MARTINGALE SOLUTIONS OF NEMATIC LIQUID CRYSTALS DRIVEN BY PURE JUMP NOISE IN THE MARCUS CANONICAL FORM

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ABSTRACT. In this work we consider a stochastic evolution equation which describes the system governing the nematic liquid crystals driven by a pure jump noise in the Marcus canonical form. The existence of a martingale solution is proved for both 2D and 3D cases. The construction of the solution relies on a modified Faedo-Galerkin method based on the Littlewood-Paley-decomposition, compactness method and the Jakubowski version of the Skorokhod representation theorem for non-metric spaces. We prove that in the 2-D case the martingale solution is pathwise unique and hence deduce the existence of a strong solution.

1. INTRODUCTION

1.1. **The Deterministic Model.** The obvious states of matter are the solid, the liquid and the gaseous state. The liquid crystal is an intermediate state of a matter, in between the liquid and the crystalline solid, i.e. it must possess some typical properties of a liquid as well as some crystalline properties. The nematic liquid crystal phase is characterised by long-range orientational order, i.e. the molecules have no positional order but tend to align along a preferred direction. Much of the interesting phenomenology of liquid crystals involves the geometry and dynamics of the preferred axis, which is defined by a vector $\mathbf{d}$. This vector is called a director. Since the sign as well as the magnitude of the director has no physical significance, it is taken to be unity.

A complete description of the physical relevance of liquid crystals has been illustrated in Chandrasekhar [17], Warner and Terentjev [49] and Gennes and Prost [21]. In the 1960's, Ericksen [20] and Leslie [32] demonstrated the hydrodynamic theory of liquid crystals. Moreover, they expanded the continuum theory which has been widely used by most researchers to design the dynamics of the nematic liquid crystals. Inspired by this theory, the most fundamental form of dynamical system representing the motion of nematic liquid crystals has been procured by Lin and Liu [35]. This system can be derived as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = -\lambda\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big), \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{d}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{d} = \gamma\left(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\right), \tag{1.3}$$

$$|\mathbf{d}|^2 = 1. \tag{1.4}$$

This holds in $\mathbb{O}_T := (0,T]\times\mathbb{O}$, where $\mathbb{O}\subset\mathbb{R}^{\mathbf{n}}, \mathbf{n}=2,3$.
Here the vector fields $\mathbf{u} := \mathbf{u}(t,x)$, $(t,x)\in(0,T]\times\mathbb{O}$, and resp. $\mathbf{d} := \mathbf{d}(t,x)$, $(t,x)\in(0,T]\times\mathbb{O}$, denote the velocity, resp. the director field, of the fluid, while $p = p(t,x)$, $(t,x)\in(0,T]\times\mathbb{O}$, denotes the scalar pressure. The symbol $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ denotes function with values in the space of

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$\mathbf{n}\times\mathbf{n}$-matrices with the entries

$$\big[\nabla\mathbf{d}\odot\nabla\mathbf{d}\big]_{i,j}(t,x)=\sum_{k=1}^{\mathbf{n}}\partial_{x_i}\mathbf{d}^{(k)}(t,x)\partial_{x_j}\mathbf{d}^{(k)}(t,x),\quad (t,x)\in(0,T]\times\mathbb{O},\quad i,j=1,\cdots,\mathbf{n}.$$

We equip the system with the initial and boundary conditions respectively as follows

$$\mathbf{u}(0)=\mathbf{u}_0\ \text{ and }\ \mathbf{d}(0)=\mathbf{d}_0, \tag{1.5}$$

$$\mathbf{u}=0\ \text{ and }\ \frac{\partial\mathbf{d}}{\partial n}=0\ \text{ on }\ \partial\mathbb{O}, \tag{1.6}$$

where $n(x)$ is the outward unit normal vector at each point x of $\mathbb{O}$.

It is the most simple mathematical model one can acquire without disrupting the basic nonlinear structure. Though (1.1)-(1.4) is a much simplified version of the equations used in Ericksen-Leslie theory, it preserves many crucial physical attributes of the nematic liquid crystals. Since

$$\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}=\mathbf{d}\times\big(\Delta\mathbf{d}\times\mathbf{d}\big),$$

we obtain non-parabolicity in (1.4). Also we have high nonlinearity in (1.1) due to the term $\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big)$. So the problem (1.1)-(1.6) form a fully nonlinear system of Partial Differential Equations with constraint. Since the system (1.1)-(1.6) comprise of the Navier-Stokes equations as a subsystem, in general one can not expect any superior results than those for the Navier-Stokes equations.

To overcome the difficulty, we have a closely related system of (1.1)-(1.6), which eases the constraint $|\mathbf{d}|^2=1$ and the gradient nonlinearity $|\nabla\mathbf{d}|^2\mathbf{d}$, due to the suggestion of Lin and Liu [35] in 1995. They have worked on the following model

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\Delta\mathbf{u}+\nabla p=-\lambda\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big)\ \text{ in }\ (0,T]\times\mathbb{O}, \tag{1.7}$$

$$\nabla\cdot\mathbf{u}=0\ \text{ in }\ [0,T]\times\mathbb{O}, \tag{1.8}$$

$$\frac{\partial\mathbf{d}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{d}=\gamma\left(\Delta\mathbf{d}-\frac{1}{\varepsilon^2}\big(|\mathbf{d}|^2-1\big)\mathbf{d}\right)\ \text{ in }\ (0,T]\times\mathbb{O}, \tag{1.9}$$

$$\mathbf{u}(0)=\mathbf{u}_0\ \text{ and }\ \mathbf{d}(0)=\mathbf{d}_0\ \text{ in }\ \mathbb{O}. \tag{1.10}$$

Where $\epsilon>0$ is an arbitrary constant. Though it is a much simpler version of the previous system (1.1)-(1.6), still it is a captivating as well as a toilsome problem. Many have done meticulous work on the systems (1.1)-(1.6) and (1.7)-(1.10) (e.g. see [22, 34, 35, 36, 33, 24, 19, 46], to name a few).

1.2. **The Stochastic Problem.** When we study SDEs, the Itô formula acts as an essential tool. Then we notice that one of the major disadvantages of the Itô formula is the usual integration by parts is not applicable and it fails to serve the usual chain rule (Newton-Leibniz type) of differential calculus. If we consider SDEs and corresponding flows on smooth manifolds then the Itô integral is not invariant under local coordinate changes and so is not a crude geometric entity. There we employ integral in the Stratonovich sense as perturbation of the Itô integral, which can be treated according to the conventional rules of integration. However the wonderful properties of Stratonovich integral are violated if the driving process has jumps. In the above considered model, in order to maintain the constraint condition on the director field, the noise must preserve the invariance property under coordinate transformation. Since we consider the stochastic integral with respect to compensated Poisson random measure, we observe that the Stratonovich integral will no longer provide us with a Newton-Leibniz type chain rule. So we require a more subtle approach to take care of the jumps. S. Marcus [37] fixed this problem by introducing an SDE of a new type whose solution pertains the characteristics incident to the Stratonovich calculus in the continuous case. There are very few noteworthy mathematics literature available on the Marcus equation (also known as canonical equation), see Chechkin and Pavyukevich [18], Applebaum [3] and Kunita [29] for details.

The probabilistic exposition of the Marcus integral is as follows. The Marcus map generates a fictitious time with respect to which, at each jump time, the process moves at an infinite speed along a curve connecting the starting point and the finishing point. This method can help us understanding many other constraint PDEs (e.g. harmonic map flow, nonlinear Schrödinger equation on a compact Riemannian manifold, Landau-Lifshitz-Gilbert Equations) driven by Lévy noise.

In this paper, we analyse the stochastic version of the problem (1.7)-(1.10). We instigate pure jump noise in (1.7) and pure jump Lévy noise in Marcus sense in (1.9). Moreover, we set $\mu = \lambda = \gamma = 1$, as well as, we supersede the Ginzburg-Landau bounded function $\chi_{|\mathbf{d}|\leq 1}(|\mathbf{d}|^2 - 1)\mathbf{d}$ by a general polynomial function $f(\mathbf{d})$. The system is given by

$$\begin{aligned} &d\mathbf{u}(t) + \big[(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) - \Delta\mathbf{u}(t) + \nabla p\big]\,dt \\ &\qquad = -\nabla\cdot\big(\nabla\mathbf{d}(t)\odot\nabla\mathbf{d}(t)\big)dt + \int_Y F(t,\mathbf{u}(t-);y)\,\tilde{\eta}_1(dt,dy), \qquad (1.11)\\ &\nabla\cdot\mathbf{u}(t) = 0, \qquad (1.12)\\ &d\mathbf{d}(t) + \big[(\mathbf{u}(t)\cdot\nabla)\mathbf{d}(t)\big]dt = \left(\Delta\mathbf{d}(t) - \frac{1}{\varepsilon^2}f(\mathbf{d}(t))\right)dt + \sum_{i=1}^{N}\big(\mathbf{d}(t)\times\mathbf{h}_i\big)\diamond dL_i(t). \qquad (1.13)\end{aligned}$$

