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SHOCK STRUCTURES DESCRIBED BY HYPERBOLIC BALANCE LAWS*

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Abstract. In this paper we consider shock structures that arise in systems of hyperbolic balance laws, i.e., hyperbolic systems of conservation laws with source terms. We show how the Whitham criterion for the existence of such shock structures can be extended to systems with more than one relaxation variable. In addition, we develop a method that is useful for determining the stability of the equilibrium states of such systems. The utility of this method is illustrated by a number of examples.

Key words. shock structure, system of balance laws, stability

AMS subject classifications. 35Q35, 35Q85

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1. Introduction.

There are many physical systems that can be described by hyperbolic conservation laws with the addition of source terms that represent relaxation processes. Such systems are usually called hyperbolic balance laws. Whitham [17] gives a number of examples: flood waves, chromatography, magnetohydrodynamics, relaxation effects in gases, etc. He shows that the nature of the solutions can be understood by considering two systems: a frozen system that is valid when the wavelengths are so short that the source terms are negligible and an equilibrium system for wavelengths that are long enough for the source terms to be in equilibrium. Both these systems are described by hyperbolic conservation laws without source terms. Whitham's analysis was mainly concerned with the conditions required for a shock of the equilibrium system to have a smooth structure determined by the source terms of the full system.

The theory developed by Whitham only considered a single relaxation process in the sense that the equilibrium system has one less dependent variable than the frozen system. However, since there are many physical systems in which there are several relaxation processes, it would obviously be useful to extend the theory to such cases. Here we show that this can be done and that many of Whitham's ideas still apply. In particular, his method of determining the stability of equilibrium states can be extended, although the analysis is somewhat more complicated.

In his analysis Whitham exploited the connection between the conditions for the stability of the equilibrium state and the existence of smooth shock structures. More recent work (e.g., Chen, Levermore, and Liu [3]; Boillat and Ruggeri [1, 2]; Kawashima and Yong [11]; Yong [18]) has been based on the existence of a strictly convex entropy function for the full system. Although there are many systems that have this property, there are some, such as multidimensional elastodynamics, that do not (Dafermos [5]). However, if such a system also has an extra conservation law, called an involution, that holds for any solution for which it is true for the initial data,

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then the entropy satisfies a weaker notion of convexity. This guarantees hyperbolicity and the uniqueness of the Cauchy problem, which suggests that many of the results that depend on a strictly convex entropy also hold for these systems. In this paper we adopt a different approach, which is more akin to that of Liubarskii [12] in that it uses the Hermite–Biehler theorem to study the stability of the equilibrium system and does not make direct use of the existence of a convex entropy. Whereas arguments that depend on such an entropy yield general results, our method is more useful for analyzing particular systems, as we show by applying it to a number of examples.

2. Hyperbolic balance laws.

Consider a system of hyperbolic balance laws in $1+1$ dimensions that can be written in the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u}), \tag{2.1}$$

where $\mathbf{u} = (u_1, \cdots u_n)^t$ are a set of n conserved quantities, $\mathbf{f}(\mathbf{u}) = (f_1, \cdots f_n)^t$ are the associated fluxes, and $\mathbf{s}(\mathbf{u}) = (s_1, \cdots s_n)^t$ are source terms depending upon $\mathbf{u}$. Here the superfix t denotes the transpose.

It is standard practice to assume that the source term is of the form

$$\mathbf{s}(\mathbf{u}) = (s_1(\mathbf{u}), \cdots s_r(\mathbf{u}), 0, \cdots 0)^t \tag{2.2}$$

with $1 \leq r < n$ (e.g., Liubarskii [12]; Chen, Levermore, and Liu [3]; Boillat and Ruggeri [2]; Yong [18]). This is not too restrictive an assumption since there are many physical systems of this form.

2.1. Frozen system.

For sufficiently short wavelengths, the source terms become negligible and (2.1) reduces to

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = 0, \tag{2.3}$$

which constitutes a set of conservation laws for $u_1, \cdots u_n$. In regions where $\mathbf{u}(x,t)$ is differentiable, (2.3) can be written as

$$\partial_t \mathbf{u} + \mathsf{J} \partial_x \mathbf{u} = 0,$$

where

$$\mathsf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

is the Jacobian of $\mathbf{f}$ w.r.t. $\mathbf{u}$. We shall suppose that the system is hyperbolic, i.e., the matrix J has a complete set of linearly independent eigenvectors and real eigenvalues (not necessarily distinct).

2.2. Equilibrium system.

For sufficiently large wavelengths, the source term dominates and $\mathbf{u}$ is restricted to the equilibrium manifold $\mathbf{s}(\mathbf{u}) = 0$ (so long as this solution is stable). Let this manifold have dimension $n-r$ with $1 \leq r < n$. We assume that this manifold may be parameterized by a set of $n-r$ variables, $\mathbf{u}_e$, which can be chosen to be the last $n-r$ elements of $\mathbf{u}$. The relations, $\mathbf{u} = \mathbf{u}(\mathbf{u}_e)$ can then be substituted into equations (2.1) to give the evolution equations for this equilibrium system. The condition (2.2) ensures that this is a system of conservation laws. The equilibrium equations are therefore of the form

$$\partial_t \mathbf{u}_e + \partial_x \mathbf{f}_e(\mathbf{u}_e) = 0. \tag{2.4}$$

We shall also assume that this is a hyperbolic system so that the equilibrium Jacobian,

$$\mathsf{J}_e = \frac{\partial \mathbf{f}_e}{\partial \mathbf{u}_e},$$

also has real eigenvalues and a complete set of linearly independent eigenvectors. Chen, Levermore, and Liu [3] have shown that this is true provided the full system has a strictly convex entropy function. As we have already pointed out, Dafermos [5] has shown that this is also true for systems with a nonconvex entropy provided they possess an involution.

3. Shock structures. If the initial data satisfies $\mathbf{s}(\mathbf{u}) = 0$ and the length scale over which it varies is sufficiently large, then the solution will stay on the equilibrium manifold and satisfy equations (2.4) as long as the length scale of the solution remains large enough. However, if equations (2.4) are nonlinear, then their solutions will develop shocks for generic initial data and will then be described by the full equations, (2.1). The question then arises as to the conditions which determine whether these shocks have a smooth structure or contain subshocks, i.e., shocks of the frozen system, (2.3).

3.1. Shock structure equations. The length scale of a shock structure determined by the full equations will be of the order of that induced by the source terms which, by assumption, is short compared to that associated with the initial data. We are therefore looking at shock structures described by the steady versions of equations (2.1), which for a shock with speed V are

$$\frac{d}{d\xi}(\mathbf{f} - V\mathbf{u}) = (\mathsf{J} - V\mathsf{I})\frac{d\mathbf{u}}{d\xi} = \mathbf{s}, \tag{3.1}$$

where $\xi = x - Vt$ and I is the identity matrix.

