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Semigroups with finitely generated universal left congruence

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Abstract
We consider semigroups such that the universal left congruence ω^ℓ is finitely generated. Certainly a left noetherian semigroup, that is, one in which all left congruences are finitely generated, satisfies our condition. In the case of a monoid the condition that ω^ℓ is finitely generated is equivalent to a number of pre-existing notions. In particular, a monoid satisfies the homological finiteness condition of being of type left-FP_1 exactly when ω^ℓ is finitely generated.

Our investigations enable us to classify those semigroups such that ω^ℓ is finitely generated that lie in certain important classes, such as strong semilattices of semigroups, inverse semigroups, Rees matrix semigroups (over semigroups) and completely regular semigroups. We consider closure properties for the class of semigroups such that ω^ℓ is finitely generated, including under morphic image, direct product, semi-direct product, free product and 0-direct union. Our work was inspired by the stronger condition, stated for monoids in the work of White, of being pseudo-finite. Where appropriate, we specialise our investigations to pseudo-finite semigroups and monoids. In particular, we answer a question of Dales and White concerning the nature of pseudo-finite monoids.

Keywords Monoids · Semigroups · Left congruences · Finitely generated · FP_1 · Pseudo-finite

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1 Introduction

A *finitary condition* for a class of (universal) algebras is a condition, defined in the appropriate language, that is satisfied by at least all finite members of the class. The concept was introduced and developed in the early part of the last century by Noether and Artin in their seminal work. Subsequently, finitary conditions have been of enormous importance in understanding the structure and behaviour of rings, groups, semigroups and many other kinds of algebras.

The classes of algebras we examine here are those of semigroups and monoids. The two finitary conditions we focus on may be stated in many different ways and arise from a variety of sources, as we explain in Sects. 2 and 3. The simplest way of approaching them is via the universal relation on a semigroup S, regarded as a left congruence. We remark that left ideals of semigroups are associated with left congruences, but, unlike the case for rings, not every left congruence comes from a left ideal. Left congruences on a monoid determine all monogenic representations by left actions on sets, in the standard way. We denote the universal left congruence relation on a semigroup S by ω_S^ℓ; on occasion, where S is not named, we write more simply ω^ℓ. The finitary conditions for S that are the subject of this article are those of ω_S^ℓ being finitely generated (as a left congruence) and the stronger condition of S being pseudo-finite. Intuitively, the latter condition puts a bound on the length of the derivation required to relate two words, using only a finite set of relations; we give a precise definition in Sect. 2. It is well known and easy to see that if G is a group, then ω_G^ℓ is finitely generated if and only if G is a finitely generated group. The work of [24] shows that G is pseudo-finite if and only if G is finite. For arbitrary monoids and semigroups, the situation is much more complex.

A semigroup S is *left noetherian* if *every* left congruence is finitely generated; certainly then ω_S^ℓ is finitely generated. The study of left noetherian semigroups was introduced by Hotzel in [11] and is still a developing topic [19]. At least in the monoid case, such a condition has been much exploited in the theory of acts over monoids. For example, if a monoid M is left noetherian, then every finitely generated left M-act is finitely presented [20]. As shown in [17], if S is left noetherian, then so is every subgroup of S. Our condition that ω_S^ℓ is finitely generated may clearly be seen to be weaker than being left noetherian: it is easy to see that ω_M^ℓ is finitely generated for any monoid M with zero. More significantly, our characterisations of semigroups S with ω_S^ℓ being finitely generated have conditions that only refer to some 'top part' (merely the identity in the case of a monoid) and properties of a minimum ideal I, including conditions on the *subgroups* of I.

We remark that many other finitary conditions have been important in the study of semigroups and monoids, naturally including the properties of being finitely generated or finitely presented [23], and other finitary conditions on the lattices of one sided congruences, for example [7,8], both of which arise from model-theoretic considerations.

In the case of a monoid M the condition that ω_M^ℓ is finitely generated is equivalent to a number of notions that have arisen from a variety of sources. From a homological standpoint the concept of being of type left-FP$_n$ was introduced for groups by Bieri [2] and later extended to monoids [4]. Much of the work into the general property of being of type left-FP$_n$ has been in the case of groups, although a recent shift to monoids can be seen in [9,10,15,16,21]. Any monoid which possesses a finite complete rewriting system was shown by Anick [1] to be of type left-FP$_n$ for each n. We refer the reader to [5] for a wider study. Using work of Kobayashi [15], we show that a monoid M is of type left-FP$_1$ exactly when ω_M^ℓ is finitely generated. Moreover, ω_M^ℓ being finitely generated may be stated in terms of the notion of ancestry introduced by White [24]. We present further formulations in Sect. 3.

The notion of being pseudo-finite was introduced in [24] in the language of ancestry. Theorem 1.7 of [24] shows that for a monoid M the augmentation ideal $\ell_1^0(M)$ is finitely-generated if and only if M is pseudo-finite. The work in [24] was motivated by the Dales-Żelazko conjecture, which states that a unital Banach algebra in which every maximal left ideal is finitely generated is necessarily finite dimensional. Through constructing links between the conjecture and ancestry, White showed the conjecture to be true for Banach algebras of the form $\ell^1(M)$ where M is a weakly right cancellative monoid. In fact, it was a question posed to the second author by Dales and White [6], concerning the nature of pseudo-finite monoids, that led to this article. We answer the question in the negative in Example 7.7.

The objective of this paper is to make a comprehensive study of those semigroups S such that ω_S^ℓ is finitely generated, or S is pseudo-finite. Our results often divide into two kinds: those for semigroups and those for monoids. On the way we consider numerous constructions, some of which have been considered before in the monoid case—see [10] for direct products, [18,22] for retracts and [9] for Clifford monoids. An important word of warning: the property of being of type left-FP$_1$ does not apply to semigroups. In [9] a semigroup is said to be of type left-FP$_1$ if S^1, the monoid obtained from S by adjoining an identity if necessary, is of type left-FP$_1$. However, for semigroups, the property of ω_S^ℓ being finitely generated and that of $\omega_{S^1}^\ell$ being finitely generated differ considerably; see, for example, Corollaries 6.4 and 6.6.

The paper is organised as follows. In Sect. 2 we list some basic properties for a semigroup S such that ω_S^ℓ is finitely generated, and formally introduce the stronger property of being pseudo-finite. In Sect. 3 we explain the relationships between the conditions that ω_S^ℓ is finitely generated, or S is pseudo-finite, and a number of other notions such as ancestry, connected right Cayley graphs, type left-FP$_1$ and right unitary generation by a subset. In Sect. 4 we consider the closure properties of the class of semigroups S with ω_S^ℓ being finitely generated, or of S being pseudo-finite, under standard algebraic constructions: morphisms, direct products, semidirect products, free products and 0-direct unions. In Sect. 5 we focus on inverse semigroups, and establish several equivalent conditions for an inverse semigroup S for ω_S^ℓ to be finitely generated, or for S to be pseudo-finite. We end the section by considering our properties for Brandt semigroups. Section 6 extends the results for Brandt semigroups to the much wider context of arbitrary Rees matrix semigroups over semigroups, with and without a zero and with or without an adjoined identity; the difference in the cases is striking. In Sect. 7 we return to considering constructions, this time two that are typical to

semigroups: those of strong semilattices of semigroups and of Bruck–Reilly extensions. We also answer the question of Dales and White by constructing a particular semilattice of groups. Finally, in Sect. 8, we take a more global view. We consider semigroups possessing a completely simple minimum ideal (without further restriction), and then specialise to the class of completely regular semigroups (semigroups which are unions of groups) and then to bands (idempotent semigroups).

We have attempted to make the paper relatively self contained. We give a brief introduction to Green's relations in Sect. 2. Many of the algebraic constructions we use will be standard, such as those of direct product or morphism. The details of those particular semigroups may be found in [13] or in the case of free products, in [3]. Our canonical notation for a semigroup is S and for a monoid is M; the identity of a monoid M is denoted by 1. We write $\overline{X}$ to denote a set $\{(1, x) : x \in X\}$ for a subset X of a monoid M. For any $X \subseteq S$ we denote by X^2 both the set of pairs $X \times X$ and the set $\{xy : x, y \in S\}$, depending on the context, which we will always endeavour to make clear. The identity relation on any set X is denoted by ι or ι_X. Given a subset A of S^2 for a semigroup S, the left congruence on S generated by A will be denoted by either ρ_A or $\langle A\rangle$. The set of idempotents of a semigroup S is denoted by $E(S)$.

2 Preliminaries

We make some initial observations surrounding the condition that ω^ℓ is finitely generated. This leads naturally to the point where we can define the property of being pseudo-finite. We start with the following well-known result.

Lemma 2.1 [14, Lemma I.4.37] *Let S be a semigroup and A be a subset of S^2. Then, for any $a, b \in S$, $a\ \rho_A\ b$ if and only if either $a = b$ or there exists a sequence*

$$a = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = b$$

where $t_i \in S^1$ and $(c_i, d_i) \in A \cup A^{-1}$ for all $1 \leq i \leq n$.

The sequence in the above lemma is referred to as an *A-sequence of length n*; if $n = 0$, we interpret this sequence as being $a = b$.

Given a pair of left congruences δ and δ' on S, we say that *δ is a principal extension of δ'* if $\delta \supset \delta'$ and there exists $(a, b) \in S^2$ such that $\delta = \langle \delta' \cup \{(a, b)\}\rangle$. Clearly, if δ covers δ' in the lattice of left congruences on S then δ is a principal extension of δ', but the converse is not true. This may be easily seen by observing that for any monoid M with 0, we have that $\omega^\ell_M = \langle \iota \cup \{1, 0\}\rangle$, but M may have non-trivial proper left congruences.

The proof of the next lemma is straightforward.

Lemma 2.2 *Let S be a semigroup. Then the following are equivalent:*

(1) *ω^ℓ_S is finitely generated;*
(2) *there is a finite chain $\iota = \delta_0 \subset \delta_1 \subset \cdots \subset \delta_n = \omega^\ell_S$ of left congruences on S where each δ_i is a principal extension of δ_{i-1} for all $1 \leq i \leq n$;*

(3) *there exists a finite subset X of S such that $\omega_S^\ell = \langle X^2 \rangle$;*
(4) *there exists a finite subset X of S such that for any $x \in X$, $\omega_S^\ell = \langle \{x\} \times X \rangle$;*
(5) *for any $u \in S$ there exists a finite subset X of S such that $u \in X$ and $\omega_S^\ell = \langle \{u\} \times X \rangle$.*

It follows from Lemma 2.2 that, for a semigroup S with ω_S^ℓ being finitely generated, we always have a generating set for ω_S^ℓ of the form X^2 for some finite subset $X \subseteq S$. Moreover, in the case when S is a monoid, we always have a generating set of the form $\overline{X} = \{(1, x) : x \in X\}$ for some finite $X \subseteq S \backslash \{1\}$. We shall make use of this observation throughout the paper without reference.

Definition 2.3 Let S be a semigroup with ω_S^ℓ being generated by A, where $A \subseteq S^2$ is finite. We say that S is *pseudo-finite with respect to A* if there exists $n \in \mathbb{N}$ such that for any $a, b \in S$, there is an A-sequence from a to b of length at most n.

We say that S is *pseudo-finite with respect to X* if $X \subseteq S$ is finite and S is pseudo-finite with respect to X^2.

Remark 2.4 Let S be a semigroup with $A \subseteq S^2$ finite. Then S is pseudo-finite with respect to A if and only if ω_S^ℓ is the union of a finite chain of reflexive, symmetric relations ρ_A^n where $a \, \rho_A^n \, b$ if there is an A-sequence of length at most n relating a to b.

The following lemma shows that the property of ω_S^ℓ being generated by a finite set, or of S being pseudo-finite with respect to a finite generating set, is independent of the given set of generators.

Lemma 2.5 *Let S be a semigroup and let ω_S^ℓ be finitely generated by $H \subseteq S^2$. Suppose $\omega_S^\ell = \langle K \rangle$ for some $K \subseteq S^2$. Then there exists a finite subset K' of K such that $\omega_S^\ell = \langle K' \rangle$.*

Further, if there exists $m \in \mathbb{N}$ such that for any $a, b \in S$, there is an H-sequence from a to b of length at most m, then there is an $m' \in \mathbb{N}$ such that for any $a, b \in S$, there is a K'-sequence from a to b of length at most m'.

Proof The first statement is well known, but we give a short proof here for completeness and convenience.

We are given that $\omega_S^\ell = \langle H \rangle = \langle K \rangle$. Let $(h, k) \in H$. Then there is a K-sequence of length $n := n(h, k)$

$$h = t_1 c_1, t_1 d_1 = t_2 c_2, \ldots, t_n d_n = k,$$

where $(c_i, d_i) \in K \cup K^{-1}$ and $t_i \in S^1$.

Let

$$K_{(h,k)} = \left(\{(c_1, d_1), \ldots, (c_n, d_n)\} \cup \{(c_1, d_1), \ldots, (c_n, d_n)\}^{-1} \right) \cap K,$$

so that $K_{(h,k)} \subseteq K$, $|K_{(h,k)}| < \infty$ and $(h, k) \in \langle K_{(h,k)} \rangle$. Let

$$K' = \bigcup_{(h,k) \in H} K_{(h,k)}.$$

Since H is finite, K' is a finite subset of K and it is clear that $H \subseteq \langle K' \rangle$. Thus

$$\omega_S^\ell = \langle H \rangle \subseteq \langle K' \rangle \subseteq \omega_S^\ell,$$

giving $\omega_S^\ell = \langle K' \rangle$ as required.

Further, suppose there exists $m \in \mathbb{N}$ such that for any $a, b \in S$, there exists an H-sequence from a to b of length at most m, that is,

$$a = s_1 h_1, s_1 k_1 = s_2 h_2, \ldots, s_m k_m = b,$$

where $(h_i, k_i) \in H \cup H^{-1}$ and $s_i \in S^1$. Notice that for each $(h, k) \in H$, there is a K'-sequence of length $n(h, k)$ for some $n \in \mathbb{N}$ connecting h to k, and hence a sequence of the same length connecting uh to uk, for any $u \in S^1$. Replace each $(s_i h_i, s_i k_i)$, $1 \leq i \leq m$ in above sequence with a K'-sequence of length $n(h_i, k_i)$. Let

$$m' = m \times \max\{n(h, k) : (h, k) \in H\}.$$

Then there is a K'-sequence from a to b of length at most m'. □

We make use of Lemma 2.5 in the definition below.

Definition 2.6 A semigroup S is *pseudo-finite* if it is pseudo-finite with respect to some finite $A \subseteq S^2$, or equivalently, if it is pseudo-finite with respect to some finite $X \subseteq S$.