This holds in $\mathbb{O}_T := (0,T]\times\mathbb{O}$, where $\mathbb{O}\subset\mathbb{R}^{\mathbf{n}}, \mathbf{n} = 2,3$. Here $\tilde{\eta}_1$ represents a time homogeneous compensated Poisson random measure with a compensator $\mathrm{Leb}\otimes\nu_1$. And $L(t) := (L_1(t),\cdots,L_N(t))$ is a $\mathbb{R}^N$- valued Lévy process with pure jump i.e., $L_c = 0$,

$$L(t) = \int_0^t\int_{\mathbb{B}} l\,\tilde{\eta}_2(ds,dl) + \int_0^t\int_{\mathbb{B}^c} l\,\eta_2(ds,dl) \qquad (1.14)$$

where $\mathbb{B} := \mathbb{B}(0,1)\subset\mathbb{R}^N$; $l\in\mathbb{R}^N$; η_2, $\tilde{\eta}_2$ represent respectively a time homogeneous Poisson random measure and the corresponding time homogeneous compensated Poisson random measure with a compensator $\mathrm{Leb}\otimes\nu_2$, i.e. $\tilde{\eta}_2 := \eta_2 - \mathrm{Leb}\otimes\nu_2$. Precise definition of the symbol $\diamond$ will be stated later. For $\mathbf{n} = 2,3$, $\mathbf{h}_i : \mathbb{O}\to\mathbb{R}^{\mathbf{n}}$, $i = 1,2,\dots N$ are given bounded functions. The initial and boundary conditions are respectively as follows

$$\mathbf{u}(0) = \mathbf{u}_0 \;\text{ and }\; \mathbf{d}(0) = \mathbf{d}_0, \qquad (1.15)$$

$$|\mathbf{d}_0(x)|_{\mathbb{R}^{\mathbf{n}}} = 1 \quad \forall\, x\in\mathbb{O}, \qquad (1.16)$$

$$\mathbf{u} = 0 \;\text{ and }\; \frac{\partial\mathbf{d}}{\partial n} = 0 \;\text{ on }\; \partial\mathbb{O}, \qquad (1.17)$$

where the vector $n(x)$ is the outward unit normal vector at each point x of $\mathbb{O}$.

We assume that the initial value of the director field $\mathbf{d}$ satisfies the saturation condition (1.16). However, since $\mathbf{d}$ solves equation (1.9) (for $\varepsilon = 1$) with $\chi_{|\mathbf{d}|\leq 1}(|\mathbf{d}|^2-1)\mathbf{d}$ replaced by $f(\mathbf{d})$, this saturation condition is not satisfied for $t > 0$. We intend to study the problem with equation (1.3) as a limit, as $\varepsilon\searrow$, of the Ginzburg-Landau approximations (1.9) (with $\chi_{|\mathbf{d}|\leq 1}(|\mathbf{d}|^2-1)\mathbf{d}$ replaced by $f(\mathbf{d})$) and then we will show that the saturation condition is also satisfied for $t > 0$. For that purpose the Marcus form of the jump noise will prove essential, see e.g. a recent work by the first two named authours on the stochastic Landau-Lifshitz-Gilbert Equations [13] (see also [12]). This is the main motivation for studying the problem in the Marcus form. We should point out that all our results remain true for the classical, i.e. non-Marcus, equations.

1.3. **Relevant Literature.** Most of the physical systems confront dynamical instabilities. The instability befalls at some critical value of the control parameter (which is in our case some random external noise) of the system. In our situation the dynamics are quite complicated because the evolution of the director field $\mathbf{d}(t,x)$ is coupled to the velocity field $\mathbf{u}(t,x)$. San Miguel [45], has studied the stationary orientational correlations of the director field of a nematic liquid crystal near the Fréedericksz transition. In this transition the molecules tend to reorient due to some random external perturbations. It has been studied by Sagués and Miguel [44] that the decay time, required for the system is shortened by the field fluctuations to leave an unstable state, which is built by switching on the field to a value beyond instability point. See also Horsthemke and Lefever[25] and references there in, for more details. A nematic drifts very much like a typical organic liquid with molecules of indistinguishable size. Since, the transitional motions are coupled to inner, orientational motions of the molecules, in most cases the flow muddles the alignment.

Conversely, by implementation of an external field, a change in the alignment will generate a flow in the nematic. So we are interested in the study of (1.11)-(1.13), which characterise the flow of nematic liquid crystals, effected by altering external forces.

There are few notable works available on the stochastic version of the problem (1.7)-(1.10). The authors in [9] have studied the Ginzburg-Landau approximation of the above system governing the nematic liquid crystals under the influence of fluctuating external forces. In their paper they have proved the existence and uniqueness of local maximal solution for both 2 and 3 dimensional cases using the Banach fixed point theorem. Also they have proved in the 2 dimensional case this local maximal solution is de facto global. Later, the same authors in [8] have considered the same model as in [9] but with a multiplicative Gaussian noise and replaced the Ginzburg-Landau function by a general polynomial, under suitable assumptions on it. In that paper they proved the existence of a global martingale solution for dimension $\mathbf{n} = 2, 3$ and showed the pathwise uniqueness of the solution in the 2 dimensional case. Hence, in this case, by means of the Yamada-Watanabe type theorem, as in a recent paper [6], the authors established the existence and uniqueness of a strong global solution.

In this work, we consider the same model which describes the dynamics of the nematic liquid crystal, but we have replaced the multiplicative Gaussian noise with pure jump Lévy noise represented by a time homogeneous Poisson random measure. Hence our paper is a generalisation of the earlier works [8, 9] by the first named author, Hausenblas and Razafimandimby. In fact many preliminary results, especially the results about the deterministic part of the model has been taken from that paper. We are interested in showing the existence (in both 2 and 3 dimensional cases) and pathwise uniqueness (only in the 2 dimensional case) of the martingale solution of the problem (1.11)-(1.13) subject to (1.15)-(1.17). Motyl [40] has proved the existence of martingale solutions of the stochastic Navier-Stokes equations driven by Lévy noise.

This paper is organised as follows. In Section 2 we define various functional spaces, its embeddings and some useful operators which are used throughout in our paper. Also we have listed all the assumptions at the end of this particular section. In Section 3 we define the martingale solution and strong solution for our problem in the view of operators defined in Section 2. Also we state the main result of our paper in this section. In Section 4 we state compactness results and tightness criterion for both $\mathbf{u}$ and $\mathbf{d}$. In Section 5 we derive several important a-priori energy estimates of the approximating sequences $(\mathbf{u}_n, \mathbf{d}_n)$, obtained by the modified Faedo-Galerkin method. Then the Sections 6 and 7 are devoted to the proof of tightness of the above approximating solutions and the existence of martingale solution respectively. For the existence of such solution we use Skorokhod embedding theorems stated in Section 4. As a consequence of this theorem, finally in the end of the Section 7 we show the convergence of the new processes to the corresponding limiting processes. In Section 8 we show the pathwise uniqueness of the solution, but only in the two dimensional case. Also we discuss about the existence of a strong solution. In Section 9 we give the proof of the main result. Finally in Appendix we prove various results and estimates which are used in the derivation of a priori estimates as well as in the proof of existence of martingale solution.

2. Functional Setting of the Model

2.1. **Basic Definitions and Functional Spaces.** Let $\mathbb{O} \subset \mathbb{R}^{\mathbf{n}}$, $\mathbf{n} = 2, 3$, be a bounded domain with smooth boundary $\partial\mathbb{O}$. For any $p \in [1, \infty)$ and $k \in \mathbb{N}$, $L^p(\mathbb{O})$ and $W^{k,p}(\mathbb{O})$ are well-known Lebesgue and Sobolev spaces of $\mathbb{R}^{\mathbf{n}}$-valued functions respectively. For $p = 2$, put $W^{k,2} = H^k$.

For instance, $H^1(\mathbb{O}; \mathbb{R}^{\mathbf{n}})$ is the Sobolev space of all $\mathbf{u} \in L^2(\mathbb{O}; \mathbb{R}^{\mathbf{n}})$, for which there exist weak derivatives $\frac{\partial \mathbf{u}}{\partial x_i} \in L^2(\mathbb{O}; \mathbb{R}^{\mathbf{n}}), i = 1, 2, \cdots, \mathbf{n}$. It is a Hilbert space with the scalar product given by

$$(\mathbf{u}, \mathbf{v})_{H^1} := (\mathbf{u}, \mathbf{v})_{L^2} + (\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}, \quad \mathbf{u}, \mathbf{v} \in H^1(\mathbb{O}, \mathbb{R}^{\mathbf{n}}).$$

Let us define the following spaces

$$\mathcal{V} := \{\mathbf{u} \in \mathcal{C}_c^\infty(\mathbb{O};\mathbb{R}^\mathbf{n}) : div\ \mathbf{u} = 0\},$$
$$\mathrm{H} := \text{the closure of } \mathcal{V} \text{ in } L^2(\mathbb{O};\mathbb{R}^\mathbf{n}),$$
$$\mathrm{V} := \text{the closure of } \mathcal{V} \text{ in } H^1(\mathbb{O};\mathbb{R}^\mathbf{n}).$$

One can use also an equivalent characterisation of these two spaces based on the trace (or Stokes) Theorem [48, Theorem I.1.2], see Theorems I.1.4 and I.1.6 therein.

In the space H we consider the scalar product and the norm inherited from $L^2(\mathbb{O};\mathbb{R}^\mathbf{n})$ and denote them by $(\cdot,\cdot)_\mathrm{H}$ and $|\cdot|_\mathrm{H}$, respectively, i.e.,

$$(\mathbf{u},\mathbf{v})_\mathrm{H} := (\mathbf{u},\mathbf{v})_{L^2}, \qquad |\mathbf{u}|_\mathrm{H} := |\mathbf{u}|_{L^2} := |\mathbf{u}|, \qquad \mathbf{u},\mathbf{v} \in \mathrm{H}. \tag{2.1}$$

In the space V we consider the scalar product inherited from the Sobolev space $H^1(\mathbb{O};\mathbb{R}^\mathbf{n})$ i.e.,

$$(\mathbf{u},\mathbf{v})_\mathrm{V} := (\mathbf{u},\mathbf{v})_{L^2} + ((\mathbf{u},\mathbf{v})), \tag{2.2}$$

where

$$((\mathbf{u},\mathbf{v})) := (\nabla\mathbf{u},\nabla\mathbf{v})_{L^2} = \sum_{i=1}^{\mathbf{n}} \int_\mathbb{O} \frac{\partial \mathbf{u}}{\partial x_i}\cdot\frac{\partial \mathbf{v}}{\partial x_i}\,dx, \qquad \mathbf{u},\mathbf{v}\in\mathrm{V}. \tag{2.3}$$

and the norm

$$|\mathbf{u}|_\mathrm{V}^2 := |\mathbf{u}|_\mathrm{H}^2 + \|\mathbf{u}\|^2, \tag{2.4}$$

where

$$\|\mathbf{u}\|^2 := |\nabla\mathbf{u}|_{L^2}^2. \tag{2.5}$$

Note that since $\mathbb{O}$ is a bounded domain, the Poincaré inequality holds on it, and therefore the norms $|\cdot|_\mathrm{V}$ and $\|\cdot\|$ are equivalent (on V).

