Equivalence of two sets of Hamiltonians associated with the rational BC_n Ruijsenaars-Schneider-van Diejen system

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Abstract

The equivalence of two complete sets of Poisson commuting Hamiltonians of the (super)integrable rational BC_n Ruijsenaars-Schneider-van Diejen system is established. Specifically, the commuting Hamiltonians constructed by van Diejen are shown to be linear combinations of the Hamiltonians generated by the characteristic polynomial of the Lax matrix obtained recently by Pusztai, and the explicit formula of this invertible linear transformation is found.

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1   Introduction

Integrable many-body systems of Calogero-Moser type crop up in a wide range of physical applications and are intimately related to important fields of mathematics [1, 2, 3, 4, 5]. They occur in rational, trigonometric/hyperbolic and elliptic families according to the functional form of the Hamiltonian that inherently also involves a crystallographic root system. A further significant feature is the existence of interesting deformations and extensions maintaining integrability, as is exemplified by relativistic [6] and spin Calogero-Moser systems [7]. In this paper we shall deal with the BC_n generalization of the relativistic rational Ruijsenaars-Schneider system, which is the simplest member of the systems discovered by van Diejen [8, 9]. We shall stay at the level of classical mechanics, where the BC_n rational 'RSvD system' is defined by the Hamiltonian[1]

$$H(\lambda, \theta) = \sum_{j=1}^{n} \cosh(\theta_j) \left[1 + \frac{\nu^2}{\lambda_j^2} \right]^{\frac{1}{2}} \left[1 + \frac{\kappa^2}{\lambda_j^2} \right]^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left[1 + \frac{\mu^2}{(\lambda_j - \lambda_k)^2} \right]^{\frac{1}{2}} \left[1 + \frac{\mu^2}{(\lambda_j + \lambda_k)^2} \right]^{\frac{1}{2}}$$

$$+ \frac{\nu\kappa}{\mu^2} \prod_{j=1}^{n} \left[1 + \frac{\mu^2}{\lambda_j^2} \right] - \frac{\nu\kappa}{\mu^2}. \tag{1.1}$$

Here μ, ν, κ are real parameters for which we impose the conditions $\mu \neq 0$, $\nu \neq 0$ and $\nu\kappa \geq 0$. The generalized momenta $\theta = (\theta_1, \ldots, \theta_n)$ run over $\mathbb{R}^n$ and the 'particle positions' $\lambda = (\lambda_1, \ldots, \lambda_n)$ vary in the Weyl chamber

$$\mathfrak{c} = \{ x \in \mathbb{R}^n \mid x_1 > \cdots > x_n > 0 \}. \tag{1.2}$$

In the work [9, 10] first the commutativity of n quantum Hamiltonian difference operators was proved. It was then shown [10, 11] that the classical limit yields a Poisson commuting family having the right functional rank for a Liouville integrable system. Except for the rational case, it is still an open problem to generate the classical Hamiltonians of van Diejen from a Lax matrix, which would provide a useful tool for analyzing the dynamics of these systems. A Lax matrix whose trace is the rational RSvD Hamiltonian (1.1) and whose higher spectral invariants provide n independent commuting Hamiltonians was recently found by Pusztai [12]. In the papers [12, 13] the action-angle duality between the hyperbolic BC_n Sutherland system and the rational BC_n RSvD system was also explored together with the scattering properties of these systems.

The question we answer in this paper is the following. What is the relationship between the commuting Hamiltonians introduced by van Diejen and the ones generated by Pusztai's Lax matrix? Both commuting families contain the 'main Hamiltonian' (1.1) and exhibit rational dependence on the positions and exponential dependence on the momenta. Thus one strongly expects that these two sets of commuting Hamiltonians can be expressed in terms of each other. Nevertheless, the question appears to be non-trivial since the Hamiltonian H is maximally superintegrable [14], which entails that it is the member of several inequivalent families of n functionally independent functions in involution.

