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A REMARK ON THE DIXMIER CONJECTURE

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Abstract. The Dixmier Conjecture says that every endomorphism of the (first) Weyl algebra A_1 (over a field of characteristic zero) is an automorphism, i.e., if $PQ-QP=1$ for some $P,Q\in A_1$ then $A_1=K\langle P,Q\rangle$. The Weyl algebra A_1 is a $\mathbb{Z}$-graded algebra. We prove that the Dixmier Conjecture holds if the elements P and Q are sums of no more than two homogeneous elements of A_1 (there is no restriction on the total degrees of P and Q).

Key Words: the Weyl algebra, the Dixmier Conjecture, automorphism, endomorphism, a $\mathbb{Z}$-graded algebra.

Mathematics subject classification 2010: 16S50, 16W20, 16S32, 16W50.

1. Introduction

In the paper, K is a field of characteristic zero and $K^*:=K\setminus\{0\}$. The algebra $A_1:=K\langle X,Y\mid [Y,X]=1\rangle$ is called the *first Weyl algebra* where $[Y,X]=YX-XY$. The n'th tensor power of A_1, $A_n:=A_1^{\otimes n}=\underbrace{A_1\otimes\cdots\otimes A_1}_{n\text{ times}}$, is called the *n'th Weyl algebra*. The algebra A_n is a simple Noetherian domain of Gel'fand-Kirillov dimension $\mathrm{GK}(A_n)=2n$, it is canonically isomorphic to the algebra of polynomial differential operators $K\langle X_1,\ldots,X_n,\partial_1,\ldots,\partial_n\rangle$ (where $\partial_i=\frac{\partial}{\partial x_i}$) via $X_i\mapsto X_i$, $Y_i\mapsto\partial_i$ for $i=1,\ldots,n$.

In his seminal paper [9], Dixmier (1968) found explicit generators for the group $G=\mathrm{Aut}_K(A_1)$ of K-automorphisms of the Weyl algebra A_1. Namely, the group G is generated by the obvious automorphisms:

$$(X,Y)\mapsto(X,Y+\lambda X^n),\quad (X,Y)\mapsto(X+\lambda Y^n,Y),\quad (X,Y)\mapsto(\mu X,\mu^{-1}Y)$$

where $\lambda\in K$, $\mu\in K^*$ and $n\in\mathbb{N}_+:=\{1,2,\ldots\}$.

In [9], Dixmier posed six problems: The first problem of Dixmier (in the list) asks if *every endomorphism of the Weyl algebra A_1 is an automorphism*, i.e., given elements P,Q of A such that $[P,Q]=1$, do they generate the algebra A_1? A similar problem but for the n'th Weyl algebra is called the *Dixmier Conjecture*. Problems 3 and 6 have been solved by Joseph [10] (1975), Problem 5 and Problem 4 (in the case of homogeneous elements) have been solved by Bavula [4] (2005).

The Dixmier Conjecture implies the *Jacobian Conjecture* (see [2]) and the inverse implication is also true (see [11] and [8]); a short proof is given in [6]; see also [1]).

In [5], it is shown that for each K-endomorphism $\phi : A_n \to A_n$ its image is very large, i.e., the left A_{2n}-module ${}^{\phi}A_n{}^{\phi}$ is a holonomic A_{2n}-module (where for all $a, b \in A_n$ and $c \in {}^{\phi}A_n{}^{\phi}$, $a \cdot c \cdot b := \phi(a)c\phi(b)$). In particular, it has finite length with simple holonomic factors over A_{2n} (see [5] for details). To prove that the Dixmier Conjecture holds for the Weyl algebra A_n it remains to show that the length is 1. Note, that the Gel'fand-Kirillov dimension of a simple A_{2n}-module can be $2n, 2n+1, \dots, 4n-1$, and the last case is the generic case.

In [7], it is shown that every algebra endomorphism of the algebra $\mathbb{I}_1 = K\langle x, \partial, \int \rangle$ of polynomial integro-differential operators is an automorphism and it is conjectured that the same result holds for $\mathbb{I}_n := \mathbb{I}_1^{\otimes n} = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle$.