It is also known that V is dense in H and the embedding is continuous. We have

$$\mathrm{V} \underset{j_1}{\hookrightarrow} \mathrm{H} \cong \mathrm{H}' \underset{j_1'}{\hookrightarrow} \mathrm{V}'. \tag{2.6}$$

The above spaces are the most used spaces to describe the fluid's velocity. To describe the fluid's director field, we will use spaces

$$L^2 := L^2(\mathbb{O},\mathbb{R}^3), \quad H^1 := H^1(\mathbb{O},\mathbb{R}^3) \text{ and } H^2 := H^2(\mathbb{O},\mathbb{R}^3).$$

Note that elements of these spaces take values in the three-dimensional Euclidean space $\mathbb{R}^3$, irrespectively of the spatial dimension $\mathbf{n}$.

2.2. **Bilinear Operators.** Let us consider the following trilinear form, see Temam [47],

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} \int_\mathbb{O} \mathbf{u}^{(i)}\partial_{x_i}\mathbf{v}^{(j)}\mathbf{w}^j\,dx, \quad \mathbf{u}\in L^p, \mathbf{v}\in W^{1,q}, \mathbf{w}\in L^r, \tag{2.7}$$

where $p,q,r\in[1,\infty]$ satisfying

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1. \tag{2.8}$$

We will recall the fundamental properties of the form b that are valid for both bounded and unbounded domains. By the Sobolev embedding Theorem, see Adams [1], and the Hölder inequality, there exists a positive constant c such that

$$|b(\mathbf{u},\mathbf{w},\mathbf{v})| \le c|\mathbf{u}|_\mathrm{V}|\mathbf{w}|_\mathrm{V}|\mathbf{v}|_\mathrm{V}, \qquad \mathbf{u},\mathbf{w},\mathbf{v}\in\mathrm{V}.$$

The form b is continuous on V. In particular, we define a bilinear map B by $B(\mathbf{u},\mathbf{w}) := b(\mathbf{u},\mathbf{w},\cdot)$, then we infer that $B(\mathbf{u},\mathbf{w})\in\mathrm{V}'$ for all $\mathbf{u},\mathbf{w}\in\mathrm{V}$ and the following inequality holds

$$|B(\mathbf{u},\mathbf{w})|_{\mathrm{V}'} \le c\ |\mathbf{u}|_\mathrm{V}|\mathbf{w}|_\mathrm{V}, \qquad \mathbf{u},\mathbf{w}\in\mathrm{V}. \tag{2.9}$$

Moreover, the mapping $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ is bilinear and continuous. The form b also has the following properties, see [47],

$$b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -b(\mathbf{u}, \mathbf{v}, \mathbf{w}), \qquad \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathrm{V}.$$

In particular,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{V}.$$

Hence

$$\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle = -\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle, \quad \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathrm{V}$$

and

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{V}. \tag{2.10}$$

Moreover, for all $\mathbf{u} \in \mathrm{V}, \mathbf{v} \in H^1$, using the notation (2.5), we have

$$|B(\mathbf{u}, \mathbf{v})|_{\mathrm{V}'} \le c\ |\mathbf{u}|^{1-\frac{\mathbf{n}}{4}} \|\mathbf{u}\|^{\frac{\mathbf{n}}{4}} |\mathbf{v}|^{1-\frac{\mathbf{n}}{4}} \|\mathbf{v}\|^{\frac{\mathbf{n}}{4}}, \quad \mathbf{n} \in \{2, 3\}, \tag{2.11}$$

For the proof, we refer to Section 1.2 of Temam [47].

We will use the following notation, $B(\mathbf{u}) := B(\mathbf{u}, \mathbf{u})$. Also note that the map $B : \mathrm{V} \to \mathrm{V}'$ is Lipschitz continuous on balls.

One can define a bilinear map $\tilde{B}$ defined on $H^1 \times H^1$ with values in $(H^1)'$ such that [1]

$$\langle \tilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \qquad \mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1$$

With an abuse of notation, we again denote by $\tilde{B}(\cdot, \cdot)$ the restriction of $\tilde{B}(\cdot, \cdot)$ to $\mathrm{V} \times H^2$, which maps continuously $\mathrm{V} \times H^2$ into L^2. Using the Gagliardo-Nirenberg inequalities one can show there exists a positive constant C such that for $\mathbf{n} \in \{2, 3\}$,

$$\left|\tilde{B}(\mathbf{u}, \mathbf{d})\right| \le C\ |\mathbf{u}|^{1-\frac{\mathbf{n}}{4}} \|\mathbf{u}\|^{\frac{\mathbf{n}}{4}} |\nabla \mathbf{d}|^{1-\frac{\mathbf{n}}{4}} |\Delta \mathbf{d}|^{\frac{\mathbf{n}}{4}}, \qquad \mathbf{u} \in \mathrm{V}, \mathbf{d} \in H^2. \tag{2.12}$$

Moreover, using Young's inequality one can get

$$\left|\tilde{B}(\mathbf{u}, \mathbf{d})\right| \le C\ \|\mathbf{u}\| |\mathbf{d}|_{H^2}, \tag{2.13}$$

We also have

$$\langle \tilde{B}(\mathbf{u}, \mathbf{d}), \mathbf{d}\rangle = 0, \qquad \mathbf{u} \in \mathrm{V}, \mathbf{d} \in H^2. \tag{2.14}$$

For the proof, we refer to Section 1.2 of Temam[47].

Let m be the trilinear form defined by

$$m(\mathbf{d}_1, \mathbf{d}_2, \mathbf{u}) = -\sum_{i,j,k=1}^{\mathbf{n}} \int_{\mathbb{O}} \partial_{x_i} \mathbf{d}_1^{(k)} \partial_{x_j} \mathbf{d}_2^{(k)} \partial_{x_j} \mathbf{u}^{(i)}\, dx, \quad \mathbf{d}_1 \in W^{1,p}, \mathbf{d}_2 \in W^{1,q}, \mathbf{u} \in W^{1,r},$$

with $p, q, r \in (1, \infty)$ satisfying condition (2.8). Since $\mathbf{n} \in \{2, 3\}$, the above integral is well defined, when $\mathbf{d}_1, \mathbf{d}_2 \in H^2$ and $\mathbf{u} \in \mathrm{V}$. We also have the following Lemma, where we use the notation (2.5).

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|m(\mathbf{d}_1, \mathbf{d}_2, \mathbf{u})| \le C\ |\nabla \mathbf{d}_1|_{L^2}^{1-\frac{\mathbf{n}}{4}} |\Delta \mathbf{d}_1|^{\frac{\mathbf{n}}{4}} |\nabla \mathbf{d}_2|_{L^2}^{1-\frac{\mathbf{n}}{4}} |\Delta \mathbf{d}_2|^{\frac{\mathbf{n}}{4}} \|\mathbf{u}\|, \quad \mathbf{d}_1, \mathbf{d}_2 \in H^2, \mathbf{u} \in \mathrm{V}.$$

For proof see [8]. Now we state the following Lemma.

Lemma 2.2. *There exists a bilinear operator $M : H^2 \times H^2 \to \mathrm{V}'$ such that*

$$\left\langle M(\mathbf{d}_1, \mathbf{d}_2), \mathbf{u}\right\rangle = m(\mathbf{d}_1, \mathbf{d}_2, \mathbf{u}), \quad \mathbf{d}_1, \mathbf{d}_2 \in H^2, \mathbf{u} \in \mathrm{V}.$$

Furthermore, there exists $C > 0$ such that

$$|M(\mathbf{d}_1, \mathbf{d}_2)|_{\mathrm{V}'} \le C\ |\nabla \mathbf{d}_1|_{L^2}^{1-\frac{\mathbf{n}}{4}} |\Delta \mathbf{d}_1|^{\frac{\mathbf{n}}{4}} |\nabla \mathbf{d}_2|_{L^2}^{1-\frac{\mathbf{n}}{4}} |\Delta \mathbf{d}_2|^{\frac{\mathbf{n}}{4}}, \quad \mathbf{d}_1, \mathbf{d}_2 \in H^2. \tag{2.15}$$

For a proof we refer to [8]. We will use the following notation, $M(\mathbf{d}) := M(\mathbf{d}, \mathbf{d})$.