Since this is a shock of the equilibrium system, (2.4), the shock relations are

$$V(\mathbf{u}_{e,l} - \mathbf{u}_{e,r}) = \mathbf{f}(\mathbf{u}_{e,l}) - \mathbf{f}(\mathbf{u}_{e,r}),$$

where the suffixes l and r denote the left and right states. These are equilibrium states and are therefore related uniquely to frozen states that satisfy $\mathbf{s}(\mathbf{u}_l) = \mathbf{s}(\mathbf{u}_r) = 0$, using the relation $\mathbf{u} = \mathbf{u}(\mathbf{u}_e)$. For given V, these relations determine one of the states on either side of the shock given the other. Since the large-scale solution is governed by the equilibrium equations, (2.4), uniqueness requires that a shock associated with the kth characteristic of the equilibrium system satisfies the evolutionary (Lax) conditions

$$\begin{aligned} &\lambda_1^e(\mathbf{u}_l) < \cdots < \lambda_{k-1}^e(\mathbf{u}_l) < V < \lambda_k^e(\mathbf{u}_l) < \cdots < \lambda_{n-r}^e(\mathbf{u}_l), \\ &\lambda_1^e(\mathbf{u}_r) < \cdots < \lambda_k^e(\mathbf{u}_r) < V < \lambda_{k+1}^e(\mathbf{u}_r) < \cdots < \lambda_{n-r}^e(\mathbf{u}_r), \end{aligned} \tag{3.2}$$

where λ_i^e are the wave speeds of the equilibrium system (e.g., Smoller [13]).

An acceptable shock solution is a heteroclinic solution of equations (3.1) that connects the equilibrium points $\mathbf{u}_l$ and $\mathbf{u}_r$, i.e., it satisfies the boundary conditions

$$\mathbf{u} \to \begin{cases} \mathbf{u}_l & \text{as } \xi \to -\infty, \\ \mathbf{u}_r & \text{as } \xi \to +\infty. \end{cases} \tag{3.3}$$

Since the source term is of the form (2.2), the last $n-r$ equations can be integrated immediately to give $n-r$ invariants

$$f_i - Vu_i = f_i(\mathbf{u}_l) - Vu_{il} = f_i(\mathbf{u}_r) - Vu_{ir}, \quad i = r+1 \cdots n. \tag{3.4}$$

We shall assume that these can be used to eliminate $n - r$ of the variables and so reduce the system (3.1) to r equations.

It is clear that the nature of the equilibrium points plays a role in determining whether there exist heteroclinic solutions of equations (3.1). That this is so can be seen by the following argument (see, e.g., Liubarskii [12]; Smoller [13]).

Consider a system of differential equations

$$\frac{d\mathbf{u}}{d\xi} = \mathbf{s} \tag{3.5}$$

with dimension n that has two equilibrium points, $\mathbf{u}_l$ and $\mathbf{u}_r$. Let L_u, L_s be the unstable/stable manifolds of the point $\mathbf{u}_l$ and R_u, R_s the unstable/stable manifolds of the point $\mathbf{u}_r$. Then the trajectories in L_u and R_s are described by $\dim(L_u) - 1$ and $\dim(R_s) - 1$ parameters, respectively. Since any trajectory that lies in both has to satisfy $n - 1$ matching conditions, this means that, in general, there will only be a unique trajectory connecting $\mathbf{u}_l$ and $\mathbf{u}_r$ if $\dim(L_u) + \dim(R_s) = n + 1$. If $\dim(L_u) + \dim(R_s) > n + 1$, the trajectory may not be unique, whereas if $\dim(L_u) + \dim(R_s) < n + 1$, any trajectory that does exist can be destroyed by perturbations of $\mathbf{u}_l$, $\mathbf{u}_r$, i.e., it is not structurally stable.

Since the system (3.1) can be reduced to one with dimension r, this would suggest that we must have

$$\dim(L_u) + \dim(R_s) = r + 1, \tag{3.6}$$

but there is a complication due to fact that we cannot write equations (3.1) in the form (3.5) at points where the matrix $\mathsf{J} - V\mathsf{I}$ is singular. This happens when one of the wave speeds of the frozen system vanishes in the shock frame. In general this means that there is no smooth solution and the shock structure must contain shocks of the frozen system. However, for $r \geq 2$ there are systems that possess heteroclinic solutions that pass through singularities of $\mathsf{J} - V\mathsf{I}$ and are structurally stable even when (3.6) is not satisfied (Roberge and Draine [8]). We shall not consider such cases here.

It is evident that if $\mathsf{J} - V\mathsf{I}$ is nonsingular, then (3.6) is a necessary condition for structurally stable heteroclinic solutions and that it is also sufficient for $r = 1$ if there are only two equilibrium points. We now show that there is an intimate connection between the nature of the equilibrium points and the linear stability of the states at the equilibrium points.

3.2. Linear stability. Let $\mathbf{u}(x,t) = \mathbf{u}_0 + \mathbf{v}(x,t)$, where $\mathbf{v}$ is a small perturbation about a uniform equilibrium state, $\mathbf{u}_0$. The linearized version of equations (2.1) is then

$$\partial_t \mathbf{v} + \mathsf{J}_0 \partial_x \mathbf{v} = \mathsf{D}_0 \mathbf{v}. \tag{3.7}$$

Here $\mathsf{J}_0 = \mathsf{J}(\mathbf{u}_0)$ and $\mathsf{D}_0 = \mathsf{D}(\mathbf{u}_0)$, where

$$\mathsf{D} = \frac{\partial \mathbf{s}}{\partial \mathbf{u}}$$

is the Jacobian of $\mathbf{s}$ w.r.t. $\mathbf{u}$. D_0 has rank r and, for a source term of the form (2.2), only the first r rows have nonzero entries.

Since we are interested in relaxation processes, it is reasonable to insist that the system returns to the equilibrium manifold after being subjected to a perturbation with infinite wavelength. This requires that the eigenvectors of D_0 span the complement of its null space and that its r nonzero eigenvalues are negative.

We shall see that it is convenient to write D_0 in the form

$$\mathsf{D}_0 = \left\{ \begin{array}{cccccc} -K_1 & K_1 d_{12} & \cdots & K_1 d_{1r} & \cdots & K_1 d_{1n} \\ K_2 d_{21} & -K_2 & \cdots & K_2 d_{2r} & \cdots & K_2 d_{2n} \\ & & & \cdot & & \\ & & & \cdot & & \\ K_r d_{r1} & K_r d_{r2} & \cdots & -K_r & \cdots & K_r d_{rn} \\ & & & 0 & & \end{array} \right\}, \tag{3.8}$$

where $K_i = K_i(\mathbf{u}_0)$.

In the usual applications, the source terms represent r different relaxation processes and we can assume that each element of $\mathbf{s}$ and hence each row of D_0 describes a different process. If this is not true in the original variables, we can always choose variables so that it is true for the linear system. The rate constant associated with the ith such process is clearly K_i and, if each of these processes is individually stable, then we must have $K_i > 0$ for each i. Even if the relaxation processes cannot be separated in this way, we shall see later that it can be useful to write D_0 in this form. The K_i then have no physical significance, instead they are merely artificial parameters whose role is to simplify the analysis. In either case, it seems reasonable to insist that the equilibrium state is stable for all $K_i > 0$.

We now look for a solution of (3.7) of the form

$$\mathbf{v} = \mathbf{v}_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)},$$

where $\mathbf{v}_0$ is a constant. The equilibrium state is stable in the sense that there are no growing waves if $\Im(\omega) \geq 0$ for all real k.