The Weyl algebra $A_1 = \oplus_{i \in \mathbb{Z}} A_{1,i}$ is a $\mathbb{Z}$-graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where $A_{1,0} = K[H]$, $H = YX$ and, for $i \geq 1$, $A_{1,i} = K[H]X^i$ and $A_{1,-i} = K[H]Y^i$. For a nonzero element a of A_1, the number of *nonzero homogeneous* components is called the *mass* of a, denoted by $m(a)$. For example, $m(\alpha X^i) = 1$ for all $\alpha \in K[H] \setminus \{0\}$ and $i \geq 1$. The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) \leq 2$ and $m(Q) \leq 2$. If $[P, Q] = 1$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \mathrm{Aut}_K(A_1)$.*

2. Proof of Theorem 1.1

The Weyl algebra is a generalized Weyl algebra. Let D be a ring with an automorphism σ and a central element a. The **generalized Weyl algebra** $A = D(\sigma, a)$ of degree 1, is the ring generated by D and two indeterminates X an Y subject to the relations [3]:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, \; YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \oplus_{n \in \mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), $v_0 = 1$. It follows from the defining relations that

$$v_n v_m = (n, m) v_{n+m} = v_{n+m} < n, m >$$

for some elements $(n, m) = \sigma^{-n-m}(< n, m >) \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m \;:\; (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \; (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$
$$n \leq m \;:\; (n, -m) = \sigma^n(a) \cdots \sigma(a), \; (-n, m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n, m) = 1$.

Let $K[H]$ be a polynomial ring in a variable H over the field K, $\sigma : H \to H - 1$ be the K-automorphism of the algebra $K[H]$ and $a = H$. The first Weyl algebra $A_1 = K\langle X, Y \mid YX - XY = 1 \rangle$ is isomorphic to the generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, H), \; X \mapsto X, \; Y \mapsto Y, \; YX \mapsto H.$$

We identify both these algebras via this isomorphism, that is $A_1 = K[H](\sigma, H)$ and $H = YX$.

If $n>0$ and $m>0$ then

$$n\geq m \ : \ (n,-m)=(H-n)\cdots(H-n+m-1),\ (-n,m)=(H+n-1)\cdots(H+n-m),$$
$$n\leq m \ : \ (n,-m)=(H-n)\cdots(H-1),\ (-n,m)=(H+n-1)\cdots H,$$

in other cases $(n,m)=1$.

The localization $B=S^{-1}A_1$ of the Weyl algebra A_1 at the Ore subset $S=K[H]\backslash\{0\}$ of A_1 is the *skew Laurent polynomial ring* $B=K(H)[X,X^{-1};\sigma]$ with coefficients from the field $K(H)=S^{-1}K[H]$ of rational functions where $\sigma\in \mathrm{Aut}_K K(H)$ and $\sigma(H)=H-1$. The map $A_1\to B$, $a\mapsto a/1$ is an algebra monomorphism. We identify the algebra A_1 with its image in the algebra B via $A_1\to B,\ \ X\mapsto X,\ \ Y\mapsto HX^{-1}$. The algebra $B=\oplus_{i\in\mathbb{Z}}B_i$ is a $\mathbb{Z}$-graded algebra where $B_i=K(H)X^i$. The algebra A_1 is a $\mathbb{Z}$-graded subalgebra of B.

A polynomial $f(H)=\lambda_nH^n+\lambda_{n-1}H^{n-1}+\cdots+\lambda_0\in K[H]$ of degree n is called a *monic* polynomial if the *leading coefficient* λ_n of $f(H)$ is 1. A rational function $h\in K(H)$ is called a *monic* rational function if $h=f/g$ for some monic polynomials f,g. A homogeneous element $u=\alpha x^n$ of B is called *monic* if α is a monic rational function. We can extend the concept of degree of polynomial to the field of rational functions by the rule $\deg h=\deg f-\deg g$ where $h=f/g\in K[H]$. If $h_1,h_2\in K(H)$ then $\deg h_1h_2=\deg h_1+\deg h_2$ and $\deg(h_1+h_2)\leq\max\{\deg h_1,\deg h_2\}$. We denote by $\mathrm{sign}(n)$ and by $|n|$ the *sign* and the *absolute value* of $n\in\mathbb{Z}$, respectively.