Here, we shall demonstrate that the n Hamiltonians of van Diejen are linear combinations of the coefficients of the characteristic polynomial of the Lax matrix of [12]. The transformation between the two sets will be shown to be invertible, and its explicit form will be

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[1]A deformation parameter $\beta > 0$ can be introduced by setting $H_\beta(\lambda, \theta) := H(\beta^{-1}\lambda, \beta\theta)$. Taking Taylor expansion of H_β in β, the leading term reproduces the usual BC_n rational Calogero-Moser Hamiltonian.

given as well. Our arguments will rely on the action-angle map constructed with the help of Hamiltonian reduction in [12]. This is captured by a symplectomorphism

$$\mathcal{S} \colon \mathfrak{c} \times \mathbb{R}^n \to \mathfrak{c} \times \mathbb{R}^n, \quad (q,p) \mapsto (\lambda, \theta), \qquad \mathcal{S}^* \left(\sum_{k=1}^n d\lambda_k \wedge d\theta_k \right) = \sum_{k=1}^n dq_k \wedge dp_k, \qquad (1.3)$$

such that $H \circ \mathcal{S}$ depends only the action variables q_k. (The notation fits the fact that the components of q serve as position variables for the dual system.) Although an explicit formula of the action-angle map is not available, we can compute the action-angle transform of the commuting Hamiltonians of interest by utilizing that [13] the H-trajectory $(\lambda(t), \theta(t))$ with initial condition $(\lambda, \theta) \in \mathfrak{c} \times \mathbb{R}^n$ has the $t \to \infty$ asymptotics

$$\lambda_k(t) \sim t \sinh(q_k) - p_k \quad \text{and} \quad \theta_k(t) \sim q_k, \qquad k = 1, \dots, n, \qquad (1.4)$$

with $(q,p) = \mathcal{S}^{-1}(\lambda, \theta)$. This will allow us to eventually show that the two sets of Hamiltonians at issue correspond to two generating sets of the Weyl group invariant polynomials in the variables $e^{\pm q_k}$ (restricted to the Weyl chamber $\mathfrak{c}$). As was already mentioned, we shall also find the explicit relationship. As a byproduct, we obtain an algebraic formula for the characteristic polynomial of Pusztai's Lax matrix, which generalizes well-known determinant identities for Cauchy-like matrices.

Section 2 describes the two families of Hamiltonians in play and specifies how they share the main Hamiltonian H (1.1). Our contribution is given by Proposition 1, Proposition 2 and Remark 3 in Section 3. Section 4 offers a short discussion of the results and open problems. There is also an appendix, where a useful formula of [9] is presented.

2 Two families of commuting Hamiltonians

2.1 Hamiltonians due to van Diejen

In [8, 11] the following complete set of Poisson commuting Hamiltonians was given:

$$H_l(\lambda, \theta) = \sum_{\substack{J \subset \{1, \dots, n\}, \ |J| \leq l \\ \varepsilon_j = \pm 1, \ j \in J}} \cosh(\theta_{\varepsilon J}) V_{\varepsilon J; J^c}^{1/2} V_{-\varepsilon J; J^c}^{1/2} U_{J^c, l - |J|}, \quad l = 1, \dots, n, \qquad (2.1)$$

with

$$\theta_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \theta_j,$$

$$V_{\varepsilon J; K} = \prod_{j \in J} w(\varepsilon_j \lambda_j) \prod_{\substack{j, j' \in J \\ j < j'}} v^2(\varepsilon_j \lambda_j + \varepsilon_{j'} \lambda_{j'}) \prod_{\substack{j \in J \\ k \in K}} v(\varepsilon_j \lambda_j + \lambda_k) v(\varepsilon_j \lambda_j - \lambda_k),$$

$$U_{K,p} = (-1)^p \sum_{\substack{I \subset K, \ |I| = p \\ \varepsilon_i = \pm 1, \ i \in I}} \left(\prod_{i \in I} w(\varepsilon_i \lambda_i) \prod_{\substack{i, i' \in I \\ i < i'}} v(\varepsilon_i \lambda_i + \varepsilon_{i'} \lambda_{i'}) v(-\varepsilon_i \lambda_i - \varepsilon_{i'} \lambda_{i'}) \right. \qquad (2.2)$$

$$\left. \times \prod_{\substack{i \in I \\ k \in K \setminus I}} v(\varepsilon_i \lambda_i + \lambda_k) v(\varepsilon_i \lambda_i - \lambda_k) \right).$$

It is worth noting that J^c in (2.1) denotes the complementary set, and the contribution to H_l coming from $J = \emptyset$ is $U_{\emptyset^c, l}$. The relatively simple form of $U_{K,p}$ above was found in [11].

