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Switching tube-based MPC: characterization of minimum dwell-time for feasible and robustly stable switching

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***Abstract*—We study the problem of characterizing mode dependent dwell-times that guarantee safe and stable operation of disturbed switching linear systems in an MPC framework. We assume the switching instances are not known a-priori, but instantly at the moment of switching. We first characterize dwell-times that ensure feasible and stable switching between independently designed robust MPC controllers by means of the well established exponential stability result available in the MPC literature. Then, we employ the concept of multi-set invariance to improve on our previous results, and obtain an exponential stability guarantee for the switching closed-loop dynamics. The theoretical findings are illustrated via a numerical example.**

***Index Terms*—Model predictive control, switching systems, dwell-time, robust control, stability.**

I. Introduction

Model predictive control (MPC) is a well established control technique that handles state and input constraints explicitly, while optimizing the system performance on-line [1]. However, the synthesis of stabilizing and admissible MPC controllers relies largely on knowing a model of the plant that is being controlled. On the other hand, many plants are better represented by a collection of models (or modes) and a logic based switching scheme [2]. This modelling framework is referred to as switching systems and it poses serious theoretical challenges for the design of admissible and stabilizing MPC controllers; for example, guaranteeing constraint satisfaction despite the switching between modes with different constraints.

To tackle some of these issues, many authors have focused on the concept of invariance for a prescribed dwell-time [3]–[8]. A dwell-time is a period of time during which the plant behaves as a single fixed mode, and it is easy to show that short dwell-times may result in unstable closed-loop dynamics even for locally stabilizing controllers [2]. In [3], [4] coupled returnable sets are introduced for a linear discrete time switching system subject to bounded additive disturbances. These sets are used in [5] to design a stabilizing and admissible MPC controller, albeit assuming homogeneity of disturbances and constraints, and requiring the solution of a min-max problem at each time instant. In [6], [7] a similar problem is solved through the computation of inter-reachable sets. Constraint satisfaction is guaranteed by design but stability is established only when future switches are known a-priori.

Another approach is to consider the dwell-time as a design variable, such as in [9], [10] where the goal is to compute a minimum dwell-time to ensure feasibility and stability of a switched MPC control architecture. Standard (robust) MPC controllers are designed for each linear mode and the concept of set reachability is employed to compute a minimum mode dependent dwell-time (MDT) that guarantees feasible switching. A contraction requirement in the computation of the reachability sets guarantees asymptotic stability.

A conceptually different technique is presented in [11], [12], where Lyapunov functions of neighbouring modes are compared in order to compute minimum MDTs that guarantee exponential stability of the switched closed-loop dynamics. Admissibility through a switch is ensured by intersecting sub-level sets of the corresponding Lyapunov functions of each mode. The latter are not invariant, but guarantee constraint satisfaction by construction (similar to [4], [7], [10]).

In this note we propose an approach for the off-line computation of admissible and stabilizing MDTs, for heterogeneous modes, based on the exponential stability result thoroughly established in the MPC literature. First we employ the exponential decay of the state trajectories to characterize a simple set that contains the corresponding closed-loop dynamics. This set is then used to compute MDTs that allow for admissible switching given (robust) controllers independently designed for each mode. The latter is in contrast to [6], [7] where the reachability between neighbouring invariant sets has to be guaranteed and [10] where the coupled invariant sets [3], [4] are employed. Furthermore, by using a simple set, we are able to compute the corresponding MDTs without the need for the explicit computation of reachable sets of the MPC-controlled system (as required in [10]).

In the nominal case we establish exponential stability of the origin by comparing Lyapunov functions, improving on the asymptotic stability result found in [6], [9], [10]. In the robust case, and provided a sufficiently long MDT, we guarantee finite time convergence (exponentially fast) to a neighbourhood of the origin. However, given the switching dynamics, this neighbourhood is larger than the one related to the robust control of single-mode uncertain dynamics. In order to improve on the latter, we present a second set of results that employ the concept of invariant multi-sets [8]. These sets remain invariant after a switch in a neighbour-to-neighbour framework, thus allowing for an exponential stability guarantee for a reduced neighbourhood around the origin in the robust case.

A. Notation

For $\mathbb{C},\mathbb{D} \subset \mathbb{R}^n$, $\mathbb{C} \oplus \mathbb{D}$ and $\mathbb{C} \ominus \mathbb{D}$ are, the Minkowski sum and Pontryagin difference respectively [13]. The 1-norm ball centred at the origin with radius r is $\mathcal{B}_r$ and conv $\{\cdot\}$ is the convex hull operator. The set of positive integers including 0 is $\mathbb{N}_0$. For $x \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$, $||x||_Q^2$ is shorthand for $x^\top Qx$, $|x|_p$ represents the p-norm of x and $Q > 0$ means that Q is positive definite. The identity matrix of dimension n is $\mathcal{I}_n$. For $a > b \in \mathbb{N}_0$, $a : b$ is the sequence of integers from a to b. A polytope is a compact polyhedron.

II. Preliminaries

A. Switching dynamics

We consider a general class of discrete-time switching linear systems subject to bounded additive disturbances and constraints, represented in state space form by

$$\begin{aligned} x(t+1) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + w(t) \\ x(t) &\in \mathbb{X}_{\sigma(t)} \subset \mathbb{R}^{n_x} \\ u(t) &\in \mathbb{U}_{\sigma(t)} \subset \mathbb{R}^{n_u} \\ w(t) &\in \mathbb{W}_{\sigma(t)} \subset \mathbb{R}^{n_x}, \end{aligned} \tag{1}$$

where $x(t)$ and $u(t)$ are respectively the state and input of the system at time t. The switching signal $\sigma(\cdot)$ is a piecewise constant function

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that, at each sampling time, takes values in the finite set $\mathcal{M} = \{1, \ldots, M\}$, and indicates the currently active mode. We require the following assumptions.

Assumption 1. $\mathbb{X}_m$, $\mathbb{U}_m$ and $\mathbb{W}_m$ are convex polytopes. $\mathbb{X}_m$ and $\mathbb{U}_m$ contain the origin in their interior and $\mathbb{W}_m$ contains the origin.

Assumption 2. The pair (A_m, B_m) is stabilizable $\forall m \in \mathcal{M}$.

Assumption 2 implies the existence, for each mode, of a linear gain K_m that renders $\bar{A}_m = A_m + B_m K_m$ Schur. The switching instances are $\{t_0, t_1, \ldots, t_k, \ldots\}$ with $t_0 = 0$ and $t_k \geq t_{k-1} + 1$, thus $\sigma(t)$ is constant in $[t_{k-1}, t_k)$ for all $k \geq 1$. Furthermore we assume that the switching signal is unknown a-priori but known instantly at each time t, and that the switching and sample times coincide. The latter allows us to define the concept of mode dependent dwell-time (MDT).

Definition 1. The MDT associated to mode $m \in \mathcal{M}$, say τ_m, is the minimum amount of time during which the switching system remains in mode m before leaping into another allowable mode. It follows that $t_{k+1} - t_k \geq \tau_m$ for any $k \in \mathbb{N}_0$ such that $\sigma(t_k) = m$.

In many applications only certain switches are allowed. In this case we refer to $\sigma(\cdot)$ as a constrained switching signal (CSS). A CSS can be precisely represented by a directed graph $\mathcal{G}(\mathcal{M}, \mathcal{E})$, where $\mathcal{M}$ is the set of nodes, and $\mathcal{E} = \{(s, d) \,|\, s, d \in \mathcal{M}\}$ the set of edges that link the nodes together. Each edge represents an allowed switch and for each $(s, d) \in \mathcal{E}$, s represents the source node and d the destination node. In other words, at each time instant t

$$\sigma(t) \in \mathcal{M}_{\sigma(t-1)} = \{d \in \mathcal{M} |\, (\sigma(t-1), d) \in \mathcal{E}\} \subseteq \mathcal{M}.$$

We focus on the regulation problem, i.e. the design of a control law $u(t) = \kappa(x(t))$ that admissibly stabilizes the origin (or a neighbourhood of it) given a CSS. Although it might be trivial to design robustly stabilizing MPC controllers for each mode, it can be shown that mode-stabilizing controllers can destabilize the switching system if the switches happen too rapidly. Furthermore, the heterogeneity of the constraints may result in constraint violation at the moment of switching. In this note, we propose a solution to these issues that relies on characterizing the minimum MDT required, by each mode, to guarantee stable and admissible switching between mode-stabilizing tube-based MPC controllers.

