fig. 5(a)) and flow discharge (Fig. (5b)) errors computed at $t = 5$ s with both MWDG-GN and DG-GN solvers for all the settings except the coarsest one with {1,7}, which was not included to save space. With increasingly finer resolution of the uniform DG-GN solver, the adaptive MWDG-GN solver requires increasingly smaller threshold values to keep the same error magnitudes. As compared to a NSW solver for numerical modelling of non-dispersive flows, a GN solver necessitates finer grid resolution to ensure capturing both dispersive and nonlinear features[11]. This implies that the adaptive MWDG-GN solver would require smaller threshold values compared to an adaptive MWDG-NSW solver[47,49] to accommodate finer resolution needs. For this test where Δx_L is near 10^{-1}, settings {3,7}-{5,7} are identified as appropriate for the adaptive MWDG-GN solver in combination with threshold values in the neighbourhood of $\varepsilon = 10^{-4}$. This seems to suggest a threshold value ε that is at least 2 to 3 orders of magnitude smaller than Δx_L to meet the uniform resolution accuracy required for an equivalent DG-GN solver, in line with an increase in relative wave amplitude a/h_0.

To evaluate efficiency of the adaptive MWDG-GN solver with reference to the same range of threshold values, its compression rate (decrease in the number of cells due to use of wavelet adaptivity, in percent) and speed up ratio (CPU time ratio of uniform to adaptive solvers) are measured after completing the full 30 s simulation. Fig. 6a and Fig. 6b show speedup and compression rate against the threshold values for settings {3,7}-{5,7}, both showing an increase in speed up ratio and compression rate with decreasing threshold values. In the neighbourhood of $\varepsilon = 10^{-4}$, the observed speedup ratio with setting {5,7} shows maximum efficiency (around 30 times) while setting {3,7} shows minimum efficiency (around 18 times). Nonetheless, compression rates are noted to be consistently closer, in the range of 75-80%. These suggest that more costs are entailed due to wavelet adaptivity overhead with decreasing size of the baseline mesh. Overall, a threshold value around $\varepsilon = 10^{-4}$ is an appropriate choice in this test for the adaptive MWDG-GN solver to preserve the accuracy of an equivalent DG-

GN solver on the finest uniform resolution accessible to the adaptive MWDG-GN solver, while being up to 30 times more efficient to run.

6.1.2 Mesh convergence analysis of accuracy and efficiency

To quantify the extent to which the adaptive MWDG-GN solver converges to the uniform DG-GN solver, an error convergence analysis is performed considering both accuracy and efficiency. Accuracy convergence is evaluated by plotting the L^2 errors of water height (Fig. 7(a)) and discharge (Fig. 7(b)) against the finest grid sizes corresponding to settings $\{N_b, L\}$ from $\{1,7\}$ to $\{5,7\}$ and computed at time $t = 5$ s. The uniform DG-GN solver delivers optimal convergence rates in the order of 2.5, and the adaptive MWDG-GN solver is observed to converge to the same asymptotic behaviour of the uniform solver showing slightly larger errors with coarsening in grid resolution. In terms of efficiency, the same L^2 errors are re-examined but with respect to the maximum number of cells entailed in the adaptive MWDG-GN and the uniform DG-GN solvers at the same output time $t = 5$ s (Fig. 8). As can be seen from Fig. 8, the rate of efficiency convergence of the adaptive MWDG-GN model is much faster than the uniform DG-GN solver in terms of yielding errors of the same order but with considerably fewer cells.

6.1.3 Qualitative comparisons and analysis of refinement levels

The predicted numerical profile of the solitary wave at different instants using the adaptive MWDG-GN model with the two settings $\{3,7\}$ and $\{5,7\}$ are compared with the exact solution and the predictions associated with their equivalent uniform DG-GN solvers (see Fig. 9). In order to distinguish possible difference among the adaptive and uniform solver predictions, only zoom-in portions near the wave crest are plotted in the sub-figures forming Fig. 9. For setting $\{5,7\}$, which allows up to a maximum of 640 cells, the adaptive MWDG-GN and uniform DG-GN solver predictions are seen to provide the best agreement with the exact solution throughout the 30 s simulation. For setting $\{3,7\}$, the maximum number of cells within the adaptive and uniform solvers is roughly halved, which is probably the main reason why these solvers consistently provided slightly poorer agreement with the exact solution. In particular, by $t = 20$ s (Fig. 9c), discrepancies become clearly visible and eventually

intensify, by t = 30 s (Fig. 9d), to form small amplitude (unphysical) tails. This deficiency detected in setting {3,7} could be indicating that the finest grid resolution allowed by this setting, i.e. $\Delta x_L = 0.52\ m$, may not be enough for the uniform DG-GN to fully capture the wave nonlinearities at a relative wave amplitude close to $a/h_0 = 0.2$ [11], hence for the adaptive MWDG-GN solvers too.