The following result is essentially folklore (see [24]):

Proposition 2.7 *Let G be a group and $A \subseteq G^2$. Then:*

(1) *ω_G^ℓ is generated by A if and only if G is generated by $\{a^{-1}b : (a, b) \in A\}$;*
(2) *G is pseudo-finite if and only if it is finite.*
Let M be monoid and let $B \subseteq M$. Then:
(3) *if M is generated by B, then ω_M^ℓ is generated by $B \times \{1\}$.*

Note that in Proposition 2.7, if A is finite then certainly so is $\{a^{-1}b : (a, b) \in A\}$. On the other hand, if we have a finite set C of generators of G, then $C \times \{1\}$ finitely generates ω_G^ℓ using (3). We now give a way of extending Proposition 2.7(3) to semigroups.

Lemma 2.8 *Let S be a semigroup generated by $X \subseteq S$. Then ω_S^ℓ is generated by W_X^2, where $W_X = X \cup \{xy : x, y \in X\}$. Moreover, if X is finite, then so is W_X.*

Proof Fix $x \in X$. For $x_{i_1} x_{i_2} \cdots x_{i_k} \in S$ where $x_{i_j} \in X$, $1 \leq j \leq k$, we have

$$\begin{aligned} &x_{i_1} \cdots (x_{i_{k-1}} x_{i_k})\, \rho_{W_X^2}\, x_{i_1} \cdots x_{i_{k-1}} \\ &\quad = x_{i_1} \cdots (x_{i_{k-2}} x_{i_{k-1}})\, \rho_{W_X^2} \cdots \rho_{W_X^2}\, x_{i_1} x_{i_2}\, \rho_{W_X^2}\, x_{i_1}\, \rho_{W_X^2}\, x. \end{aligned}$$

□

In the above result, if $S = \bigcup_{1 \leq i \leq n} X^i$ for $n \in \mathbb{N}$, then clearly there is a bound on the length of $\rho_{W_X^2}$-sequences needed to connect elements of S, so that if X is finite, then S is finite and demonstrably, S is pseudo-finite. The same comments apply to Proposition 2.7(3). However, we show in Example 5.5 that there exists a finitely generated monoid which is not pseudo-finite. On the other hand, if M is any monoid with 0, it is clear that M is pseudo-finite with respect to $\{(1, 0)\}$.

For what follows, it is convenient to recall a few details of Green's relations on a semigroup S, and their associated pre-orders; for further information, we recommend [13]. The relation $\leq_{\mathcal{L}}$ is defined on a semigroup S by the rule

$$a \leq_{\mathcal{L}} b \Leftrightarrow S^1 a \subseteq S^1 b.$$

It is easy to see that $\leq_{\mathcal{L}}$ is a right compatible pre-order, so that the associated equivalence relation $\mathcal{L}$ is a right congruence. The relations $\leq_{\mathcal{R}}$ and $\mathcal{R}$ are defined dually, and the relations $\leq_{\mathcal{J}}$ and $\mathcal{J}$ are obtained using principal two-sided ideals. The relation $\mathcal{H}$ is defined as $\mathcal{L} \cap \mathcal{R}$; any $\mathcal{H}$-class containing an idempotent e is a maximal subgroup, denoted by H_e.

We now make some observations which will be very useful for later sections.

Lemma 2.9 *Let S be a non-trivial semigroup such that $\omega_S^\ell = \langle A \rangle$ for some $A \subseteq S^2$. Let $c(A) = \{x : \exists (x, y) \in A \cup A^{-1}\}$. Then for every $s \in S$ there exists some $x \in c(A)$ such that $s \leq_{\mathcal{L}} x$.*

Proof As S is non-trivial we have $A \neq \emptyset$ and then $c(A) \neq \emptyset$. Let $s \in S$ and choose $u \in S$ with $s \neq u$. Since $s \rho_A u$, we have $s = tx$ for some $t \in S^1$ and $x \in c(A)$, giving $s \leq_{\mathcal{L}} x$. □

Proposition 2.10 *Let S be a semigroup. Then ω_S^ℓ is finitely generated (S is pseudo-finite) if and only if $\omega_{S^1}^\ell$ is finitely generated (S^1 is pseudo-finite) and there is a finite set $U \subseteq S$ such that for every $a \in S$ we have $a \leq_{\mathcal{L}} u$ for some $u \in U$.*

Proof It suffices to consider the case where S is not a monoid.

Suppose first that ω_S^ℓ is finitely generated by $A \subseteq S^2$. The subset U exists by Lemma 2.9 and clearly $\omega_{S^1}^\ell$ is finitely generated by $A \cup \{(1, u)\}$ for any $u \in S$.

Conversely, suppose that U exists as given and $\omega_{S^1}^\ell$ is finitely generated. Without loss of generality we can assume that $\overline{X}$ generates $\omega_{S^1}^\ell$ for some finite $X \subseteq S$. Let $Y = X^2 \cup \{(u, ux) : u \in U, x \in X\}$ so that $Y \subseteq S^2$ and is finite.

Let $s, t \in S$. We know that $s \rho_{\overline{X}} t$, so there exists an $\overline{X}$-sequence

$$s = t_1 c_1, t_1 d_1 = t_2 c_2, \ldots, t_n d_n = t$$

from s to t. Let $z_i = t_i c_i$ for $1 \leq i \leq n$. Suppose first that no $z_i = 1$ and consider $(t_i c_i, t_i d_i)$ for $1 \leq i \leq n$. Clearly $t_i \neq 1$ and so $t_i = pu$ for some $p \in S^1$ and $u \in U$. Then $(t_i c_i, t_i d_i) = (puc_i, pud_i)$ and $(uc_i, ud_i) \in Y$. It follows that $s \rho_Y t$ in this case. The other situation is where $z_i = 1$ for some $1 < i \leq n$. Let i be least such that $z_i = 1$. We must have that $t_{i-1} = 1 = t_i = d_{i-1} = c_i$, so that $(c_{i-1}, d_{i-1}) = (x, 1)$

and $(c_i, d_i) = (1, y)$ for some $x, y \in X$. Then $s \rho_Y t_{i-1}c_{i-1} = x \rho_Y y = d_i t_i = z_{i+1}$, interpreting z_{n+1} by t. An inductive argument now completes the proof that $s \rho_Y t$.

The claim involving pseudo-finiteness is clear from the argument above. □

In dealing with semigroups such that ω_S^ℓ is finitely generated we may, where convenient, call up the above description involving the monoid S^1.

The following result will be particularly useful when considering semigroups with a minimum ideal.

Lemma 2.11 *Let S be a semigroup and let I be a left ideal of S. Then ω_S^ℓ is finitely generated if and only if there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$ and $\omega_I^\ell = \rho_{X^2}|_{I\times I}$. In addition, S is pseudo-finite if and only if there exists $n \in \mathbb{N}$ such that for any $a, b \in I$, there is an X^2-sequence from a to b of length at most n.*

Proof Suppose that $\omega_S^\ell = \langle X^2 \rangle$ for some finite subset X of S. Clearly, $\omega_I^\ell = \rho_{X^2}|_{I\times I}$ and it follows from Lemma 2.9 that for every $a \in S$ there exists some $x \in X$ such that $a \leq_{\mathcal{L}} x$. Conversely, let X be a finite subset of S satisfying the required properties. Fix some $u \in I$ and let $Y = X \cup \{u\}$. For each $a \in S$, there exists $x \in X$ such that $a \leq_{\mathcal{L}} x$ and so $a = tx$ for some $t \in S^1$. It follows from the assumption $\omega_I^\ell = \rho_{X^2}|_{I\times I}$ that $a = tx \rho_{Y^2} tu \rho_{Y^2} u$ as $u, tu \in I$, so that $\omega_S^\ell = \rho_{Y^2}$.

The second statement now follows from Lemma 2.5. □

Corollary 2.12 *Let S be a monoid and let I be a left ideal of S. Then ω_S^ℓ is finitely generated if and only if there exists a finite subset X of S such that $\omega_I^\ell = \rho_{X^2}|_{I\times I}$. In addition, S is pseudo-finite if and only if there exists $n \in \mathbb{N}$ such that for any $a, b \in I$, there is an X^2-sequence from a to b of length at most n.*

As a consequence of Lemma 2.11 we obtain the following extension of an existing result for monoids [9, Proposition 6]. This result also follows from the corresponding result for monoids in [9], together with Proposition 2.10. In the monoid case clearly we may drop the condition on the relation $\leq_{\mathcal{L}}$ below. Note also that the converse does not hold in general, as seen in [9, Example 1].

Corollary 2.13 *Let S be a semigroup and I be a left ideal of S. Suppose that ω_I^ℓ is finitely generated (I is pseudo-finite) and there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$. Then ω_S^ℓ is finitely generated (S is pseudo-finite).*

Corollary 2.14 *The following are equivalent for a semigroup S with zero:*

(1) *there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$;*
(2) *ω_S^ℓ is finitely generated;*
(3) *S is pseudo-finite.*

Proof Clearly, $\{0\}$ forms an ideal of S, and hence the required result holds by Lemma 2.11. □

Note that in the above, ω_S^ℓ is generated by $X \times \{0\}$. In the monoid case, this reduces to $\{(1, 0)\}$, so, as we have already observed:

Corollary 2.15 *Any monoid with zero is pseudo-finite.*

3 Alternative conditions for ω^ℓ to be finitely generated

In this section we give a variety of alternative conditions for semigroups and monoids such that ω^ℓ is finitely generated. For monoids these conditions are already known, involving the notions of ancestry [24], right Cayley graphs, the homological condition called type left-FP$_1$, and of being right unitarily generated [15]. We remind the reader that in [9] a semigroup S is said to be of type left-FP$_1$ if and only if the monoid S^1 is of type left-FP$_1$. Although we do not attempt to consider here the property of being type left-FP$_1$ for semigroups, we take an essentially different approach to [9]. Namely, for a semigroup S, we consider the property that ω^ℓ_S is finitely generated, which is a stronger property than $\omega^\ell_{S^1}$ being finitely generated. Indeed, according to Corollary 2.15, $\omega^\ell_{S^1}$ is finitely generated for any semigroup with 0. On the other hand if N is an infinite null semigroup, then as it has infinitely many maximal left ideals, Corollary 2.14 tells us that ω^ℓ_N is not finitely generated.

Our first condition involves the notion of a left M-act over a monoid M.[1] Let S be a semigroup and let A be a non-empty set. Then A is a left S-act if there is a map $S \times A \to A$ where $(s, a) \mapsto s \cdot a$, such that for all $s, t \in S$ and $a \in A$ we have $s \cdot (t \cdot a) = (st) \cdot a$. If $S = M$ is a monoid then we also insist that $1 \cdot a = a$ for all $a \in A$. The notion of a left M-act A being finitely presented is the standard one from universal algebra, that is, A is isomorphic to a finitely generated free left M-act factored by a finitely generated left M-act congruence. The free left M-act on one generator is isomorphic to M regarded as a left ideal of itself, and as such, a left M-act congruence is precisely a left congruence on M. For further details of monoid and semigroup acts we refer the reader to the monograph [14].

Proposition 3.1 *Let M be a monoid. Then ω^ℓ_M is finitely generated if and only if the trivial left M-act $\Theta_M = \{z\}$ is finitely presented.*

Proof If ω^ℓ_M is finitely generated then $\Theta_M \cong M/\omega^\ell_M$ is finitely presented.

Conversely, if Θ_M is finitely presented, then by a standard result of algebra, it has finite presentation with respect to any set of generators. Thus ω^ℓ_M, which is the kernel of the M-act morphism $M \to \Theta_M$, is finitely generated. □

For the corresponding result for semigroups we need a little more care. Let S be a semigroup. We say that an S-act A is *quasi-free* if A is a disjoint union of copies of S, where S is regarded as a left ideal of itself. To ease notation, we write $A = \bigcup_{i \in I} S_i$, where $S_i = \{s_i : s \in S\}$, $s_i = t_j$ if and only if $i = j$ and $s = t$, and for any $s_i \in A$ and $t \in S$ we have $t \cdot s_i = (ts)_i$.

Proposition 3.2 *Let S be a semigroup. Then ω^ℓ_S is finitely generated if and only if the trivial S-act Θ_S is isomorphic to a quasi-free S-act $A = \bigcup_{i \in I} S_i$ where I is finite, factored by a finitely generated congruence.*

Proof If ω^ℓ_S is finitely generated then with $I = \{i\}$ and identifying $A = S_1$ with S we have $S \to \Theta_S$ has kernel ω^ℓ_S.

[1] The authors are grateful to Professor Nik Ruškuc for this observation.

Conversely, suppose that $\Theta_S \cong A/\rho$ where $A = \bigcup_{i\in I} S_i$, $\rho = \langle H\rangle$ and I and H are finite. Putting $K = \{(u, v) : \exists(u_i, v_j) \in H \text{ for some } i, j \in I\}$, it is easy to see that $\rho_K = \omega_S^\ell$. □

We next consider the notion of *ancestry* in a monoid due to White [24].

Definition 3.3 [24] Let M be a monoid and let X be a non-empty subset of M. We say that an element $a \in M$ has an *ancestry of length n* with respect to X if there exists a finite sequence $(z_i)_{i=1}^n$ of length n in M such that $z_1 = a$, $z_n = 1$ and for each $1 < i \leq n$ there exists $x \in X$ such that either $z_i x = z_{i-1}$ or $z_i = z_{i-1}x$.

Lemma 3.4 *Let M be a monoid and let X be a finite subset of M. Then ω_M^ℓ is (finitely) generated by $\overline{X}$ (M is pseudo-finite with respect to $\overline{X}$) if and only if every element of M has an ancestry (of bounded length) with respect to X.*

Proof Suppose that $\omega_M^\ell = \rho_{\overline{X}}$. For any $a \in M$ we have $a\ \rho_{\overline{X}}\ 1$, so that there exists a sequence $a = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = 1$, where $t_i \in S^1$ and $(c_i, d_i) \in \overline{X} \cup \overline{X}^{-1}$ for all $1 \leq i \leq n$. Let $z_1 = a$, $z_n = 1$ and $z_i = t_id_i$ for all $1 < i < n$. If $(c_i, d_i) = (1, x) \in \overline{X}$ we have $z_{i-1} = t_ic_i = t_i$ and so $z_i = t_id_i = z_{i-1}x$. On the other hand, if $(d_i, c_i) = (1, x) \in \overline{X}$ then $z_i = t_id_i = t_i$ and so $z_{i-1} = t_ic_i = z_ix$. Hence $(z_i)_{i=1}^n$ is an ancestry of a with respect to X of length n.