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Equation (2.1) makes sense for $l = 0$, as well, giving $H_0 \equiv 1$. In the rational case the functions v and w take the following form[2]

$$v(x) = \frac{x + \mathrm{i}\mu}{x}, \quad w(x) = \left[\frac{x + \mathrm{i}\nu}{x}\right]\left[\frac{x + \mathrm{i}\kappa}{x}\right]. \tag{2.3}$$

Up to irrelevant constants, H_1 reproduces the Hamiltonian H (1.1). Indeed, one can check that $H_1 = 2(H - n)$.

Take any $(\lambda, \theta) \in \mathfrak{c} \times \mathbb{R}^n$, set $(q, p) = \mathcal{S}^{-1}(\lambda, \theta)$ and consider the H-trajectory $(\lambda(t), \theta(t))$ with initial condition (λ, θ). Notice that the Hamiltonian H_l (2.1) is constant along the H-trajectory. By utilizing the asymptotics (1.4), one can readily check that

$$(\mathcal{S}^* H_l)(q, p) = \lim_{t \to \infty} H_l(\lambda(t), \theta(t)) = \sum_{\substack{J \subset \{1, \dots, n\}, \, |J| \leq l \\ \varepsilon_j = \pm 1, \, j \in J}} (-2)^{l - |J|} \binom{n - |J|}{l - |J|} \cosh(q_{\varepsilon J}). \tag{2.4}$$

From now on we let $\mathcal{H}_l$ stand for the pullback $\mathcal{S}^* H_l$ just computed, and stress that it depends only on the variable q.

2.2 Hamiltonians obtained from the Lax matrix

We recall some relevant objects of [12]. First, prepare the $2n \times 2n$ Hermitian, unitary matrix

$$C = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \tag{2.5}$$

and the $2n \times 2n$ Hermitian matrix

$$h(\lambda) = \begin{bmatrix} a(\mathrm{diag}(\lambda)) & b(\mathrm{diag}(\lambda)) \\ -b(\mathrm{diag}(\lambda)) & a(\mathrm{diag}(\lambda)) \end{bmatrix} \tag{2.6}$$

containing the smooth functions $a(x), b(x)$ given on the interval $(0, \infty) \subset \mathbb{R}$ by

$$a(x) = \frac{\sqrt{x + \sqrt{x^2 + \kappa^2}}}{\sqrt{2x}}, \quad b(x) = \mathrm{i}\kappa \frac{1}{\sqrt{2x}} \frac{1}{\sqrt{x + \sqrt{x^2 + \kappa^2}}}. \tag{2.7}$$

Then introduce the vectors $z(\lambda) \in \mathbb{C}^n$, $F(\lambda, \theta) \in \mathbb{C}^{2n}$ by the formulae

$$z_l(\lambda) = -\left[1 + \frac{\mathrm{i}\nu}{\lambda_l}\right] \prod_{\substack{m=1 \\ (m \neq l)}}^{n} \left[1 + \frac{\mathrm{i}\mu}{\lambda_l - \lambda_m}\right]\left[1 + \frac{\mathrm{i}\mu}{\lambda_l + \lambda_m}\right], \tag{2.8}$$

and

$$F_l(\lambda, \theta) = e^{-\frac{\theta_l}{2}} |z_l(\lambda)|^{\frac{1}{2}}, \quad F_{n+l}(\lambda, \theta) = \overline{z_l(\lambda)} F_l(\lambda, \theta)^{-1}, \tag{2.9}$$