[1]To be precise, the form b should be replaced by a form $\tilde{b}$ defined by formula (2.7) but for $\mathbb{R}^3$-valued vector fields $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

2.3. **Linear Operators, Its Properties and Important Embeddings.** Now we will recall operators and their properties used in [15]. Consider the natural embedding $j : \mathrm{V} \hookrightarrow \mathrm{H}$ and its adjoint $j' : \mathrm{H} \hookrightarrow \mathrm{V}$. Since the range of j is dense in H, the map j' is one-to-one. Let us put

$$\mathscr{A}\mathbf{u} := ((\mathbf{u}, \cdot)), \qquad \mathbf{u} \in \mathrm{V}, \tag{2.16}$$

where $((\cdot,\cdot))$ is defined in (2.3). If $\mathbf{u} \in \mathrm{V}$, then $\mathscr{A}\mathbf{u} \in \mathrm{V}'$. Since we have the following inequalities

$$|((\mathbf{u},\mathbf{v}))| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\| \leq \|\mathbf{u}\|(\|\mathbf{v}\|^2 + |\mathbf{v}|_{\mathrm{H}}^2)^{\frac{1}{2}} = \|\mathbf{u}\| \cdot |\mathbf{v}|_{\mathrm{V}}, \quad \mathbf{v} \in \mathrm{V}.$$

we infer that

$$|\mathscr{A}\mathbf{u}|_{\mathrm{V}'} \leq \|\mathbf{u}\|, \qquad \mathbf{u} \in \mathrm{V}. \tag{2.17}$$

The Neumann Laplacian acting on $\mathbb{R}^{\mathbf{n}}$-valued function will be denoted by $\mathcal{A}$, i.e.,

$$D(\mathcal{A}) := \left\{ \mathbf{d} \in H^2 : \frac{\partial \mathbf{d}}{\partial n} = 0 \ on \ \partial\mathbb{O} \right\}, \tag{2.18}$$

$$\mathcal{A}\mathbf{d} := -\sum_{i=1}^{\mathbf{n}} \frac{\partial^2 \mathbf{d}}{\partial x_i^2}, \quad \mathbf{d} \in D(\mathcal{A}).$$

It is known that $\mathcal{A}$ is a non-negative self-adjoint operator in L^2. As we are working on a bounded domain, $\mathcal{A}$ has compact resolvent.

Also we have the dense embeddings

$$H^2 \underset{j_2}{\hookrightarrow} H^1 \hookrightarrow L^2. \tag{2.19}$$

Assumption 2.3. *(A) Assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration, and the probability space satisfies the usual conditions.*

(B) $\tilde{\eta}_1$ is a compensated time homogeneous Poisson random measure on a measurable space $(Y, \mathscr{B}(Y))$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a σ-finite intensity measure ν_1. See Appendix for definitions and more details.

(C) Assume that $(L(t))_{t\geq 0}$ is a $\mathbb{R}^N$-valued, $(\mathcal{F}_t)$-adapted Lévy process of pure jump type defined over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the corresponding time homogeneous Poisson random measure η_2 on a measurable space $(\mathbb{B}, \mathscr{B}(\mathbb{B}))$ (See Appendix for definition). Also assume that the corresponding intensity measure ν_2 is such that Supp $\nu_2 \subset \mathbb{B}$, *where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^N$.*

(D) Let $F : [0,T] \times \mathrm{H} \times Y \to \mathrm{H}$ is a measurable function and there exists a constant L such that

$$\int_Y \left|F(t,\mathbf{u}_1;y) - F(t,\mathbf{u}_2;y)\right|_{\mathrm{H}}^2 \nu(dy) \leq L\left|\mathbf{u}_1 - \mathbf{u}_2\right|_{\mathrm{H}}^2, \quad \mathbf{u}_1, \mathbf{u}_2 \in \mathrm{H}, t \in [0,T]. \tag{2.20}$$

and for each $p \geq 1$ there exists a constant C_p such that

$$\int_Y \left|F(t,\mathbf{u};y)\right|_{\mathrm{H}}^p \nu(dy) \leq C_p\big(1 + |\mathbf{u}|_{\mathrm{H}}^p\big), \quad \mathbf{u} \in \mathrm{H}, \quad t \in [0,T], \tag{2.21}$$

(E) Assume that $\mathbf{h}_i \in L^\infty \cap H^1$, for each $i = 1, 2, \cdots, N$.

(F) Let $\mathbb{I}_{\mathbf{n}}$ be the set defined by

$$\mathbb{I}_{\mathbf{n}} = \begin{cases} \mathbb{N} := \{1,2,3,\cdots\} & if\ \mathbf{n} = 2, \\ \{1\}, & if\ \mathbf{n} = 3. \end{cases} \tag{2.22}$$

For $N \in \mathbb{I}_{\mathbf{n}}$ and numbers $b_j, j = 0, \cdots, N$, with $b_N > 0$ we define a function $\tilde{f} : [0,\infty) \to \mathbb{R}$ by

$$\tilde{f}(r) = \sum_{j=0}^{N} b_j r^j, for\ any\ r \in \mathbb{R}_+.$$

We define a map $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ by

$$f(\mathbf{d}) = \tilde{f}(|\mathbf{d}|^2)\mathbf{d}. \tag{2.23}$$

Let $\mathrm{F} : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}$ *be a Fréchét differentiable map such that for any* $\mathbf{d} \in \mathbb{R}^{\mathbf{n}}$ *and* $\mathbf{g} \in \mathbb{R}^{\mathbf{n}}$

$$\mathrm{F}'(\mathbf{d})[\mathbf{g}] = f(\mathbf{d}) \cdot \mathbf{g}.$$

Let also $\tilde{F}$ *be an antiderivative of* $\tilde{f}$ *such that* $\tilde{F}(0) = 0$*. We have*

$$\tilde{F}(r) = a_{N+1} r^{N+1} + \mathcal{U}(r),$$

where $\mathcal{U}$ *is a polynomial function of at most degree* N *such that* $\mathcal{U}(0) = 0$ *and* $a_{N+1} > 0$*.*

(G) *There is a strictly positive self-adjoint operator* S *on* L^2 *with compact resolvent commuting with* $\mathcal{A}$ *and* $D(S^k) \hookrightarrow H^1$ *for some* $k \in \mathbb{N}$*. Moreover, we assume that* S *has generalised Gaussian* $(1, \infty)$*-bounds, i.e. for all* $t > 0$ *there is a measurable function* $q(t, \cdot, \cdot) : \mathbb{O} \times \mathbb{O} \to \mathbb{R}$ *with*

$$(e^{-tS} h)(x) = \int_{\mathbb{O}} q(t, x, y) h(y) \mu(dy), \quad t > 0, \quad a.e.\ x \in \mathbb{O}$$

for all $h \in L^2$ *and*

$$|q(t, x, y)| \le \frac{C}{\mu(B(x, t^{\frac{1}{2}}))} \exp\left\{ \frac{-c|x - y|^2}{t} \right\}, \tag{2.24}$$

for all $t > 0$ *and almost all* $(x, y) \in \mathbb{O} \times \mathbb{O}$ *with constants* $c, C > 0$*. In particular,* e^{-tS} *can be extended to a* C_0*-semigroup on* $L^p(\mathbb{O})$ *for all* $p \in [1, \infty)$*.*

Remark 2.4. *We have the following results*

$$|\tilde{f}(r)| \le l_1(1 + r^N) \quad and \quad |\tilde{f}'(r)| \le l_2(1 + r^{N_1}), \quad r > 0.$$

for some $l_1, l_2 > 0$*. And there exist positive constants* $c, \tilde{c}$ *such that*

$$|f(\mathbf{d})|_{\mathbb{R}^{\mathbf{n}}} \le c\left(1 + \left|\mathbf{d}\right|_{\mathbb{R}^{\mathbf{n}}}^{2N+1}\right) \quad and \quad |f'(\mathbf{d})|_{\mathbb{R}^{\mathbf{n}}} \le \tilde{c}\left(1 + \left|\mathbf{d}\right|_{\mathbb{R}^{\mathbf{n}}}^{2N}\right), \quad \mathbf{d} \in \mathbb{R}^{\mathbf{n}}. \tag{2.25}$$

We have the following interesting properties of the polynomial functions $\tilde{F}$ and f.

Lemma 2.5. *Let* $\tilde{F}$ *and* f *be polynomial functions as above. Then there exists a constant* $C > 0$*, depending on* N*, and constants* $C_1 > 0$ *and* $C_2 > 0$*, depending on* N *and* $|\mathbb{O}|$*, such that for all* $\mathbf{d} \in L^{2N+2}(\mathbb{O})$*, we have*

$$|\mathbf{d}|_{L^{2N+2}}^{2N+2} \le C \int_{\mathbb{O}} \tilde{F}(|\mathbf{d}(x)|^2) dx + C|\mathbf{d}|_{L^2}^2, \tag{2.26}$$

$$|f(\mathbf{d})|_{L^{\frac{2N+2}{2N+1}}} \le C_1\left(1 + |\mathbf{d}|_{L^{2N+2}}^{2N+1}\right), \tag{2.27}$$

$$|f'(\mathbf{d})|_{L^{\frac{2N+2}{2N+1}}} \le C_2\left(1 + |\mathbf{d}|_{L^{2N+2}}^{2N}\right). \tag{2.28}$$

Proof. The proof of (2.26) follows from Lemma 8.7 of [14]. The proof of (2.27) follows directly from (2.25). Proof of (2.28) is also direct from (2.25) and the embedding $L^{\frac{2N+2}{2N}} \hookrightarrow L^{\frac{2N+2}{2N+1}}$. □

Remark 2.6. *It is straightforward to see that* (2.26) *can also be written in the following form*

$$|\mathbf{d}|_{L^{2N+2}}^{2N+2} \le C \int_{\mathbb{O}} \mathrm{F}(\mathbf{d}(x)) dx + C|\mathbf{d}|_{L^2}^2. \tag{2.29}$$

Remark 2.7. *As a straightforward consequences of Assumption 2.3(G) (see* [10, 11]*), there is an orthonormal basis* $\{\varsigma_n\}_{n\in\mathbb{N}}$ *of* L^2 *consisting of the eigenfunctions of the Neumann Laplacian* $\mathcal{A}$*, and a nondecreasing sequence* $\{\lambda_n\}_{n\in\mathbb{N}}$ *with* $\lambda_n > 0$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$ *and*

$$Sx = \sum_{n=1}^{\infty} \lambda_n \left(x, \varsigma_n\right)_{L^2} \varsigma_n, \quad x \in D(S) = \left\{ x \in L^2 : \sum_{n=1}^{\infty} \lambda_n^2 |\left(x, \varsigma_n\right)_{L^2}|^2 < \infty \right\}.$$

2.4. **The Marcus Mapping.** Define a bounded linear map

$$g_i : H^1 \ni \mathbf{d} \mapsto \mathbf{d} \times \mathbf{h}_i \in H^1. \tag{2.30}$$

The map g_i is bounded because of the Sobolev emebdding $H^1 \hookrightarrow L^6$ and Assumption 2.3 (E).