B. Single Tube MPC

In order to achieve robust regulation of the constrained switching system (1) we employ the robust control technique known as tube MPC (TMPC) [14], and the exponential stability result available for it. We now recall some standard definitions and a brief description of the TMPC technique applied to a single mode $m \in \mathcal{M}$ (see [14], [15] for a detailed description).

Definition 2 (Invariant sets)**.** Consider the dynamics in (1) for a single mode m and control law $u(t) = K_m x(t)$. A set $\mathbb{S}_m$ is robust positive invariant (RPI) for mode m if $\bar{A}_m \mathbb{S}_m \oplus \mathbb{W} \subseteq \mathbb{S}_m$, and positive invariant (PI) if $\mathbb{W}_m = \{0\}$. Furthermore, $\mathbb{S}_m$ is an admissible RPI (PI) set if $\mathbb{S}_m \subset \mathbb{X}_m$ and $K_m \mathbb{S}_m \subset \mathbb{U}_m$.

TMPC relies on the regulation of artificial undisturbed (also called nominal) trajectories represented by $(\bar{x}(t), \bar{u}(t))$, subject to constraints tightened by an RPI set to account for the effect of the disturbances. At each time instant, the optimal control problem solved by the m-TMPC controller is

$$\mathbb{P}_{N_m}(x(t)) \colon \quad \min_{\bar{\boldsymbol{u}}, \bar{x}_0} J_{N_m}(\bar{\boldsymbol{u}}, \bar{x}_0) \tag{2a}$$

s.t. (for $k = 0, \ldots, N_m - 1$)

$$x(t) - \bar{x}_0 \in \mathbb{S}_m \tag{2b}$$

$$\bar{x}_{k+1} = A_m \bar{x}_k + B_m \bar{u}_k \tag{2c}$$

$$\bar{x}_k \in \bar{\mathbb{X}}_m \subseteq \mathbb{X}_m \ominus \mathbb{S}_m \tag{2d}$$

$$\bar{u}_k \in \bar{\mathbb{U}}_m \subseteq \mathbb{U}_m \ominus K_m \mathbb{S}_m \tag{2e}$$

$$\bar{x}_{N_m} \in \bar{\mathbb{X}}_{f,m} \subseteq \bar{\mathbb{X}}_m, \tag{2f}$$

where $(\bar{x}_k, \bar{u}_k)$ are the nominal predictions, updated at each time instant to account for the newly measured true state, N_m is the prediction horizon, and $\bar{\boldsymbol{u}} = \{\bar{u}_0, \ldots, \bar{u}_{N_m - 1}\}$ is the input sequence to be optimized. The sets $\mathbb{S}_m$ and $\bar{\mathbb{X}}_{f,m}$ are respectively an admissible RPI and an admissible PI set for the uncertain and nominal dynamics (2c) of mode m for a given stabilizing K_m according to Definition 2. These sets can be computed using several different approaches such as [13], [16]–[18].

In standard tube MPC implementations, the cost function is designed to approximate the infinite horizon LQR cost

$$J_{N_m}(\bar{\boldsymbol{u}}, \bar{x}_0) = \sum_{k=0}^{N-1} \left(||\bar{x}_k||^2_{Q_m} + ||\bar{u}_k||^2_{R_m}\right) + ||\bar{x}_N||^2_{P_m},$$

with $Q_m, R_m > 0$ and $\bar{A}_m^\top P_m \bar{A}_m + Q_m + K_m^\top R_m K_m - P_m = 0$. Define

$$\left(\bar{\boldsymbol{u}}^*(x(t)), \bar{x}_0^*(x(t))\right) = \arg \mathbb{P}_{N_m}(x(t))$$
$$V_{N_m}(x(t)) = J_{N_m}\left(\bar{\boldsymbol{u}}^*(x(t)), \bar{x}_0^*(x(t))\right),$$

then the nominal trajectories are updated with $(\bar{x}(t), \bar{u}(t)) = (\bar{x}_0^*(x(t)), \bar{u}_0^*(x(t)))$. Let $\bar{\mathcal{X}}_{N_m}$ be the set of all the states for which $\mathbb{P}_{N_m}(x)$ is feasible when constraint (2b) is replaced by $\bar{x}_0 = x(t)$, then the following result holds [14], [15].

Proposition 1. If (i) Assumptions 1 and 2 hold, (ii) the sets $\mathbb{S}_m$ and $\bar{\mathbb{X}}_{f,m}$ are convex polytopes with the origin in their interior, and (iii) the loop is closed with $u(t) = \kappa_m(x(t)) = \bar{u}_0^*(x(t)) + K_m(x(t) - \bar{x}_0^*(x(t)))$, then (a) the optimization problem (2) is recursively feasible with feasibility region $\mathcal{X}_{N_m} = \mathbb{S}_m \oplus \bar{\mathcal{X}}_{N_m}$ (b) the sets $\bar{\mathcal{X}}_{N_m}$ and $\bar{\mathcal{X}}_{N_m - 1}$ are convex polytopes with the origin in their interior and invariant under $\bar{u}_0^*(x(t))$, (c) state and input constraints are met at all times despite the disturbance, and (d) there exist constant scalars $b_m, d_m, f_m > 0$ such that for all $x \in \mathcal{X}_{N_m}$ and $w \in \mathbb{W}_m$ it holds that:

$$b_m |\bar{x}_0^*(x)|_2^2 \leq V_{N_m}(x) \leq d_m |\bar{x}_0^*(x)|_2^2 \tag{3a}$$

$$V_{N_m}(A_m x + B_m \kappa_m(x) + w) - V_{N_m}(x) \leq -f_m |\bar{x}_0^*(x)|_2^2. \tag{3b}$$

Corollary 1. The system of inequalities (3) implies that there exist constant scalars $c_m > 0$ and $\lambda_m \in (0, 1)$ such that for all $x(0) \in \mathbb{S}_m \oplus \bar{\mathcal{X}}_{N_m}$, it holds that

$$|\bar{x}(t)|_2 \leq c_m \lambda_m^t |\bar{x}(0)|_2. \tag{4}$$

Therefore the origin is exponentially stable for the optimized nominal trajectories of mode m.

Proofs for Proposition 1 and Corollary 1 can be found in [14], [15].

III. Switching Tube MPC

Mode-admissible and mode-stabilizing controllers can result in constraint violation and unstable behaviour if the switch between modes occurs too quickly [2]. To prevent these issues, given the switching system in (1) and a collection of m independent robust controllers like the one described in Section II-B, we propose the characterization of minimum MDTs for admissible switching and minimum MDTs for stabilizing switching, both depending on the exponential stability constants presented in Proposition 1 and Corollary 1. The core idea of our approach to finding MDTs for admissible switching is to bound the state of the m mode closed-loop system within a simple set whose Chebyshev radius decreases exponentially over time. Whenever this set is inside the feasibility region of a neighbouring mode, say l, a switch is admissible. Corollary 1 bounds the 2-norm of the nominal state trajectories, however we propose to use a 1-norm ball as a bounding set because the latter is a convex polyhedron. This results in the set operations needed in our approach – for example, the intersection of two convex sets – being greatly simplified. We now recast the exponential decay in (4) to account for the 1-norm.

$$|\bar{x}(t)|_1 \leq \sqrt{n_x} c_m \lambda_m^t |\bar{x}(0)|_2 \tag{5}$$

A drawback of using a 1-norm ball is that its radius (the right hand side of (5)) is larger than the radius of the corresponding 2-norm ball (right hand side of (4)). This introduces conservativeness since a possibly larger number of time steps is required for the admissibility inclusion to be verified. However, the difference between both radii, and thus the associated conservativeness, decreases exponentially fast.

A. Minimum MDT for admissible switching: known $\mathcal{X}_{N_m}$

Admissibility of the m-TMPC controller depends on whether the current state lies inside the feasibility region $\mathcal{X}_{N_m}$. Proposition 1 guarantees that, given an appropriate design, these feasibility sets are convex polytopes. These sets are also often called the N_m-stabilizable sets to $\bar{\mathbb{X}}_{f,m}$ since they contain all the states that can be feasibly driven to the terminal set with a sequence of N_m control actions. Following this, and according to [18], the computation of $\bar{\mathcal{X}}_{N_m}$ requires N_m iterations of the backwards reachability operator, starting in $\bar{\mathbb{X}}_{f,m}$. If the dimension of the plant is large, the number of defining half-spaces may grow prohibitively fast throughout the iterations, making it computationally expensive to reach N_m. Our first set of results assume that $\bar{\mathcal{X}}_{N_m}$ and $\bar{\mathcal{X}}_{N_m-1}$ are known for all $m \in \mathcal{M}$, but in view of the previous discussion, Section III-B provides an alternative for when that is not the case.