$l = 1, \dots, n$. With these notations at hand, the $2n \times 2n$ matrix

$$A_{j,k}(\lambda, \theta) = \frac{\mathrm{i}\mu F_j \overline{F_k} + \mathrm{i}(\mu - 2\nu) C_{j,k}}{\mathrm{i}\mu + \Lambda_j - \Lambda_k}, \quad j, k \in \{1, \dots, 2n\}, \tag{2.10}$$

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[2]The parameters appearing in [8, 11] can be recovered by introducing β as in footnote 1 and then writing μ, μ_0, μ_0' for $\beta\mu, \beta\nu, \beta\kappa$, respectively. In the convention of [13], our μ, θ and q correspond to $2\mu, 2\theta$ and $2q$.

with $\Lambda = \text{diag}(\lambda, -\lambda)$ is used to define the 'RSvD Lax matrix' [12]:

$$L(\lambda, \theta) = h(\lambda)^{-1} A(\lambda, \theta) h(\lambda)^{-1}. \tag{2.11}$$

The matrices h, A, and L are invertible and satisfy the relations

$$ChC = h^{-1}, \quad CAC = A^{-1}, \quad CLC = L^{-1}. \tag{2.12}$$

Their determinants are

$$\det(h) = \det(A) = \det(L) = 1. \tag{2.13}$$

Let K_m denote the coefficients of the characteristic polynomial of L (2.11),

$$\det(L(\lambda, \theta) - x \mathbf{1}_{2n}) = K_0(\lambda, \theta) x^{2n} + K_1(\lambda, \theta) x^{2n-1} + \cdots + K_{2n-1}(\lambda, \theta) x + K_{2n}(\lambda, \theta). \tag{2.14}$$

An immediate consequence of (2.12),(2.13) is that

$$K_{2n-m} \equiv K_m, \quad m = 0, 1, \ldots, n, \tag{2.15}$$

thus the functions $K_0 \equiv 1, K_1, \ldots, K_n$ fully determine the characteristic polynomial (2.14). The first non-constant member of this family is proportional to H (1.1), that is $K_1 = -2H$. The asymptotic form of the Lax matrix L (2.13) is the diagonal matrix

$$\text{diag}(e^{-q}, e^q), \tag{2.16}$$

hence the action-angle transforms of the functions K_m ($m = 0, 1, \ldots, n$) can be easily computed to be

$$(\mathcal{S}^* K_m)(q, p) = (-1)^m \sum_{a=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\substack{J \subset \{1,\ldots,n\}, \ |J|=m-2a \\ \varepsilon_j = \pm 1, \ j \in J}} \binom{n - |J|}{a} \cosh(q_{\varepsilon J}). \tag{2.17}$$

Of course, we used the asymptotics (1.4) and that K_m is constant along the flow of H. Now we introduce the shorthand $\mathcal{K}_m := \mathcal{S}^* K_m$, and observe that it only depends on q.

3   Relation between the two families of Hamiltonians

It is worth emphasizing that finding a formula relating the families $\{H_l\}_{l=0}^n$ and $\{K_m\}_{m=0}^n$ is equivalent to finding a relation between their action-angle transforms $\{\mathcal{H}_l\}_{l=0}^n$ and $\{\mathcal{K}_m\}_{m=0}^n$.

Proposition 1. *There exists an invertible linear relation between the two families $\{\mathcal{H}_l\}_{l=0}^n$ and $\{\mathcal{K}_m\}_{m=0}^n$.*

Proof. Let us introduce the auxiliary functions

$$\mathcal{M}_k(q) = \sum_{\substack{J \subset \{1,\ldots,n\}, \ |J|=k \\ \varepsilon_j = \pm 1, \ j \in J}} \cosh(q_{\varepsilon J}), \quad q \in \mathbb{R}^n, \quad k = 0, 1, \ldots, n. \tag{3.1}$$

For any $l \in \{0, 1, \ldots, n\}$ the Hamiltonian $\mathcal{H}_l$ (2.4) is a linear combination of $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_l$,

$$\mathcal{H}_l(q) = \sum_{k=0}^l (-2)^{l-k} \binom{n-k}{l-k} \mathcal{M}_k(q). \tag{3.2}$$