Let us define a generalized Marcus mapping

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times H^1 \to H^1$$

such that for each fixed $l = (l_1, l_2, \cdots, l_N) \in \mathbb{R}^N$, $\mathbf{d}_0 \in H^1$, the function

$$t \mapsto \Phi(t, l, \mathbf{d}_0)$$

is the solution of the following ordinary differential equation

$$\frac{d}{dt}\mathbf{d}(t) = \sum_{i=1}^{N} l_i g_i(\mathbf{d}(t)),\ t \geq 0, \tag{2.31}$$

$$\mathbf{d}(0) = \mathbf{d}_0. \tag{2.32}$$

i.e.,

$$\Phi(t, l, \mathbf{d}_0) = \Phi(0, l, \mathbf{d}_0) + \int_0^t \sum_{i=1}^{N} l_i g_i(\Phi(s, l, \mathbf{d}_0))\, ds,\ t \geq 0. \tag{2.33}$$

Observe that since $\mathbf{h}_i \in L^\infty$, the maps g_i are also bounded linear from L^2 to L^2 and more generally from L^p to L^p for any $p \geq 2$. Hence the map Φ is well defined as a map $\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times L^p \to L^p$ for every $p \geq 2$.

Notation: We fix $t = 1$ now onward in this paper and consider Φ as the function of last variable for fixed t and l. Denote $\Phi(l, \cdot) := \Phi(1, l, \cdot)$.

Given an $\mathcal{F}_0$-measurable random variable $\mathbf{d}_0$, the equation (1.13) with the notation $\diamond$ is defined in the integral form as follows

$$\begin{aligned}
\mathbf{d}(t) = \mathbf{d}_0 &- \int_0^t \left[(\mathbf{u}(t)\cdot\nabla)\mathbf{d}(t) - \Delta\mathbf{d}(t) + \frac{1}{\varepsilon^2} f(\mathbf{d}(t))\right] dt \\
&+ \int_0^t \int_{\mathbb{B}} [\Phi(l, \mathbf{d}(s-)) - \mathbf{d}(s-)]\, \tilde{\eta}_2(ds, dl) + \int_0^t \int_{\mathbb{B}^c} [\Phi(l, \mathbf{d}(s-)) - \mathbf{d}(s-)]\, \eta_2(ds, dl) \\
&+ \int_0^t \int_{\mathbb{B}} \left\{ \Phi(l, \mathbf{d}(s)) - \mathbf{d}(s) - \sum_{i=1}^{N} l_i g_i(\mathbf{d}(s)) \right\} \nu_2(dl) ds
\end{aligned} \tag{2.34}$$

In view of Theorem IV.9.1 of [26], we will concentrate only with the case when $\eta = 0$ on $\mathbb{B}^c$. In other words, in this study we ignore the large jumps.
Let us also introduce three auxiliary functions

$$G(l, z) := \Phi(l, z) - z, \quad l \in \mathbb{R}^N, z \in H^1, \tag{2.35}$$

$$K(l, z) := \Phi(l, z) - z - \sum_{i=1}^{N} l_i g_i(z), \quad l \in \mathbb{R}^N, z \in H^1, \tag{2.36}$$

and

$$b(z) := \int_{\mathbb{B}} \left[\Phi(l, z) - z - \sum_{i=1}^{N} l_i g_i(z)\right] \nu_2(dl) := \int_{\mathbb{B}} K(l, z)\, \nu_2(dl), \quad z \in H^1. \tag{2.37}$$

With the above notation, equation (2.34) can be written in the following more compact form

$$\begin{aligned}
\mathbf{d}(t) = \mathbf{d}_0 &- \int_0^t \left[(\mathbf{u}(s)\cdot\nabla)\mathbf{d}(s) - \Delta\mathbf{d}(s) + \frac{1}{\varepsilon^2} f(\mathbf{d}(s))\right] ds \\
&+ \int_0^t \int_{\mathbb{B}} G(l, \mathbf{d}(s-))\, \tilde{\eta}_2(ds, dl) + \int_0^t b(\mathbf{d}(s))\, ds, \quad t \geq 0.
\end{aligned} \tag{2.38}$$

This can be furhter written as

$$\mathbf{d}(t) = \mathbf{d}_0 - \int_0^t \big[\mathcal{A}\mathbf{d}(s) + \tilde{B}(\mathbf{u}(s), \mathbf{d}(s)) + \frac{1}{\varepsilon^2} f(\mathbf{d}(s))\big]\, ds \\ + \int_0^t \int_{\mathbb{B}} G(l, \mathbf{d}(s-))\, \tilde{\eta}_2(ds, dl) + \int_0^t b(\mathbf{d}(s))\, ds. \tag{2.39}$$

Let us define $\mathcal{L}(H^1)$ be the space of all bounded linear operators from H^1 to H^1. Given a fixed $l \in \mathbb{R}^N$, we can define a linear operator

$$\mathcal{R} : H^1 \ni \mathbf{d} \mapsto \sum_{i=1}^N l_i g_i(\mathbf{d}) = \mathbf{d} \times \sum_{i=1}^N l_i \mathbf{h}_i := \mathbf{d} \times \bar{\mathbf{h}} \in H^1. \tag{2.40}$$

Then $\mathcal{R} = \sum\limits_{i=1}^N l_i g_i$ and $\|\mathcal{R}\|_{\mathcal{L}(H^1)} \leq |l|_{\mathbb{R}^N} \|g\|_{\mathcal{L}(H^1)}$, where we have denoted

$$\|g\|^2_{\mathcal{L}(H^1)} := \sum_{i=1}^N |g_i|^2_{\mathcal{L}(H^1)}.$$

In a similar manner, we abbreviate

$$\|g\|^2_{\mathcal{L}(L^2)} := \sum_{i=1}^N |g_i|^2_{\mathcal{L}(L^2)}, \qquad \|g\|^2_{\mathcal{L}(L^p)} := \sum_{i=1}^N |g_i|^2_{\mathcal{L}(L^p)}, \quad p \geq 2.$$

Note that the function $y(t) := \Phi(t, l, x),\ t \geq 0$, solves

$$\begin{cases} \dfrac{dy}{dt} = \mathcal{R}y, \quad y(0) = x. \end{cases} \tag{2.41}$$

Hence $y(t) = e^{t\mathcal{R}}x = \sum\limits_{k=0}^\infty \dfrac{t^k}{k!}\mathcal{R}^k x$, as $\mathcal{R}$ is linear.

Let us also formulate the following fundamental result.

Lemma 2.8. *Let $\psi : L^2 \ni z \mapsto |z|^p_{L^2} \in \mathbb{R},\ p \geq 2$. If $\mathscr{N} := \int_0^1 \|e^{s\mathcal{R}}\|^p ds$, then*

(1) $|\psi(\Phi(l, x)) - \psi(x)| \leq \mathscr{N} p\, |l|_{\mathbb{R}^N} \|g\|_{\mathcal{L}(L^2)}\, |x|^p_{L^2}$.
(2) $|\psi(\Phi(l, x)) - \psi(x) - \psi'(x)\mathcal{R}x| \leq \mathscr{N} p^2\, |l|^2_{\mathbb{R}^N} \|g\|^2_{\mathcal{L}(L^2)} |x|^p_{L^2}$.

Proof. See Lemma 2.2 in [13] for the details of the proof. □

Using the notations defined in the previous sections we rewrite the equations (1.11)-(1.13) in the differential form for $\varepsilon = 1$ as,

$$d\mathbf{u}(t) + \big[\mathscr{A}\mathbf{u}(t) + B(\mathbf{u}(t)) + M(\mathbf{d}(t))\big]\, dt = \int_Y F(t, \mathbf{u}(t); y)\, \tilde{\eta}_1(dt, dy), \tag{2.42}$$

$$d\mathbf{d}(t) = -\big[\mathcal{A}\mathbf{d}(t) + \tilde{B}(\mathbf{u}(t), \mathbf{d}(t)) + f(\mathbf{d}(t))\big] dt + \int_{\mathbb{B}} G(l, \mathbf{d}(t))\, \tilde{\eta}_2(dt, dl) + b(\mathbf{d}(t))\, dt. \tag{2.43}$$

3. Statement of the Main Result

Let us recall the definition of a martingale solution.

Definition 3.1. *A* **martingale solution** *of the problem*(2.42)-(2.43) *is a system $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{u}}, \bar{\mathbf{d}}, \bar{\eta}_1, \bar{\eta}_2\big)$, where*

(1) $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}\big)$ is a filtered probability space with a filtration $\bar{\mathbb{F}} = \big(\bar{\mathcal{F}}_t\big)_{t\geq 0}$,
(2) $\bar{\eta}_1$ is a time homogeneous Poisson random measure on $(Y, \mathscr{B}(Y))$ over $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}\big)$ with the intensity measure ν_1 and $\bar{\eta}_2$ is a time homogeneous Poisson random measure on $(\mathbb{B}, \mathscr{B}(\mathbb{B}))$ with the intensity measure ν_2.