The dispersion relation is

$$|\omega \mathsf{I} - k\mathsf{J}_0 + \mathrm{i}\mathsf{D}_0| = 0. \tag{3.9}$$

This can be written in terms of products of the K_i as follows:

$$\begin{aligned} P = P_0(\lambda) - \frac{\mathrm{i}}{k}\sum_{i=1}^{r} K_i P_i(\lambda) - \frac{1}{k^2}\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} K_i K_j P_{ij}(\lambda) \\ \cdots + \left(\frac{-\mathrm{i}}{k}\right)^r K_1 \cdots K_r P_{12\cdots r}(\lambda) = 0. \end{aligned} \tag{3.10}$$

Here $\lambda = \omega/k$ and P_0, $P_1 \cdots P_r$, $P_{12} \cdots P_{r-1r}$, etc., are real polynomials of degree n, $n-1$, $n-2$, etc. Following Liubarskii [12], we shall call these the auxiliary system polynomials. They represent systems in different limits of the relaxation rates and generally correspond to the principal subsystems considered in Boillat and Ruggeri [1]. The only difference is that, unlike the principal subsystems, an auxilliary polynomial may represent an artificial unphysical system as in the examples in sections 4.2 and 4.3 below.

In order to determine the conditions for linear stability, we use the Hermite–Biehler theorem (see, e.g., Liubarskii [12]; Bhattacharyya, Chapellat, and Keel [9]). Equation (3.10) is of the form

$$M_r(\lambda) - \frac{\mathrm{i}}{k} M_i(\lambda) = 0, \tag{3.11}$$

where M_r and M_i are real polynomials of degree n and $n-1$, respectively.

The Hermite–Biehler theorem uses the principle of the argument to show that the necessary and sufficient conditions for there to be no roots of (3.11) in the lower half-plane if $k > 0$ and none in the upper half-plane if $k < 0$ are that

1. the coefficients of the highest powers of λ in M_r and M_i are positive,
2. the roots of M_r and M_i are real,
3. the roots of M_i interleave with those of M_r.

It follows that if the conditions of the Hermite–Biehler theorem are satisfied, then $\Im(\omega) \geq 0$ and the linear system is stable. Note that (3.11) satisfies condition 1 since the K_i are positive and it is evident from (3.8), (3.9) that the coefficient of the highest power of λ is unity in P_0 and $K_1 \cdots K_r$ in $P_1 \cdots P_r$.

We now show how this can be used to relate the linear stability of the equilibrium states to the structure condition, (3.6).

3.3. Equilibrium points of the shock structure equations. Let $\mathbf{u} = \mathbf{u}_0$ be an equilibrium point of the shock structure equations (3.1). Linearizing about $\mathbf{u}_0$ gives

$$(\mathsf{J}_0 - V\mathsf{I})\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\xi} = \mathsf{D}_0\mathbf{v}, \tag{3.12}$$

where $\mathbf{u}(\xi) = \mathbf{u}_0 + \mathbf{v}(\xi)$ with $\mathbf{v}$ a small perturbation and $\mathsf{J}_0 = \mathsf{J}(\mathbf{u}_0)$, $\mathsf{D}_0 = \mathsf{D}(\mathbf{u}_0)$, as in equations (3.7).

The solutions of (3.12) are of the form $\mathbf{v} = \mathbf{v}_0\mathrm{e}^{\mu\xi}$, where the eigenvalues, μ, are given by

$$|(V\mathsf{I} - \mathsf{J}_0)\mu + \mathsf{D}_0| = 0. \tag{3.13}$$

This is an nth order polynomial which becomes identical to (3.9) if we set $\mu = -\mathrm{i}k$ and $V = \omega/k$. From (3.10), it therefore follows that (3.13) can be written

$$\begin{aligned} P = P_0(V)\mu^n - \sum_{i=1}^{r} K_i P_i(V)\mu^{n-1} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} K_i K_j P_{ij}(V)\mu^{n-2} \\ \cdots + (-1)^r K_1 \cdots K_r P_{12\cdots r}(V)\mu^{n-r} = 0. \end{aligned} \tag{3.14}$$

As expected, this has $n - r$ zero roots corresponding to the $n - r$ invariants (3.4) and the remaining roots determine the nature of the equilibrium point. The following theorem gives sufficient conditions for shocks of the equilibrium system to satisfy the structure condition (3.6).

THEOREM 3.1. *Consider a shock of the equilibrium system with left and right states $\mathbf{u}_l$ and $\mathbf{u}_r$ that satisfy the evolutionary condition* (3.2) *for some k. If there exists a path from $\mathbf{u}_l$ to $\mathbf{u}_r$ that lies entirely in the equilibrium manifold along which*

1. *the states are strictly linearly stable in the sense that all the roots of* (3.10) *have strictly positive imaginary part for all $K_i > 0$ and $k \neq 0$,*
2. *none of the frozen system wave speeds in the shock frame change sign,*

then the states on either side of the shock satisfy the structure condition, (3.6).

Proof. We first show that condition 1 on the stability of the states along the path from $\mathbf{u}_l$ to $\mathbf{u}_r$ ensures that none of the r nonzero roots of (3.14) are purely imaginary. As we have already seen, (3.14) becomes identical to (3.10) if we set $\mu = -\mathrm{i}k$ and $V = \omega/k$. But stability of the equilibrium states demands that the roots for ω/k of the real and imaginary parts of this are real and interleave for all real k, i.e., for all imaginary μ. The real and imaginary parts therefore do not vanish simultaneously and so there are no purely imaginary roots of (3.14). This means that the dimension of the center manifold of the equilibrium points of the r dimensional system is zero.

We therefore have

$$\dim(L_s) + \dim(L_u) = \dim(R_s) + \dim(R_u) = r,$$

which means that the structure condition (3.6) will be satisfied if

$$\dim(R_s) = \dim(L_s) + 1. \tag{3.15}$$

Along the path from $\mathbf{u}_l$ to $\mathbf{u}_r$, the real part of a root of (3.14) can change sign either by going to infinity and reappearing on the other side of the imaginary axis or by crossing the imaginary axis. The first possibility requires that P_0 vanishes, which is excluded by condition 2. Since there are no purely imaginary roots, the second possibility requires that a root pass through the origin, which can occur only when $P_{1\cdots r}$ vanishes.

Let the path from the left to the right state be given by $\mathbf{u} = \mathbf{u}(\epsilon)$ with $\mathrm{u}(0) = \mathbf{u}_l$ and $\mathrm{u}(1) = \mathbf{u}_r$. Now write (3.14) in the form

$$P_0\mu^r - Q_1\mu^{r-1} + Q_2\mu^{r-2} \cdots (-1)^r Q_r = 0, \tag{3.16}$$

where the Q_i are linear combinations of the auxiliary system polynomials. Differentiating this w.r.t. ϵ and setting $\mu = 0$ gives

$$\frac{d\mu}{d\epsilon} = \frac{1}{Q_{r-1}} \frac{dQ_r}{d\epsilon}. \tag{3.17}$$

If Q_{r-1} always has the same sign at a particular root of Q_r, then μ must pass through the origin in one direction when $dQ_r/d\epsilon > 0$ and in the other direction when $dQ_r/d\epsilon < 0$.

In order to show that Q_{r-1} does indeed have this property, we make use of condition 1. If, for some j, we let $K_i \to \infty$ for all $i \neq j$, then (3.10) reduces to

$$T_j - \mathrm{i}\frac{K_j}{k} P_{1\cdots r} = 0. \tag{3.18}$$

Here T_j is the auxiliary polynomial of degree $n - r + 1$ in (3.10) that is not multiplied by K_j.