Conversely, suppose that an element $a \in M$ has an ancestry of length n with respect to X. Then there exists a finite sequence $(z_i)_{i=1}^n$ in M such that $z_1 = a$, $z_n = 1$ and for $1 < i \leq n$, there exists $x \in X$ such that $z_ix = z_{i-1}$ or $z_i = z_{i-1}x$. In the first case, $(z_{i-1}, z_i) = (z_ix, z_i) = (z_i, z_i)(x, 1)$ and in the second, $(z_{i-1}, z_i) = (z_{i-1}, z_{i-1}x) = (z_{i-1}, z_{i-1})(1, x)$. We thus obtain that $z_1 = a, z_2, \ldots, z_n = 1$ is a $\rho_{\overline{X}}$-sequence of length n from a to 1. Hence $\omega_M^\ell = \rho_{\overline{X}}$.

The statement involving pseudo-finiteness is clear from the above. □

We may restate the concept of ancestry by using right Cayley graphs.

Definition 3.5 Let M be a monoid and X a subset of M (we do not assume that X is a generating set for M). The *right Cayley graph* $\Gamma^r(M, X)$ of M with respect to X is defined as follows:

(1) the vertex set is M;
(2) there is a directed edge labelled by $x \in X$ from a to b, denoted by $a \xrightarrow{x} b$, if $b = ax$.

By definition, a right Cayley graph $\Gamma^r(M, X)$ is directed. However, underlying any right Cayley graph is an undirected labelled graph $\Gamma_u^r(M, X)$. We say that $\Gamma_u^r(M, X)$ is of *bounded width* if there is $n \in \mathbb{N}$ such that any two distinct vertices are joined by a path of length no greater than n; note that this implies $\Gamma_u^r(M, X)$ is connected.

Proposition 3.6 *Let M be a monoid and let X be a finite subset of M. Then ω_M^ℓ is finitely generated by $\overline{X}$ if and only if $\Gamma_u^r(M, X)$ is connected. Moreover, M is pseudo-finite with respect to $\overline{X}$ if and only if $\Gamma_u^r(M, X)$ has bounded width.*

Proof Suppose that $\omega_M^\ell = \rho_{\overline{X}}$. Let $a, b \in M$. Then there exists an $\overline{X}$-sequence

$$a = t_1 c_1, t_1 d_1 = t_2 c_2, \ldots, t_n d_n = b$$

where $(c_i, d_i) \in \overline{X} \cup \overline{X}^{-1}$ and $t_i \in M$ for all $1 \leq i \leq n$. Notice that for every pair $(t_i c_i, t_i d_i)$ we have a path in $\Gamma^r(M, X)$ of the form $t_i c_i = t_i \xrightarrow{d_i} t_i d_i$ if $c_i = 1$, or $t_i c_i \xleftarrow{c_i} t_i = t_i d_i$ if $d_i = 1$, so that a is connected to b in $\Gamma_u^r(M, X)$.

Conversely, suppose that $\Gamma_u^r(M, X)$ is connected. Suppose there exists an edge between a pair of elements a and b of S, so that we must have $a = bx$ or $b = ax$ for some $x \in X$ and $(a, b) \in \rho_{\overline{X}}$. Since any two vertices are connected by a path it follows that $\rho_{\overline{X}} = \omega_M^\ell$.

The second claim of the proposition is clear from the above proof. □

Corollary 3.7 *Let S be a semigroup. Then ω_S^ℓ is finitely generated (pseudo-finite) if and only if there exists some finite subset X of S such that $\Gamma_u^r(S^1, X)$ of S^1 is connected (has bounded width), and there is a finite set $U \subseteq S$ such that for every $a \in S$ we have $a \leq_{\mathcal{L}} u$ for some $u \in U$.*

Proof This follows from Propositions 2.10 and 3.6, together with the fact we may assume that if $\Gamma_u^r(S^1, X)$ is connected, then $1 \notin X$. □

Another way to characterise a monoid M with ω_M^ℓ being finitely generated is to use a connection between right Cayley graphs and the property of being right unitarily generated. This equivalence was established by Kobayashi [15], together with the equivalence to the property of M being type left-FP_1, as we now briefly explain. Further details may be found in [15].

Definition 3.8 [15] Let M be a monoid and N be a submonoid of M. Then N is said to be *right unitary* if for any $n \in N$ and $m \in M$,

$$mn \in N \Rightarrow m \in N.$$

Let X be a subset of M and let $U^r(X)$ denote the smallest right unitary submonoid of M containing X. If $M = U^r(X)$, then M is said to be *right unitarily generated* by X.

Definition 3.9 [15] Let M be a monoid and $\mathbb{Z}M$ be the monoid ring of M over the integers $\mathbb{Z}$. For $n \geqslant 0$, M is of *type left*-FP_n if there is a resolution

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$, regarded as a left $\mathbb{Z}M$-module with trivial action, such that $A_0, A_1, \ldots, A_n$ are finitely generated left $\mathbb{Z}M$-modules.

Our next result, collecting together the equivalent conditions for ω_M^ℓ to be finitely generated, follows from Proposition 3.2, Lemma 3.4 and Proposition 3.6, with the remaining conditions coming from [15, Proposition 2.4, Theorem 2.6].

Theorem 3.10 *Let M be a monoid with a finite subset X. The following conditions are equivalent:*

(1) *ω_M^ℓ is finitely generated by $\overline{X}$;*
(2) *the trivial M-act Θ_M is isomorphic to $M/\rho_{\overline{X}}$ and so is finitely presented;*
(3) *each element of M has an ancestry with respect to X;*
(4) *the undirected right Cayley graph $\Gamma_u^r(M, X)$ of M with respect to X is connected;*
(5) *M is right unitarily generated by X.*

Further, M is of type left-FP$_1$ if and only if any (all) of these conditions hold.

Analogous omnibus results also hold for semigroups, and for pseudo-finite monoids and semigroups. In the case of being right unitarily generated, we would require a further concept of the number of steps involved in the generation of M.

4 Standard constructions

The aim of this section is to consider several standard constructions and their behaviour with respect to ω^ℓ being finitely generated, and of being pseudo-finite. As usual, there are two kinds of questions one can ask: whether the class of semigroups with ω^ℓ being finitely generated is closed under a particular construction, and whether the fact that ω^ℓ is finitely generated passes down to components of the construction. The properties we look at include morphisms, direct products, semidirect products, free products and 0-direct unions. We also provide a number of examples.

Our first result is known in the case of a retract of a monoid [21, Theorem 3].

Proposition 4.1 *Let S be a semigroup and let T be a morphic image of S. If ω_S^ℓ is finitely generated (S is pseudo-finite), then ω_T^ℓ is finitely generated (T is pseudo-finite).*

Proof Suppose that $\omega_S^\ell = \rho_A$ for some finite subset A of S^2 and $\varphi : S \longrightarrow T$ is an epimorphism. For any $u, v \in T$, there exists $s, t \in S$ such that $s\varphi = u$ and $t\varphi = v$. Hence there exists a sequence $s = s_1a_1, s_1b_1 = s_2a_2, \ldots, s_nb_n = t$ where $s_i \in S^1$ and $(a_i, b_i) \in A \cup A^{-1}$ for all $1 \leq i \leq n$. By applying φ to the above sequence, we have

$$u = s\varphi = (s_1\varphi)(a_1\varphi), (s_1\varphi)(b_1\varphi) = (s_2\varphi)(a_2\varphi), \ldots, (s_n\varphi)(b_n\varphi) = t\varphi = v$$

giving that $\omega_T^\ell = \rho_{A\varphi}$ where $A\varphi = \{(u\varphi, v\varphi) : (u, v) \in A\}$. Clearly, if S is pseudo-finite with respect to A, then T is pseudo-finite with respect to $A\varphi$. □

In [10], a direct product of a pair of monoids M and N is shown (in the context of being of type left-FP$_1$) to have the property that $\omega_{M\times N}^\ell$ is finitely generated if and only if both ω_M^ℓ and ω_N^ℓ are finitely generated. We now extend this result to cover pseudo-finiteness. The first part of the lemma below follows immediately from Proposition 4.1, by applying the projection morphisms.

Proposition 4.2 *Let S and T be semigroups. If $\omega_{S\times T}^\ell$ is finitely generated ($S \times T$ is pseudo-finite) then both ω_S^ℓ and ω_T^ℓ are finitely generated (pseudo-finite). If S and T are monoids, then the converse is true.*

Proof Let S and T be monoids such that $\omega_S^\ell = \rho_{X^2}$ and $\omega_T^\ell = \rho_{Y^2}$ for some finite subsets $X \subseteq S$ and $Y \subseteq T$. For any $(s,t), (u,v) \in S \times T$, we have $s = s_1a_1, s_1b_1 = s_2a_2, \ldots, s_mb_m = u$ where $m \in \mathbb{N}^0$, $s_i \in S^1$, $(a_i, b_i) \in X^2$ for all $1 \leq i \leq m$, and $t = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = v$ where $n \in \mathbb{N}^0$, $t_i \in T^1$, $(c_i, d_i) \in Y^2$ for all $1 \leq i \leq n$. If $n \geqslant m$, then we put $s_{m+1} = \cdots = s_n = s_m$ and $a_{m+1} = b_{m+1} = \cdots = a_n = b_n = b_m$. Then

$$(s,t) = (s_1a_1, t_1c_1), (s_1b_1, t_1d_1) = (s_2a_2, t_2c_2), \ldots, (s_nb_n, t_nd_n) = (u,v),$$

so that

$$(s,t) = (s_1,t_1)(a_1,c_1), (s_1,t_1)(b_1,d_1) = (s_2,t_2)(a_2,c_2), \ldots, (s_n,t_n)(b_n,d_n) = (u,v).$$

A similar discussion holds for $n < m$, so that $\omega_{S\times T}^\ell = \rho_{(X\times Y)^2}$. Since the length of the $(X \times Y)^2$-sequence required is no greater than the maximum of m and n, it is clear that the statement on pseudo-finiteness also holds. □

The converse of Proposition 4.2 does not necessarily hold if we remove the condition that S and T are monoids.

Example 4.3 Let C be the infinite cyclic monoid generated by a. It follows from Proposition 2.7 we have $\omega_C^\ell = \rho_A$ where $A = \{(a, a^2)\}$. For any $i \in \mathbb{N}$, if $(a, a^i) = s(a^j, a^k)$ where $s \in C^2$ we have $s = (1,1)$ and $(a, a^i) = (a^j, a^k)$. Lemma 2.9 now says $\omega_{C\times C}^\ell$ is not finitely generated.

Let S and T be monoids such that T acts on S. If $t \cdot (ss') = (t \cdot s)(t \cdot s')$ for all $t \in T$ and $s, s' \in S$, then we say that T acts on S by *morphisms*. In this case we can form the *semidirect product* $S \rtimes T$ with underlying set $S \times T$ and binary operation given by $(s,t)(s',t') = (s(t \cdot s'), tt')$. It is easy to see that $S \rtimes T$ is a semigroup, and if T acts *monoidally*, that is, if $t \cdot 1_S = 1_S$ for all $t \in T$, then $S \rtimes T$ is a monoid with identity $(1,1) := (1_S, 1_T)$.

Proposition 4.4 *Let S and T be monoids such that $\omega_S^\ell = \langle U^2 \rangle$ and $\omega_T^\ell = \langle V^2 \rangle$ for some finite subsets $U \subseteq S$ and $V \subseteq T$. Suppose T acts monoidally on S by morphisms. Then $\omega_{S\rtimes T}^\ell$ is finitely generated. Moreover, if S and T are pseudo-finite, then so is $S \rtimes T$.*

Proof Let $W = P \cup Q$ where $P = \{((s,1),(s',1)) : s, s' \in U\}$ and $Q = \{((1,t),(1,t')) : t, t' \in V\}$. We claim that $\omega_{S\rtimes T}^\ell = \langle W \rangle$. For this we notice that if $u \in S$ is connected to $1 \in S$ via a U^2-sequence of length n, then $(u,1)$ is connected to $(1,1)$ in $S \rtimes T$ via a P-sequence of the same length. A similar statement holds for T and Q. Now let $(s,t) \in S \rtimes T$. Then

$$(s,t) = (s,1)(1,t) \; \rho_W \; (s,1)(1,1) = (s,1) \; \rho_W \; (1,1),$$

and so the required results hold. □

Since T is a morphic image of $S \rtimes T$, if $\omega^\ell_{S\rtimes T}$ is finitely generated ($S \rtimes T$ is pseudo-finite), then ω^ℓ_T is finitely generated (T is pseudo-finite). In the case of a direct product, we know that if $\omega^\ell_{S\times T}$ is finitely generated then so is ω^ℓ_S. For arbitrary semidirect products, this need not hold, as we will see in Example 5.6.

The following example shows that in Proposition 4.4, the action of T on S being monoidal is necessary.

Example 4.5 Let X be a finite set and let X^* be the free monoid generated by X. By Proposition 2.7, $\omega^\ell_{X^*}$ is finitely generated. Let $\mathcal{Y}$ be the join semilattice consisting of all finite subsets of X^* including the empty word ϵ under union and let $\mathcal{Y}^0 = \mathcal{Y} \cup \{0\}$. Since $\mathcal{Y}^0$ has identity $\{\epsilon\}$ and a zero, $\omega^\ell_{\mathcal{Y}^0}$ is finitely generated by Corollary 2.15. Define an action of X^* on $\mathcal{Y}^0$ as follows:

$$w \cdot 0 = 0,\ w \cdot A = wA = \{wa : a \in A\} \text{ for } w \in X^* \text{ and } A \in \mathcal{Y}.$$

It is easy to check that this is an action by morphisms, but is not a monoid action because $w \cdot \{\epsilon\} = \{w\} \neq \{\epsilon\}$. Put $T = \mathcal{Y}^0 \rtimes X^*$. Let $x \in X$ and, for each $i \in \mathbb{N}$, suppose $(\{\epsilon\}, x^i) = (B, u)(Y, w_i)$ for some $(B, u) \in T^1$ and $(Y, w_i) \in T$. Then $\{\epsilon\} = B \cup (u \cdot Y)$, so that $B = u \cdot Y = \{\epsilon\}$, giving $u = \epsilon$, and $w_i = x^i$. Hence T has infinitely many maximal $\mathcal{L}$-classes, and ω^ℓ_T is not finitely generated by Lemma 2.9.

For the next result it is convenient to have the notion of length $|w|$ of an element w in the free product $S * T$ of semigroups S and T. We say that w has *length* n if $w = a_1 a_2 \ldots a_n$ where, if $a_i \in S$ (respectively, T), then $a_{i+1} \in T$ (respectively, S). Notice that $|ww'| = |w| + |w'|$ or $|w| + |w'| - 1$.