Suppose $m, l \in \mathcal{M}$; a switch from mode m to mode l is feasible at time t_k if and only if $x(t_k) \in \mathcal{X}_{N_l}$. However the heterogeneity of the modes may result in $\mathcal{X}_{N_m} \not\subseteq \mathcal{X}_{N_l}$ thus, even though $x(t_k - 1) \in \mathcal{X}_{N_m}$ implies $x(t_k) \in \mathcal{X}_{N_m - 1}$ with $\mathcal{X}_{N_m-1}$ invariant for the closed-loop, it does not necessarily imply $x(t_k) \in \mathcal{X}_{N_l}$. Note that we imposed $t_k \geq t_{k-1} + 1$, thus the dwell-times are necessarily no less than one. In view of this, define

$$\alpha_m = \max_{x \in \bar{\mathcal{X}}_{N_m - 1}} |x|_2, \tag{6}$$

and set $r_m(\tau) = \sqrt{n_x} c_m \lambda_m^\tau \alpha_m$. Then, the following result holds.

Proposition 2. Define

$$\mathbb{B}_{r_m(t - t_{k-1} - 1)} = \bar{\mathcal{X}}_{N_m - 1} \cap \mathcal{B}_{r_m(t-t_{k-1}-1)}. \tag{7}$$

If mode m became active at the last switching instant t_{k-1} (feasibly), and the loop is closed with $\kappa_m(\cdot)$, the nominal state trajectory of the switching system fulfils $\bar{x}(t) \in \mathbb{B}_{r_m(t-t_{k-1}-1)}$ for all $t \geq t_{k-1} + 1$.

Proof. If $x(t_{k-1}) \in \mathcal{X}_{N_m}$ and the loop is closed with $\kappa_m(\cdot)$, then $\bar{x}(t) \in \bar{\mathcal{X}}_{N_m}$ for all $t \geq t_{k-1}$. Particularly $\bar{x}(t_{k-1}+1) \in \bar{\mathcal{X}}_{N_m - 1}$. The rest follows from the exponential stability result in Corollary 1 and inequality (5). ■

Note that $r_m(t - t_{k-1} - 1)$ is a conservative radius for the ball that contains the state at time t because it is computed with α_m instead of the current state norm $|x(t_{k-1}+1)|_2$, however this allows Proposition 2 to be independent of the initial state and of the specific times in which a switch takes place. Nevertheless, given the effect of the additive disturbance, the m-TMPC controller can only guarantee stability of the set $\mathbb{S}_m$, thus feasible switching needs the following assumption.

Assumption 3. For all $m \in \mathcal{M}$ it holds that $\mathbb{S}_m \subset \mathbb{S}_l \oplus \bar{\mathcal{X}}_{N_l}$ for all $l \in \mathcal{M}_m$.

Theorem 1. Consider any pair $m, l \in \mathcal{M}$ with $m \neq l$, $\sigma(t_{k-1}) = m$ and $l \in \mathcal{M}_m$. If $\tau^f_{m,l}$ is such that $\mathbb{S}_m \oplus \mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \mathbb{S}_l \oplus \bar{\mathcal{X}}_{N_l}$, then a switch to mode l is feasible at any time t_k that fulfils $t_k - t_{k-1} - 1 \geq \tau^f_{m,l}$.

Proof. If $t_k \geq t_{k-1} + 1 + \tau^f_{m,l}$, then $x(t_k) \in \mathbb{S}_m \oplus \mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \mathcal{X}_{N_l}$, thus the optimization problem $\mathbb{P}_{N_l}(x(t_k))$ is feasible. ■

Corollary 2. If $\sigma(\cdot)$ is CSS, then the minimum MDT that guarantees feasible switching out of mode m is τ^f_m defined by

$$\tau^f_m = 1 + \max_{l \in \mathcal{M}_m} \tau^f_{m,l}.$$

Since we use 1-norm balls to bound the state trajectories, instead of accurate reachable sets such as in [10], we expect to obtain longer (more conservative) feasibility MDTs. However, the computation of exact reachable sets requires the explicit characterization of $\kappa_m(\cdot)$; although this is possible for low-dimensional systems, it requires the implementation of multi-parametric programming [19]. Theorem 1, on the other hand, only requires the computation of $\bar{\mathcal{X}}_{N_m}$, which is achievable by the recursive application of the backwards reachability operation [18].

B. Minimum MDT for admissible switching: unknown $\mathcal{X}_{N_m}$

The computation of feasibility regions does not scale well with the dimension of the plant, however, as the number of defining half-spaces of the backwards reachability sets may grow prohibitively large. Given the invariance of the terminal set, it can be shown [18] that the i_m-step stabilizable sets to $\bar{\mathbb{X}}_{f,m}$ are consecutively inclusive, hence any $\bar{\mathcal{X}}_{i_m}$ with $i_m \in [1, N_m)$ represents a feasible set for (2). Nevertheless, if $\bar{\mathcal{X}}_{N_m}$ is not tractable, then the computation of $\bar{\mathcal{X}}_{i_m}$ may also not be, even for $i_m = 1$. In order to avoid computing any such set note that for a feasible switch it is sufficient that $\mathbb{S}_m \oplus \mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \mathcal{X}_{N_l}$, which is readily met if $\mathbb{S}_m \oplus \mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \mathbb{S}_l \oplus \Theta_l$ for any $\Theta_l \subseteq \bar{\mathcal{X}}_{N_l}$.

Proposition 3. Assume that $\bar{\mathcal{X}}_{N_m}$ has a non-empty interior and define the vertices of $\bar{\mathbb{X}}_m$ by $\{v^i_m\}$ for $i = 1, \ldots, n_m$. For all $i = 1, \ldots, n_m$ there exist $\bar{\beta}^i_m \in (0, 1]$ such that $\mathbb{P}_{N_m}(\beta^i_m v^i_m)$, with constraint (2b) replaced by $\bar{x}_0 = \beta^i_m v^i_m$, is feasible for $\beta^i_m \in (0, \bar{\beta}^i_m]$ but infeasible for $\beta^i_m > \bar{\beta}^i_m$. Furthermore, $\Theta_m = \operatorname{conv}\{\bar{\beta}^i_m v^i_m\} \subseteq \bar{\mathcal{X}}_{N_m}$.

Proof. If $\bar{\mathcal{X}}_{N_m}$ has a non-empty interior, then there exists $r > 0$ such that $\mathcal{B}_r \subseteq \bar{\mathcal{X}}_{N_m}$. It follows from the compactness of $\bar{\mathbb{X}}_m$ that there exists $\beta \in (0, 1]$ such that $\beta \bar{\mathbb{X}}_m \subseteq \mathcal{B}_r$, which completes the proof. ■

The set Θ_m described in Proposition 3 serves as a replacement to $\bar{\mathcal{X}}_{N_m}$ when the latter is not available. In order to compute the feasibility MDTs replace α_m in (6) by

$$\bar{\alpha}_m = \max_{x \in \Theta_m} |x|_2, \tag{8}$$

and $\bar{\mathcal{X}}_{N_m-1}$ in (7) by $\bar{\mathbb{X}}_m$, then the following holds.

Theorem 2. Consider any pair $m, l \in \mathcal{M}$ with $m \neq l$, $\sigma(t_{k-1}) = m$, $l \in \mathcal{M}_m$ and Θ_l from Proposition 3. If $\bar{\tau}^f_{m,l}$ is such that $\mathbb{S}_m \oplus \mathbb{B}_{r_m(\bar{\tau}^f_{m,l})} \subseteq \mathbb{S}_l \oplus \Theta_l$, then a switch to mode l is feasible at any time t_k that fulfils $t_k - t_{k-1} \geq \bar{\tau}^f_{m,l}$.