The roots of the real and imaginary parts of this must be real and interleave because condition 1 demands that this system be stable. They can have no roots in common since we are insisting on strict stability, i.e., (3.18) has no real roots. The coefficients of the highest powers of λ in T_j and $P_{1\cdots r}$ must also have the same sign, which we can choose to be positive. Since this is true for all $j = 1 \cdots r$, it follows that the roots of the T_j lie in disjoint intervals I_i ($i = 1 \cdots r - 1$) that interleave with the roots of $P_{1\cdots r}$, which are the same as those of Q_r. The T_j are polynomials of the same degree for which the coefficient of the highest power of λ is positive. They therefore have the same sign outside the intervals I_i. But the Q_{r-1} in (3.16) is a linear combination of the T_j with positive coefficients and therefore has no roots outside these intervals. It is a continuous function that is positive at one end of I_i and negative at the other end, which means that it must have at least one root in I_i.

However, since it is a polynomial of degree $r-1$, it can only have one root in each of the $r-1$ intervals I_i. The roots of Q_{r-1} therefore interleave with those of Q_r at all points of the path from $\mathbf{u}_l$ to $\mathbf{u}_l$, which guarantees that Q_{r-1} always has the same sign when we pass through a root of Q_r.

Since the shock is evolutionary, (3.2) must be satisfied for some k, i.e., $V < \lambda_k^e(\mathbf{u}_l)$ and $V > \lambda_k^e(\mathbf{u}_r)$. It follows that if, for $i \neq k$, there is a point on the path where $\lambda_i^e = V$ with $dQ_r/d\epsilon > 0$, then there is also such a point $dQ_r/d\epsilon < 0$. Equation (3.17) then tells us that the vanishing of Q_r at these points does not lead to a change in the sign of the real part of a root of (3.14).

The same argument applies to any points on the path where $\lambda_k^e = V$, except for the last such point, i.e., the one for which $\mathbf{u}$ is closest to $\mathbf{u}_r$. First suppose that at this point $Q_r(\lambda) > 0$ for $\lambda_k^e < \lambda < \lambda_{k+1}^e$. Since the roots of Q_{r-1} interleave with those of Q_r, this means that $Q_{r-1}(\lambda_k^e) < 0$. From (3.2) we must therefore have $dQ_r/d\epsilon > 0$ so that at this point

$$\frac{d\mu}{d\epsilon} < 0$$

from (3.17). The real part of a root of (3.14) therefore changes from positive to negative at this point and remains negative along the rest of the path. The same argument shows that this is also true if $Q_r(\lambda) < 0$ for $\lambda_k^e < \lambda < \lambda_{k+1}^e$. We have therefore proved that the sign of the real part of one of the roots of (3.14) changes from positive to negative along the path from $\mathbf{u}_l$ to $\mathbf{u}_r$, which means that (3.15) holds and hence that structure condition, (3.6), is satisfied. ◻

If the conditions of Theorem 3.1 are satisfied, then a smooth shock structure exists; otherwise it does not. This is the result obtained by Whitham [16, 17] for the case $r = 1$. Equation (3.14) then becomes

$$P_0(V)\mu^n - K_1 P_1(V)\mu^{n-1} = 0,$$

so that the only nonzero root is $\mu = K_1 P_1(V)/P_0(V)$. It is clear that (3.15) is satisfied provided P_0 does not change sign and P_1 changes sign in the manner required by the evolutionary condition.

For systems with $r \geq 1$, Boillat and Ruggeri [2] used the existence of a convex entropy to show that a smooth structure does not exist if the shock speed exceeds the fastest wave speed in the frozen system. One can see that this also follows from Theorem 3.1. If the shock is faster than the fastest wave speed in the frozen system, then all wave speeds ahead of the shock are negative in the frame of the shock. However, the evolutionary condition (3.2) requires one of the equilibrium wave speeds to be positive behind the shock, which implies that the highest wave speed is also positive there and has therefore changed sign. The theorem then tells us that a smooth shock structure is not possible.

The need for a path in the equilibrium manifold from $\mathbf{u}_l$ to $\mathbf{u}_r$ for $r > 1$ might seem an unnatural requirement, but it has a sound physical basis. It seems reasonable to suppose that the shocks in any physical system are part of a continuum that includes infinitely weak shocks. This notion can be expressed more precisely.

For any left state $\mathbf{u}_l$, let $\mathbf{u}_r$ be a right state that can be connected to $\mathbf{u}_r$ by a shock of the equilibrium system that satisfies the evolutionary conditions (3.2) for a particular k. The strength of these shocks can be parametrized by

$$s = \begin{cases} \left|\dfrac{V}{\lambda_k^e(\mathbf{u}_r)}\right| - 1 \;\; \text{if } V > 0, \\[2ex] \left|\dfrac{V}{\lambda_k^e(\mathbf{u}_l)}\right| - 1 \;\; \text{if } V < 0, \end{cases}$$

so that $s \geq 0$ with $s = 0$ corresponding to a shock with zero strength. If such shocks exist for every $s \in [0, s_m]$ for some $s_m > 0$ and P_0 has the same sign on either side of the shock for all members of the family, then one can obviously construct an appropriate path from $\mathbf{u}_l$ to $\mathbf{u}_r$ that consists of members of the family.

The conditions of the theorem are more complicated versions of those for shocks whose structure is determined by dissipative effects that depend upon gradients, such as viscosity, heat conduction, etc., (e.g., Smoller [13]; Falle and Komissarov [10]).

3.4. Hierarchical interleaving and linear stability. Theorem 3.1 gives the connection between the structure condition and the linear stability of the linear system, but that still leaves us with the problem of showing that a given equilibrium state is stable. Chen, Levermore, and Liu [3] have shown that the equilibrium system is stable provided that the full system possesses a strictly convex entropy function and is dissipative in the sense that this entropy function does not decrease. This is a very powerful result, since many systems do indeed have this property. However, it is not always a simple matter to show that an arbitrary system is dissipative.

Here we present an alternative approach based on the Hermite–Biehler theorem. This theorem tells us that stability depends upon whether the roots of the real and imaginary parts of (3.11) interleave, but it is not easy to show that this is true in any particular case. However, as we shall see, the decomposition (3.10) of the dispersion relation provides a useful way of doing this.

It is evident that the real roots of the auxiliary system polynomials give the wave speeds in the various limits:

P_0	wave speeds for $K_1 = K_2 \cdots = K_r = 0$ (the frozen system)
P_1	wave speeds for $K_1 \to \infty$, $K_2, \cdots K_r = 0$
	.
	.
$P_{12\cdots r}$	wave speeds for $K_1 \cdots K_r \to \infty$ (the equilibrium system).

It is clear from (3.10) that these polynomials determine the stability of the linear system (3.7). Unfortunately, it is not possible to derive conditions on these polynomials that guarantee stability for $r > 2$, but it is possible for $r \leq 2$.