Proposition 4.6 *Let S and T be semigroups. Then ω^ℓ_{S*T} is finitely generated if and only if ω^ℓ_S and ω^ℓ_T are finitely generated. The semigroup $S * T$ is never pseudo-finite.*

Proof Suppose first that $\omega^\ell_S = \rho_{U^2}$ and $\omega^\ell_T = \rho_{V^2}$ where U and V are finite. Fix $u \in U$ and $v \in V$ and put $W = U \cup V$; note that $u\ \rho_{W^2}\ v$. We claim that $\omega^\ell_{S*T} = \rho_{W^2}$. If w is an element of length 1 in $S * T$ then either $w \in S$, so that as $w\ \rho_{U^2}\ u$ in S, we also have $w\ \rho_{W^2}\ u$ in $S * T$; similarly if $w \in T$. Suppose now that $n > 1$ and any element of length $n-1$ is ρ_{W^2}-related to u (or, equivalently, to v). Let $w = a_1 a_2 \ldots a_n \in S * T$ have length n. By our inductive hypothesis, $w\ \rho_{W^2}\ a_1 u$ and $w\ \rho_{W^2}\ a_1 v$. Since one of $a_1 u, a_1 v$ has length 1, we have completed our claim.

Conversely, suppose that ω^ℓ_{S*T} is finitely generated by P^2. Consider the monoid S^1. Clearly S embeds into S^1 and there is the trivial morphism $T \to S^1$ that takes every element of T to 1. By the nature of free products, these morphisms can be simultaneously extended to a morphism from $S * T$ to S^1 that is clearly onto. By Proposition 4.1, $\omega^\ell_{S^1}$ is finitely generated. Either S is trivial, so that clearly ω^ℓ_S is finitely generated, or we can find distinct $s, u \in S$. In the latter case there exists a P^2-sequence connecting s to u, of which the first step gives $s = tc$ for some $t \in (S * T)^1$ and $c \in P$. It follows that (with a natural identification) we have $t \in S^1$ and $c \in S$. From Proposition 2.10 we deduce ω^ℓ_S is finitely generated, and similarly for ω^ℓ_T.

To see that $S * T$ is never pseudo-finite, suppose for contradiction that $S * T$ is pseudo-finite with respect to P^2, and the bound on the length of the P^2-sequences

needed to connect elements of $S * T$ is k. Let h^t, h^b be the lengths of the longest and shortest elements of P, respectively. Consider $t \in (S * T)^1$ and $c, d \in P$. Then

$$| \, |tc| - |td| \, | = | \, |t| + |c| - |t| - |d| \pm 1 \, | = | \, |c| - |d| \pm 1 \, | \leq h^t - h^b + 1.$$

Let $w \in S * T$ have length r where $r > k(h^t - h^b + 1)$. It is now clear that w cannot be related to any element of length 1 by a P^2-sequence of length k. Thus $S * T$ cannot be pseudo-finite. □

Turning our attention to the case of the monoid free product of two monoids $M *^m N$ (where, of course, the monoid identities are identified, so that $M *^m N = M * N / \langle (1_M, 1_N) \rangle$), we note from [16, Proposition 4.1] that if M and N are finitely presented, then $\omega^\ell_{M *^m N}$ is finitely generated if ω^ℓ_M and ω^ℓ_N are. A similar argument to that in Proposition 4.6 allows us to extend this to arbitrary monoids. The new complication is that although length is still well defined, the length of a product of w and v may be less than $|w| + |v| - 1$, since right cancellative elements at the end of w may cancel with left cancellative elements at the start of v. Bearing this in mind, we can adapt the proof of Proposition 4.6 to show:

Corollary 4.7 *Let M and N be monoids. Then $\omega^\ell_{M *^m N}$ is finitely generated if and only if ω^ℓ_M and ω^ℓ_N are finitely generated. The monoid $M *^m N$ is never pseudo-finite unless one of M, N is pseudo-finite and the other is trivial.*

Proof The first statement follows as in Proposition 4.6. For the second, suppose for contradiction that $M *^m N$ is pseudo-finite with respect to P^2, and the bound on the length of the P^2-sequences needed to connect elements of $M *^m N$ is k. Let h be the length of the longest element of P. Notice if $a \in M *^m N$ and $b \in P$ then $|ab| = |a| + p$ where $-h \leq p \leq h$. Consider $t \in M *^m N$ and $c, d \in P$. Then there exists $-h \leq p, q \leq h$ such that $|tc| = |t| + p$ and $|td| = |t| + q$, and so

$$| \, |tc| - |td| \, | = | \, (|t| + p) - (|t| + q) \, | = | \, p - q \, | \leq 2h.$$

The proof then follows that of Proposition 4.6 by taking w to have length $r > 2kh$.

Finally, if neither monoid is trivial then an argument as in Proposition 4.6, adjusted as indicated above, gives that $M *^m N$ is not pseudo-finite. Without loss of generality suppose that N is trivial. Then $M *^m N$ is isomorphic to M and so is pseudo-finite if and only if M is. □

We end this section by considering 0-direct unions. We only need consider semigroups, since any monoid with 0 is pseudo-finite.

Proposition 4.8 *Let S and T be semigroups with zero and let $P = S \cup T$ be the 0-direct union of S and T. Then ω^ℓ_P is finitely generated if and only if both ω^ℓ_S and ω^ℓ_T are finitely generated. Moreover, P is pseudo-finite if and only if both S and T are pseudo-finite.*

Proof Recall from Corollary 2.14 that for a semigroup S with zero, ω^ℓ_S is finitely generated if and only if S is pseudo-finite.

If S and T are pseudo-finite, then there are finite subsets U of S (V of T) such that for every $a \in S$ ($b \in T$) there is some $u \in U$ ($v \in V$) such that $a \leq_{\mathcal{L}} u$ ($b \leq_{\mathcal{L}} v$). Let $W = U \cup V$; it is then clear that for every $p \in P$ we have $p \leq_{\mathcal{L}} w$ for some $w \in W$. Corollary 2.14 gives that P is pseudo-finite.

For the converse, we need only remark that S and T are morphic images of P, and invoke Proposition 4.1. □

By a simple induction argument we note that the result above holds for the 0-direct union of finitely many semigroups.

5 Inverse semigroups

Inverse monoids M such that ω_M^ℓ is finitely generated were briefly considered in [9] in the case where M has a least idempotent. In the case where M is a semilattice of groups, this latter condition follows from the fact that ω_M^ℓ is finitely generated, however, as we show, it is not necessary for M to have a least idempotent in order that ω_M^ℓ is finitely generated. Our focus in this section is to give a complete characterisation of those inverse semigroups S such that (i) ω_S^ℓ is finitely generated, and (ii) S is pseudo-finite.

We begin by remarking that if S is an inverse semigroup and ω_S^ℓ is generated by $A \subseteq S^2$, then the universal relation, regarded as a right congruence and denoted by ω_S^r, is generated by $A_1 = \{(a^{-1}, b^{-1}) : (a, b) \in A\}$ and so ω_S^ℓ is finitely generated if and only if ω_S^r is finitely generated.

Theorem 5.1 *Let S be an inverse semigroup and $E(S)$ be the set of idempotents of S. Then the following statements are equivalent:*

(1) *ω_S^ℓ is finitely generated;*
(2) *(i) there is a finite set $U \subseteq E(S)$ such that for every $e \in E(S)$ we have $e \leq u$ for some $u \in U$; and*
(ii) there is a finitely generated inverse subsemigroup W of S such that for all $a \in S$ and $e \in E(W)$, there exists $w \in W$ with $aw = ew^{-1}w$;
(3) *(i) there is a finite set $U \subseteq E(S)$ such that for every $e \in E(S)$ we have $e \leq u$ for some $u \in U$; and*
(iii) there is a finitely generated inverse subsemigroup W of S such that for all $a \in S$ there exists $w \in W$ with $aw \in E(W)$.

Proof (1) $\Rightarrow$ (2) Condition (i) follows easily from Proposition 2.10 and the fact that S is inverse.

To show (ii), suppose that $\omega_S^\ell = \langle X^2 \rangle$ for some finite subset $X \subseteq S$. Let W be the subsemigroup of S generated by $X \cup X^{-1}$ where X^{-1} is the set of all inverses of elements in X. First, we show that, for any $a, b \in S$, a finite sequence

$$b = s_1 p_1, s_1 q_1 = s_2 p_2, \ldots, s_n q_n = a$$

where $s_i \in S^1$, $p_i, q_i \in X$ for $1 \leq i \leq n$, gives $as = bs^{-1}s$ with $s = q_n^{-1}p_n \cdots q_1^{-1}p_1$. If $n = 1$, then

$$as = aq_1^{-1}p_1 = s_1q_1q_1^{-1}p_1 = s_1p_1p_1^{-1}q_1q_1^{-1}p_1 = bp_1^{-1}q_1q_1^{-1}p_1 = bs^{-1}s.$$

Suppose that the result holds for $n = k$. We show the result also holds for $n = k + 1$. Here we have

$$b = s_1p_1, s_1q_1 = s_2p_2, \ldots, s_kq_k = s_{k+1}p_{k+1}, s_{k+1}q_{k+1} = a.$$

Put $s_kq_k = s_{k+1}p_{k+1} = c$. Then, by induction,

$$c(q_k^{-1}p_k \cdots q_1^{-1}p_1) = b(q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}(q_k^{-1}p_k \cdots q_1^{-1}p_1).$$

Now we have

$$\begin{aligned}
&a(q_{k+1}^{-1}p_{k+1}q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= s_{k+1}q_{k+1}(q_{k+1}^{-1}p_{k+1}q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= s_{k+1}p_{k+1}(p_{k+1}^{-1}q_{k+1}q_{k+1}^{-1}p_{k+1})(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= s_{k+1}p_{k+1}(q_k^{-1}p_k \cdots q_1^{-1}p_1)(q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}\\
&\quad (p_{k+1}^{-1}q_{k+1}q_{k+1}^{-1}p_{k+1})(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= c(q_k^{-1}p_k \cdots q_1^{-1}p_1)(q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}\\
&\quad (p_{k+1}^{-1}q_{k+1}q_{k+1}^{-1}p_{k+1})(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= b(q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&\quad (q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}(p_{k+1}^{-1}q_{k+1}q_{k+1}^{-1}p_{k+1})(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= b(q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}(p_{k+1}^{-1}q_{k+1}q_{k+1}^{-1}p_{k+1})(q_k^{-1}p_k \cdots q_1^{-1}p_1)\\
&= b(q_{k+1}^{-1}p_{k+1}q_k^{-1}p_k \cdots q_1^{-1}p_1)^{-1}(q_{k+1}^{-1}p_{k+1}q_k^{-1}p_k \cdots q_1^{-1}p_1).
\end{aligned}$$

Choosing $b = e \in E(W)$ we have $as = es^{-1}s$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Suppose that W is a finitely generated inverse subsemigroup of S with a finite set $Y = Y^{-1}$ of generators, and U is the set of all idempotents guaranteed by (i). We show that $\omega_S^\ell = \langle H^2 \rangle$ where $H = Y \cup U$. Let $w_1w_2 \cdots w_k$ be a finite product of elements in Y. Then for each $1 \leq i \leq k$ there exists some $e_i \in U$ such that $w_i = w_ie_i$, and so

$$\begin{aligned}
e_1 \ \rho_{H^2} \ w_1 &= w_1e_1 \ \rho_{H^2} \ w_1w_2 \ \rho_{H^2} \ \cdots \rho_{H^2} \ w_1w_2 \cdots w_{k-1}\\
&= w_1w_2 \cdots w_{k-1}e_{k-1} \ \rho_{H^2} \ w_1w_2 \cdots w_{k-1}w_k.
\end{aligned}$$

For any $a \in S$, by assumption, there exists $w \in W$ such that $aw \in W$. By taking $e \in U$ such that $a \leq_{\mathcal{L}} e$, we have $a = ae \ \rho_{H^2} \ aw \ \rho_{H^2} \ e$, so $\rho_{H^2} = \omega_S^\ell$, as required. □

Corollary 5.2 *Let S be an inverse monoid S. Then ω_S^ℓ is finitely generated if and only if (ii) or (iii) of Theorem 5.1 hold.*

Now we specialise Theorem 5.1 to the pseudo-finite case. We first make the following observation: if S is an inverse semigroup and the semilattice $E(S)$ of idempotents of S has least element (zero) e, then $H_e = eSe$. Indeed, for any $s \in S$,

$$ese\,\mathcal{R}\,e(ses^{-1}) = e \text{ and } ese\,\mathcal{L}\,(s^{-1}es)e = e$$

so that $ese\,\mathcal{H}\,e$ and so $eSe \subseteq H_e$. Clearly, $H_e \subseteq eSe$ and so $H_e = eSe$.

Proposition 5.3 *Suppose that S is an inverse semigroup with semilattice of idempotents $E(S)$. Then S is pseudo-finite if and only if there is a finite set $U \subseteq E(S)$ such that for every $f \in E(S)$ we have $f \leq u$ for some $u \in U$; $E(S)$ has a least element e, and the group $\mathcal{H}$-class H_e is finite.*

Proof Suppose that S is pseudo-finite with respect to $X \subseteq S$. The first condition follows from Theorem 5.1. For any pair of idempotents $f, g \in E(S)$, there exists an X^2-sequence

$$g = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = f$$

where $t_i \in S, (c_i, d_i) \in X^2$ for $1 \leq i \leq n$. It follows from the proof of Theorem 5.1 that $fh = gh^{-1}h$ where $h = d_n^{-1}c_n \cdots d_1^{-1}c_1$. As S is an inverse semigroup, $fh = gh^{-1}h$ gives $fh(fh)^{-1} = gh^{-1}h(gh^{-1}h)^{-1}$, and so $fhh^{-1} = gh^{-1}h$. Note that every idempotent $f, g \in E(S)$ leads to such an idempotent $h^{-1}h$. As S is pseudo-finite, we can bound the length of the X^2-sequences required and find a finite set H consisting of those h obtained as above. Let w be the product of all $h^{-1}h$ such that $h \in H$. Then $fw = gw$, and so $fgw = gw$, giving $e = gw$ is least element for $E(S)$ and $H_e = eSe$. Further, H_e is a morphic image of S via $\varphi : S \to H_e$ defined by $s\varphi = ese$. Indeed, for any $a, b \in S$,

$$aebe = aebb^{-1}be = ab(b^{-1}eb)e = abe,$$

and so

$$(a\varphi)(b\varphi) = (eae)(ebe) = eaebe = eabe = (ab)\varphi.$$

By Proposition 4.1, H_e is pseudo-finite and hence a finite group by Proposition 2.7.