This shows that the matrix of the linear map transforming $\{\mathcal{M}_k\}_{k=0}^n$ into $\{\mathcal{H}_l\}_{l=0}^n$ is lower triangular with ones on the diagonal, hence the above relation is invertible. Similarly, any function $\mathcal{K}_m$ (2.17), $m \in \{0, 1, \ldots, n\}$ can be expressed as a linear combination of $\mathcal{M}_m, \mathcal{M}_{m-2}, \ldots, \mathcal{M}_3, \mathcal{M}_1$ or $\mathcal{M}_m, \mathcal{M}_{m-2}, \ldots, \mathcal{M}_2, \mathcal{M}_0$ depending on the parity of m, that is

$$\mathcal{K}_m(q) = (-1)^m \sum_{a=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n - (m - 2a)}{a} \mathcal{M}_{m-2a}(q). \tag{3.3}$$

Hence the linear transformation relating $\{\mathcal{M}_k\}_{k=0}^n$ to $\{\mathcal{K}_m\}_{m=0}^n$ has a lower triangular matrix with diagonal components ± 1, implying that it is invertible. This proves the existence of an invertible linear relation between the two families $\{\mathcal{H}_l\}_{l=0}^n$ and $\{\mathcal{K}_m\}_{m=0}^n$. $\qquad\square$

Now, we prove an explicit formula expressing $\mathcal{H}_l$ as linear combination of $\{\mathcal{K}_m\}_{m=0}^l$.

Proposition 2. *For any fixed $n \in \mathbb{N}$, $l \in \{1, \ldots, n\}$ and $q \in \mathbb{R}^n$ we have*

$$(-1)^l \mathcal{H}_l(q) = \mathcal{K}_l(q) + \sum_{m=0}^{l-1} \frac{2(n - m)}{2(n - m) - (l - m)} \binom{(n - l) + (n - m)}{l - m} \mathcal{K}_m(q). \tag{3.4}$$

Proof. Substitute $\mathcal{K}_m$ (2.17) into the right-hand side of the expression above to obtain

$$\sum_{k=0}^{l-1} \sum_{a=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{\substack{J \subset \{1,\ldots,n\}, \; |J|=k-2a \\ \varepsilon_j = \pm 1, \; j \in J}} (-1)^k \frac{2(n - k)}{2(n - k) - (l - k)} \binom{(n - l) + (n - k)}{l - k} \times$$

$$\times \binom{n - (k - 2a)}{a} \cosh(q_{\varepsilon J}) + \sum_{a=0}^{\lfloor \frac{l}{2} \rfloor} \sum_{\substack{J \subset \{1,\ldots,n\}, \; |J|=l-2a \\ \varepsilon_j = \pm 1, \; j \in J}} (-1)^l \binom{n - (l - 2a)}{a} \cosh(q_{\varepsilon J}). \tag{3.5}$$

Since $k = |J| + 2a$ it is obvious that $(-1)^k = (-1)^{-|J|}$. Multiply (3.5) by $(-1)^l$ and change the order of summations over a and J to get

$$\sum_{\substack{J \subset \{1,\ldots,n\}, \; |J|<l \\ \varepsilon_j = \pm 1, \; j \in J}} (-1)^{l-|J|} \sum_{a=0}^{\lfloor \frac{l-|J|}{2} \rfloor} \frac{2[n - (|J| + 2a)]}{2[n - (|J| + 2a)] - [l - (|J| + 2a)]} \times$$

$$\times \binom{(n - l) + (n - (|J| + 2a))}{l - (|J| + 2a)} \binom{n - |J|}{a} \cosh(q_{\varepsilon J}) + \sum_{\substack{J \subset \{1,\ldots,n\}, \; |J|=l \\ \varepsilon_j = \pm 1, \; j \in J}} \cosh(q_{\varepsilon J}). \tag{3.6}$$

Now, comparison of (3.2) with (3.6) leads to a relation equivalent to (3.4),

$$\sum_{a=0}^{\lfloor \frac{l-|J|}{2} \rfloor} \frac{2[n - (|J| + 2a)]}{2[n - (|J| + 2a)] - [l - (|J| + 2a)]} \times$$

$$\times \binom{2n - (l + |J| + 2a)}{l - (|J| + 2a)} \binom{n - |J|}{a} \Big/ \binom{n - |J|}{l - |J|} = 2^{l-|J|}. \tag{3.7}$$