Theorem 3.2. *For $r \leq 2$, the necessary and sufficient conditions for the equilibrium state to be stable for all $K_i > 0$ are that the auxiliary system polynomials have the following properties:*

1. *the coefficients of the highest power of λ are positive,*
2. *their roots are real,*
3. *they satisfy the hierarchical interleaving condition: for any integer s with $n - r < s \leq n$, the roots of the polynomials with degree $s - 1$ interleave with those of degree s.*

Proof. For $r = 1$ the proof is simple. Equation (3.10) then becomes

$$P_0 - \mathrm{i}\frac{K_1}{k}P_1 = 0,$$

so that the conditions for the Hermite–Biehler theorem reduce to 1, 2, and 3. Hence these conditions are also sufficient for $r = 1$.

For $r = 2$, (3.10) becomes

$$P_0 - \frac{\mathrm{i}}{k}(K_1P_1 + K_2P_2) - \frac{1}{k^2}K_1K_2P_{12} = 0. \tag{3.19}$$

We first show that stability requires conditions 1 and 2. Since all positive values of K_i are allowed, we can always choose the K_i such that (3.19) degenerates to one of the auxiliary system polynomials. This is a real polynomial, so that if it has complex roots, they must occur in complex conjugate pairs, one of which corresponds to instability ($\Im(\omega) < 0$). Conditions 1 and 2 are therefore necessary for the system to be stable.

To show that condition 3 is a necessary condition for the roots of the real and imaginary parts of (3.19) to interleave, we consider a pair of auxiliary polynomials Q_s, Q_{s-1} with degree s and $s-1$. For any such pair, we can always choose the K_i such that (3.19) degenerates to

$$Q_s - \mathrm{i}\frac{K}{k}Q_{s-1} = 0,$$

where K is now a product of some of the K_i and is therefore positive. Hence by the Hermite–Biehler theorem, a necessary condition for stability is that the roots of Q_s and Q_{s-1} interleave and that the coefficients of the highest power of λ in Q_s and Q_{s-1} have the same sign. Applying this to every such pair proves the necessity of conditions 1 and 3.

To prove that the roots of the imaginary part of (3.19) are real, let $\lambda_1^1 \le \lambda_2^1 \cdots \lambda_{n-1}^1$ and $\lambda_1^2 \le \lambda_2^2 \cdots \lambda_{n-1}^2$ be the roots of P_1 and P_2, respectively. Let I_j be the smallest interval that contains λ_j^1 and λ_j^2. Condition 3 ensures that these intervals are disjoint since they are separated by the roots of P_0. As in the proof of Theorem 3.1, the imaginary part is a continuous function that is positive at one end of I_j and negative at the other end. It must therefore have at least one root in each of the $n-1$ intervals I_j, but it can only have one since it is a polynomial of degree $n-1$. Hence all the roots of the imaginary part are real.

To prove that the roots of the real part of (3.19) are real and interleave with those of the imaginary part, let $\lambda_1^0 \le \lambda_2^0 \cdots \lambda_n^0$ and $\lambda_1^{12} \le \lambda_2^{12} \cdots \lambda_{n-2}^{12}$ be the roots of P_0 and P_{12}, respectively. Condition 3 means that $\lambda_1^0 < \lambda_1^{12}$ and $\lambda_n^0 > \lambda_{n-2}^{12}$. Since the coefficients of the highest power of these polynomials are positive and their degrees differ by 2, they have the same sign in $\lambda < \lambda_1^0$ and in $\lambda > \lambda_n^0$. From this and the fact that the magnitude of P_0 increases faster than that of P_{12} as $\lambda \to \pm\infty$, it follows that the real part has one root in $\lambda < \lambda_1^0$ and another in $\lambda > \lambda_n^0$. Condition 3 guarantees that these roots do not coincide with those of the imaginary part.

As for the imaginary part, let R_j be the smallest interval that contains λ_{j+1}^0 and λ_j^{12} ($j = 1 \cdots n-2$). We can now use exactly the same argument as for the imaginary part to prove that each of these intervals contains a root of the real part. Together with the roots in $\lambda < \lambda_1^0$ and $\lambda > \lambda_n^0$, this gives n real roots. Furthermore, condition 3 ensures that the intervals R_j interleave with the intervals I_j for the imaginary part.

We have therefore proved that the conditions of the Hermite–Biehler theorem are satisfied and hence that the equilibrium state is stable if $k \neq 0$.

For $k = 0$ (3.10) reduces to

$$\omega^n - \mathrm{i}(K_1 + K_2)\omega^{n-1} - K_1 K_2 c_{12}\omega^{n-2} = 0,$$

where $c_{12} > 0$ is the coefficient of the highest power in P_{12}. It is clear that the nonzero roots of this have positive imaginary part if $c_{12} > 0$. The system is therefore also stable for $k = 0$, although it is not strictly stable in this limit. □

In the course of their stability proof, Chen, Levermore, and Liu [3] obtain a condition on the wave speeds of a stable system that is related to the hierarchical interleaving condition and is valid for $r \geq 1$. This says that in a stable system each equilibrium wave speed, λ_i^e, lies in the closed interval $[\lambda_i^0, \lambda_{i+r}^0]$, where λ_i^0 are the frozen wave speeds (the roots of P_0). This condition is necessary but not sufficient for stability of the equilibrium system, whereas the stricter hierarchical interleaving condition is both necessary and sufficient for $r \leq 2$. Our requirement that the equilibrium state be stable for all $K_i \geq 0$ seems reasonable on physical grounds, but removing it simply means that the condition is sufficient but not necessary.

4. Examples. In order to demonstrate the utility of the theory described above, we now apply it to some examples. We shall see that Theorem 3.2 can be useful for establishing the stability of systems with $r \leq 2$. Furthermore, it is possible to use the form (3.10) for the dispersion relation to prove stability even when there are repeated and coincident roots.

4.1. Ideal gas with two damped internal degrees of freedom. At finite temperatures, the internal energy of a molecular gas comprises separate components corresponding to translational, rotational, and vibrational degrees of freedom. When the gas temperature changes, it takes a finite time for these individual components to come into equilibrium. As a simple model of this process, we consider equations of the form (2.1) with

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho(\epsilon + u^2/2) \\ \rho\epsilon_1 \\ \rho\epsilon_2 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u(\epsilon + p/\rho + u^2/2) \\ \rho u\epsilon_1 \\ \rho u\epsilon_2 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s_4 \\ s_5 \end{pmatrix}. \tag{4.1}$$

Here ρ, u, and p are the gas density, velocity, and pressure. ϵ is the total internal energy per unit mass and is given by

$$\epsilon = \epsilon_{\mathrm{T}} + \epsilon_1 + \epsilon_2, \tag{4.2}$$

where ϵ_T is the energy associated with the translational degrees of freedom. The pressure is given by

$$p = (\gamma_{\mathrm{T}} - 1)\rho\epsilon_{\mathrm{T}}, \tag{4.3}$$

where $\gamma_{\mathrm{T}} = 5/3$ since it corresponds to the translational degrees of freedom. The frozen system is then standard gas dynamics, with two additional passively advected degrees of freedom.