Conversely, suppose that $E(S)$ has a least element e, $H_e = eSe$ is finite and U exists as given; put $Y = H_e \cup U$. For any $a \in S$,

$$e = (a^{-1}ea)e = a^{-1}(eaa^{-1}ae) = a^{-1}(eaea^{-1}a) = a^{-1}e(aea^{-1})a = a^{-1}ea$$

giving $ae = aa^{-1}ea = ea$ and so $eae = ae$. Moreover, there exists $f \in U$ such that $a \leq_{\mathcal{L}} f$, so that $a = af\ \rho_{Y^2}\ ae = eae\ \rho_{Y^2}\ e$. Clearly then S is pseudo-finite with respect to Y. □

Clearly in the case of a monoid we may drop the first condition on Proposition 5.3. We also have the following consequence of Theorem 5.1.

Corollary 5.4 [9, Corollary 4]. *Let M be an inverse monoid with a least idempotent e. Then ω_M^ℓ is finitely generated if and only if H_e is finitely generated.*

Our next example shows that the existence of a least idempotent is not necessary for ω^ℓ to be finitely generated.

Example 5.5 Consider the Bicyclic monoid B. Then ω_B^ℓ is finitely generated but B is not pseudo-finite.

Proof Regarding the underlying set of B as $\mathbb{N}^0 \times \mathbb{N}^0$, it is clear that B is finitely generated by $\{(0, 1), (1, 0)\}$ and so ω_B^ℓ is finitely generated by Proposition 2.7. The idempotents of B form an infinite descending chain, and thus B is not pseudo-finite by Corollary 5.4. □

For any monoid M with a finitely generated minimum ideal, we have by [9, Proposition 6] and Proposition 2.7 that ω_M^ℓ is finitely generated. We have seen that a pseudo-finite inverse semigroup possesses a minimum ideal which is finitely generated (in fact finite). The following example shows that the same is not true for inverse semigroups S satisfying the weaker condition that ω_S^ℓ is finitely generated.

Example 5.6 Let G be the infinite cyclic group on g and let $Y = \{e_i : i \in \mathbb{Q}\} \cup \{1\}$ be a semilattice, with $e_i e_j = e_k$ where $k = \min\{i, j\}$ and 1 is the identity. Define an action of G on Y by

$$g^i \cdot 1 = 1,\ g^i \cdot e_j = e_{i+j}.$$

It is easy to see that this is an action by morphisms. Let $T = Y \rtimes G$, noting that T forms an E-unitary inverse monoid with identity $(1, g^0)$. Hence Sect. 5.9 of [13] applies, so that $(1, g^n)^{-1} = (1, g^{-n})$ and $(e_q, g^n)^{-1} = (g^{-n} \cdot e_q, g^{-n}) = (e_{q-n}, g^{-n})$ for each $q \in \mathbb{Q}$ and $n \in \mathbb{Z}$. Let W be the inverse subsemigroup of S generated by $\{(1, g), (e_1, g)\}$. Then it is a simple exercise to show that

$$W = \langle (1, g), (1, g^{-1}), (e_1, g), (e_0, g^{-1}) \rangle = \{(1, g^n), (e_m, g^n) : m, n \in \mathbb{Z}\},$$

with $E(W) = \{(1, g^0), (e_n, g^0) : n \in \mathbb{Z}\}$. Let $(A, g^n) \in S$. If $A = e_p$ for some $p \in \mathbb{Q}$ then take $k \in \mathbb{Z}$ such that $n + k < p$, and if $A = 1$ then take any $k \in \mathbb{Z}$. Then $(e_k, g^{-n}) \in W$ and

$$(A, g^n)(e_k, g^{-n}) = (Ae_{n+k}, g^{n-n}) = (e_{n+k}, g^0) \in E(W).$$

Hence ω_S^ℓ is finitely generated by Corollary 5.2.

Note that $(A, g^n)\ \mathcal{J}\ (B, g^m)$ if and only if there exist $z, t \in \mathbb{Z}$ such that $g^z \cdot A \leq B$ and $g^t \cdot B \leq A$. If $A = e_p$ and $B = e_q$ then this can be satisfied for any $z, t < \min\{p - q, q - p\}$, while if $A = B = 1$ then this is satisfied for any z, t. It follows that the $\mathcal{J}$-classes of T are $J = \{(1, g^n) : n \in \mathbb{Z}\}$ and $S \backslash J$. Hence $S \backslash J$ is the unique proper ideal of S, and is not finitely generated.

Note that Example 5.6 gives the desired example of a semidirect product $S \rtimes T$ with $\omega^\ell_{S\rtimes T}$ being finitely generated, but such that ω^ℓ_S is not.

Corollary 5.7 *Let E be a semilattice. Then the following statements are equivalent:*

(1) *ω^ℓ_E is finitely generated;*
(2) *E is pseudo-finite;*
(3) *E has a least element and there is a finite set $U \subseteq E$ such that for every $e \in E$ we have $e \leq u$ for some $u \in U$.*

Proof (1) $\Rightarrow$ (3) follows from Theorem 5.1 and the fact a finitely generated subsemigroup of a semilattice is finite. (3) $\Rightarrow$ (2) is immediate from Proposition 5.3 and (2) $\Rightarrow$ (1) follows by definition. □

Corollary 5.8 [9, Theorem 9] *Let E be a semilattice with identity 1. Then ω^ℓ_E is finitely generated if and only if E is pseudo-finite if and only if E has a least element.*

It follows from Theorem 5.1 and Proposition 5.3 that:

Corollary 5.9 *Let $S = \mathcal{B}^0(G, I)$ be a Brandt semigroup over a group G. Then the following statements are equivalent:*

(1) *I is finite;*
(2) *ω^ℓ_S is finitely generated;*
(3) *S is pseudo-finite.*

Notice that we find another approach to Corollary 5.9 in the next section when dealing with arbitrary Rees matrix semigroups.

6 Rees matrix semigroups

In this section we examine Rees matrix semigroups $\mathcal{M} = \mathcal{M}[S; I, \Lambda; P]$ and Rees matrix semigroups with zero $\mathcal{M}^0 = \mathcal{M}^0[S; I, \Lambda; P]$ over a semigroup S. Note that we make no restriction on the elements of $P = (p_{\lambda i})$. Of course, if $S = G$ is a group, then $\mathcal{M} = \mathcal{M}[G; I, \Lambda; P]$ is completely simple, and if every row/column of P contains a non-zero entry then $\mathcal{M}^0 = \mathcal{M}^0[G; I, \Lambda; P]$ is completely 0-simple. We recall from [9] that completely simple semigroups of type left-FP$_1$ were considered, but the convention in [9] is that one considers the property for the corresponding monoid obtained by adjoining an identity.

There are four cases for us to consider that arise from the existence or otherwise of an identity, and the existence or otherwise of a zero:

(1) $T = \mathcal{M}^0[S; I, \Lambda; P]^1$;
(2) $T = \mathcal{M}[S; I, \Lambda; P]$;
(3) $T = \mathcal{M}^0[S; I, \Lambda; P]$;
(4) $T = \mathcal{M}[S; I, \Lambda; P]^1$.

We discuss each of them in turn and specialise to the case where S is a group. We conclude with some remarks on the pseudo-finite case.

The first case is trivial, because any monoid with zero is pseudo-finite by Corollary 2.15.

We now consider the second case.

Theorem 6.1 *Let $T = \mathcal{M}[S; I, \Lambda; P]$ be a Rees matrix semigroup over a semigroup S. Then ω_T^ℓ is finitely generated if and only if the following conditions hold:*

(1) *I and Λ are finite;*
(2) *there is a finite set $V \subseteq S$ such that with*

$$H = \{(p_{vi}a, p_{vj}b) : v \in \Lambda, i, j \in I, a, b \in V\},$$

every element of S is ρ_H-related to an element of V.

Proof Suppose that $\omega_T^\ell = \langle U^2 \rangle$ for some finite set $U = I' \times V \times \Lambda'$ where I', V, Λ' are finite subsets of I, S and Λ, respectively. If T is finite we can take $T = U$ and $V = S$ and we are done. Otherwise, let $(i, a, \lambda), (j, b, \mu) \in T$ be distinct. Then there exists a U^2-sequence

$$(i, a, \lambda) = t_1(i_1, a_1, \lambda_1), t_1(j_1, b_1, \mu_1) = t_2(i_2, a_2, \lambda_2), \ldots, t_n(j_n, b_n, \mu_n) = (j, b, \mu)$$

where $t_m \in T^1$ and $((i_m, a_m, \lambda_m), (j_m, b_m, \mu_m)) \in U^2$ for all $1 \leq m \leq n$. Clearly $\lambda = \lambda_1 \in \Lambda'$, so that $\Lambda = \Lambda'$ is finite. Furthermore, if I is infinite, then we can pick i and j above as being distinct elements of $I \setminus I'$. It is then easy to see that each $t_i \in T$ and reading from left to right the first co-ordinate of each t_k is equal to i, which is not possible in view of the final equality. Hence I is finite. On the other hand, choosing $i = j$, $\lambda = \mu$, and $b \in V$, then taking $a \in S$ to be arbitrary we have that a is connected to an element $b' \in V$ via an H-sequence. Indeed, either $t_k \in T$ for $1 \leq k \leq n$, in which case $b = b'$, or letting k be least with $t_k = 1$, we have $a \, \rho_H \, a_k = b' \in V$.

Conversely, suppose that (1) and (2) hold. Let $W = \{a \in S : (a, b) \in H \text{ for some } b \in S\}$. We claim that $\omega_T^\ell = \langle Q^2 \rangle$ where $Q = I \times (V \cup W) \times \Lambda$. To see this, let $(i, a, \lambda) \in T$. If $(i, a, \lambda) \in Q$ we are done. Otherwise, a is connected via an H-sequence

$$a = k_1 a_1, k_1 b_1 = k_2 a_2, \ldots, k_n b_n = b$$

where $b \in V$. Fix our notation as $(a_l, b_l) = (p_{\eta_l i_l} u_l, p_{\eta_l j_l} v_l) \in H$ for all $1 \leq l \leq n$. If every $k_\ell \in S$, then we have a Q^2-sequence

$$\begin{aligned}(i, a, \lambda) &= (i, k_1, \eta_1)(i_1, u_1, \lambda), (i, k_1, \eta_1)(j_1, v_1, \lambda) \\ &= (i, k_2, \eta_2)(i_2, u_2, \lambda), \ldots, (i, k_n, \eta_n)(j_n, v_n, \lambda) = (i, b, \lambda).\end{aligned}$$

On the other hand, if ℓ is the least occurrence of $k_\ell = 1$, we have a Q^2-sequence

$$\begin{aligned}(i, a, \lambda) &= (i, k_1, \eta_1)(i_1, u_1, \lambda), (i, k_1, \eta_1)(j_1, v_1, \lambda) \\ &= (i, k_2, \eta_2)(i_2, u_2, \lambda), \ldots, (i, k_{\ell-1}, \eta_{\ell-1})(j_{\ell-1}, v_{\ell-1}, \lambda) = (i, c, \lambda)\end{aligned}$$

where $c \in W$. In either case, (i, a, λ) is Q^2-related to an element of Q, completing the proof. □

Consequently, for the Rees matrix semigroup $T = \mathcal{M}[S; I, \Lambda; P]$, if ω_T^ℓ is finitely generated, then P is a finite matrix and ω_S^ℓ is finitely generated. The following example shows that the converse needs not be true.

Example 6.2 Let S be an infinite monoid with zero, so that ω_S^ℓ is finitely generated by Corollary 2.15. Let P be the 1×1 matrix $P = (0)$. Then $T = \mathcal{M}[S; 1, 1; P]$ is an infinite null semigroup, and thus ω_T^ℓ is not finitely generated.

Corollary 6.3 *Let $T = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup over a group G. Then ω_T^ℓ is finitely generated if and only if I, Λ are finite, and G is finitely generated.*

Proof Suppose that ω_T^ℓ is finitely generated so that (1) and (2) of Theorem 6.1 hold. With the notation of that result, for any $g \in G$ we have an H-sequence connecting g to some element $v \in V$, so that by a standard argument, $v^{-1}g \in \langle K \rangle$ and hence $g \in \langle K \cup V \rangle$ where $K = \{u^{-1}v : (u, v) \in H\}$. Hence G is finitely generated.

Conversely, if I, Λ are finite and G is finitely generated by L, say, then with $\overline{L} = \{(1, l) : l \in L\}$ we have that $\overline{L}$ generates ω_G^ℓ. Since $\overline{L} = \{(pp^{-1}, pp^{-1}l) : p \in P, l \in L\}$ we have $L' = \{(p_{vi}a, p_{vj}b) : v \in \Lambda, i \in I, a, b \in Q\}$ generates ω_G^l where $Q = \{p^{-1}, p^{-1}l : p \in P, l \in L\}$. The result now follows from Theorem 6.1. □

Since a rectangular band is isomorphic to a Rees matrix semigroup over the trivial group it follows that:

Corollary 6.4 *Let B be a rectangular band. Then ω_B^ℓ is finitely generated if and only if B is pseudo-finite if and only if B is finite.*

Theorem 6.5 *Let $T = \mathcal{M}[S; I, \Lambda; P]^1$ be a Rees matrix semigroup over a semigroup S, with identity adjoined. Then ω_T^ℓ is finitely generated if and only if the following conditions hold:*

(1) *I is finite;*
(2) *there is a finite set $V \subseteq S$ and a finite subset Q of entries of $P = (p_{\lambda,i})$ such that any element a of S is ρ_U-related to an element of V via the left congruence ρ_U defined on S^1, where*

$$U = \{(p_{vi}a, p_{vj}b) : v \in \Lambda, i, j \in I, a, b \in V\} \cup \{(1, pv), (pv, 1) : p \in Q, v \in V\},$$

via a U-sequence of the form

$$a = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_kd_k = v,$$

where $t_i \in S$ and $(d_j, c_{j+1}) \neq (1, 1)$ for any $1 \leq j < k$.

Proof Suppose that $\omega_T^\ell = \langle X^2 \rangle$ for some finite set $X = (I' \times V \times \Lambda') \cup \{1\}$.

If I is infinite, then we can pick $i \in I \backslash I'$ and choosing $(i, a, \lambda) \in T$ we have $(i, a, \lambda)\, \rho_{X^2}\, 1$, so that there exists an X^2-sequence

$$(i, a, \lambda) = z_1 r_1, z_1 s_1 = z_2 r_2, \ldots, z_n s_n = 1$$

where $z_m \in T$ and $(r_m, s_m) \in X^2$ for all $1 \leq m \leq n$. As $i \in I \backslash I'$, we have $z_1 \in \{i\} \times S \times \Lambda$, implying $z_2, \ldots, z_n \in \{i\} \times S \times \Lambda$, but $z_n s_n = 1$ forces $z_n = 1$, a contradiction.