Proof. First notice that Θ_m is feasible but not invariant under the m-TMPC control law, unlike $\bar{\mathcal{X}}_{N_m-1}$. In view of this, and given that $\bar{x}(t) \in \bar{\mathbb{X}}_m$ at all times by construction, Proposition 2 holds with $\bar{\mathcal{X}}_{N_m-1}$ replaced by $\bar{\mathbb{X}}_m$ in (7). Furthermore, if $t_k \geq t_{k-1} + \bar{\tau}^f_{m,l}$, then $x(t_k) \in \mathbb{S}_m \oplus \mathbb{B}_{r_m(\bar{\tau}^f_{m,l})} \subseteq \mathbb{S}_l \oplus \Theta_l \subseteq \mathcal{X}_{N_l}$, thus the optimization problem $\mathbb{P}_{N_l}(x(t_k))$ is feasible. ■

Corollary 3. If $\sigma(\cdot)$ is CSS, then the minimum MDT that guarantees feasible switching out of mode m is $\bar{\tau}^f_m$ defined by

$$\bar{\tau}^f_m = \max_{l \in \mathcal{M}_m} \bar{\tau}^f_{m,l}.$$

Remark 1. In order to obtain the values of $\bar{\beta}^i_m$ first note that we only need to verify whether $\mathbb{P}_{N_m}\left(\bar{\beta}^i{}_m v^i_m\right)$ has a feasible solution, rather than finding the optimal. The exact values of $\bar{\beta}^i_m$ in Proposition 3 can then be easily found by solving, for each vertex, the linear program

$$\max_{\bar{\boldsymbol{u}}, \bar{\beta}^i_m} \bar{\beta}^i_m$$

subject to constraints (2c)–(2f) and $\bar{x}_0 = \bar{\beta}^i{}_m v^i_m$.

Remark 2. If the vertices of $\bar{\mathbb{X}}_m$ are not available, we can replace $\bar{\mathbb{X}}_m$ in Proposition 3 with any convex polytope in $\mathbb{R}^{n_x}$. The conservativeness of the resulting collection of sets Θ_m depends on the number of vertices of the unknown $\bar{\mathcal{X}}_{N_m}$, but Theorem 2 and Corollary 3 hold without changes.

Note that, although $\Theta_l \subseteq \bar{\mathcal{X}}_{N_l}$, Theorem 2 does not necessarily lead to longer feasibility MDTs. This is because $\bar{\alpha}_m$ in (8) is also computed with respect to $\Theta_m \subseteq \bar{\mathcal{X}}_{N_m}$. However, this approach does result in a smaller region of attraction. Interestingly, a similar trade-off is observed in [20]. To address this issue, and recover the full region of attraction, assume that $x(0)$ is known before initializing the plant. Depending on the application only an estimate might be available, however a worst case scenario approach can be observed. Even if $x(0) \notin \mathbb{S}_m \oplus \Theta_m$, we can easily test $x(0) \in \mathcal{X}_{N_m}$ just by solving $\mathbb{P}_{N_m}(x(0))$ off-line. In view of this we can define a supplementary MDT $\tau_{m,0}$ for the initial state, such that $\mathbb{B}_{r_{m,0}(\tau_{m,0})} \subseteq \Theta_m$ with $r_{m,0}(\tau) = \sqrt{n_x} c_m \lambda^\tau_m |x(0)|_2$. After the initialization MDT has passed, it is guaranteed that the $x(\tau_{m,0}) \in \mathbb{S}_m \oplus \Theta_m$. Thereafter, the feasibility MDTs computed by Theorem 2 and Corollary 3 guarantee admissible switching, thus practically recovering the full size of the region of attraction. If the initial mode $\sigma(0)$ is known, then $\tau_{\sigma(0),0}$ is enforced as initialization MDT. However, since the switching signal is not available for design, enforcing the maximum initialization MDT among all modes guarantees admissible switching independent of the initialization mode.

C. MDT for robustly stabilizing switching

The exponential stability of the nominal trajectories described in Corollary 1 (valid for a single mode) relies primarily upon the optimal value function $V_{N_m}(\cdot)$ being a Lyapunov function for the closed-loop system [15]. However, when a switch happens, two different cost functions come into play, thus the rate of change (3b) is not necessarily negative. This implies that, although an MDT greater or equal to τ^f_m (or $\bar{\tau}^f_m$) ensures feasible switching and the nominal trajectories are not affected by disturbances, they could oscillate around the edges of the feasibility regions, and never approach the origin.

In [11] a multiplicative difference is employed, in an undisturbed set-up, to relate the optimal value functions of different modes and compute a minimum MDT required to maintain nominal stability. However, a similar approach is not valid here because the bounds in (3) depend on $\bar{x}^*_0(x)$, which is an optimization variable, and so it does not necessarily take the same value for different controllers at a given state x.

In fact, it can be shown that,

$$\begin{aligned} V_{N_l}(x(t_{k+1})) - V_{N_m}(x(t_k)) \leq & \mathcal{G}_{l,m} |\bar{x}^*_{0,m}(x(t_k))|^2_2 \\ & + d_l |\bar{x}^*_{0,l}(x(t_{k+1}))|^2_2 \end{aligned} \tag{9}$$

where $\bar{x}^*_{0,m}(x(t_k))$ solves $\mathbb{P}_{N_m}(x(t_k))$, $\bar{x}^*_{0,l}(x(t_{k+1}))$ is the solution to $\mathbb{P}_{N_l}(x(t_{k+1}))$ and $\mathcal{G}_{l,m}$ is a negative monotonically decreasing function of the bounds in (3) and $t_{k+1} - t_k$. Although $\bar{x}^*_{0,l}(x(t_{k+1}))$ is also a function of $t_{k+1} - t_k$, it can only be defined via the explicit characterization of the control law $\kappa_l(\cdot)$. Furthermore, if $\mathbb{S}_m \nsubseteq \mathbb{S}_l$, a switch from mode m to mode l could result in an increase of optimal value function after it had become zero, thus we cannot guarantee exponential stability of the origin for the nominal closed-loop switched trajectories.

Instead of directly comparing optimal value functions, we make use of the robust invariance property of the feasibility regions in order to compute a collection of robustly stabilizing MDTs. First assume that a collection of sets $\{\Omega_m\}_{m \in \mathcal{M}}$ that fulfil the following assumptions is available.

Assumption 4. For all $m \in \mathcal{M}$ the set Ω_m is a PI set for the m nominal closed-loop dynamics $\bar{x}(t+1) = A_m \bar{x}(t) + B_m \bar{u}^*_0(x(t))$.

Assumption 5. For all $l \in \mathcal{M}$, the set Ω_l is large enough such that $\mathbb{S}_m \subset \mathbb{S}_l \oplus \Omega_l$ holds for all m such that $l \in \mathcal{M}_m$.

The goal of the collection of sets Ω_m is to provide a robust stability result despite the switching, at the expense of increasing the size of the set that is shown to be stable (when compared to a non-switching implementation). In view of Assumption 3, which is required for this overall approach to computing MDTs to be applicable, Assumptions 4 and 5 are met with $\Omega_m = \bar{\mathcal{X}}_{N_m}$. However, we seek to characterize the smallest possible neighbourhood of $\mathbb{S}_m$ that can be rendered stable despite the switching. In general, finding the minimal set that is invariant under the m-TMPC nominal control law and that fulfils Assumption 5 is not simple, since it requires the characterization of sub-level sets of the optimal value function. For unconstrained linear systems stabilized by a linear control law, these sets are characterized by simple ellipsoids (given the quadratic cost); but state constraints yield an implicit and non-linear MPC control law, resulting in that the sub-level sets need to be obtained numerically [15].

Nevertheless, there exist two simple candidates for Ω_m that fulfil Assumption 4 and may meet Assumption 5, although without any minimality guarantees . A first alternative is $\bar{\mathcal{X}}_{N_m-1}$ which according to Proposition 1 remains invariant under the nominal control law. A second alternative is a scaling of the corresponding terminal set. Indeed, if K_m is set to the corresponding LQR gain and $\bar{\mathbb{X}}_{f,m}$ as the maximal admissible PI set, then $\delta \bar{\mathbb{X}}_{f,m}$ also meets Assumption 4 for any $\delta \in [0, 1)$. Considering then, that the terminal gain associated to $\bar{\mathbb{X}}_{f,m}$ does not need to be set equal to the tube gain, the design of the former could account for the fulfilment of Assumption 5.