For $n = l$ in (3.7) one obtains

$$
\begin{cases}
2 \displaystyle\sum_{a=0}^{\lfloor \frac{l-|J|}{2} \rfloor} \binom{l-|J|}{a} = 2^{l-|J|}, & \text{if } l-|J| \text{ is odd}, \\[2em]
2 \displaystyle\sum_{a=0}^{\frac{l-|J|}{2}-1} \binom{l-|J|}{a} + \binom{l-|J|}{\frac{l-|J|}{2}} = 2^{l-|J|}, & \text{if } l-|J| \text{ is even},
\end{cases}
\tag{3.8}
$$

which are well-known identities for the binomial coefficients. This means that (3.4) holds for $l = n$ for all $n \in \mathbb{N}$, which implies that if we consider $n + 1$ variables it is sufficient to check the cases $l < n + 1$. With that in mind let us progress by induction on n and suppose that (3.4) is verified for all $1 \leq l \leq n$ for some $n \in \mathbb{N}$.

First, notice that the Hamiltonians $\mathcal{H}_l$ (2.4) satisfy the following recursion

$$
\mathcal{H}_l(q_1, \ldots, q_n, q_{n+1}) = \mathcal{H}_l(q_1, \ldots, q_n) + 4\sinh^2(\tfrac{q_{n+1}}{2})\mathcal{H}_{l-1}(q_1, \ldots, q_n).
\tag{3.9}
$$

This can be checked either directly or by utilizing that $\mathcal{H}_l$ is the l-th elementary symmetric function with variables $\sinh^2(\frac{q_i}{2})$ (see Appendix A). Similarly, the functions $\mathcal{K}_k$ (2.17) satisfy

$$
\mathcal{K}_k(q_1, \ldots, q_n, q_{n+1}) = \mathcal{K}_k(q_1, \ldots, q_n) - 2\cosh(q_{n+1})\mathcal{K}_{k-1}(q_1, \ldots, q_n) + \mathcal{K}_{k-2}(q_1, \ldots, q_n),
\tag{3.10}
$$

with $\mathcal{K}_{-1} \equiv 0$. Let us introduce some shorthand notation, such as the $\mathbb{R}^{l+1}$ vectors

$$
\vec{\mathcal{H}}(n) := (\mathcal{H}_0, -\mathcal{H}_1, \ldots, (-1)^l\mathcal{H}_l)^\top \quad \text{and} \quad \vec{\mathcal{K}}(n) := (\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_l)^\top
\tag{3.11}
$$

and the $\mathbb{R}^{(l+1)\times(l+1)}$ matrices

$$
\mathcal{A}(n)_{j+1,k+1} := \begin{cases}
\dfrac{2(n-k)}{2(n-k)-(j-k)} \dbinom{(n-j)+(n-k)}{j-k}, & \text{if } j \geq k, \\[1.5em]
0, & \text{if } j < k,
\end{cases}
\tag{3.12}
$$

where $j, k \in \{0, \ldots, l\}$ and

$$
\mathcal{H}(n, n+1) := \mathbf{1}_{l+1} - 4\sinh^2(\tfrac{q_{n+1}}{2})\mathcal{I}_{-1}, \quad \mathcal{K}(n, n+1) := \mathbf{1}_{l+1} - 2\cosh(q_{n+1})\mathcal{I}_{-1} + \mathcal{I}_{-2}
\tag{3.13}
$$

with $(\mathcal{I}_{-m})_{j+1,k+1} := \delta_{j,k+m}$, $m > 0$. The relations (3.9) and (3.10) can be written in the concise form

$$
\vec{\mathcal{H}}(n+1) = \mathcal{H}(n, n+1)\vec{\mathcal{H}}(n), \quad \vec{\mathcal{K}}(n+1) = \mathcal{K}(n, n+1)\vec{\mathcal{K}}(n)
\tag{3.14}
$$

and our assumption is condensed into

$$
\vec{\mathcal{H}}(n) = \mathcal{A}(n)\vec{\mathcal{K}}(n).
\tag{3.15}
$$

Using this notation it is clear that the desired induction step is equivalent to the matrix equation

$$
\mathcal{H}(n, n+1)\mathcal{A}(n) = \mathcal{A}(n+1)\mathcal{K}(n, n+1).
\tag{3.16}
$$