Let A be an algebra and $a\in A$. The subalgebra of A, $C_A(a)=\{b\in A\mid ab=ba\}$, is called the *centralizer* of the element a in A.

Proposition 2.1 ([4], Proposition 2.1)**.** (Centralizer of a Homogeneous Element of the Algebra B)

(1) *Let $u=\alpha X^n$ be a monic element of B_n with $n\neq 0$. Then the centralizer $C_B(u)=K[v,v^{-1}]$ is a Laurent polynomial ring for a unique element $v=\beta X^{\mathrm{sign}(n)s}$ where s is the least positive divisor of n for which there exists an element $\beta=\beta_s\in K(H)$, necessarily monic and uniquely defined, such that*

$$\beta\,\sigma^s(\beta)\,\sigma^{2s}(\beta)\cdots\sigma^{(n/s-1)s}(\beta)=\alpha,\ \ \text{if}\ \ n>0, \tag{1}$$

$$\beta\,\sigma^{-s}(\beta)\,\sigma^{-2s}(\beta)\cdots\sigma^{-(|n|/s-1)s}(\beta)=\alpha,\ \ \text{if}\ \ n<0. \tag{2}$$

(2) *Let $u\in K(H)\backslash K$. Then $C_B(u)=K(H)$.*

Let $A_{1,+}:=K[H][X;\sigma]$ and $A_{1,-}:=K[H][Y;\sigma^{-1}]$. The algebras $A_{1,+}$ and $A_{1,-}$ are (skew polynomial) subalgebras of A_1.

Lemma 2.2 ([4])**.** *If $u\in A_{1,\pm}\setminus\{0\}$ then $C_A(u)\subseteq A_{1,\pm}$.*

The K-automorphism of the Weyl algebra A_1,

$$\xi:A_1\to A_1,\ X\mapsto Y,\ Y\mapsto -X, \tag{3}$$

reverses the $\mathbb{Z}$-grading of the Weyl algebra A_1, that is

$$\xi(A_{1,i}) = A_{1,-i} \text{ for all } z \in \mathbb{Z}. \tag{4}$$

By the *degree* of an element of A_1 we mean its *total degree* with respect to the canonical generators X and Y of A_1. Let $A_{1,\leq i} := \{p \in A \mid \deg(p) \leq i\}$ for $i \in \mathbb{N}$. Then $\{A_{1,\leq i}\}_{i\in\mathbb{N}}$ is the standard filtration of the algebra A_1 associated with the generators X and Y. For all $i \in \mathbb{Z} \setminus \{0\}$ and $f \in K[H] \setminus K$,

$$\deg \sigma^i(f) = \deg f \text{ and } \deg(1-\sigma^i)(f) = \deg f - 1. \tag{5}$$

Proof of Theorem 1.1: *(i) If $P, Q \in A_{1,\leq 1}$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some $\tau \in \mathrm{Aut}_K(A_1)$*: Clearly, $P = aY + bX + \lambda$ and $Q = cY + dX + \mu$ for some $a, b, c, d, \lambda, \mu \in K$. Then $1 = [P, Q] = ad - bc$. So, the automorphism τ can be chosen of the form

$$\tau(Y) = aY + bX + \lambda \text{ and } \tau(X) = cY + dX + \mu.$$

So, till the end of the proof we assume that at least one of the polynomials P or Q does not belong to the space $A_{1,\leq 1}$. In view of the relation $1 = [P, Q] = [-Q, P]$, we can assume that $P \notin A_{1,\leq 1}$. In view of Equation (4), we can assume that the highest homogeneous part of P, say $P_p \in A_{1,p}$, satisfies the condition that $p \geq 2$. Since $m(P) \leq 2$, either $P = P_p$ (if $m(P) = 1$) or otherwise $P = P_r + P_p$ for some nonzero $P_r \in A_{1,r}$ where $r < p$.