If Assumption 3 holds, Theorem 1 guarantees the existence of a collection of $\tau^f_{m,l}$ such that $\mathbb{S}_m \oplus \mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \mathbb{S}_l \oplus \bar{\mathcal{X}}_{N_l}$ for all $m \in \mathcal{M}$ and $l \in \mathcal{M}_m$. Accordingly, if the true MDTs are defined following Corollary 2 then the set

$$\mathcal{O} = \bigcup_{m \in \mathcal{M}} \left(\mathbb{S}_m \oplus \bar{\mathcal{X}}_{N_m}\right)$$

is an RPI set for the switched closed-loop dynamics. In view of this we have the following result.

Proposition 4. If Assumptions 4 and 5 hold, then there exists $\tau_{m,l}^{g} \geq 1$ such that

$$\mathbb{S}_m \oplus \left(\Omega_m \cap \mathcal{B}_{\hat{r}_m(\tau_{m,l}^{g})}\right) = \mathbb{S}_m \oplus \hat{\mathbb{B}}_{\hat{r}_m(\tau_{m,l}^{g})} \subseteq \mathbb{S}_l \oplus \Omega_l, \quad (10)$$

where $\hat{r}_m(\tau) = \sqrt{n_x} c_m \lambda_m^{\tau} \max_{x\in\Omega_m} |x|_2$. Furthermore, if the feasibility MDTs are set to $\max\left(\tau_{m,l}^{f}, \tau_{m,l}^{g}\right)$, the set

$$\mathcal{O}_g = \bigcup_{m\in\mathcal{M}} (\mathbb{S}_m \oplus \Omega_m) \quad (11)$$

is an RPI set for the switched closed-loop dynamics.

Proof. Given Assumption 4 and Corollary 1, for any $\bar{x}(0) \in \Omega_m$ the nominal closed-loop fulfils $\bar{x}(t) \in \hat{\mathbb{B}}_{\hat{r}_m(t)}$. In view of this and Assumption 5, it follows that there exists a positive $\tau_{m,l}^{g}$ such that (10) holds. Moreover, if the feasibility MDTs are set to the maximum between $\tau_{m,l}^{f}$ and $\tau_{m,l}^{g}$, it follows from Assumptions 4, the proof of Theorem 2 and the recursive feasibility guarantee in Proposition 1 that once the state reaches $\mathcal{O}_g$ in (11) it remains there forever, independent of the disturbance and the switching signal, thus $\mathcal{O}_g$ is an RPI set for the switching closed-loop dynamics. ∎

Define now

$$\tau_m^{g} = \max_{l\in\mathcal{M}_m} \tau_{m,l}^{g},$$

in view of Proposition 4, the following result holds.

Theorem 3. If Assumptions 4 and 5 hold, the feasibility MDTs are $\hat{\tau}_m^{f} = \max\left\{\tau_m^{f}, \tau_m^{g}\right\}$ for all $m \in \mathcal{M}$ and, for at least one $\bar{m} \in \mathcal{M}$ the stability MDT $\tau_{\bar{m}}^{s}$ is such that $\mathbb{B}_{r_{\bar{m}}(\tau_{\bar{m}}^{s})} \subseteq \hat{\mathbb{B}}_{\hat{r}_{\bar{m}}(\tau_{\bar{m}}^{g})}$, then, as soon as $\sigma(t_k) = \bar{m}$, the true state enters $\mathcal{O}_g$ in finite time posterior to the switch into mode $\bar{m}$ and remains therein for all future time instances.

Proof. Since there exists at least one $\bar{m}$ such that $\mathbb{B}_{r_{\bar{m}}(\tau_{\bar{m}}^{s})} \subseteq \hat{\mathbb{B}}_{\hat{r}_{\bar{m}}(\tau_{\bar{m}}^{g})}$, it follows that if $\sigma(t_k) = \bar{m}$, then for any $t \in [t_k + \tau_{\bar{m}}^{s}, t_{k+1})$ the state fulfils $x(t) \in \mathbb{S}_{\bar{m}} \oplus \mathbb{B}_{\hat{r}_m(\tau_{\bar{m}}^{g})}$, thus $x(t) \in \mathcal{O}_g$. Moreover, $x(t) \in \mathbb{S}_l \oplus \Omega_l$ for all $l \in \mathcal{M}_{\bar{m}}$, thus given Assumptions 4 and 5, if the feasibility MDTs fulfil $\hat{\tau}_m^{f} \geq \tau_m^{g}$, Proposition 4 holds and the set $\mathcal{O}_g$ is RPI set for the switched closed-loop dynamics, thus $x(t) \in \mathcal{O}_g$ for all $t \geq t_k + \tau_{\bar{m}}^{s}$. ∎

Remark 3. In Assumption 5 the set Ω_l can be arbitrarily small only if $\mathbb{S}_m \subseteq \mathbb{S}_l$, for all $m \in \mathcal{M}$ such that $l \in \mathcal{M}_m$. Otherwise, the size of Ω_l is lower bounded so that $\mathbb{S}_m \subset \mathbb{S}_l \oplus \Omega_l$ holds.

Theorem 3 guarantees robust stability of the set $\mathcal{O}_g$ by means of the recursive feasibility property associated to the different TMPC optimization problems. In this context the stability MDT $\tau_{\bar{m}}^{s}$ is nothing more than a large enough feasibility MDT so that mode $\bar{m}$ reaches $\mathbb{B}_{\hat{r}_m(\tau_{m,l}^{g})} \subseteq \Omega_{\bar{m}}$, rendering $\mathcal{O}_g$ invariant for all subsequent switches.

D. Robustly stabilizing and admissible switching MDT

If the feasibility regions $\mathcal{X}_{N_m}$ have been computed, it follows from Corollary 2 and Theorem 3 that the minimum MDT required to achieve robustly stabilizing and constraint admissible closed-loop dynamics with the switching TMPC controllers is τ_m defined by

$$\tau_m = \max\left\{\max\left\{\tau_m^{f}, \tau_m^{g}\right\}, \tau_m^{s}\right\}.$$

If the feasibility regions are not available, then it follows from Corollary 3 and Theorem 3 that the aforementioned MDT is

$$\tau_m = \max\left\{\max\left\{\bar{\tau}_m^{f}, \tau_m^{g}\right\}, \tau_m^{s}\right\}.$$

IV. Undisturbed dynamics

The undisturbed case, i.e. when $\mathbb{W}_m = \{0\}$ for all $m \in \mathcal{M}$, can be seen as a special instance of the general MDT problem analysed in Section III. The undisturbed MPC optimization can be obtained from the tube one presented in Section II-B by setting $\mathbb{S}_m = \{0\}$; this results in the nominal trajectories equating the true ones with $\bar{x}_0 = x(t)$, effectively reducing the number of optimization variables.

With the above modifications, the results pertaining the computation of minimum feasibility MDTs described in Sections III-A and III-B hold, however, the stability MDT results can be strengthened due to the fact that $\bar{x}_0^*(x)$ is not an optimization variable, and therefore does not change from mode to mode at the same state x.

A. MDT for stabilizing switching

As discussed before, when a switch takes place two different cost functions must be compared, thus the rate of change (3b) is not necessarily negative. In order to account for a switch, for all pairs $m, l \in \mathcal{M}$ with $l \in \mathcal{M}_m$ define $\mu_{l,m} \geq d_l - b_m$, where d_l and b_m are those in (3a). It follows that

$$V_{N_l}(x) - V_{N_m}(x) \leq \mu_{l,m}|x|_2^2 \quad \forall x \in \mathcal{X}_{N_m} \cap \mathcal{X}_{N_l}. \quad (12)$$

Equation (12) provides an additive bound on the change of the optimal value functions at the same state when a switch takes place. Note that the main difference between (9) and (12) is that the right hand side of the latter does not depend on the optimization of two different controllers, but on the fixed value of the current state. In view of (12), the following result holds.

Theorem 4. For any two switching instances $(t_k, \sigma(t_k) = m)$ and $(t_{k+1}, \sigma(t_{k+1}) = l \in \mathcal{M}_m)$ that fulfil the associated feasibility MDT, if $t_{k+1} - t_k \geq \tau_{m,l}^{s}$ with $\tau_{m,l}^{s}$ such that

$$\frac{\mu_{l,m} c_m^2 \lambda_m^{2\tau_{m,l}^{s}}}{d_m} < 1 - \left(1 - \frac{f_m}{d_m}\right)^{\tau_{m,l}^{s}} = \mathcal{F}(f_m, d_m, \tau_{m,l}^{s}), \quad (13)$$

then the origin is exponentially stable for the switched closed-loop, with respect to the switching instants.