ϵ_1 and ϵ_2 are the internal energies associated with the other two sets of degrees of freedom. The source terms are

$$s_4 = K_1 \rho \left(\epsilon_{\mathrm{T}} - \frac{\gamma_1 - 1}{\gamma_{\mathrm{T}} - 1} \epsilon_1 \right), \qquad s_5 = K_2 \rho \left(\epsilon_{\mathrm{T}} - \frac{\gamma_2 - 1}{\gamma_{\mathrm{T}} - 1} \epsilon_2 \right). \tag{4.4}$$

From these equations, it is clear that the equilibrium system and the auxiliary systems are each also ideal gas hydrodynamics. For the equilibrium system, $(\gamma_{\mathrm{T}} - 1)\epsilon_{\mathrm{T}} = (\gamma_1 - 1)\epsilon_1 = (\gamma_2 - 1)\epsilon_2$ and the effective adiabatic constant γ'_{12} is given by

$$\frac{1}{\gamma'_{12} - 1} = \frac{1}{\gamma_{\mathrm{T}} - 1} + \frac{1}{\gamma_1 - 1} + \frac{1}{\gamma_2 - 1}, \tag{4.5}$$

while for the auxiliary systems γ'_1, γ'_2 are given by

$$\frac{1}{\gamma'_i - 1} = \frac{1}{\gamma_{\mathrm{T}} - 1} + \frac{1}{\gamma_i - 1}. \tag{4.6}$$

These equations correspond to the usual thermodynamic equipartition principle, that at thermal equilibrium each independent quadratic mode has energy $\frac{1}{2} k_{\mathrm{B}} T$, where k_{B} is the Boltzmann constant. This means that $\gamma_i = 1 + 2/n_i$, where n_i is the number of modes. Each of the auxiliary systems corresponds to perfect gas dynamics with a modified ratio of specific heats and fewer passively advected variables than the frozen system. Equations (4.5) and (4.6) give $\gamma_{\mathrm{T}} > \gamma'_i > \gamma'_{12}$, which means that the hierarchical interleaving condition is satisfied. Since $r = 2$, the system is therefore stable by Theorem 3.2.

Steady shocks satisfy the usual Rankine–Hugoniot conditions for the equilibrium system. The $r = 2$ equations (3.1) for the shock structure are then

$$\frac{d\epsilon_1}{dx} = \frac{\lambda_1}{u - V} \left(\epsilon_{\mathrm{T}} - \frac{\gamma_1 - 1}{\gamma_{\mathrm{T}} - 1} \epsilon_1 \right), \qquad \frac{d\epsilon_2}{dx} = \frac{\lambda_2}{u - V} \left(\epsilon_{\mathrm{T}} - \frac{\gamma_2 - 1}{\gamma_{\mathrm{T}} - 1} \epsilon_2 \right), \tag{4.7}$$

and the $n - r = 3$ invariants in (3.4) are

$$\begin{aligned} \rho(u - V) &= \Phi = \text{const}, \\ p + \rho(u - V)^2 &= \Pi = \text{const}, \\ \left[\epsilon + \frac{p}{\rho} + \frac{1}{2}(u - V)^2 \right] &= \frac{\mathcal{E}}{\Phi} = \text{const}. \end{aligned} \tag{4.8}$$

These result in the quadratic equation

$$\frac{\gamma_{\mathrm{T}} + 1}{2} \Phi (u - V)^2 - \gamma_{\mathrm{T}} \Pi (u - V) + (\gamma_{\mathrm{T}} - 1) \left[\mathcal{E} - \Phi(\epsilon_1 + \epsilon_2) \right] = 0, \tag{4.9}$$

which is sufficient to determine ϵ_{T} and $v - V$ from ϵ_1, ϵ_2 in the absence of shocks of the frozen system.

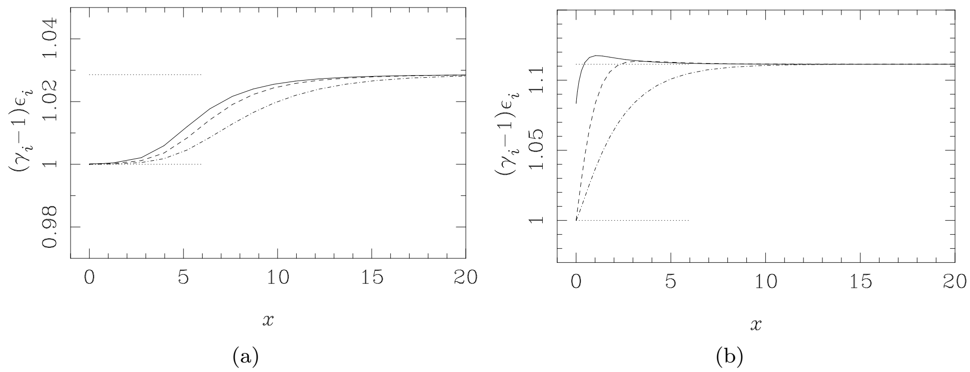

FIG. 1. *Internal structure of shocks in the system of hydrodynamics with two damped internal degrees of freedom.* (a) *Speed into shock is below the frozen sound speed, $u-V=1.2$;* (b) *speed into shock is above the frozen sound speed, $u-V=1.4$. The solid line shows $(\gamma_T-1)\epsilon_T$, the dashed line $(\gamma_1-1)\epsilon_1$, and the dot-dashed line $(\gamma_2-1)\epsilon_2$. The dotted lines show the upstream and downstream equilibrium values. In the supersonic case, the curve for ϵ_T jumps through an unresolved shock of the frozen system, and then overshoots its equilibrium postshock value; in this case, ϵ_1 also overshoots.*

Note that shocks of the equilibrium system may also include unresolved subshocks of the frozen system. Indeed for fixed ϵ_1, ϵ_2, equations (4.8) are exactly the Rankine–Hugoniot conditions for the frozen system. In fact, it is obvious that the shocks of the equilibrium system which pass through a frozen shock speed must include a subshock of the frozen system since the frozen system sound speed is the fastest signal propagation speed.

As an example, consider the case of a diatomic molecule at a temperature large enough for quantum effects to be negligible, for which $n_1=2$ for the two independent rotational modes and $n_2=2$ for the separate kinetic and potential energy components of the vibrational energy. $\gamma_T=5/3$ since it corresponds to the three translational modes, $\gamma_1=\gamma_2=2$ and $\gamma'_{12}=9/7$.

In Figure 1, we show the internal structure of two shocks in this system, for which we have taken $K_1=1$, $K_2=0.3$, and $p=\rho=1$ in the upstream state. Figure 1(a) shows a smooth shock structure in which the shock speed is below the frozen sound speed and Figure 1(b) one that contains a subshock since the shock velocity exceeds the frozen sound speed.

4.2. Two fluid isothermal magnetohydrodynamics. In dense molecular clouds, the density of charged particles can be so low that the plasma cannot be assumed to be perfectly conducting. The properties of such a plasma can be modeled by assuming that it consists of two fluids: a perfectly conducting fluid corresponding to the charged particles and a nonconducting fluid corresponding to the neutral particles, which interact via a friction force (see, e.g., Draine and McKee [7]). Here we consider a simplified version of the oblique shock problem described in Wardle and Draine [15] in which we assume that both fluids are isothermal at the same temperature.