Spelling this out at some arbitrary (j, k)-th entry gives us

$$\frac{A+B}{A}\binom{A}{B} - 4\sinh^2\left(\frac{\alpha}{2}\right)\frac{A+B}{A+1}\binom{A+1}{B-1} =$$

$$= \frac{A+B+2}{A+2}\binom{A+2}{B} - 2\cosh(\alpha)\frac{A+B}{A+1}\binom{A+1}{B-1} + \frac{A+B-2}{A}\binom{A}{B-2}, \quad (3.17)$$

where

$$A := 2n - j - k, \quad B := j - k, \quad \alpha := q_{n+1}. \tag{3.18}$$

A simple direct calculation shows that (3.17) indeed holds implying that (3.4) is also true for $n+1$ for any $l \leq n$. The case $l = n+1$ is given by the argument preceding induction. This completes the proof. $\qquad\square$

Remark 3. We showed in Proposition 1 that the relation (3.4) is invertible. Without spending space on the proof, we note that the inverse relation can be written explicitly as

$$(-1)^m \mathcal{K}_m(q) = \sum_{l=0}^m \binom{2(n-l)}{m-l}\mathcal{H}_l(q). \tag{3.19}$$

4 Discussion

In this paper we demonstrated that the commuting Hamiltonians of the rational RSvD system constructed originally by van Diejen are linear combinations of the coefficients of the characteristic polynomial of the Lax matrix found recently by Pusztai, and vice versa. The derivation utilized the action-angle map and the scattering theory results of [12, 13]. Our Proposition 2 gives rise to a determinant representation of the somewhat complicated expressions H_l in (2.1). It could be of some interest to provide a purely algebraic proof of the resulting formula of the characteristic polynomial of the Lax matrix.

The configuration space $\mathfrak{c}$ (1.2) is an open Weyl chamber associated with the Weyl group $W(\mathrm{BC}_n)$, and after extending this domain all Hamiltonians that we dealt with enjoy $W(\mathrm{BC}_n)$ invariance. In particular, the sets $\{\mathcal{H}_l\}_{l=0}^n$, $\{\mathcal{K}_l\}_{l=0}^n$ and $\{\mathcal{M}_l\}_{l=0}^n$ represent different free generating sets of the invariant polynomials in the functions $e^{\pm q_k}$ $(k = 1, \ldots, n)$ of the action variables q_k acted upon by the sign changes and permutations that form $W(\mathrm{BC}_n)$. In order to verify this, it is useful to point out that the $W(\mathrm{BC}_n)$ invariant polynomials in the variables $e^{\pm q_k}$ are the same as the ordinary symmetric polynomials in the variables $\cosh(q_k)$. The statement that $\{\mathcal{H}_l\}_{l=0}^n$ is a free generating set for these polynomials then follows, for example, from the identity presented in Appendix A.

Analogous statements hold obviously also for the different real form of the complex rational RSvD system studied in [15], which is also superintegrable.

An interesting open problem for future work is to extend the considerations reported here to the hyperbolic RSvD system having five independent coupling parameters.

A $\mathcal{H}_l$ as elementary symmetric function

Fix an arbitrary $n \in \mathbb{N}$ and $l \in \{0, 1, \ldots, n\}$ and let e_l stand for the l-th elementary symmetric polynomial in n variables $x_1, \ldots, x_n$, i.e., $e_0(x_1, \ldots, x_n) = 1$ and for $l \geq 1$

$$e_l(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_l \leq n} x_{j_1} \cdots x_{j_l}. \tag{A.1}$$

In the text, we referred to the following useful result due to van Diejen ([9] Proposition 2.3). For convenience, we present it together with a direct proof.