(ii) $(m(P), m(Q)) \neq (1, 1)$: Suppose that $m(P) = m(Q) = 1$, we seek a contradiction. Then $P = \alpha X^p$ and $Q = \beta Y^p$ for some nonzero polynomials $\alpha, \beta \in K[H]$. Then

$$1 = [P, Q] = \alpha\sigma^p(\beta)(p, -p) - \beta\sigma^{-p}(\alpha)(-p, p)$$
$$= \alpha\sigma^p(\beta)(p, -p) - \beta\sigma^{-p}(\alpha)\sigma^{-p}((p, -p)) = (1 - \sigma^{-p})(\alpha\sigma^p(\beta)(p, -p)).$$

Since $p \geq 2$ (or $P \notin A_{1,\leq 1}$),

$$0 = \deg 1 = \deg\,(1 - \sigma^{-p})(\alpha\sigma^p(\beta)(p, -p)) = \deg\alpha + \deg\beta + \deg\,(p, -p) - 1$$

(by Equation (5)) $\geq 0 + 0 + p - 1 \geq 2 - 1 = 1$, a contradiction.

(iii) $(m(P), m(Q)) \neq (1, 2)$: Suppose that $m(P) = 1$ and $m(Q) = 2$. Then $P = \alpha X^p$ for some $p \geq 2$ and $Q = Q_s + Q_q$ where $Q_s \in A_{1,s}$, $Q_q \in A_{1,q}$ and $s < q$. By Lemma 2.2, the equality $[P, Q] = 1$ implies that $[P, Q_s] = 1$ and $[P, Q_q] = 0$. By the case (ii), this is not possible.

(iv) Suppose that $m(P) = 2$ and $m(Q) = 1$. Then $P = P_r + P_p$ and $Q = Q_q$. By Lemma 2.2 the equality $[P, Q] = 1$ implies that $[P_p, Q_q] = 0$ and $[P_r, Q_q] = 1$. Then, $q \geq 0$, by Lemma 2.2. The case $q = 0$ is not possible since then both $P_r, Q_q \in K[H]$ and this would contradict the equality $[P_r, Q_q] = 1$. Therefore, $q > 0$. Then $P_r = \beta Y^q$ and $Q_q = \alpha X^q$ for some nonzero elements $\beta, \alpha \in K[H]$. Then

$$-1 = [Q_q, P_r] = (1 - \sigma^{-q})(\alpha\sigma^p(\beta)(q, -q))$$

implies that

$$0 = \deg(-1) = \deg\,(1 - \sigma^{-q})(\alpha\sigma^p(\beta)(q, -q)) = \deg\alpha + \deg\beta + q - 1,$$

by Equation (5). Hence, $q=1$, $\alpha,\beta\in K^*$ and $\beta=-\alpha^{-1}$. Then $P,Q\in A_{1,\leq 1}$, and, by the statement (i), the pair (P,Q) is obtained from the pair (Y,X) by applying an automorphism of A_1.

(v) $(m(P),m(Q))\neq(2,2)$: Since $m(P)=m(Q)=2$, we can write $P=P_r+P_p$ and $Q=Q_s+Q_q$ as sums of homogeneous elements where $r<p$, $P_r\in A_{1,r}$, $P_p\in A_{1,p}$ and $s<q$, $Q_s\in A_{1,s}$, $Q_q\in A_{1,q}$. The equality $[P,Q]=1$ implies that

$$[P_r,Q_s]=0 \text{ and } [P_p,Q_q]=0,$$

see Lemma 2.2. By Lemma 2.2, the elements r and s have the same sign (i.e., either $r<0,s<0$ or $r=s=0$ or $r>0,s>0$) and also the elements p and q have the same sign. Since $p\geq 2$, we must have $q>0$.