(3) $\bar{\mathbf{u}} : [0,T] \times \bar{\Omega} \to \mathrm{H}$ *is a progressively measurable process with* $\bar{\mathbb{P}}$*-a.e. paths*

$$\bar{\mathbf{u}}(\cdot,\omega) \in \mathbb{D}([0,T];\mathrm{H}_w) \cap L^2(0,T;\mathrm{V}) \tag{3.1}$$

such that for all $t \in [0,T]$ *and all* $v \in \mathrm{V}$ *the following identity holds* $\bar{\mathbb{P}}$*-a.s.*

$$\begin{aligned}(\bar{\mathbf{u}}(t),v)_{\mathrm{H}} &+ \int_0^t \langle \bar{\mathbf{u}}(s), \mathscr{A} v\rangle\, ds + \int_0^t \langle B(\bar{\mathbf{u}}(s)), v\rangle\, ds \\ &+ \int_0^t \langle M(\bar{\mathbf{d}}(s)), v\rangle\, ds = (\mathbf{u}_0, v)_{\mathrm{H}} + \int_0^t \int_Y \big(F(s,\bar{\mathbf{u}}(s);y), v\big)_{\mathrm{H}} \tilde{\bar{\eta}}_1(ds,dy).\end{aligned} \tag{3.2}$$

(4) $\bar{\mathbf{d}} : [0,T] \times \bar{\Omega} \to H^1$ *is a progressively measurable process with* $\bar{\mathbb{P}}$*-a.e. paths*

$$\bar{\mathbf{d}}(\cdot,\omega) \in \mathbb{D}([0,T];H^1_w) \cap L^2(0,T;D(\mathcal{A})) \tag{3.3}$$

such that for all $t \in [0,T]$ *and all* $v \in D(\mathcal{A})$ *the following identity holds* $\bar{\mathbb{P}}$*-a.s.*

$$\begin{aligned}(\bar{\mathbf{d}}(t),v)_{L^2} &+ \int_0^t \big(\bar{\mathbf{d}}(s), \mathcal{A}v\big)_{L^2} ds + \int_0^t \big(\tilde{B}(\bar{\mathbf{u}}(s),\bar{\mathbf{d}}(s)), v\big)_{L^2} ds \\ = (\mathbf{d}_0, v)_{L^2} &- \int_0^t \big(f(\bar{\mathbf{d}}(s)), v\big)_{L^2} ds + \int_0^t \int_{\mathbb{B}} \big(G(l,\bar{\mathbf{d}}(s)), v\big)_{L^2} \tilde{\bar{\eta}}_2(ds,dl) + \int_0^t \big(b(\bar{\mathbf{d}}(s)), v\big)_{L^2} ds.\end{aligned} \tag{3.4}$$

Definition 3.2. *It is said that the problem* (2.42)-(2.43) *has a* **strong solution** *if and only if for every stochastic basis* $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$ *with a filtration* $\mathbb{F} = \big\{\mathcal{F}_t\big\}_{t\ge 0}$ *and a time homogeneous Poisson random measure* $\bar{\eta}_1$ *on* $(Y,\mathscr{B}(Y))$ *with the intensity measure* ν_1 *and a time homogeneous Poisson random measure* $\bar{\eta}_2$ *on* $(\mathbb{B},\mathscr{B}(\mathbb{B}))$ *with the intensity measure* ν_2*, there exist an* $\mathbb{F}$*-progressively measurable process* $\mathbf{u} : [0,T] \times \Omega \to \mathrm{H}$ *with* $\mathbb{P}$*-a.e. paths*

$$\mathbf{u}(\cdot,\omega) \in \mathbb{D}([0,T];\mathrm{H}) \cap L^2(0,T;\mathrm{V}) \tag{3.5}$$

and progressively measurable process $\mathbf{d} : [0,T] \times \Omega \to H^1$ *with* $\mathbb{P}$*-a.e. paths*

$$\mathbf{d}(\cdot,\omega) \in \mathbb{D}([0,T];H^1) \cap L^2(0,T;D(\mathcal{A})) \tag{3.6}$$

such that for all $t \in [0,T]$ *and* $v \in \mathrm{V}$ *the following identity holds* $\mathbb{P}$*-a.s.*

$$\begin{aligned}(\mathbf{u}(t),v)_{\mathrm{H}} &+ \int_0^t \langle \mathbf{u}(s), \mathscr{A} v\rangle\, ds + \int_0^t \langle B(\mathbf{u}(s)), v\rangle\, ds \\ &+ \int_0^t \langle M(\mathbf{d}(s)), v\rangle\, ds = (\mathbf{u}_0, v)_{\mathrm{H}} + \int_0^t \int_Y \big(F(s,\mathbf{u}(s);y), v\big)_{\mathrm{H}} \tilde{\eta}(ds,dy).\end{aligned} \tag{3.7}$$

and for all $v \in D(\mathcal{A})$ *the following identity holds* $\mathbb{P}$*-a.s.*

$$\begin{aligned}(\mathbf{d}(t),v)_{L^2} &+ \int_0^t \big(\mathbf{d}(s), \mathcal{A}v\big)_{L^2} ds + \int_0^t \big(\tilde{B}(\mathbf{u}(s),\mathbf{d}(s)), v\big)_{L^2} ds \\ = (\mathbf{d}_0, v)_{L^2} &- \int_0^t \big(f(\mathbf{d}(s)), v\big)_{L^2} ds + \int_0^t \int_{\mathbb{B}} \big(G(l,\mathbf{d}(s)), v\big)_{L^2} \tilde{\bar{\eta}}_2(ds,dl) + \int_0^t \big(b(\mathbf{d}(s)), v\big)_{L^2} ds.\end{aligned} \tag{3.8}$$

The main result we are going to prove in this paper is as follows:

Theorem 3.3. *Let the Assumption 2.3 holds. Let* $\mathbf{n} = 2,3$ *and* $(\mathbf{u}_0,\mathbf{d}_0) \in \mathrm{H} \times H^1$*. Then there exists a martingale solution to the* (2.42)-(2.43) *in the sense of Definition 3.1, such that the following inequalities are satisfied*

$$\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]} \big|\bar{\mathbf{u}}(t)\big|^2_{\mathrm{H}} + \int_0^T \big|\bar{\mathbf{u}}(t)\big|^2_{\mathrm{V}}\, ds\Big] < \infty, \tag{3.9}$$

and

$$\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\bar{\mathbf{d}}(t)\big|^2_{H^1}+\int_0^T\big|\bar{\mathbf{d}}(t)\big|^2_{D(\mathcal{A})}\Big]<\infty. \tag{3.10}$$

Moreover, the pathwise uniqueness (see Definition 8.3) holds in two dimension. In particular, in this case, problem (2.42)-(2.43) *has a strong solution in the sense of Definition 3.2.*

Remark 3.4. *We will see later in Remark 8.2 that the nonlinear terms appearing in the equations* (3.2) *and* (3.4) *satisfy the following conditions depending on* $\mathbf{n}$*.*
If $\mathbf{n}=2$*, then*

$$\bar{\mathbb{E}}\int_0^T\Big[\big|B(\bar{\mathbf{u}}(s))\big|^2_{\mathrm{V}'}+\big|M(\bar{\mathbf{d}}(s))\big|^2_{\mathrm{V}'}+\big|\tilde{B}(\bar{\mathbf{u}}(s),\bar{\mathbf{d}}(s))\big|^2_{L^2}+\big|f(\bar{\mathbf{d}}(s))\big|^2_{L^2}\Big]\,ds<\infty, \tag{3.11}$$

while if $\mathbf{n}=3$ *then the above holds but with exponent* 2 *being repalced by* $\frac{4}{3}$*, i.e.*

$$\bar{\mathbb{E}}\int_0^T\Big[\big|B(\bar{\mathbf{u}}(s))\big|^{\frac{4}{3}}_{\mathrm{V}'}+\big|M(\bar{\mathbf{d}}(s))\big|^{\frac{4}{3}}_{\mathrm{V}'}+\big|\tilde{B}(\bar{\mathbf{u}}(s),\bar{\mathbf{d}}(s))\big|^{\frac{4}{3}}_{L^2}\Big]\,ds<\infty. \tag{3.12}$$

This can be compared with classical results for deterministic NSEs, see Lemma 3.3.4 and Theorem 3.3.3 in the monograph [48] *by Temam. See also Remark on p. 3179 in* [16].

Remark 3.5. *If* $\mathbf{n}=3$*, then*

$$\bar{\mathbb{E}}\int_0^T\big|f(\bar{\mathbf{d}}(s))\big|_{L^2}\,ds<\infty.$$

To prove this result, we observe from (2.22) *that for* $\mathbf{n}=3$*,* $N=1$*. Hence by* (2.25)*, there exists a constant* $c>0$ *such that*

$$|f(\bar{\mathbf{d}})|_{\mathbb{R}^3}\le c\,\big(1+|\bar{\mathbf{d}}|^3_{\mathbb{R}^3}\big),\quad \bar{\mathbf{d}}\in\mathbb{R}^3.$$

The rest follows from the embedding $H^1\hookrightarrow L^6$*.*

4. Compactness and Tightness Criterion

4.1. **Compactness Results.** Let $(\mathbb{M},\rho)$ be a complete separable metric space. Let $\mathbb{D}([0,T];\mathbb{M})$ be the space of all $\mathbb{M}$-valued càdlàg functions defined on $[0,T]$. This space is endowed with the Skorokhod topology.