Fix $(i, a, \mu) \in T$ where $\mu \in \Lambda'$. We have an X^2-sequence, which we may assume to be of minimum length (hence $(d_j, c_{j+1}) \neq (1, 1)$ for any $1 \leq j < n$)

$$(i, a, \mu) = t_1 c_1, t_1 d_1 = t_2 c_2, \ldots, t_n d_n = 1,$$

so that $t_i c_i \neq 1$ for $1 \leq i \leq n$. Notice that $t_n = 1$. Suppose that $t_1, \ldots, t_k \neq 1$ and $t_{k+1} = 1$ for some $k \in \{1, \ldots, n-1\}$. We therefore have a sequence

$$a = t'_1 c'_1, t'_1 d'_1 = t'_2 c'_2, \ldots, t'_k d'_k = a'_{k+1}$$

where if $t_l = (g_l, w_l, \eta_l)$ we have $t'_l = w_l$, and

$$c'_l = \begin{cases} 1 & \text{if } c_l = 1 \\ p_{\eta_l h_l} a_l & \text{if } c_l = (h_l, a_l, \lambda_l) \end{cases}$$

and

$$d'_l = \begin{cases} 1 & \text{if } d_l = 1 \\ p_{\eta_l k_l} b_l & \text{if } d_l = (k_l, b_l, \mu_l). \end{cases}$$

Let $Q = \{p_{\nu j} : \nu \in \Lambda', j \in I\}$. We show by induction on the length of the sequence that for $1 \leq l \leq k$ if $c_l = 1$ (respectively, $d_l = 1$), then $t_l \in I \times S \times \Lambda'$ and $d'_l \in QV$ (respectively, $c'_l \in QV$).

Starting with $l = 1$, if $c_1 = 1$ (so $d_1 \neq 1$) then $t_1 = (i, a, \mu) = (g_1, w_1, \eta_1)$ and $d'_1 = p_{\eta_1 k_1} b_1 = p_{\mu k_1} b \in QV$. On the other hand, if $d_1 = 1$ (so $c_1 \neq 1$) then, noticing that $c_2 \neq 1$ (else our sequence could be replaced by a shorter one) we have $t_1 = t_2 c_2$ so that $\eta_1 \in \Lambda'$ and $c_1 = p_{\eta_1 h_1} a_1 \in QV$. Suppose for induction that $2 \leq m \leq k$ and the result holds for all sequences of length strictly less than m, and let our sequence be of length m. Consider $t_{m-1} d_{m-1} = t_m c_m$ which, by our inductive assumption, must lie in $I \times S \times \Lambda'$. If $c_m = 1$ (so $d_m \neq 1$) then $t_m = t_{m-1} d_{m-1} \in I \times S \times \Lambda'$ and $d'_m = p_{\eta_m k_m} b_m \in QV$. On the other hand, if $d_m = 1$ (so $c_m \neq 1$), then $c_{m+1} \neq 1$, so that $t_m = t_{m+1} c_{m+1} \in I \times S \times \Lambda'$ and $c'_m = p_{\eta_m h_m} a_m \in QV$. Thus (2) holds, with U as defined.

Suppose for the converse that (1) and (2) hold. Let Λ' be a finite subset of Λ chosen such that $Q \subseteq V = \{p_{\lambda i} : \lambda \in \Lambda', i \in I\}$. Let

$$W = (I \times V \times \Lambda') \cup \{1\}.$$

We show that $\omega_T^\ell = \langle W^2 \rangle$.

Let $(i, a, \lambda) \in T$ and pick any $b \in V$, $\mu \in \Lambda'$. Let $s = ap_{\lambda i}b$. Notice that

$$(i, a, \lambda) = (i, a, \lambda)1\, \rho_{W^2}(i, a, \lambda)(i, b, \mu) = (i, s, \mu).$$

By assumption, there is a U-sequence in S^1

$$s = t_1'c_1', t_1'd_1' = t_2'c_2', \ldots, t_n'd_n' = v$$

for some $v \in V$, where no $t_l' = 1$ and no pair $(c_l', d_{l+1}') = (1, 1)$. We aim to 'lift' this sequence to a W^2-sequence

$$(i, s, \kappa) = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = (i, v, \tau)$$

in T^1, for some $\kappa, \tau \in \Lambda'$. For each $1 \leq l \leq n$ we have $(c_l', d_l') = (p_{\eta_l h_l} a_l, p_{\eta_l k_l} b_l)$, or $(c_l', d_l') = (1, p_{\eta_l k_l} b_l)$ or $(c_l', d_l') = (p_{\eta_l h_l} a_l, 1)$. Fix $\mu \in \Lambda'$, put $t_l = (i, t_l', \eta_l)$ for $1 \leq l \leq n$ and define

$$c_j = \begin{cases} 1 & \text{if } c_j' = 1 \\ (h_j, a_j, \eta_{j-1}) & \text{if } d_{j-1} = 1 \\ (h_j, a_j, \mu) & \text{otherwise} \end{cases}$$

and

$$d_j = \begin{cases} 1 & \text{if } d_j' = 1 \\ (k_j, b_j, \eta_{j+1}) & \text{if } c_{j+1} = 1 \\ (k_j, b_j, \mu) & \text{otherwise.} \end{cases}$$

It is then clear that with $\kappa = \eta_1$ if $c_1' = 1$, and $\kappa = \mu$ else, and $\tau = \eta_n$ if $d_n = 1$ and $\tau = \mu$ else, we have a W^2-sequence connecting (i, s, κ) to (i, v, τ). Since $(i, a, \lambda)\, \rho_{W^2}\, (i, s, \kappa)$ and $(i, v, \tau) \in W$, we are done. □

Corollary 6.6 *Let B be a rectangular band. Then $\omega_{B^1}^\ell$ is finitely generated if and only if B^1 is pseudo-finite if and only if B has finitely many $\mathcal{R}$-classes.*

Proof We need only consider the issue of pseudo-finiteness. If B has finitely many $\mathcal{R}$-classes $\{R_{a_i} : 1 \leq i \leq n\}$, then letting $X = \{1, a_i : 1 \leq i \leq n\}$, for any $b \in B$ we have $b \in R_{a_i}$ for some i and then $b = b1\, \rho_{X^2}\, ba_i = a_i\, \rho_{X^2}\, 1$, so that B is pseudo-finite. □

In the case where S is a monoid in the hypothesis of Theorem 6.5 we obtain little simplification beyond the obvious ability to drop the condition that $t_i \in S$; the condition on the pair (d_j, c_{j+1}) is redundant, as it merely says the sequence cannot be reduced by setting $t_jc_j = t_{j+1}d_{j+1}$. Where S is a group we may simplify considerably.

Suppose S has been normalised in row 1 and column 1, that is, there exists $1 \in I \cap \Lambda$ such that $p_{1i} = p_{\lambda 1} = 1$ for all $i \in I$ and $\lambda \in \Lambda$. Let $G^P = \{p_{\lambda i} : i \in I, \lambda \in \Lambda\}$. By [12],

$$\langle E(S) \rangle = \mathcal{M}[\langle G^P \rangle; I, \Lambda; P]. \tag{6.1}$$

Corollary 6.7 *Let $T = \mathcal{M}[G; I, \Lambda; P]^1$ be a normalised Rees matrix semigroup over a group G. Then ω_T^ℓ is finitely generated if and only if the following conditions hold:*

(1) *I is finite;*
(2) *$G = \langle G^P \cup V \rangle$ where V is a finite set.*

Proof Suppose that ω_T^ℓ is finitely generated. Thus Theorem 6.5 holds and there are finite sets $V \subseteq G$, $Q \subseteq G^P$, and $U \subseteq G \times G$ as in (2) of that result. Augment V by $\{q^{-1} : q \in Q\}$ and, noticing that $(1, qv) = (qq^{-1}, qv)$ and $(qv, 1) = (qv, qq^{-1})$ we have that

$$U = \{(p_{vi}a, p_{vj}b) : v \in \Lambda, i, j \in I, a, b \in V\}.$$

Hence by previous remarks we have that

$$G = \langle \{p_{vi}ab^{-1}p_{vj}^{-1} : a, b \in V, v \in \Lambda, i, j \in I\} \rangle = \langle G^P \cup V \rangle,$$

noting that $1 \in I$, and so (2) holds.

Conversely, if (1) and (2) hold then it is easy to see that ω_G^l has a set of generators of the form

$$\{(p_{vi}, p_{vj}) : v \in \Lambda, i, j \in I\} \cup \{(1, v), (1, v^{-1}) : v \in V\}$$

and hence of the form

$$\{(p_{vi}, p_{vj}) : v \in \Lambda, i, j \in I\} \cup \{(p_{\tau k}p_{\tau k}^{-1}, p_{\tau k}p_{\tau k}^{-1}v), (p_{\tau k}p_{\tau k}^{-1}, p_{\tau k}p_{\tau k}^{-1}v^{-1}) : v \in V\},$$

where $p_{\tau k}$ is chosen and fixed, and hence (by suitably augmenting V to give a finite set W) a set of generators of the form

$$\{(p_{vi}a, p_{vj}b) : v \in \Lambda, i, j \in I, a, b \in W\}.$$

In view of earlier comments, the result now follows from Theorem 6.5. □

Recall that in [9], a semigroup is said to be of type left-FP_1 if the monoid S^1 has this property. In light of (6.1) we may now recover Theorems 4 and 7 of [9].

Corollary 6.8 *Let $T = U^1$ be a completely simple semigroup with an identity adjoined and let G be a maximal subgroup of U. Then ω_T^ℓ is finitely generated if and only if U has finitely many $\mathcal{R}$-classes and G is generated by a set $\langle (\langle E(S) \rangle \cap G) \cup V \rangle$ where V is finite. If in addition U has finitely many $\mathcal{L}$-classes, then ω_T^ℓ is finitely generated if and only if U has finitely many $\mathcal{R}$-classes and G is finitely generated.*

Finally we consider the case when $T = \mathcal{M}^0[S; I, \Lambda; P]$.

Proposition 6.9 *Let $T = \mathcal{M}^0[S; I, \Lambda; P]$ be a Rees matrix semigroup with zero over a semigroup S. Then ω_T^ℓ is finitely generated if and only if:*

(1) *Λ is finite; and*
(2) *(a) I is finite and there is a finite set $V \subseteq S$ such that for every $a \in S \backslash V$, we have $S^1 a \subseteq Sp_{\nu i} v$ for some $\nu \in \Lambda, i \in I$ and $v \in V$, or*
(b) I is infinite and there is a finite set $V \subseteq S$ and a finite set $I' \subseteq I$ such that for every $a \in S$, we have $S^1 a \subseteq Sp_{\nu i} v$ for some $\nu \in \Lambda, i \in I'$ and $v \in V$.

Proof Suppose that $\omega_T^\ell = \langle U^2 \rangle$ for some finite set $U = (I' \times V \times \Lambda') \cup \{0\}$. If T is finite then (1) and (2)(a) trivially hold. Otherwise, we can find distinct $(i, a, \lambda), (j, b, \mu) \notin U$. By assumption, there exists a U^2-sequence

$$(i, a, \lambda) = t_1 w_1, t_1 q_1 = t_2 w_2, \ldots, t_n q_n = (j, b, \mu)$$

where $t_m \in T^1$ and $(w_m, q_m) \in U^2$ for all $1 \leq m \leq n$. Notice that w_1 must be some $(i_1, a_1, \lambda_1) \in U \backslash \{0\}$, implying $\lambda = \lambda_1 \in \Lambda'$, so that $\Lambda = \Lambda'$ is finite and (1) holds.

Suppose that I is finite. For any $a \notin V$ we have $(i, a, \lambda) \notin U$ and then (since T is not finite) we can begin a sequence as above, yielding $t_1 = (i, c, \nu) \in T$. We then have that $a = cp_{\nu i_1} a_1$, giving $S^1 a \subseteq Sp_{\nu i_1} v$ as required, and so (2)(a) holds. On the other hand, if I is infinite then for any $a \in S$ we can find an element $(i, a, \lambda) \notin U$ and a similar argument shows that (2)(b) holds.

Conversely, suppose that (1), and (2)(a) or (2)(b) hold. If T is finite we are done. Otherwise, let $U = \{0\} \cup \{(i, v, \lambda) : i \in I', v \in V, \lambda \in \Lambda\}$, where in case (2)$(a)$ we take $I' = I$. We show that $\omega_T^\ell = \langle U^2 \rangle$. To see this, let $(i, a, \lambda) \in T$. If $(i, a, \lambda) \in U$ we are done. Otherwise, in case (2)(a), since $a \notin V$ we have $a = xp_{vj} v$ for some $\nu \in \Lambda$, $j \in I'$, $v \in V$ and $x \in S$, giving

$$(i, a, \lambda) = (i, x, \nu)(j, v, \lambda)\, \rho_{U^2}\, (i, x, \nu)0 = 0.$$

In case (2)(b) we can write any $a \in S$ as $a = xp_{vj} v$ for some $\nu \in \Lambda$, $j \in I'$, $v \in V$ and $x \in S$, and achieve our aim. □

Again we may simplify the proposition above for groups.

Corollary 6.10 *Let $T = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with zero over a group G. Then ω_T^ℓ is finitely generated if and only if either T is finite, or Λ is finite and T is non-null.*

Proof Recall that T is non-null if and only if the sandwich matrix has a non-zero element. □

Remark 6.11 We end by considering again our four cases of whether or not our Rees matrix semigroup has a zero or 1 in the context of being pseudo-finite. In cases (1) and (3) we have a semigroup with zero, hence by Corollary 2.14, the universal relation of our semigroup is finitely generated if and only if it is pseudo-finite. In Theorems 6.1 and 6.5 and their corollaries we merely need to impose a bound on the length of the sequences to achieve the criterion for being pseudo-finite (so that in Corollaries 6.3 and 6.8, the group G is required to be finite).

Open Question 6.12 Let $T = \mathcal{M}[S; I, \Lambda; P]$. Fixing $(i, \lambda) \in I \times \Lambda$ we have that $K_{i\lambda} = \{(i, s, \lambda) : s \in S\}$ becomes a variant of S with sandwich element $p_{\lambda i}$. In view of the results of this section we ask, what is the connection between ω^ℓ_S being finitely generated and $\omega^\ell_{S^a}$ being finitely generated, for a variant S^a of S?