Suppose that $r\geq 0$, we seek a contradiction. Then $s\geq 0$ and so the elements P and Q are elements of the subring $A_{1,+}=\bigoplus_{i\geq 0}K[H]X^i$. Now,

$$K[H]\ni 1=[P,Q]\in[A_{1,+},A_{1,+}]\subseteq\oplus_{i\geq 1}K[H]X^i,$$

a contradiction. Therefore, $r<0$ and $s<0$.

The equality $1=[P,Q]=[P_r,Q_q]+[P_p,Q_s]$ and Lemma 2.2 imply that $r+q=0$ and $p+s=0$, that is $r=-q$ and $s=-p$. So,

$$P=P_{-q}+P_p \text{ and } Q=Q_{-p}+Q_q.$$

The elements P_p and P_{-q} are homogeneous elements of the Weyl algebra A_1. The Weyl algebra A_1 is a homogeneous subalgebra of the algebra

$$K(H)[X,X^{-1};\sigma]=K(H)[Y,Y^{-1};\sigma^{-1}]$$

where $K(H)$ is the field of rational functions in the variable H and the automorphism σ of $K(H)$ is given by the rule $\sigma(H)=H-1$. By [4, Proposition 2.1(1)], the centralizer $C_B(P_p)$ of the element P_p in B is a Laurent polynomial algebra

$$K[\alpha X^n,(\alpha X^n)^{-1}]$$

for some nonzero element $\alpha\in K(H)$ and $n\geq 1$. In general, $\alpha\notin K[H]$. Similarly,

$$C_B(P_{-q})=K[\beta Y^m,(\beta Y^m)^{-1}]$$

for some nonzero element $\beta\in K(H)$ and $m\geq 1$.

Since $[P_p,Q_q]=0$, $Q_q\in C_B(P_p)$ and

$$P_p=\lambda(P_p)(\alpha X^n)^i=\lambda(P_p)\alpha\sigma^n(\alpha)\cdots\sigma^{n(i-1)}(\alpha)X^{ni}=\alpha_{n,i}X^p,$$

$$Q_q=\lambda(Q_q)(\alpha X^n)^j=\lambda(Q_q)\alpha\sigma^n(\alpha)\cdots\sigma^{n(j-1)}(\alpha)X^{nj}=\alpha'_{n,j}X^q,$$

for some nonzero scalars $\lambda(P_p),\lambda(Q_q)\in K^*$ and some $i\geq 1$ and $j\geq 1$ where

$$\alpha_{n,i}=\lambda(P_p)\alpha\sigma^n(\alpha)\cdots\sigma^{n(i-1)}(\alpha)\in K[H],\ p=ni,$$

$$\alpha'_{n,j}=\lambda(Q_q)\alpha\sigma^n(\alpha)\cdots\sigma^{n(j-1)}(\alpha)\in K[H],\ q=nj.$$

Since $[P_{-p},Q_{-p}]=0$, $Q_{-p}\in C_B(P_{-q})$ and

$$P_{-q}=\lambda(P_{-q})(\beta Y^m)^s=\lambda(P_{-q})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(s-1)}(\beta)Y^{ms}=\beta_{m,s}Y^p,$$

$$Q_{-p}=\lambda(Q_{-p})(\beta Y^m)^t=\lambda(Q_{-p})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(t-1)}(\beta)Y^{mt}=\beta'_{m,t}Y^q,$$

for some nonzero scalars $\lambda(P_{-q}), \lambda(Q_{-p}) \in K^*$ and some $s \geq 1$ and $t \geq 1$ where

$$\beta_{m,s} = \lambda(P_{-q})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(s-1)}(\beta) \in K[H],\ p = ms,$$
$$\beta'_{m,t} = \lambda(Q_{-p})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(t-1)}(\beta) \in K[H],\ q = mt.$$