Let the densities of the fluids be ρ_n, ρ_c, their velocity components in the x, y directions be u_n, u_c, v_n, v_c, and a be the isothermal sound speed. The equations are

$$\frac{\partial \mathbf{u}_n}{\partial t}+\frac{\partial \mathbf{f}_n}{\partial x}=\mathbf{s}_n, \quad \frac{\partial \mathbf{u}_c}{\partial t}+\frac{\partial \mathbf{f}_c}{\partial x}=\mathbf{s}_c \tag{4.10}$$

with

$$\mathbf{u}_n = \begin{pmatrix} \rho_n \\ \rho_n u_n \\ \rho_n v_n \end{pmatrix}, \quad \mathbf{f}_n = \begin{pmatrix} \rho_n u_n \\ \rho_n u_n^2 + a^2 \rho_n \\ \rho_n u_n v_n \end{pmatrix}, \quad \mathbf{s}_n = \begin{pmatrix} 0 \\ F_x \\ F_y \end{pmatrix}, \tag{4.11}$$

and

$$\mathbf{u}_c = \begin{pmatrix} \rho_c \\ \rho_c u_c \\ \rho_c v_c \\ B_y \end{pmatrix}, \quad \mathbf{f}_c = \begin{pmatrix} \rho_c u_c \\ \rho_c u_c^2 + a^2 \rho_c + B_y^2/2 \\ \rho_c u_c v_c - B_x B_y \\ u_c B_y - v_c B_x \end{pmatrix}, \quad \mathbf{s}_c = \begin{pmatrix} 0 \\ -F_x \\ -F_y \\ 0 \end{pmatrix}. \tag{4.12}$$

Note that $B_x = \text{const}$ for a plane-parallel system because of $\nabla \cdot \mathbf{B} = 0$. There is no need to introduce z components of either the velocity or the field since one can show that these vanish in the shock structure if they do so in the upstream state. Hence $B_z = w = 0$ throughout.

The friction force, $\mathbf{F} = (F_x, F_y)$, must depend upon the relative velocity of the two fluids, but its actual form is unimportant for our present purposes so long as it ensures that the two fluids have the same velocity in the equilibrium state. Draine [6] assumes that

$$\mathbf{F} = K \rho_n \rho_c (\mathbf{q}_c - \mathbf{q}_n), \tag{4.13}$$

where $\mathbf{q}_n = (u_n, v_n)$, $\mathbf{q}_c = (u_c, v_c)$ are the fluid velocities and K is a coupling constant.

In the absence of the source terms, the neutral system is ordinary isothermal compressible flow and the conducting system is isothermal coplanar magnetohydrodynamics. The wave speeds of the frozen system, $\mathbf{s}_n = \mathbf{s}_c = 0$, are

$$\lambda_1 = u_n - a, \ \lambda_2 = u_n, \ \lambda_3 = u_n + a \tag{4.14a}$$

for the neutral fluid and

$$\lambda_4 = u_c - c_{fc}, \ \lambda_5 = u_c - c_{sc}, \ \lambda_6 = u_c + c_{sc}, \ \lambda_7 = u_c + c_{fc} \tag{4.14b}$$

for the conducting fluid. Here c_{fc} and c_{sc} are the frozen fast and slow speeds given by

$$c_{fc}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_c} + a^2 + \left\{\left(\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_c} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_c}\right\}^{1/2}\right], \tag{4.15a}$$

$$c_{sc}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_c} + a^2 - \left\{\left(\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_c} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_c}\right\}^{1/2}\right]. \tag{4.15b}$$

The equilibrium system is

$$\frac{\partial \mathbf{u}_e}{\partial t} + \frac{\partial \mathbf{f}_e}{\partial x} = 0 \tag{4.16}$$

with

$$\mathbf{u}_e = \begin{pmatrix} \rho_e \\ \rho_e u_e \\ \rho_e v_e \\ B_y \\ \rho_c \end{pmatrix}, \quad \mathbf{f}_e = \begin{pmatrix} \rho_e u_e \\ \rho_e u_e^2 + a^2 \rho_e + B_y^2/2 \\ \rho_e u_e v_e - B_x B_y \\ u_e B_y - v_e B_x \\ \rho_c u_e \end{pmatrix}, \tag{4.17}$$

where $\rho_e = \rho_c + \rho_n$, $u_e = u_c = u_n$, and $v_e = v_c = v_n$. Since this is also isothermal coplanar magnetohydrodynamics with passive advection of the density of the charged fluid, the wave speeds are

$$\lambda_1 = u_e - c_{fe},\ \lambda_2 = u_e - c_{se},\ \lambda_3 = u_e,\ \lambda_4 = u_e + c_{se},\ \lambda_4 = u_e + c_{fe}. \tag{4.18}$$

Here c_{se} and c_{fe} are given by (4.15a) and (4.15b) with ρ_c replaced byρ_e.

If we write the friction force as

$$\begin{aligned} F_x &= K_1 \rho_n \rho_c (u_c - u_n), \\ F_y &= K_2 \rho_n \rho_c (v_c - v_n), \end{aligned} \tag{4.19}$$

then the source term is of the form (2.2). This is obviously artificial in this case since the real system always has $K_1 = K_2 = K$, but this does not matter since we shall show that the system satisfies the hierarchical interleaving theorem. In fact this is an example of the utility of introducing artificial relaxation parameters.

There is no loss of generality in setting $u_e = 0$, in which case the roots of the polynomials P_0, P_1, P_2, and P_{12} that appear in (3.19) are

$$\begin{array}{ll} P_0 & -c_{fc},\ -a,\ -c_{sc},\ 0,\ c_{sc},\ a,\ c_{fc}, \\ P_1 & -c_{f1},\ -c_{s1},\ 0,\ 0,\ c_{s1},\ c_{f1}, \\ P_2 & -c_{f2},\ -a,\ -c_{s2},\ c_{s2},\ a,\ c_{f2}, \\ P_{12} & -c_{fe},\ -c_{se},\ 0,\ c_{se},\ c_{fe}. \end{array} \tag{4.20}$$

The mixed system characteristic speeds are given by

$$c_{f1}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_e} + a^2 + \left\{\left(\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_e} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_c}\right\}^{1/2}\right], \tag{4.21a}$$

$$c_{s1}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_e} + a^2 - \left\{\left(\frac{B_x^2}{\rho_c} + \frac{B_y^2}{\rho_e} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_c}\right\}^{1/2}\right], \tag{4.21b}$$

$$c_{f2}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_e} + \frac{B_y^2}{\rho_c} + a^2 + \left\{\left(\frac{B_x^2}{\rho_e} + \frac{B_y^2}{\rho_c} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_e}\right\}^{1/2}\right], \tag{4.22a}$$

$$c_{s2}^2 = \frac{1}{2}\left[\frac{B_x^2}{\rho_e} + \frac{B_y^2}{\rho_c} + a^2 - \left\{\left(\frac{B_x^2}{\rho_e} + \frac{B_y^2}{\rho_c} + a^2\right)^2 - \frac{4B_x^2 a^2}{\rho_e}\right\}^{1/2}\right]. \tag{4.22b}$$

Since $\rho_e > \rho_c$, the wave speeds in P_1 can be regarded as those for a system with a smaller value of B_y than P_0 and the wave speeds in P_2 as those for a system with a smaller value of B_x. By considering the derivatives of these speeds w.r.t. B_y and B_x, one can readily show that

$$c_{fc} > c_{f2} > c_{f1} > c_{fe} > a > c_{s1} > c_{sc} > c_{se} > c_{s2}. \tag{4.23}$$

These conditions ensure that the polynomials satisfy the hierarchical interleaving condition, but the zero roots in P_0 and P_1 mean that the system is not strictly stable if $K_1 \neq 0$ and $K_2 = 0$. However, in this case the K_i are artificial parameters and it makes no physical sense for them to be different. As long as K_1 and K_2 are nonzero, the imaginary part of (3.19) has two roots symmetric about the origin in the interval $[-c_{s2}, c_{s2}]$, which interleave with the zero root of the real part and the system is strictly stable. Furthermore, one can readily show that the evolutionary shocks of the equilibrium system always satisfy the structure condition (3.6). This is a more general version of the result derived in the appendix in Wardle and Draine [15].