Proof. First of all note that (12) puts a finite bound on the increase of the optimal value function produced by a switch at any given state. Secondly, from (12) and algebraic manipulation of (3) and (4) it follows that

$$\begin{aligned} V_{N_l}(x(t_{k+1})) - V_{N_m}(x(t_k)) \leq & -d_m \mathcal{F}(f_m, d_m, \tau_{m,l}^{s})|x(t_k)|_2^2 \\ & + \mu_{l,m} c_m^2 \lambda_m^{2(t_{k+1}-t_k)}|x(t_k)|_2^2, \end{aligned} \quad (14)$$

thus if (13) holds, the left hand side of (14) is negative. Furthermore, since (3) holds for all $m \in \mathcal{M}$ it follows that there exists $b, d, f > 0$ such that the candidate function $V(x(t_k)) = V_{\sigma(t_k)}(x(t_k))$ fulfils

$$\begin{aligned} b|x(t_k)|_2^2 \leq & V(x(t_k)) \leq d|x(t_k)|_2^2 \\ V(x(t_{k+1})) - & V(x(t_k)) \leq -f|x(t_k)|_2^2. \end{aligned}$$

In turn, this implies that there exists constants $c > 0$ and $\gamma \in (0, 1)$ such that $|x(t_k)|_2 \leq c\gamma^k |x(0)|_2$, therefore the origin is exponentially stable for the switched closed-loop dynamics. ∎

Since $V_{N_m}(\cdot)$ is a Lyapunov function for all $m \in \mathcal{M}$, it must happen that $f_m < d_m$. In view of this, the right hand side of the inequality in (13) is positive, monotonically increasing on $\tau_{m,l}^{s}$ and bounded above by 1. Also, the left hand side is negative if $\mu_{l,m} < 0$, and positive but monotonically decreasing on $\tau_{m,l}^{s}$ and bounded below by 0 if $\mu_{l,m} \geq 0$, thus a finite $\tau_{m,l}^{s}$ always exists such that (13) is met.

The bound in (12), however explicit, might lead to unnecessary conservativeness given that at any switching time t_k there is only one MPC controller active. An alternative is to compare the corresponding optimal value functions at dynamically adjacent states. For all pairs $m, l \in \mathcal{M}$ with $l \in \mathcal{M}_m$ define $\bar{\mu}_{l,m} \geq \mu_{l,m} - f_l$. It follows that

$$V_{N_l}\left(A_m x + B_m \kappa_m(x)\right) - V_{N_m}(x) \leq \bar{\mu}_{l,m}|x|^2$$
$$\forall \left(A_m x + B_m \kappa_m(x)\right) \in \mathcal{X}_{N_m} \cap \mathcal{X}_{N_l}, \;\; x \in \mathcal{X}_{N_m},$$

which provides an additive bound on the change of the optimal value functions at dynamically adjacent states. In view of this, we have the following result analogous to Theorem 4.

Proposition 5. If $\bar{\tau}^s_{m,l}$ is such that

$$\frac{\bar{\mu}_{l,m} c_m^2 \lambda_m^{2\left(\bar{\tau}^s_{m,l}-1\right)}}{d_m} < 1 - \left(1 - \frac{f_m}{d_m}\right)^{\bar{\tau}^s_{m,l}-1}, \qquad (15)$$

the origin is exponentially stable for the switched closed-loop, with respect to the switching instants.

Proposition 5 follows from the same arguments than Theorem 4. Analogously to (13), there exists a finite $\bar{\tau}^s_{m,l}$ such that (15) is met. However, whether (15) is less stringent than (12) cannot be determined without specifying the values of the various bounding constants in (3). In view of this, we propose to compute both and compare them in order to obtain the less conservative stabilizing MDT.

Corollary 4. If $\sigma(t)$ is a CSS, then the minimum MDT that guarantees exponential stability throughout a switch out of mode m is τ^s_m defined by

$$\tau^s_m = \max_{l \in \mathcal{M}_m} \min\left\{\tau^s_{m,l}, \bar{\tau}^s_{m,l}\right\}.$$

In parallel to the robust case, if the feasibility regions $\mathcal{X}_{N_m}$ have been computed, it follows from Corollary 2, Theorem 4 and Proposition 5 that the minimum MDT required to achieve stabilizing and constraint admissible closed-loop dynamics with the switching MPC controllers is τ_m defined by

$$\tau_m = \max\left\{\tau^f_m, \tau^s_m\right\},$$

and if the feasibility regions are not available, Corollary 3, Theorem 4 and Proposition 5 yield a minimum MDT of

$$\tau_m = \max\left\{\bar{\tau}^f_m, \tau^s_m\right\}.$$

V. Switching multi-set Tube MPC

The main feature that allowed us, in Section IV-A, to improve on the stability MDT results of Section III-C is the fact that the nominal trajectories are not re-optimized at each time instant (because they represent the true plant states). In order to obtain similar results in the robust set-up we propose to employ an alternative version of TMPC in which the nominal trajectories are allowed to evolve independently after initialization [15, Chapter 3]. At time $t = 0$ the optimal problem $\mathbb{P}_{N_m}$ is solved, but for any $t > 0$ constraint (2b) is replaced by $\bar{x}_0 = \bar{x}^*_1(x(t-1))$. Therefore, the nominal state at time t is the one step ahead optimal prediction made at time $t-1$, or simply

$$\bar{x}_0 = \bar{x}(t) = A_m \bar{x}(t-1) + B_m \bar{u}^*_0(t-1), \qquad (16)$$

thus the nominal state and the true state evolve separately, and the cost function now depends only on the nominal trajectories. We now recast Proposition 1 to reflect such modifications. In what follows we refer to the modified optimization problem as $\bar{\mathbb{P}}_{N_m}$

Proposition 6. If Assumptions 1 and 2 hold, the sets $\mathbb{S}_m$ and $\bar{\mathbb{X}}_{f,m}$ are convex polytopes with the origin in their interior, the loop is closed with $u(t) = \kappa_m(x(t)) = \bar{u}^*_0(\bar{x}(t)) + K_m\left(x(t) - \bar{x}(t)\right)$ and constraint (2b) is replaced by (16) then (a) the optimization problem $\bar{\mathbb{P}}_{N_m}$ is recursively feasible with feasibility region $\mathcal{X}_{N_m} = \mathbb{S}_m \oplus \bar{\mathcal{X}}_{N_m}$ (b) the sets $\bar{\mathcal{X}}_{N_m}$ and $\bar{\mathcal{X}}_{N_m-1}$ are convex polytopes with the origin in their interior and invariant under $\bar{u}^*_0(x(t))$, (c) state and input constraints are met at all times despite the disturbance, and (d) there exists constant scalars $b_m, d_m, f_m > 0$ such that for all $\bar{x} \in \bar{\mathcal{X}}_{N_m}$ it holds that:

$$b_m|\bar{x}|_2^2 \leq V_{N_m}(\bar{x}) \leq d_m|\bar{x}|_2^2 \qquad (17a)$$
$$V_{N_m}\left(A_m\bar{x} + B_m\bar{u}^*_0(\bar{x})\right) - V_{N_m}(\bar{x}) \leq -f_m|\bar{x}|_2^2. \qquad (17b)$$

Furthermore, there exists constant scalars $c_m > 0$ and $\lambda_m \in (0,1)$ such that for all $\bar{x}(0) \in \bar{\mathcal{X}}_{N_m}$, it holds that

$$|\bar{x}(t)|_1 \leq \sqrt{n_x} c_m \lambda_m^t |\bar{x}(0)|_2. \qquad (18)$$

Therefore the origin is exponentially stable for the nominal trajectories of mode m when in closed-loop with $\bar{\kappa}_m(x(t)) = \bar{u}^*_0(\bar{x}(t))$.

The proof to Proposition 6 can be found in [15].