To analyse resolution prediction ability of adaptive MWDG-GN solver, Fig. 10 illustrates the associated spatial refinement levels in line with the free-surface elevations over the full domain, considered at the same output times as in Fig. 9. For both {5,7} and {3,7} settings, the adaptive MWDG-GN solver is found to favourably select the finest resolution (i.e. at the maximum level L = 7) around the crest of the solitary wave. Therein, a wider extent of fine resolution prediction (i.e. so-called over-refinement) is observed with setting {3,7} than with setting {5,7}, and this can be attributed to the aforementioned discrepancies in terms of small amplitude tails at t = 20 s and t = 30 s that could have exaggerated wavelet coefficients, thereby causing spurious over-refinement. In the regions of quiescent flow, the adaptive MWDG-GN solver selects the coarsest resolution (i.e. levels L = 1 to 2) in both settings.

To further analyse the efficiency of the MWDG-GN solver over the full 30 s simulation, its instantaneous number of cells have been recorded. Fig. 11 shows the time variation of the number of cells used by the adaptive solver for both settings {5,7} and {3,7}, as well as the constant cell numbers entailed in their associated DG-GN solvers. As previously shown in Fig. 6, the setting {3,7} is found less efficient than the finer setting {5,7} despite having a compression rate of the same order. Fig. 11 further shows that these similar compression rates hold during the full length of the simulation, with the number of cells in the adaptive grid being about 15-18% of the number of cells in the uniform counterparts, for both settings {3,7} and {5,7}. The relative decrease in efficiency with setting {3,7} is likely to have been caused by deficiencies observed previously in Fig. 10 (i.e. the trailing unphysical fluctuations), which lead to over-refinements therein, in turn causing extra costs associated with wavelet adaptivity overhead. In can be therefore concluded from this test, that choosing a right setting, even if based on finer resolution, is central to meet both accuracy and efficiency needs within the adaptive MWDG-GN solver.

6-2- Run-up and run-down of a solitary wave over a sloped beach

To examine the performance of the adaptive MWDG-GN model in dealing with wet/dry fronts and topography, it is also tested for solitary wave run up on a sloping beach supported by the experimental work of Synolakis[58]. The computational domain consists of a channel with the initial free-surface elevation of 1 m approaching a sloped beach (1:19.85). In order for the solitary wave to be initially located in the channel, the computational domain is extended to 77 m, which is longer than the actual experiments (Fig. 12). Here, a solitary wave with a non-breaking wave with weak nonlinearity is selected, which has a lower relative amplitude than in test 6.1 ($a/h_0 = 0.019$). Based on the domain size, the adaptive MWDG-GN model is run for $t = 25\ s$ using a setting {4,7}, which allows up to 512 cells ($\Delta x_L = 0.15$ m). The uniform DG-GN simulation is also run on the finest resolution grid to allow for a relative comparison. Following the observations in test 6.1, the following three threshold values are selected and tested with the adaptive MWDG-DG solver: $\varepsilon = 10^{-3}, \varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$ (i.e. being 2-3 orders of magnitude smaller than Δx_L and in the neighbourhood of 10^{-4}).

The numerical free-surface elevation profiles produced by both MWDG-GN and DG-GN solvers at different (normalized) output times $t^* = t(g/h_0)^{1/2}$ are compared in Fig. 13 with reference to the experimental profiles of Synolakis[58]. The wave profiles computed by the adaptive MWDG-GN solver closely match the profiles computed by the uniform DG-GN on the fine grid, while remaining in a good agreement with the experimental data during the run-up and run-down phases. Using smaller threshold values ($\varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$) only makes improvements in certain areas of the flow, e.g. the wet/dry front at run-up phase at $t^* = 45$. In other areas, the adaptive MWDG-GN solver predictions based on the largest threshold value ($\varepsilon = 10^{-3}$) found similar to those relative to the smallest threshold ($\varepsilon = 10^{-4}$). With $\varepsilon = 10^{-3}$, the MWDG-GN solver used lower resolution levels while preserving close predictive accuracy even with smaller ε, and hence is here the most efficient option. Outside the vicinity of the wet/dry fronts, during the run-up and run-down phases, the adaptive MWDG-GN solver predicted relatively coarse-to-moderate resolution levels, varying between $L = 2$ to 4. For this test, all adaptive MWDG-GN solvers did not excessively use the finest resolution level around the wave crest

423 barely. This can be attributed to the relatively weak nonlinearity of the solitary wave in contrary to the
424 moderately nonlinear wave explored in test 6.1.

425 To analyse the efficiency of the adaptive MWDG-GN solver in relation to the choice of the
426 threshold value, the time evolution of their mesh size is plotted in Fig 14, which also contains the size
427 of the uniform grid relative to the DG-GN. This figure reinforces that $\varepsilon = 10^{-3}$ provides the most
428 efficient option with the adaptive MWDG-GN solver: it consistently activated around 8% (i.e. 45 cells)
429 of the cells accessible to it, while delivering predictions as close as the other adaptive MWDG-GN
430 solvers and the DG-GN simulation using 512 cells. With $\varepsilon = 5 \times 10^{-4}$ and $\varepsilon = 10^{-4}$, the adaptive
431 MWDG-GN is seen to activate higher cell percentage. However, the percentage of active cells required
432 did not exceed 18% in this test, even at $\varepsilon = 10^{-4}$. In terms of speedup, it is found between 30 to 55 in
433 this test, which is expected given the weak magnitude of wave nonlinearity and less dispersive effects
434 as compared to test 6.1.