Now,

$$1 = [P, Q] = [P_p, Q_{-p}] + [P_{-q}, Q_q] = [\alpha_{n,i}X^p, \beta'_{m,t}Y^p] + [\beta_{m,s}Y^q, \alpha'_{n,j}X^q]$$
$$= \alpha_{n,i}\sigma^p(\beta'_{m,t})(p,-p) - \beta'_{m,t}\sigma^{-p}(\alpha_{n,i})(-p,p)$$
$$+\beta_{m,s}\sigma^{-q}(\alpha'_{n,j})(-q,q) - \alpha'_{n,j}\sigma^q(\beta_{m,s})(q,-q).$$

Using the equalities $(-p,p) = \sigma^{-p}((p,-p))$ and $(-q,q) = \sigma^{-q}((q,-q))$, the last equality above can be rewritten as follows 1=ab

$$1 = (1-\sigma^{-p})(a) + (1-\sigma^{-q})(b) \tag{6}$$

where $a = \alpha_{n,i}\sigma^p(\beta'_{m,t})(p,-p) \in K[H]$ and $b = \alpha'_{n,j}\sigma^q(\beta_{m,s})(q,-q) \in K[H]$.

Recall that $P = P_{-q} + P_p$, $Q = Q_{-p} + Q_q$, 2=ab

$$p = mt = ni \geq 2 \text{ and } q = ms = nj \geq 1. \tag{7}$$

Suppose that $p = q$, and so $P = P_{-p} + P_p$, $Q = Q_{-p} + Q_p$. Then $Q = \lambda P_p$ for some $\lambda \in K^*$. Notice that

$$1 = [P, Q] = [P, Q - \lambda P],\ \ m(P) = 2 \ \text{ and } \ m(Q - \lambda P) = 1.$$

By the case (iv), the pair $(P, Q - \lambda P)$ is obtained from the pair (Y, X) by applying an automorphism of the Weyl algebra A_1.

So, either $p < q$ or $p > q$. In view of (P,Q)-symmetry $(1 = [P,Q] = [-Q,P])$, it suffices to consider, say, the first case only. Since $p < q$, the equalities (7) imply that $i < j$ and $t < s$. Then, using Equation (5) and the fact that $\deg(p,-p) = p$ for all $p \geq 1$, we see that

$$\deg a = \deg \alpha_{n,i} + \deg \beta'_{m,t} + p - 1,$$
$$\deg b = \deg \alpha'_{n,j} + \deg \beta_{m,s} + q - 1.$$

Since $i < j$ and $t < s$, $\deg \alpha_{n,i} < \deg \alpha'_{n,j}$ and $\deg \beta'_{m,t} < \deg \beta_{m,s}$. In particular, $\deg a < \deg b$. This equality contradicts Equation (6) since, by Equation (5),

$$0 = \deg 1 = \deg a - 1 - \deg b + 1 = \deg a - \deg b > 0.$$

This means that the cases $p < q$ and $p > q$ are impossible. The proof of the theorem is complete. □

Corollary 2.3. *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) = 1$ or $m(Q) = 1$. If $[P,Q] = 1$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \mathrm{Aut}_K(A_1)$.*

Proof: Without loss of generality we may assume $m(Q) = 1$ and $m(P) \geq 3$. That is $Q = Q_q$ and $P = \sum_{i \in I} P_i$, where $I \subset \mathbb{Z}$ is a finite set, $q \in \mathbb{Z} \setminus \{0\}$ and the elements Q_q and P_i are homogeneous in A_1. By Equation (4), we may assume that $q > 0$. Then

$$1 = [P, Q] = \sum_i [P_i, Q_q]$$

implies that $-q \in I$, $[P_{-q}, Q_q] = 1$ and $[P_j, Q_q] = 0$ for all $j \in I$ such that $j \neq -q$. By Theorem 1.1,

$$q = 1, Q_1 = \lambda X \text{ and } P_{-1} = \lambda^{-1} Y \text{ for some } \lambda \in K^*.$$

By Lemma 2.2, $C := P - P_{-1} \in C_A(X) = K[X]$. Then $P = \tau(Y)$ and $Q = \tau(X)$ where $\tau : A_1 \to A_1$, $X \mapsto \lambda X$, $Y \mapsto \lambda^{-1} Y + C$, is an automorphism. $\square$

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