7 Strong semilattices of semigroups and Bruck–Reilly extensions

Strong semilattices of monoids which are of type left-FP$_n$, and hence of type left-FP$_1$, have been classified in [9, Theorem 9]. Our aim in this section is to generalise their result, by studying strong semilattices of semigroups, with and without an identity adjoined, and to examine the property of being pseudo-finite in this context. Using a strong semilattice of groups we are able to give a counterexample to the conjecture of Dales and White, mentioned in the introduction. We also examine Bruck–Reilly extensions $\mathrm{BR} = \mathrm{BR}(S, \theta)$ of a monoid S to determine when $\omega^\ell_{\mathrm{BR}}$ is finitely generated. For details concerning strong semilattices of semigroups, and Bruck–Reilly extensions, we refer the reader to [13]. Note that in the result below, even if our component semigroups are monoids, we are not assuming that the connecting morphisms $\varphi_{\alpha,\beta}$ are monoid morphisms.

Proposition 7.1 *Let $S = [\mathcal{Y}; S_\alpha; \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups. Then ω^ℓ_S is finitely generated (S is pseudo-finite) if and only if*

(1) *there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$;*
(2) *$\mathcal{Y}$ has a least element 0;*
(3) *$\omega^\ell_{S_0}$ is finitely generated (S_0 is pseudo-finite).*

Proof Suppose that $\omega^\ell_S = \langle X^2 \rangle$ for some finite set $X \subseteq S$. Then (1) holds by Lemma 2.9. Since there is a natural epimorphism from S to $\mathcal{Y}$, we deduce that $\omega^\ell_{\mathcal{Y}}$ is finitely generated by Proposition 4.1, so that $\mathcal{Y}$ has a least element by Corollary 5.7. Further, S_0 is a morphic image of S under $\psi : S \to S_0$ defined by $s\psi = s\varphi_{\alpha,0}$ for $s \in S_\alpha$. Thus $\omega^\ell_{S_0}$ is finitely generated; if S is pseudo-finite, then S_0 is also pseudo-finite.

Conversely, suppose (1)-(3) hold. Then S_0 forms an ideal of S, and so the result follows from Corollary 2.13. □

Proposition 7.2 *Let $S = [\mathcal{Y}; S_\alpha; \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups that is* not *a monoid. Then $\omega^\ell_{S^1}$ is finitely generated (S^1 is pseudo-finite) if and only if $\mathcal{Y}$ has a least element 0 and $\omega^\ell_{S^1_0}$ is finitely generated (S^1_0 is pseudo-finite).*

Proof Suppose that $\omega^\ell_{S^1}$ is finitely generated (S^1 is pseudo-finite). Since there is a natural epimorphism from S^1 to $\mathcal{Y}^1$, we deduce that $\omega^\ell_{\mathcal{Y}^1}$ is finitely generated, so that $\mathcal{Y}^1$ (and so $\mathcal{Y}$) has a least element 0 by Corollary 5.8. For convenience denote the identity of S^1 by 1_S and the identity of S^1_0 by 1_{S_0}. Also, S^1_0 is a morphic image of S^1 under $\psi : S^1 \to S^1_0$ defined by $s\psi = s\varphi_{\alpha,0}$ for $s \in S_\alpha$ and $1_S\psi = 1_{S_0}$. Thus $\omega^\ell_{S^1_0}$ is finitely generated (S^1_0 is pseudo-finite) by Proposition 4.1.

Conversely, suppose that $\mathcal{Y}$ has a least element 0 and $\omega^{\ell}_{S^1_0} = \langle X^2 \rangle$ for a finite set $X \subseteq S^1_0$. Fix $u \in X \cap S_0$ and let $Y = (X \cap S_0) \cup \{1_S\}$. If $t \in S_0$ then as $t\ \rho_{X^2}\ u$ in S^1_0, it is easy to see, replacing 1_{S_0} by 1_S if necessary, that $t\ \rho_{Y^2}\ u$ in S^1 via a Y^2-sequence of the same length as the original X^2-sequence. For any $s \in S$ we have $su \in S_0$, giving $s = s1_S\ \rho_{Y^2}\ su\ \rho_{Y^2}\ u\ \rho_{Y^2}\ 1_S$, so that $\rho_{Y^2} = \omega^{\ell}_{S^1}$. The statement on pseudo-finiteness follows. □

Corollary 7.3 cf. [9, Corollary 3] *Let $S = [\mathcal{Y}; G_\alpha; \varphi_{\alpha,\beta}]$ be a Clifford monoid. Then ω^{ℓ}_S is finitely generated (S is pseudo-finite) if and only if $\mathcal{Y}$ has a least element 0 and G_0 is finitely generated (finite).*

Remark 7.4 We recall that a *normal band* is a strong semilattice of rectangular bands. We may immediately apply Propositions 7.1 and 7.2 to this case, calling upon Corollaries 6.4 and 6.6. We examine bands in greater detail in Sect. 8.

Our aim now is to extend Example 5.5 by considering arbitrary Bruck–Reilly extensions of monoids:

Proposition 7.5 *Let S be a monoid with ω^{ℓ}_S being finitely generated and let $T = BR(S, \theta)$ be the Bruck–Reilly extension of S determined by θ. Then ω^{ℓ}_T is finitely generated, but T is not pseudo-finite.*

Proof Let e be the identity of S and $\omega^{\ell}_S = \langle X^2 \rangle$ for a finite set $X \subseteq S$. Let $Y = \{(1, e, 0), (0, e, 0), (0, x, 0) : x \in X\}$. Notice that, for any $(u, a, 0) \in T$,

$$(u, a, 0) = \underbrace{(1, e, 0) \cdots (1, e, 0)}_{u-1 \text{ times}}(1, a, 0).$$

As $\omega^{\ell}_S = \langle X^2 \rangle$, we have that $a = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = e$ where $t_i \in S$, $(c_i, d_i) \in X^2$ for all $1 \leq i \leq n$. This gives

$$\begin{aligned}(1, a, 0) = (1, t_1, 0)(0, c_1, 0), (1, t_1, 0)(0, d_1, 0) = (1, t_2, 0)(0, c_2, 0), \ldots \\ \cdots (1, t_n, 0)(0, d_n, 0) = (1, e, 0)\end{aligned}$$

and hence $(1, a, 0)\ \rho_{Y^2}\ (1, e, 0)\ \rho_{Y^2}\ (0, e, 0)$. Notice that $(1, a, 0)\ \rho_{Y^2}\ (0, e, 0)$ gives

$$(2, a, 0) = (1, e, 0)(1, a, 0)\ \rho_{Y^2}\ (1, e, 0)(0, e, 0) = (1, e, 0)\ \rho_{Y^2}\ (0, e, 0).$$

By induction we derive

$$(u, a, 0) = \underbrace{(1, e, 0) \cdots (1, e, 0)}_{u-1 \text{ times}}(1, a, 0)\ \rho_{Y^2}\ (0, e, 0).$$

For any $(u, a, v) \in T$ where $v \neq 0$,

$$(u, a, v - 1) = (u, a, v)(1, e, 0)\ \rho_{Y^2}\ (u, a, v)(0, e, 0) = (u, a, v).$$

Again, a simple inductive argument yields $(u, a, 0)\ \rho_{Y^2}\ (u, a, v)$, so $(u, a, v)\ \rho_{Y^2}\ (0, e, 0)$, and hence $\rho_{Y^2} = \omega_T^\ell$.

There exists an epimorphism from T onto the Bicyclic monoid given by $(m, g, n)\psi = (m, n)$, and so the final statement is immediate from Proposition 4.1 and Example 5.5. □

In the opposite direction to that in Proposition 7.5, we see that the property of ω^ℓ being finitely generated for $\mathrm{BR}(S, \theta)$ does not necessarily transfer to S.

Example 7.6 Let G be the free group on $\{a_i : i \in \mathbb{N}^0\}$ and let $\theta : G \to G$ be given by $a_i\theta = a_{i+1}$. Let $T = BR(G, \theta)$ be the Bruck–Reilly extension of G determined by θ. Clearly G is *not* finitely generated. We claim that $\omega_T^\ell = \langle X^2 \rangle$ for the finite set

$$X = \{(0, e, 0), (0, e, 1), (1, a_0, 0), (1, e, 0)\}.$$

Proof By the usual arguments, we have $(i, e, j)\ \rho_{X^2}\ (0, e, 0)$, for any $i, j \in \mathbb{N}^0$. Noting that $(1, a_0, 0) \in X$, we assume that $(0, a_i, 0)\ \rho_{X^2}\ (0, e, 0)$ for all $i \leq n$. Then

$$(0, a_n, 0)(1, e, 0)\ \rho_{X^2}\ (0, a_n, 0)(0, e, 0) = (0, a_n, 0)\ \rho_{X^2}\ (0, e, 0).$$

Now

$$(0, a_n, 0)(1, e, 0) = (1, a_n\theta, 0) = (1, a_{n+1}, 0).$$

From the above we have

$$(0, a_{n+1}, 0) = (0, e, 1)(1, a_{n+1}, 0)\ \rho_{X^2}\ (0, e, 1)(0, e, 0) = (0, e, 1)\ \rho_{X^2}\ (0, e, 0),$$

completing our inductive step. We then have

$$(0, e, 0) = (0, a_n^{-1}, 0)(0, a_n, 0)\ \rho_{X^2}\ (0, a_n^{-1}, 0)$$

for any $n \in \mathbb{N}^0$, and since the ρ_{X^2}-class of $(0, e, 0)$ is a submonoid we obtain that $(0, g, 0)\ \rho_{X^2}\ (0, e, 0)$ for all $g \in G$.

For any $u \in \mathbb{N}^0$, $g \in G$ we have

$$(u, g, 0) = (1, e, 0)^u(0, g, 0),$$

so that $(u, g, 0)\ \rho_{X^2}\ (0, e, 0)$ and then similarly to the proof of Proposition 7.5, we show via induction that $(u, g, v)\ \rho_{X^2}\ (0, e, 0)$, for any $v \in \mathbb{N}^0$. □

Every monoid with zero is pseudo-finite, as is every finite monoid and hence from Proposition 4.2, every direct product of such. Dales and White [6], see also [24], posed the question of whether the converse is true, namely, that every pseudo-finite monoid is isomorphic to a direct product of a monoid with zero by a finite monoid. We answer this question negatively by the following example.

Example 7.7 Let $\mathcal{Y} = \{\alpha, \beta\}$ be a semilattice with $\beta < \alpha$ and let $M = [\mathcal{Y}; G_\alpha; \varphi_{\alpha,\beta}]$ be a strong semilattice of groups, where $G_\alpha = G$ is an infinite group with identity 1 and no elements of order 2, $G_\beta = \{a, e\}$ is a group with identity e, and $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$ is defined by $g\varphi_{\alpha,\beta} = e$ for all $g \in G$. Then M is a pseudo-finite monoid that is not isomorphic to a direct product of a monoid with zero by a finite monoid.

Proof Clearly, M does not have zero, but as β is the least element of $\mathcal{Y}$, it follows from Corollary 7.3 that M is pseudo-finite. Suppose that U and V are monoids and $\psi : M \to U \times V$ is an isomorphism. As $|E(M)| = 2$ and $|E(U \times V)| = |E(U)| \times |E(V)|$, without loss of generality, we assume $|E(U)| = 2$ and $|E(V)| = 1$. Let $E(U) = \{1_U, f\}$ and $E(V) = \{1_V\}$ where 1_U and 1_V are the identities of U and V, respectively. Then $1\psi = (1_U, 1_V)$ and $e\psi = (f, 1_V)$. For any $w = (p, q) \in U \times V$, we have $w\psi^{-1} \mathcal{H} 1$ or $w\psi^{-1} \mathcal{H} e$, so that $w \mathcal{H} (1_U, 1_V)$ or $w \mathcal{H} (f, 1_V)$, which implies $p \mathcal{H} 1_U, q \mathcal{H} 1_V$ or $p \mathcal{H} f, q \mathcal{H} 1_V$. Hence we deduce that $V = H_{1_V}$ is a group and $H_f \times H_{1_V} \cong G_\beta = \{a, e\}$, so that $|H_f| = 1$ or $|H_{1_V}| = 1$. If $|H_{1_V}| = |V| = 1$, then $M \cong U$. If $|H_f| = 1$ and $|V| = |H_{1_V}| = 2$, then $V = \{1_V, b\}$ where $b^2 = 1_V$, so that the order of $(1_U, b)$ is 2, so is the element $(1_U, b)\psi^{-1}$ in G, but G has no elements of order 2, a contradiction. □

8 Semigroups with a minimum ideal and completely regular semigroups

In many of our examples of a semigroup S with ω_S^ℓ being finitely generated, S is required to possess a minimum ideal I such that ω_I^ℓ is finitely generated. Clifford semigroups and normal bands illustrate this point. Indeed, if $S = [\mathcal{Y}; S_\alpha; \varphi_{\alpha,\beta}]$ is a strong semilattice of semigroups, and $\mathcal{Y}$ has a least element 0, then any minimum ideal is contained in S_0, and if S is Clifford or a normal band, S_0 will be the minimum ideal exactly. In this section we give necessary and sufficient conditions for a semigroup with a completely simple minimum ideal to have ω^ℓ finitely generated. We apply our result to completely regular semigroups and bands. We remark that we make no restriction here on the number of right ideals of the minimum ideal, unlike [9, Theorem 8].

Theorem 8.1 *Let S be a semigroup with a minimum ideal S_0 that is completely simple. Then ω_S^ℓ is finitely generated if and only if the following hold:*

(1) *there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_\mathcal{L} x$;*
(2) *if G is a maximal subgroup of S_0, then $G = \langle F \cup V \rangle$ where V is finite and $F = \langle E(C_0) \rangle \cap G$ where C_0 is the union of finitely many $\mathcal{R}$-classes of S_0;*
(3) *there exists an $\mathcal{L}$-class L of S_0 and a finite subset $W \subseteq L$ such that every idempotent in L is ρ_{W^2}-related to an element of W via the left congruence ρ_{W^2} defined on S.*

Proof We first suppose that $\omega_S^\ell = \langle U^2 \rangle$ for a finite set $U \subseteq S$. Condition (1) holds by Lemma 2.15.