A. Minimum MDT for admissible switching: known $\mathcal{X}_{N_m}$

Although similar arguments to those in Section III-A can be used to bound the nominal state inside a ball of time dependent radius, Theorem 1 does not hold for this version of TMPC. To illustrate why, note that Proposition 2 holds for $\bar{\mathbb{P}}_{N_m}$, thus if mode m became active (feasibly) at time t_{k-1}, we can find a finite $\bar{t} > t_{k-1}$ such that

$$\bar{x}(\bar{t}) \in \mathbb{B}_{r_m(\bar{t}-t_{k-1}-1)} \subseteq \bar{\mathcal{X}}_{N_l} \subseteq \bar{\mathbb{X}}_l. \qquad (19)$$

Furthermore, $x(\bar{t}) - \bar{x}(\bar{t}) \in \bar{\mathbb{S}}_m$ due to the robust invariance property of $\mathbb{S}_m$. Given (19) $\bar{\mathbb{P}}_{N_l}$ is feasible at time $\bar{t}$, so if a switch takes place the input is defined by

$$u(\bar{t}) = \bar{u}(\bar{t}) + K_l\left(x(\bar{t}) - \bar{x}(\bar{t})\right) \in \bar{\mathbb{U}}_l \oplus K_l\mathbb{S}_m,$$

but since $\bar{\mathbb{U}}_l \oplus K_l\mathbb{S}_m$ is not necessarily a subset of $\mathbb{U}_l$, the true input constraints may be violated. Furthermore, even, if $u(\bar{t}) \in \mathbb{U}_l$, it is easy to show that

$$x(\bar{t}+1) - \bar{x}(\bar{t}+1) \in \bar{A}_l\mathbb{S}_m \oplus \mathbb{W}_l,$$

which is not necessarily a subset of $\mathbb{S}_l$ yielding a possible violation of the true state constraints at time $\bar{t}+1$.

We propose to address the state constraint violation issue by employing the concept of multi-set invariance proposed in [8]. Following the results presented there, we can compute a collection of sets $\{\mathbb{S}_m\}_{m\in\mathcal{M}}$ that fulfil

$$\bar{A}_l\mathbb{S}_m \oplus \mathbb{W}_l \subseteq \mathbb{S}_l, \quad \forall m \in \mathcal{M} \quad \forall l \in \mathcal{M}_m \cup \{m\}. \qquad (20)$$

In order to guarantee that the input constraints are not violated we incorporate an additional control step represented by an m, l-transition controller, characterized by the following optimization problem

$$\tilde{\mathbb{P}}^m_{N_l}(x(t)): \qquad \min_{\bar{\boldsymbol{u}}} J_{N_l}\left(\bar{\boldsymbol{u}}, \bar{x}_0\right) \qquad (21a)$$

s.t. (for $k = 0, \ldots, N_l - 1$)

$$\bar{x}_0 = \bar{x}(t) \qquad (21b)$$
$$\bar{u}_0 \in \tilde{\mathbb{U}}^m_l \subseteq \mathbb{U}_l \ominus K_l\mathbb{S}_m \qquad (21c)$$
$$\bar{x}_{k+1} = A_l\bar{x}_k + B_l\bar{u}_k \qquad (21d)$$
$$\bar{x}_k \in \bar{\mathbb{X}}_l \subseteq \mathbb{X}_l \ominus \mathbb{S}_l \qquad (21e)$$
$$\bar{u}_k \in \bar{\mathbb{U}}_l \subseteq \mathbb{U}_l \ominus K_l\mathbb{S}_l \qquad (21f)$$
$$\bar{x}_{N_l} \in \bar{\mathbb{X}}_{f,l} \subseteq \bar{\mathbb{X}}_l. \qquad (21g)$$

Define $V^m_{N_l}(x(t))$ as the optimal value function for (21) with

$$\bar{\boldsymbol{u}}^*(\bar{x}(t)) = \arg\min \tilde{\mathbb{P}}^m_{N_l}(x(t)),$$

and the set $\tilde{\mathcal{X}}^m_{N_l}$ as the set of all the states for which $\tilde{\mathbb{P}}^m_{N_l}(x)$ is feasible. The following result holds.

Proposition 7. For any pair $m, l \in \mathcal{M}$ with, $\sigma(t_{k-1}) = m$ and $l \in \mathcal{M}_m$, if the set $\mathbb{S}_l$ fulfils (20), $\bar{x}(t_k) \in \tilde{\mathcal{X}}^m_{N_l}$ and the loop is closed with $u(t_k) = \kappa_l(x(t_k)) = \bar{u}^*_0(\bar{x}(t)) + K_l\,(x(t_k) - \bar{x}(t_k))$ then (a) $\bar{\mathbb{P}}_{N_l}(x(t_k+1))$ is feasible, (b) $u(t_k) \in \mathbb{U}_l$ and $x(t_k+1) \in \mathbb{X}_l$, and (c) $V^m_{N_l}(x)$ fulfils (17) and (18) for all $\bar{x} \in \tilde{\mathcal{X}}^m_{N_l}$.

Proof. (a) If $\bar{x}(t_k) \in \tilde{\mathcal{X}}^m_{N_l}$, then the m, l-transition optimization is feasible, and so there exists a sequence of $N_l - 1$ control actions inside $\bar{\mathbb{U}}_l$ such that, starting from $\bar{x}(t_k+1)$, the state sequences reach $\bar{\mathbb{X}}_{l,f}$ without leaving $\bar{\mathbb{X}}_l$. This implies $\bar{x}(t_k+1) \in \bar{\mathcal{X}}_{N_l-1} \subseteq \bar{\mathcal{X}}_{N_l}$, thus $\bar{\mathbb{P}}_{N_l}(x(t_k+1))$ is feasible. (b) Feasibility of $\tilde{\mathbb{P}}^m_{N_l}(\bar{x}(t_k))$ implies that $u(t_k) \in \tilde{\mathbb{U}}^m_l \oplus K_l\mathbb{S}_m$, which by (21c) is a subset of $\mathbb{U}_l$. Furthermore, since $\mathbb{S}_l$ fulfils (20), then $x(t_k+1) - \bar{x}(t_k+1) \in \mathbb{S}_l$, thus

$$x(t_k+1) - \bar{x}(t_k+1) + \bar{x}(t_k+1) \in \mathbb{S}_l \oplus \bar{\mathcal{X}}_{N_l} \subseteq \mathbb{X}_l.$$

(c) Follows from the proof of Proposition 6 (see [15]). ∎

In view of Proposition 7, the following holds from Theorem 1.

Theorem 5. Consider any pair $m, l \in \mathcal{M}$ with $m \neq l$, $\sigma(t_{k-1}) = m$ and $l \in \mathcal{M}_m$. If $\tau^f_{m,l}$ is such that $\mathbb{B}_{r_m(\tau^f_{m,l})} \subseteq \tilde{\mathcal{X}}^m_{N_l}$, then a switch to mode l is feasible at any time t_k that fulfils $t_k - t_{k-1} - 1 \geq \tau^f_{m,l}$.

Corollary 5. If $\sigma(\cdot)$ is CSS, then the minimum MDT that guarantees feasible switching out of mode m is defined by

$$\tau^f_m = 1 + \max_{l \in \mathcal{M}_m} \tau^f_{m,l}.$$

Remark 4. Theorem 5 is the parallel of Theorem 1 when we employ the TMPC version in which nominal trajectories are not optimized. The results in Section III-B, i.e. when the feasibility regions are not available, are also valid in this case with the appropriate modifications to account for the m, l-transition controller.

B. Robustly stabilizing and admissible switching MDT

Since the nominal trajectories are not optimized, and given that $V^m_{N_l}(x)$ is a Lyapunov function for the m, l-transition trajectories (see Proposition 7) the results from Section IV-A related to minimum required stabilizing MDTs apply without changes to the nominal trajectories. In view of this, if the feasibility regions $\mathcal{X}_{N_m}$ have been computed, it follows from Corollaries 4 and 5 that the minimum MDT required to achieve robustly stabilizing and constraint admissible closed-loop dynamics with the switching multi-set TMPC controllers is τ_m defined by

$$\tau_m = \max\left\{\tau^f_m, \tau^s_m\right\}.$$

By employing the concept of multi-set invariance, alongside with a different variant of TMPC, we are able to guarantee exponential stability of the origin for the nominal trajectories even in the presence of heterogeneous disturbances and constraints. However, the multi-sets fulfil (20) for all $l \in \mathcal{M}_m$ additionally to the standard RPI condition (represented by $l = m$), therefore the minimal invariant multi-sets are, at least, as large as the minimal RPI sets, possibly shrinking the region of attraction of the switching multi-set TMPC controller.