435 6-3- Head-On Collision of Two Solitary Waves

436 A final test is introduced to study the behaviour of the adaptive MWDG-GN solver, when there are
437 more than one solitary wave propagating, each featured by a higher relative amplitude ($a/h_0 > 0.2$).
438 Therefore, the experimental test of Craig et al.[59] is selected as it involves the head-on collision of two
439 solitary waves propagating in opposite directions. This problem is characterised by the change of the
440 shape as well as a small phase-shift of the waves as a consequence of the nonlinearity and dispersion.
441 The setup consists of a 3.6 m long flume with still water depth of $h_0 = 5\ cm$. The left wave with an
442 amplitude of $a_1 = 1.063\ cm$ is initially located at $x = 0.5\ m$ while the right one is initially located at
443 $x = 3.1\ m$ with an amplitude of $a_2 = 1.217\ cm$. These values result in relative amplitudes a/h_0 equal
444 to 0.212 and 0.243, for the left and right waves respectively. Each of these solitary waves can be
445 considered to have moderate-to-high nonlinearity and are expected to cause an even higher nonlinearity
446 at the instant when they merge into a bigger solitary wave.

447 Adaptive MWDG-GN simulations are performed up to $t = 2.5\ s$ and based on setting {3,7} that
448 permits a maximum of 384 cells. As the flume experiment is 3.6 m long, adopting this setting means

449 the adaptive MWDG-GN solver can access a resolution as fine as $\Delta x_L = 0.0093\, m$. As before, the
450 uniform DG-GN model is also run on the grid using the finest level of resolution. For the threshold
451 value parameter, $\varepsilon = 10^{-6}$ is selected for this test informed by the analysis of test 6.1 (see Sec.6.1.1).
452 The spatial evolution of the solitary waves simulated by the adaptive MWDG-GN solver and the
453 uniform DG-GN counterpart at different output times are shown in Fig. 15, as well as experimental
454 profiles[59]. At the instant of head-on collision (around $t = 1.693\ s$), the wave amplitude reaches around
455 to a level larger than the sum of the amplitudes of the two incident solitary waves (equivalent of a ratio
456 $a/h_0 = 0.5$). After the collision (around $t = 1.865\ s$), two waves come out with reduced amplitudes,
457 returning to their initial form. As an outcome of this collision (during $t = 1.693$ to $1.824\ s$), the two
458 waves lose momentum, which results in lower amplitudes (compared to the initial values) and a phase
459 lag.

460 In terms of free-surface elevation predictions, it can be seen from Fig. 15 that the predictions
461 produced by the adaptive and uniform solvers are almost identical, both providing a close agreement
462 with experimental profiles at all output times. Due to the high nonlinearity of the two waves and the
463 strong interactions between them, the adaptive MWDG-GN solver needed resolution levels that are
464 higher compared to the previous test cases (Sec. 6.1 and 6.2). Fine resolution levels are also observed
465 in areas away from the wave crests, with at least $L = 4$. This is also evident in Fig. 16, which shows the
466 time evolution of the number of cells in adaptive and uniform schemes. Initially, the adaptive MWDG-
467 GN solver activated 57% of cells with an increasing trend with the propagation of the two solitary
468 waves, reaching a final percentage of 85%. As a consequence of these low compression rates, the final
469 speedup is here only equal to 1.6, suggesting that the adaptive MWDG-GN solver may not be ideal for
470 the problems with poor locality such as involving multiple waveforms or periodic waves.

471 **7- Conclusions**

472 In this work, we applied a multiwavelet-based grid adaptation technique to Green-Naghdi (GN)
473 equations. This is achieved by extending a previously developed uniform mesh Discontinuous Galerkin
474 (DG) solver to the GN equations (DG-GN) from an adaptive Multiwavelet-based DG (MWDG) method

for the NSW equations (MWDG-NSW). The performance of the MWDG-GN solver is demonstrated by several benchmark tests. The adaptive solver is shown to provide a robust method for driving grid adaptation, with the adaptivity being controlled only by a single threshold value and with inherent error control. For the threshold parameter, it has been verified that choosing a value between 2 to 3 orders of magnitude smaller than the size of the grid cells on the finest level of resolution, would result in an optimal combination of efficiency and accuracy to resolve small scale features of dispersive wave propagations. Therefore, the same accuracy as the uniform DG-GN solver can be achieved by the adaptive MWDG-GN solver, but with significantly fewer cells. For the case of single solitary waves, compression rates of at least 80% and speedups around 30 are achieved. It is also found that the efficiency gain of using grid adaptation depends on the amount of nonlinear and dispersive effects. Accordingly, the best performance of the proposed solver MWDG-GN solver is sought to be in case of moderate nonlinearity and dispersion. The 2D extension of the present MWDG scheme is the subject of future work.

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