Let H be a maximal subgroup in S_0 and let e be identity of H. Let $h \in H$. As $\omega_S^\ell = \langle U^2 \rangle$, we have

$$h = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = e,$$

where $t_i \in S^1$ and $(c_i, d_i) \in U^2$. Multiplying the above sequence by e from both sides, we have

$$h = et_1c_1e, et_1d_1e = et_2c_2e, \ldots, et_nd_ne = e.$$

Now $et_i \in S_0$ and $(c_ie, d_ie) \in Ue \times Ue \subseteq S_0 \times S_0$. It is convenient at this point to assume that S_0 is a Rees matrix semigroup $\mathcal{M} = \mathcal{M}[G; I, \Lambda; P]$ and that $H = H_{11}$ for some distinguished $1 \in I \cap \Lambda$, where P is normalised in row 1 and column 1. We can then write the above sequence as

$$\begin{aligned} h = (1, g, 1) &= (1, t_1', \nu_1)(i_1, c_1', 1), (1, t_1', \nu_1)(j_1, d_1', 1) = (1, t_2', \nu_2)(i_2, c_2', 1), \\ &\ldots, (1, t_n', \nu_n)(j_n, d_n', 1) = (1, f, 1) = e, \end{aligned}$$

where f is the identity of G, $(1, t_k', \nu_k) \in e\mathcal{M}$ and $((i_k, c_k', 1), (j_k, d_k', 1)) \in Ue \times Ue$. We now have

$$g = t_1'p_{\nu_1 i_1}c_1', t_1'p_{\nu_1 j_1}d_1' = t_2'p_{\nu_2 i_2}c_2', \ldots, t_n'p_{\nu_n j_n}d_n' = f,$$

which gives us $g = d_n'^{-1}p_{\nu_n j_n}^{-1}p_{\nu_n i_n}c_n'd_{n-1}'^{-1} \cdots p_{\nu_1 j_1}^{-1}p_{\nu_1 i_1}c_1'$. Let I' be the union of $\{1\}$ together with the rows indexed by the first co-ordinates of the elements of Ue and let $C_0 = \mathcal{M}[G; I', \Lambda; P']$, where $P' = (p_{\lambda i})$ is the $\Lambda \times I'$ submatrix of P. By (6.1), we have that H is generated by $(\langle E(C_0) \rangle \cap H) \cup V$ where $V = Ue$, and so (2) holds.

Note that Ue is contained in the $\mathcal{L}$-class L_e of S_0. If $f \in E(L_e)$ then there exists $c_i, d_i \in U$ and $t_i \in S^1$ $(1 \leq i \leq n)$, such that

$$f = t_1c_1, t_1d_1 = t_2c_2, \ldots, t_nd_n = e.$$

Multiplying through by e on the right, and noting that $fe = f$ we obtain (3) by augmenting U by the finite set Ue.

Conversely, suppose (1)-(3) hold. From Corollary 6.8 we know that $\omega_{C_0^1}^\ell$ is finitely generated, say by a set U^2. Let G be a maximal subgroup in L with identity e and let $F = Ue$. Then F is a finite subset of C_0, and as any two elements of G are related via a U^2-sequence in C_0^1, a now familiar argument shows that they are related in S by an F^2-sequence of the same length.

Let $Y = X \cup W \cup F \cup \{e\}$. We claim $\omega_S^\ell = \langle Y^2 \rangle$. Let $s \in S$ be such that $s \mathcal{L} e$, and suppose $s \mathcal{H} f = f^2$. Then $es \mathcal{H} e$, so that $es \rho_{F^2} e$, and so $es \rho_{Y^2} e$. Hence $fes \rho_{Y^2} fe$, and so $fes = fs = s \rho_{Y^2} f = fe$. However, $f \rho_{W^2} e$, so that $f \rho_{Y^2} e$, and thus $s \rho_{Y^2} e$. Now take any $u \in S$, and let $t \in X$ be such that $u = at$ for some $a \in S^1$. Then $u = at \rho_{Y^2} ae \rho_{Y^2} e$ as $ae \mathcal{L} e$. Hence $\omega_S^\ell = \langle Y^2 \rangle$. □

If we let the minimum ideal S_0 have finitely many $\mathcal{R}$-classes, then the result above simplifies as follows. Note that this corollary can be seen as an extension of Theorem 8 of [9] from monoids to semigroups.

Corollary 8.2 *Let S be a semigroup with a minimum ideal S_0 that is completely simple and has finitely many $\mathcal{R}$-classes. Then ω^ℓ_S is finitely generated if and only if the following hold:*

(1) *there exists a finite subset X of S such that for every $a \in S$ we have some $x \in X$ with $a \leq_{\mathcal{L}} x$;*
(2) *if G is a maximal subgroup of S_0, then $G = \langle F \cup V \rangle$ where V is finite and $F = \langle E(S_0) \rangle \cap G$.*

To search for examples of semigroups with a minimum ideal that is completely simple, a natural starting point is completely regular semigroups.

Corollary 8.3 *Let S be a completely regular semigroup, with decomposition $S = \bigcup_{\alpha \in \mathcal{Y}} S_\alpha$ into a semilattice of completely simple semigroups S_α, $\alpha \in \mathcal{Y}$. Then ω^ℓ_S is finitely generated if and only if $\mathcal{Y}$ has a least element 0, so that S_0 forms a minimum ideal that is completely simple, and conditions (1)–(3) of Theorem 8.1 hold.*

Proof As $\mathcal{J}$ is a congruence on S, Proposition 4.1 tells us that $\omega^\ell_{S/\mathcal{J}}$ is finitely generated, and so $\mathcal{Y} \cong S/\mathcal{J}$ has a least element by Corollary 5.7. Conditions (1)–(3) and the converse is immediate from Theorem 8.1. □

For the case of bands, Conditions (1)–(3) of Theorem 8.1 reduce noticeably, since maximal subgroups are trivial and $\leq_{\mathcal{L}}$ has a simplified form:

Corollary 8.4 *Let B be a band, with decomposition $B = \bigcup_{\alpha \in \mathcal{Y}} B_\alpha$ into a semilattice of rectangular bands B_α, $\alpha \in \mathcal{Y}$. Then ω^ℓ_B is finitely generated if and only if the following hold:*

(1) *$\mathcal{Y}$ has a least element 0;*
(2) *there exists a finite subset X of B such that for every $e \in B$ we have some $x \in X$ with $ex = e$;*
(3) *there exists an $\mathcal{L}$-class L of B_0 and a finite subset $W \subseteq L$ such that every element of L is ρ_{W^2}-related to an element of W via the left congruence ρ_{W^2} defined on B.*

We may use Corollary 8.2 in partnership with Corollaries 8.3 and 8.4 to specialise to completely regular semigroups, and to bands, with minimum ideal having finitely many $\mathcal{R}$-classes.

Corollary 8.5 *Let S be a completely regular semigroup, with decomposition $S = \bigcup_{\alpha \in \mathcal{Y}} S_\alpha$ into a semilattice of completely simple semigroups S_α, $\alpha \in \mathcal{Y}$. Suppose also that S has a minimum ideal S_0 with finitely many $\mathcal{R}$-classes. Then ω^ℓ_S is finitely generated if and only if*

(1) *there exists a finite subset X of $E(S)$ such that for every $a \in S$ we have some $e \in X$ with $ae = a$;*

(2) *if G is a maximal subgroup of S_0, then $G = \langle F \cup V \rangle$ where V is finite and $F = \langle E(S_0) \rangle \cap G$.*

If S is a band, (2) is redundant.

Our current examples of completely regular semigroups with finitely generated universal relation include completely simple semigroups and strong semilattices of groups or rectangular bands (possibly with identity adjoined). In each case they possess a minimum ideal with finitely many $\mathcal{R}$-classes, although we will now construct an example which dictates that this is not a general phenomenon.

The following construction will be used in the desired example, but, as it is quite general, it is convenient to state it separately.

Lemma 8.6 *Let S be a semigroup and U a left zero semigroup with $S \cap U = \emptyset$. Suppose that S acts on U on the left via $(s, u) \mapsto s \cdot u$. Let $T = S \cup U$ and define a binary operation $*$ on T, extending those on S and U, as follows:*

$$s * u = s \cdot u, \; u * s = u$$

for all $s \in S, u \in U$. Then T is a semigroup having U as minimum ideal. If S is a monoid with identity 1 and acts monoidally, then T is a monoid with identity 1.

Proof It is clear that every element of U is a left zero for the multiplication. It is easy to check that for any $s, t \in S$ and $u, v \in U$ we have

$$(s * t) * u = s * (t * u), \; (s * u) * t = s * (u * t) \text{ and } (s * u) * v = s * (u * v)$$

so that the multiplication is associative. The result follows. □

In the next example, by $\mathcal{T}_U^{\mathrm{op}}$ we mean the full transformation monoid on U with composition *right to left*.

Example 8.7 There exists a left regular band monoid with finitely generated universal relation such that the minimum ideal has infinitely many $\mathcal{R}$-classes.

Proof Let $U = \{u_i : i \in \mathbb{N}^0\}$ be a left zero semigroup and let $L = \{\ell_i : i \in \mathbb{N}\}$. Define a map $L \times U \to U$ by $\ell_i \cdot u_0 = u_0$ and $\ell_i \cdot u_j = u_i$ for any $i, j \in \mathbb{N}$. This gives us a map from L to $\mathcal{T}_U^{\mathrm{op}}$, which can be extended to a morphism from L^* to $\mathcal{T}_U^{\mathrm{op}}$ and we therefore obtain an action of L^* on U. Let M be the free left regular band monoid on L. Notice that for any $v, w \in L^*$ we have

$$vwv \cdot u_i = vw \cdot u_i \text{ for any } i \in \mathbb{N}^0,$$

so that we also have an induced action of S on U given by $\overline{w} \cdot u_j = w \cdot u_j$, where $\overline{w}$ denotes the natural image of $w \in L^*$ in S.

Let $T = S \cup U$ be made into a monoid under $*$ as in Lemma 8.6. We claim that $\omega_T^\ell = \langle X^2 \rangle$ where $X = \{1, u_0, u_1\}$. To see this we use Corollary 8.4, noting that conditions (1) and (2) are clear by construction. Let $W = \{u_0, u_1\}$, and take any $u_i \in U \backslash W$. Then $u_i = l_i u_1 \, \rho_{W^2} \, l_i u_0 = u_0$, and so Condition (3) holds, thus proving the claim. □

In the case where $\mathcal{H}$ is a congruence on a completely regular semigroup S, we may take a different approach to Corollary 8.3. Our aim is to determine the relevant property of the completely simple semigroup in terms of the band $S/\mathcal{H}$ and a maximal subgroup of the minimum ideal of S.

Theorem 8.8 *Let S be a completely regular semigroup and suppose that $\mathcal{H}$ is a congruence on S. Let $S = \bigcup_{\alpha\in\mathcal{Y}} S_\alpha$ be the decomposition of S into a semilattice of completely simple semigroups S_α, $\alpha \in \mathcal{Y}$. Then ω_S^ℓ is finitely generated if and only if the following hold:*

(1) *the universal relation on $S/\mathcal{H}$ is finitely generated;*
(2) *if G is a maximal subgroup of S_0, where 0 is the least element of $\mathcal{Y}$ (the existence of 0 follows from (1)), then $G = \langle F \cup V\rangle$ where V is finite and $F = \langle E(C_0)\rangle \cap G$ where C_0 is the union of finitely many $\mathcal{R}$-classes of S_0.*

Proof We first suppose that $\omega_S^\ell = \langle U^2\rangle$ for a finite set $U \subseteq S$. As $\mathcal{H}$ is a congruences on S, Proposition 4.1 tells us that the universal relation on the quotient $S/\mathcal{H}$ is also finitely generated. Condition (2) follows from Corollary 8.3.

Conversely, suppose that (1) and (2) hold. Let E be a finite subset of $E(S)$ such that the universal relation on $S/\mathcal{H}$ is finitely generated by $[E] \times [E]$ where $[E] = \{H_f : f \in E\}$. As in the proof of Theorem 8.1 we may assume that G is an $\mathcal{H}$-class of S_0 and there exists a finite subset W in S such that for any $g, h \in G$ we have $g\ \rho_{W^2}\ h$.

Let $X = E \cup W \cup \{e\} \cup Ee$, where e is the identity of G. We claim that $\omega_S^\ell = \langle X^2\rangle$. Let $a \in S$. Then $[a] \leq_{\mathcal{L}} [f]$ for some $f \in E$ and so $a \leq_{\mathcal{L}} f$. Now

$$a = af\ \rho_{X^2}\ ae = b\,\mathcal{L}\,e$$

and $eb \in G$, so that $eb\ \rho_{W^2}\ e$. If $g = g^2\ \mathcal{H}\ b$, then $b\ \rho_{W^2}\ ge = g$ and as $\rho_{W^2} \subseteq \rho_{X^2}$, so $b\ \rho_{X^2}\ g$. It remains to show that $g\ \rho_{X^2}\ e$.

As $[g]\ \rho_{[E]^2}\ [e]$, there exists a sequence

$$[g] = [t_1][c_1], [t_1][d_1] = [t_2][c_2], \ldots, [t_n][d_n] = [e],$$

for $[t_i] \in (S/\mathcal{H})^1$ and $([c_i], [d_i]) \in [E]^2$, $1 \leq i \leq n$. Now

$$g\ \mathcal{H}\ t_1c_1, t_1d_1\ \mathcal{H}\ t_2c_2, \ldots, t_nd_n\ \mathcal{H}\ e.$$

Multiplying by e on the right we have

$$g\ \mathcal{H}\ t_1(c_1e), t_1(d_1e)\ \mathcal{H}\ t_2(c_2e)\ \ldots, t_n(d_ne)\ \mathcal{H}\ e.$$

Now suppose $p, q \in L_e$ where $p\ \mathcal{H}\ q\ \mathcal{H}\ r = r^2$. Then $ep\ \mathcal{H}\ eq\ \mathcal{H}\ e$ and so $ep\ \rho_{W^2}\ eq$. This implies $rep\ \rho_{W^2}\ req$ and hence $p\ \rho_{W^2}\ q$. Thus

$$g\ \rho_{W^2}\ t_1(c_1e)\ \rho_{X^2}\ t_1(d_1e)\ \rho_{W^2}\ t_2(c_2e)\ \ldots\ \rho_{W^2}\ e,$$

and so $g\ \rho_{X^2}\ e$ and $\omega_S^\ell = \langle X^2\rangle$, as required. □

Remark 8.9 All the results in this section may be easily adapted to the pseudo-finite case. For example, in Theorem 8.1 the pseudo-finiteness of S guarantees that the lengths of the $(F \cup V)^2$-sequences and of the W^2-sequences in (2) and (3) respectively may be bounded, with the converse also clearly holding.

Open Question 8.10 Every pseudo-finite semigroup considered in this article has the property that it contains an ideal which is completely simple. The contrast to the weaker case where ω^J is finitely generated is highlighted by Theorem 5.1 and Proposition 5.3. In view of this we ask whether all pseudo-finite semigroups have this property? By the work of Sect. 8, a positive answer to this question would give a complete description of all pseudo-finite semigroups.

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