A sequence $(\mathbf{u}_n)\subset\mathbb{D}([0,T];\mathbb{M})$ converges to $\mathbf{u}\in\mathbb{D}([0,T];\mathbb{M})$ iff there exists a sequence (μ_n) of homeomorphisms of $[0,T]$ such that μ_n tends to the identity uniformly on $[0,T]$ and $\mathbf{u}_n\circ\mu_n$ tends to $\mathbf{u}$ uniformly on $[0,T]$.

The topology is metrizable by the following metric ϑ_T

$$\vartheta_T(\mathbf{u},\mathbf{v}):=\inf_{\mu\in\sigma_T}\Big[\sup_{t\in[0,T]}\rho\big(\mathbf{u}(t),\mathbf{v}\circ\mu(t)\big)+\sup_{t\in[0,T]}|t-\mu(t)|+\sup_{s\neq t}\Big|\log\frac{\mu(t)-\mu(s)}{t-s}\Big|\Big],$$

where σ_T is the set of increasing homeomorphisms of $[0,T]$.
Moreover, $\big(\mathbb{D}([0,T];\mathbb{M}),\vartheta_T\big)$ is a complete metric space.

Remark 4.1. *It follows from the above definition, that if* $(\mathbf{u}_n)$ *converges to* $\mathbf{u}$ *in* $\mathbb{D}([0,T];\mathbb{M})$*, then*

$$\mathbf{u}_n(0)\to\mathbf{u}(0)\ \textit{in}\ \mathbb{M}.$$

Definition 4.2. *Let* $\mathbf{u}\in\mathbb{D}([0,T];\mathbb{M})$ *and let* $\delta>0$ *be given. A* ***modulus*** *of continuity is defined by*

$$\mathcal{W}_{[0,T],\mathbb{M}}(\mathbf{u};\delta):=\inf_{\Pi_\delta}\max_{t_i\in\tilde{\omega}}\sup_{t_i\le s<t<t_{i+1}\le T}\rho\big(\mathbf{u}(t),\mathbf{u}(s)\big), \tag{4.1}$$

where Π_δ *is the set of all increasing sequences* $\tilde{\omega}=\{0=t_0<t_1<\cdots<t_n=T\}$ *with the following property*

$$t_{i+1}-t_i\ge\delta,\quad i=0,1,\cdots,n-1.$$

Analogous to the Arzelà-Ascoli Theorem for the space of continuous functions, we have the following criterion for the relative compactness of a subset of the space $\mathbb{D}([0,T];\mathbb{M})$.

Theorem 4.3. *A set $B\subset\mathbb{D}([0,T];\mathbb{M})$ has precompact iff it satisfies the following two conditions:*

(1) there exists a dense subset $J\subset[0,T]$ such that for every $t\in J$ the set $\{\mathbf{u}(t),\mathbf{u}\in B\}$ has compact closure in $\mathbb{M}$.
(2) $\lim_{\delta\to 0}\sup_{\mathbf{u}\in B}\mathcal{W}_{[0,T]}(\mathbf{u};\delta)=0$.

Proof. For details see [38]. □

Let us consider the following functional spaces.

$$\begin{aligned}
\mathbb{D}([0,T];\mathrm{V}') \;&:=\; \text{the space of càdlàg functions } \mathbf{u}:[0,T]\to\mathrm{V}' \text{ with the topology}\\
&\qquad \mathcal{T}_1 \text{ induced by the Skorokhod metric } \delta_T,\\
\mathbb{D}([0,T];L^2) \;&:=\; \text{the space of càdlàg functions } \mathbf{d}:[0,T]\to L^2 \text{ with the topology } \mathcal{T}_1',\\
L^2_w(0,T;\mathrm{V}) \;&:=\; \text{the space } L^2(0,T;\mathrm{V}) \text{ with the weak topology } \mathcal{T}_2,\\
L^2_w(0,T;D(\mathcal{A})) \;&:=\; \text{the space } L^2(0,T;D(\mathcal{A})) \text{ with the weak topology } \mathcal{T}_2',\\
L^2(0,T;\mathrm{H}) \;&:=\; \text{the space of measurable functions } \mathbf{u}:[0,T]\to\mathrm{H} \text{ with the}\\
&\qquad \text{topology } \mathcal{T}_3,\\
L^2(0,T;H^1) \;&:=\; \text{the space of measurable functions } \mathbf{d}:[0,T]\to H^1 \text{ with the}\\
&\qquad \text{topology } \mathcal{T}_3'.
\end{aligned}$$

Let H_w denote the Hilbert space H endowed with the weak topology. Let us consider the space

$$\begin{aligned}
\mathbb{D}([0,T];\mathrm{H}_w) \;&:=\; \text{the space of weakly càdlàg functions } \mathbf{u}:[0,T]\to\mathrm{H} \text{ with the}\\
&\qquad \text{weakest topology } \mathcal{T}_4 \text{ such that for all } h\in\mathrm{H} \text{ the mappings}\\
&\mathbb{D}([0,T];\mathrm{H}_w)\ni\mathbf{u}\mapsto(\mathbf{u}(\cdot),h)_{\mathrm{H}}\in\mathbb{D}([0,T];\mathbb{R}) \text{ are continuous.}
\end{aligned}$$

In particular, $\mathbf{u}_n\to\mathbf{u}$ in $\mathbb{D}([0,T];\mathrm{H}_w)$ iff for all $h\in\mathrm{H}$:

$$(\mathbf{u}_n(\cdot),h)_{\mathrm{H}}\to(\mathbf{u}(\cdot),h)_{\mathrm{H}} \text{ in the space } \mathbb{D}([0,T];\mathbb{R}).$$

Similarly we define $\mathbb{D}([0,T];H^1_w)$ with the topology $\mathcal{T}_4'$.

The following two results are due to [40], where the details of the proof can be found.

Theorem 4.4. (*Compactness Criterion for* $\mathbf{u}$) *Let us consider the space*

$$\mathcal{Z}_{T,1}=L^2_w(0,T;\mathrm{V})\cap L^2(0,T;\mathrm{H})\cap\mathbb{D}([0,T];\mathrm{V}')\cap\mathbb{D}([0,T];\mathrm{H}_w) \tag{4.2}$$

and $\mathscr{T}^1$ be the supremum of the corresponding topologies. Then a set $\mathcal{K}_1\subset\mathcal{Z}_{T,1}$ is $\mathscr{T}^1$-relatively compact if the following three conditions hold

(1) $\sup_{\mathbf{u}\in\mathcal{K}_1}\sup_{s\in[0,T]}|\mathbf{u}(s)|_{\mathrm{H}}<\infty$,
(2) $\sup_{\mathbf{u}\in\mathcal{K}_1}\int_0^T|\mathbf{u}(s)|^2_{\mathrm{V}}\,ds<\infty$, *i.e., $\mathcal{K}_1$ is bounded in* $L^2(0,T;\mathrm{V})$,
(3) $\lim_{\delta\to 0}\sup_{\mathbf{u}\in\mathcal{K}_1}\mathcal{W}_{[0,T],\mathrm{V}'}(\mathbf{u};\delta)=0$.

Theorem 4.5. (*Compactness Criterion for* $\mathbf{d}$) *Let us consider the space*

$$\mathcal{Z}_{T,2}=L^2_w(0,T;D(\mathcal{A}))\cap L^2(0,T;H^1)\cap\mathbb{D}([0,T];L^2)\cap\mathbb{D}([0,T];H^1_w) \tag{4.3}$$

and let $\mathscr{T}^2$ be the supremum of the corresponding topologies. Then a set $\mathcal{K}_2\subset\mathcal{Z}_{T,2}$ is $\mathscr{T}^2$-relatively compact if the following three conditions hold

(1) $\sup_{\mathbf{d}\in\mathcal{K}_2}\sup_{s\in[0,T]}|\mathbf{d}(s)|_{H^1}<\infty$,
(2) $\sup_{\mathbf{d}\in\mathcal{K}_2}\int_0^T|\mathbf{d}(s)|^2_{D(\mathcal{A})}\,ds<\infty$, *i.e., $\mathcal{K}_2$ is bounded in* $L^2(0,T;D(\mathcal{A}))$,
(3) $\lim_{\delta\to 0}\sup_{\mathbf{d}\in\mathcal{K}_2}\mathcal{W}_{[0,T],L^2}(\mathbf{d};\delta)=0$.

Note that $\mathcal{Z}_{T,1}$ and $\mathcal{Z}_{T,2}$ are not Polish spaces.

4.2. **The Aldous Condition.** Here $(\mathbb{M},\rho)$ is a complete, separable metric space. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a probability space with usual hypotheses. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-adapted and $\mathbb{M}$-valued processes.
The following definition is borrowed from [28]. The notation $\mathcal{W}_{[0,T],\mathbb{M}}$ is defined in (4.1).

Definition 4.6. *Let (X_n) be a sequence of $\mathbb{M}$-valued random variables. The sequence of laws of these processes is* **Tight** *if and only if*

[T] $\forall\,\varepsilon>0\ \ \forall\,\eta>0\ \ \exists\,\delta>0:$

$$\sup_{n\in\mathbb{N}}\mathbb{P}\big\{\mathcal{W}_{[0,T],\mathbb{M}}(X_n,\delta)>\eta\big\}\le\varepsilon.$$

Definition 4.7. *A sequence $(X_n)_{n\in\mathbb{N}}$ of $\mathbb{M}$-valued random variables satisfies the* **Aldous condition** *if and only if*

[A] $\forall\,\epsilon>0\ \ \forall\,\eta>0\ \ \exists\,\delta>0$ *such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n\le T$ one has*

$$\sup_{n\in\mathbb{N}}\sup_{0\le\theta\le\delta}\mathbb{P}\big\{\rho(X_n(\tau_n+\theta),X_n(\tau_n))\ge\eta\big\}\le\epsilon.$$

Lemma 4.8. *Condition* **[A]** *implies condition* **[T]**.