VI. Illustrative example

To illustrate our approach to computing MDTs we consider a switching system with $M = 5$ and a CSS represented by the graph in Figure 1. The dynamics of each mode are $A_1 = [1.5\ 0;\ 1.5\ 1]$, $A_2 = [1\ 1.5;\ 0\ 1.5]$, $A_3 = [0.7\ 0.1;\ 0.2\ 0.4]$, $A_4 = [0.8\ 0.3;\ 0.4\ 0.1]$, $A_5 = [0.2\ 0.1;\ 0.2\ 0.6]$, $B_1 = [1;\ 0.8]$, $B_2 = [1;\ 0.8]$, $B_3 = [1;\ 0.5]$, $B_4 = [0.7;\ 0.8]$ and $B_5 = [1.3;\ 0.6]$.

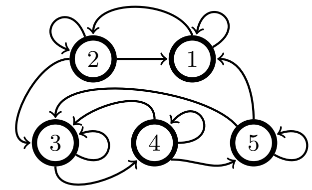

Fig. 1. Graph representing the CSS for the numerical example.

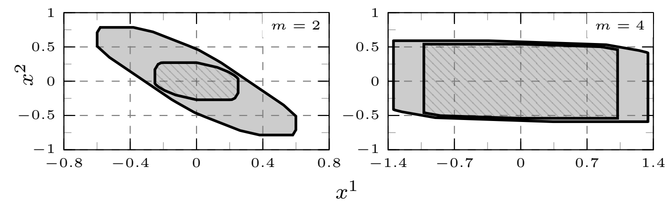

Fig. 2. Robust invariant sets: minimal multi-set, minimal RPI set.

Each mode is subject to state constraints, $\mathbb{X}_1 = \{x \in \mathbb{R}^2 \mid |x|_\infty \leq 2\}$, $\mathbb{X}_2 = {}^1\!/_2\mathbb{X}_1$, $\mathbb{X}_3 = \mathbb{X}_1$, $\mathbb{X}_4 = T\mathbb{X}_1$, $\mathbb{X}_5 = \mathbb{X}_4$, with $T = [1.5\ 0;\ 0\ 1]$, and input constraints $\mathbb{U}_2 = \{u \in \mathbb{R} \mid |u|_\infty \leq 2\}$, $\mathbb{U}_1 = {}^3\!/_2\mathbb{U}_2$, $\mathbb{U}_3 = 2\mathbb{U}_2$, $\mathbb{U}_4 = \mathbb{U}_1$, $\mathbb{U}_5 = {}^1\!/_4\mathbb{U}_1$. Finally, each mode is subject to additive uncertainties bounded by $\mathbb{W}_3 = \{w \in \mathbb{R}^2 \mid |w|_\infty \leq 1\}$, $\mathbb{W}_1 = {}^1\!/_{10}\mathbb{W}_3$, $\mathbb{W}_2 = \mathbb{W}_1$, $\mathbb{W}_4 = {}^1\!/_2\mathbb{W}_3$, $\mathbb{W}_5 = {}^7\!/_{10}\mathbb{W}_3$, which fulfils Assumption 1. Although of low order, this example incorporates a high level of heterogeneity.

For simplicity of exposition we set the cost matrices to $Q_1 = 10\mathcal{I}_2$, $Q_{2:5} = \mathcal{I}_2$, $R_{1:5} = 1$, and the MPC horizons to $N_{1:5} = 5$. By setting K_m to the corresponding LQR gain Assumptions 2 and 3 are met, therefore the implementation of independently designed TMPC controllers is feasible. Figure 2 shows the minimal RPI set and the minimal multi-set for modes 2 and 4. As expected, the minimal multi-set, being more demanding, can be larger, leading to a smaller region of attraction for certain modes.

Table I (first two columns) shows the exponential stability constants computed following the guidelines in [15, Section 2.4]. The analysis depicted therein focuses on the existence of the bounding functions in (3) and not their tightness, giving way to a conservative upper bound (i.e. a large d_m in (3a)). This in turn yields a large c_m and a $\lambda_m \approx 1$, resulting in a slow convergence rate. This has a direct impact on the shrinkage rate of the set $\mathbb{B}_{r_m}$ in (7), thereby increasing the MDTs required to guarantee a feasible switching.

Table II presents the feasibility MDTs computed following our approach and employing the exponential stability constants in Table I. As expected, given the conservative upper bound obtained from [15], some of the feasibility dwell-times are unnecessarily conservative. For example, whenever mode 5 becomes active, it must remain active for 392 time steps before we can guarantee that a switch into modes 1

TABLE I
Convergence constants for the TMPC case.

	Analytical bound [15]		Numerical bound	
Mode	c_m	λ_m	c_m	λ_m
1	4.093	0.970	1.409	0.705
2	3.310	0.953	1.902	0.851
3	2.564	0.921	1.141	0.481
4	2.670	0.927	1.287	0.629
5	7.610	0.991	1.186	0.538

TABLE II
FEASIBILITY MDTs.

	TMPC		Multi-set		MPC	
Mode	τ_m^f	$\bar{\tau}_m^f$	τ_m^f	$\bar{\tau}_m^f$	τ_m^f	$\bar{\tau}_m^f$
1	97	96	77	76	89	88
2	1	1	1	1	1	1
3	1	20	1	1	1	20
4	31	30	39	38	27	26
5	392	462	447	446	89	90

TABLE III
FEASIBILITY MDTs (NUMERICAL BOUNDS).

	TMPC		Multi-set		MPC	
Mode	τ_m^f	$\bar{\tau}_m^f$	τ_m^f	$\bar{\tau}_m^f$	τ_m^f	$\bar{\tau}_m^f$
1	7	6	9	8	6	5
2	1	1	1	1	1	1
3	1	2	1	1	1	2
4	5	4	6	5	4	3
5	4	4	5	4	3	2

or 3 is feasible (for the TMPC case with known feasibility regions).

To demonstrate the practicality of our approach in characterizing MDTs, we estimate a tighter upper bound (d_m) through Monte Carlo simulations. The corresponding optimization problem (either (2) or the modified versions discussed in Section V) is solved for 1000 randomly selected, albeit feasible, values of the state. A less conservative upper bounding scalar d_m is then obtained by comparing $V_{N_m}(x(t))$ and $d_m|\bar{x}_0^*(x)|_2^2$ at each randomly selected point. Table I shows the convergence constants resulting from these numerically obtained bounds and Table III presents the feasibility MDTs that result from using these tighter bounds. In this case, mode 5 needs to remain active only during 4 time steps to allow for a feasible switch, around 1% of the time obtained using the analytical bounds. These result indicate that our approach can obtain suitable mode dependent dwell-times given tight bounds on the optimal value function.

Finally, Table IV presents the stability MDTs obtained with the numerical bounds. In the TMPC case the stability guarantee relies on feasibility (Theorem 3), therefore the MDTs are generally larger when compared to the Multi-set case. Furthermore, the stability MDT of mode 2 is generally larger than for other modes across cases. This can be explained by the cost functions; indeed, mode 2 is allowed to switch into mode 1 (see Figure 1) however $Q_1 = 10Q_2$, therefore we need to stay a longer time in mode 2 to guarantee a cost decrease when switching to mode 1.

VII. CONCLUSIONS

In this note we presented a new approach to establishing minimum required MDTs to ensure admissible and robustly stabilizing closed-loop trajectories in a robust MPC set-up. A disadvantage of our approach is that the exponential decay rate in (4), upon which the the MDTs depend, can only be guaranteed when the MPC optimization is solved to optimality. Nevertheless, our set-up results in the corresponding optimization being a convex QP problem, for which efficient algorithms exist. However, the decay rates that are the norm in robust MPC implementations are not always tight, which, alongside the use of 1-norm balls for bounding the closed-loop trajectories, results in unnecessarily conservative MDTs. Nevertheless, our example showed that admissible and stabilizing switching can be guaranteed for considerably shorter MDTs by employing tighter bounds on the MPC optimal value function, obtained numerically in this note.

Future work will focus on the definition of less conservative upper bounds for the MPC optimal value function, and in incorporating the case in which the switch is not assumed to be detected immediately.

TABLE IV
STABILITY MDTs (NUMERICAL BOUNDS).

	TMPC		Multi-set	MPC
Mode	τ_m^g	$\tau_m^s(\bar{\tau}_m^s)$	τ_m^s	τ_m^s
1	15	19(18)	1	1
2	1	95(94)	13	14
3	1	21(20)	2	2
4	16	19(18)	4	3
5	7	12(11)	4	4

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