4.3. Magnetohydrodynamics with tensor resistivity. The last example is an extension of the case of scalar resistive magnetohydrodynamics discussed by Whitham [16] to tensor resistivity. The system is (2.1) with

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_x \\ E_y \\ E_z \\ B_y \\ B_z \end{pmatrix}, \quad \mathbf{f}_n = \begin{pmatrix} \rho u \\ \rho u^2 + a^2\rho \\ \rho u v \\ \rho u w \\ 0 \\ c^2 B_z \\ -c^2 B_y \\ -E_z \\ E_y \end{pmatrix}, \quad \mathbf{s}_n = \begin{pmatrix} 0 \\ j_y B_z - j_z B_y \\ j_z B_x - j_x B_z \\ j_x B_y - j_y B_x \\ -c^2 j_x \\ -c^2 j_y \\ -c^2 j_z \\ 0 \\ 0 \end{pmatrix}. \tag{4.24}$$

Here ρ is the fluid density, $\mathbf{q} = (u, v, w)$ the fluid velocity, (E_x, E_y, E_z) the electric field, (B_x, B_y, B_z) the magnetic field, (j_x, j_y, j_z) the current density, and c the velocity of light. For simplicity we have assumed that the fluid is isothermal with sound speed a.

The current is given by

$$\mathbf{j} = s_0(\mathbf{E}' \cdot \mathbf{B})\mathbf{B} + s_1(\mathbf{E}' \wedge \mathbf{B}) + s_2(\mathbf{E}' \wedge \mathbf{B}) \wedge \mathbf{B}, \tag{4.25}$$

where

$$\mathbf{E}' = \mathbf{E} + \mathbf{q} \wedge \mathbf{B} \tag{4.26}$$

is the electric field in the fluid frame. The coefficients are

$$s_0 = \frac{\sigma_0}{B^2}, \qquad s_1 = \frac{\sigma_1}{B}, \qquad s_2 = \frac{\sigma_2}{B^2}, \tag{4.27}$$

where σ_0 is the conductivity parallel to the magnetic field, σ_1 is the Hall conductivity, and σ_2 is the Pedersen conductivity (Cowling [4]).

When all the conductivities are zero, the system decouples into Maxwell's equations in a vacuum and isothermal fluid dynamics. When the conductivities are infinite, $\mathbf{E}'$ vanishes and one recovers isothermal magnetohydrodynamics in the limit $c \to \infty$.

Again, it is convenient to introduce artificial parameters K_1, K_2, K_3 by replacing (j_x, j_y, j_z) in the source term by $(K_1 j_x, K_2 j_y, K_3 j_z)$. There is no loss of generality in setting $\mathbf{q} = 0$, $\mathbf{E} = 0$, $\mathbf{B} = (B_x, B_y, 0)$ in the initial state. The dispersion relation is then of the form (3.10) with $r = 3$, but since we are interested in the limit $c \to \infty$, we need only consider the dispersion relation in this limit. The polynomials P_0, P_2, P_3, and P_{23} then vanish and the dispersion relation reduces to

$$Q_1(\lambda) + \frac{\mathrm{i}}{k}[K_2 Q_{12}(\lambda) + K_3 Q_{13}(\lambda)] - \frac{1}{k^2} K_2 K_3 Q_{123}(\lambda) = 0, \tag{4.28}$$

where $\lambda = \omega/k$ as before. The roots of these polynomials are

Q_1	$0,\ 0,\ \pm a,$
Q_{12}	$0,\ \pm a,\ \pm c_a,$
Q_{13}	$0,\ \pm c_s,\ \pm c_f,$
Q_{123}	$\pm c_s,\ \pm c_a,\ \pm c_f,$

with c_s and c_f the slow and fast speeds given by (4.15a)–(4.15b) with ρ_c replaced by ρ. c_a is the Alfvén speed given by

$$c_a = \frac{B_x}{\sqrt{\rho}}.$$

Note that $c_s \le c_a \le c_f$ and $c_s \le a \le c_f$. The coefficients of the highest power of λ in these polynomials involve the conductivities σ_0, σ_1, and σ_2, but they are always positive. In this case Q_{123} has degree 6, Q_{12} and Q_{13} degree 5, and Q_1 degree 4. The system is therefore stable if the real and imaginary parts of (4.28) interleave.

Although taking the limit $c \to \infty$ has reduced the system from $r = 3$ to $r = 2$, as in the previous example, we cannot apply Theorem 3.2 directly because of the coincidence of the roots of polynomials with different degrees. However, by considering the signs of the polynomials, it is not difficult to show that the roots of the real and imaginary parts of (4.28) interleave. For example, Q_{12} and Q_{13} have opposite signs in $[-c_f, -c_a]$ so that the imaginary part has a root in this interval, but the real part cannot have a root in this interval since Q_1 and Q_{123} also have opposite signs there. Similar arguments can be applied to all such intervals to show that the roots of the real and imaginary parts do indeed interleave for all positive values of K_2 and K_3. Similarly one can readily show that the structure condition (3.6) holds as long as none of the K_i vanish. Note that although the parameters K_1, K_2, and K_3 are entirely artificial in this case, they play a crucial role in the stability analysis because they can be used to decompose the dispersion relation into polynomials whose roots can easily be determined.

We can now determine the nature of the shock structure in exactly the same way as Whitham [16]. By Theorem 3.1, a smooth shock structure is possible if none of the frozen wave speeds in the shock frame changes sign across the shock, but if this is not true, then there must be a subshock of the frozen system somewhere within the shock structure. Since c is effectively infinite, the only relevant frozen speed is $a - V$. If $a - V < 0$ in the upstream state and $a - V > 0$ in the downstream state, then the shock structure must contain a gas subshock. Since this is the slowest wave of the frozen system, the subshock appears at the downstream end of the shock structure.

5. Conclusions. In this paper we have derived a number of results for shock structures described by hyperbolic systems of balance laws with more than one relaxation process. Theorem 3.1 tells us that the conditions for the existence of a smooth shock structure are very similar to those obtained by Whitham [17] for a single relaxation process. One of these conditions is the stability of the equilibrium states and Theorem 3.2 provides a useful way of determining this. Even when the conditions of Theorem 3.2 do not hold, Example 4.3 shows that it may still be possible to use these ideas to establish stability.

Although the emphasis in this paper has been on the connection between the stability of the equilibrium system and the nature of the shock structure, the decomposition of the dispersion relation into the form given by (3.10) can provide an effective method for determining stability. For example, Tytarenko, Williams, and

Falle [14] used this technique to study instabilities in a two-fluid system that arises in certain astrophysical applications. As the examples in sections 4.2 and 4.3 show, this method is not dependent upon the existence of physically meaningful rate constants associated with different relaxation processes.

Finally, this work has shown the value of the Hermite–Biehler theorem in the study of stability. Although the original motivation for this theorem was the stability analysis of control systems and it is still widely used for this purpose, its potential in other areas has not been fully exploited.

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