Proposition A. *By using* (2.4) *it can be shown that*

$$\mathcal{H}_l(q) = 4^l e_l\left(\sinh^2 \frac{q_1}{2}, \ldots, \sinh^2 \frac{q_n}{2}\right). \tag{A.2}$$

Proof. First, e_l has the equivalent form

$$e_l\left(\sinh^2 \frac{q_1}{2}, \ldots, \sinh^2 \frac{q_n}{2}\right) = \sum_{J \subset \{1, \ldots, n\}, \, |J| = l} \prod_{j \in J} \sinh^2 \frac{q_j}{2}. \tag{A.3}$$

Utilizing the identity $\sinh^2(\alpha/2) = [\cosh(\alpha) - 1]/2$ casts the right-hand side into

$$\sum_{J \subset \{1, \ldots, n\}, \, |J| = l} 2^{-l} \prod_{j \in J} [\cosh(q_j) - 1] = \sum_{J \subset \{1, \ldots, n\}, \, |J| = l} 2^{-l} \sum_{K \subset J} (-1)^{l - |K|} \prod_{k \in K} \cosh(q_k). \tag{A.4}$$

The two sums on the right-hand side can be merged into one, but the multiplicity of subsets must remain the same. This results in the appearance of a binomial coefficient

$$\sum_{J \subset \{1, \ldots, n\}, \, |J| \leq l} \frac{(-1)^{l - |J|}}{2^l} \binom{n - |J|}{l - |J|} \prod_{j \in J} \cosh(q_j) =$$

$$= \sum_{\substack{J \subset \{1, \ldots, n\}, \, |J| \leq l \\ \varepsilon_j = \pm 1, \, j \in J}} \frac{(-1)^{l - |J|}}{2^{l + |J|}} \binom{n - |J|}{l - |J|} \prod_{j \in J} \cosh(\varepsilon_j q_j), \tag{A.5}$$

where we also used that cosh is an even function and compensated the 'over-counting' of terms. Now, let us simply pull a 4^{-l} factor out of the sum to get

$$4^{-l} \sum_{\substack{J \subset \{1, \ldots, n\}, \, |J| \leq l \\ \varepsilon_j = \pm 1, \, j \in J}} (-2)^{l - |J|} \binom{n - |J|}{l - |J|} \prod_{j \in J} \cosh(\varepsilon_j q_j). \tag{A.6}$$

Recall the following identity for the hyperbolic cosine of the sum of a finite number, say N, real arguments (see [16] Art. 132 and apply $\cos(i\alpha) = \cosh(\alpha)$)

$$\cosh\left(\sum_{k=1}^{N} \alpha_k\right) = \left[\prod_{k=1}^{N} \cosh(\alpha_k)\right]\left[\sum_{m=0}^{\lfloor \frac{N}{2} \rfloor} e_{2m}(\tanh(\alpha_1), \ldots, \tanh(\alpha_N))\right], \tag{A.7}$$

where e_{2m} are now elementary symmetric functions with arguments $\tanh(\alpha_1), \ldots, \tanh(\alpha_N)$. Note that for any $m > 0$ and set of signs ε there is another one ε', such that $e_{2m}^{J, \varepsilon'} = -e_{2m}^{J, \varepsilon}$,

therefore by using (A.7) we see that (A.6) equals to

$$4^{-l} \sum_{\substack{J \subset \{1,\dots,n\}, \ |J| \leq l \\ \varepsilon_j = \pm 1, \ j \in J}} (-2)^{l-|J|} \binom{n-|J|}{l-|J|} \prod_{j \in J} \cosh(\varepsilon_j q_j) \sum_{m=0}^{\left\lfloor \frac{|J|}{2} \right\rfloor} s_{2m}^{J,\varepsilon} =$$

$$= 4^{-l} \sum_{\substack{J \subset \{1,\dots,n\}, \ |J| \leq l \\ \varepsilon_j = \pm 1, \ j \in J}} (-2)^{l-|J|} \binom{n-|J|}{l-|J|} \cosh(q_{\varepsilon J}). \quad \text{(A.8)}$$

Applying (2.4) concludes the proof. □

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