Proof. See Theorem 2.2.2 of [28]. □

Lemma 4.9. *Let $(E,|\cdot|_E)$ be a separable Banach space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of E-valued random variables. Assume that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n\le T$ and for every $n\in\mathbb{N}$ and $\theta\ge0$ the following condition holds*

$$\mathbb{E}\big[|X_n(\tau_n+\theta)-X_n(\tau_n)|_E^{\alpha}\big]\le C\theta^{\beta}\tag{4.4}$$

for some $\alpha,\beta>0$ and some constant $C>0$. Then the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies the **Aldous condtion** *in the space E.*

Proof. See [40] for the proof. □

In the view of Theorem 4.4, to show the law of $\mathbf{u}_n$ is tight, we need the following result

Corollary 4.10. *Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg $\mathbb{F}$-adapted, V'-valued processes such that*

(a')

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{s\in[0,T]}|\mathbf{u}_n(s)|_{\mathrm{H}}\Big]<\infty,$$

(b')

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\int_0^T|\mathbf{u}_n(s)|_{\mathrm{V}}^2\,ds\Big]<\infty,$$

(c') $(\mathbf{u}_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition in V'.

Let $\mathbb{P}_n^1$ be the law of $\mathbf{u}_n$ on $\mathcal{Z}_{T,1}$. Then for every $\epsilon>0$ there exists a compact subset K_ϵ^1 of $\mathcal{Z}_{T,1}$ such that

$$\mathbb{P}_n^1(K_\epsilon^1)\ge1-\epsilon.$$

Similarly in the view of Theorem 4.5, to show the law of $\mathbf{d}_n$ is tight, we need the following result

Corollary 4.11. *Let $(\mathbf{d}_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg $\mathbb{F}$-adapted, L^2-valued processes such that*

(a'')

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{s\in[0,T]}|\mathbf{d}_n(s)|_{H^1}\Big]<\infty,$$

(b'')

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\int_0^T|\mathbf{d}_n(s)|_{D(\mathcal{A})}^2\,ds\Big]<\infty,$$

(c'') $(\mathbf{d}_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition in L^2.

Let $\mathbb{P}_n^2$ be the law of $\mathbf{d}_n$ on $\mathcal{Z}_{T,2}$. Then for every $\epsilon > 0$ there exists a compact subset K_ϵ^2 of $\mathcal{Z}_{T,2}$ such that

$$\mathbb{P}_n^2(K_\epsilon^2) \geq 1 - \epsilon.$$

4.3. **Skorokhod Embedding Theorems.** We have the following Jakubowski version of the Skorokhod theorem due to [27].

Theorem 4.12. *Let $(\mathscr{G}, \tau)$ be a topological space such that there exists a sequence (g_m) of continuous functions $g_m : \mathscr{G} \to \mathbb{R}$ that separates points of $\mathscr{G}$. Let (Z_n) be a sequence of $\mathscr{G}$-valued random variables. Suppose that for every $\epsilon > 0$ there exists a compact subset $G_\epsilon \subset \mathscr{G}$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\{Z_n \in G_\epsilon\}) > 1 - \epsilon.$$

Then there exists a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$, a sequence $(X_k)_{k\in\mathbb{N}}$ of $\mathscr{G}$-valued random variables and an $\mathscr{G}$-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathscr{L}aw(Z_{n_k}) = \mathscr{L}aw(X_k), \qquad k = 1, 2, \cdots$$

and for all $\omega \in \Omega$,

$$X_k(\omega) \xrightarrow{\tau} X(\omega) \quad k \to \infty.$$

We will use the following version of the Skorokhod embedding theorem in our paper (see Corollary 5.3 of [39]), which is similar to the version stated in [40] and [7].

Theorem 4.13. *Let $\mathscr{X}_1$ be a separable complete metric space and let $\mathscr{X}_2$ be a topological space such that there exists a sequence $\{f_l\}_{l\in\mathbb{N}}$ of continuous functions $f_l : \mathscr{X}_2 \to \mathbb{R}$ separating points of $\mathscr{X}_2$. Let $\mathscr{X} := \mathscr{X}_1 \times \mathscr{X}_2$ with the Tychonoff topology induced by the projections*

$$\pi_i : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathscr{X}_i, \qquad i = 1, 2.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{X}_n : \Omega \to \mathscr{X}_1 \times \mathscr{X}_2$, $n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathscr{L}(\mathcal{X}_n), n \in \mathbb{N}\}$ is tight on $\mathscr{X}_1 \times \mathscr{X}_2$. Finally let us assume that there exists a random variable $\rho : \Omega \to \mathscr{X}_1$ such that $\mathscr{L}(\pi_1 \circ \mathcal{X}_n) = \mathscr{L}(\rho)$ for all $n \in \mathbb{N}$.

Then there exists a subsequence $\left(\mathcal{X}_{n_k}\right)_{k\in\mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a family of $\mathscr{X}_1 \times \mathscr{X}_2$-valued random variables $\{\bar{\mathcal{X}}_k, k \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and a random variable $\mathcal{X}_* : \bar{\Omega} \to \mathscr{X}_1 \times \mathscr{X}_2$ such that

(1) $\mathscr{L}(\bar{\mathcal{X}}_k) = \mathscr{L}(\mathcal{X}_{n_k})$ for all $k \in \mathbb{N}$;
(2) $\bar{\mathcal{X}}_k \to \mathcal{X}_*$ in $\mathscr{X}_1 \times \mathscr{X}_2$ a.s. as $k \to \infty$;
(3) $\pi_1 \circ \bar{\mathcal{X}}_k(\bar{\omega}) = \pi_1 \circ \bar{\mathcal{X}}_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$.

Proof. For proof see Appendix B of [40]. □

5. Energy Estimates

In order to prove the existence of a martingale solution of (2.42)-(2.43), we will employ the Galerkin method. We will disuss about the existence of this approximated system. Then we will give apriori estimates for the approximating sequences $\mathbf{u}_n$ and $\mathbf{d}_n$. The results stated in below Subsections hold for both dimensions $\mathbf{n} = 2$ and 3.

5.1. **The Faedo-Galerkin Approximation.** Our proof of existence of martingale solution depends on the Galerkin approximation. Let $\{\varrho_i\}_{i=1}^\infty$ be the orthonormal basis of H composed of eigenfunctions of the stokes operator $\mathscr{A}$. Let $\{\varsigma_i\}_{i=1}^\infty$ be the orthonormal basis of L^2 consisting of the eigenfunctions of the Neumann Laplacian $\mathcal{A}$. Let us define the following finite dimensional spaces for any $n \in \mathbb{N}$

$$\mathrm{H}_n := Linspan\{\varrho_1, \ldots, \varrho_n\},$$
$$\mathbb{L}_n := Linspan\{\varsigma_1, \ldots, \varsigma_n\}.$$

Our aim is to derive uniform estimates for the solution of the projection of (2.42)-(2.43) onto the finite dimensional space $\mathrm{H}_n \times \mathbb{L}_n$, i.e., its Galerkin approximation. For this let us denote by P_n the projection from H onto H_n and $\tilde{P}_n$ be the projection from L^2 onto $\mathbb{L}_n$. Since the operators

$\tilde{P}_n, n \in \mathbb{N}$ are not in general bounded from $L^p(\mathbb{O})$ to $L^p(\mathbb{O})$ for $p \geq 2$, we require another sequence of finite dimensional operators derived from the Littlewood-Paley decomposition, introduced in [10, 11], to cut-off the noise terms in the director field equation.

Proposition 5.1. *There exists a sequence $(S_n)_{n\in\mathbb{N}}$ of self-adjoint operators $S_n : L^2 \to \mathbb{L}_n$ for $n \in \mathbb{N}$ with $S_n\psi \to \psi$ in H^1 for $n \to \infty$ and $\psi \in H^1$ and the uniform norm estimates*

$$\sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^2)} \leq 1, \quad \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(H^1)} \leq 1, \quad \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^p)} < \infty. \tag{5.1}$$

A proof of this result can be found in Proposition 5.2 of [10] of the first named author and an alternative proof in Proposition 5.1 of [11] of the first two authors, and the proof is based on a spectral multiplier theorem by Kunstmann and Uhl [30] for operators with generalised Gaussian bounds.

We consider the following mappings:

$$\begin{aligned}
&B_n : \mathrm{H}_n \ni \mathbf{u} \mapsto P_n B(\mathbf{u},\mathbf{u}) \in \mathrm{H}_n,\\
&M_n : \mathbb{L}_n \ni \mathbf{d} \mapsto P_n M(\mathbf{d}) \in \mathrm{H}_n,\\
&f_n : \mathbb{L}_n \ni \mathbf{d} \mapsto \tilde{P}_n f(\mathbf{d}) \in \mathbb{L}_n,\\
&\tilde{B}_n : \mathrm{H}_n \times \mathbb{L}_n \ni (\mathbf{u},\mathbf{d}) \mapsto \tilde{P}_n \tilde{B}(\mathbf{u},\mathbf{d}) \in \mathbb{L}_n,\\
&g_{i_n} \ : \ \mathbb{L}_n \ni \mathbf{d} \mapsto S_n(\mathbf{d} \times \mathbf{h}_i) S_n \in \mathbb{L}_n.
\end{aligned}$$

Let $P_n\mathbf{u}_0 = \mathbf{u}_n(0) := \mathbf{u}_{0n}$ and $\tilde{P}_n\mathbf{d}_0 = \mathbf{d}_n(0) := \mathbf{d}_{0n}$. Recall that $\mathbb{B} := \{l \in \mathbb{R}^N : |l| \leq 1\}$. For $l \in \mathbb{B}$, let $\Phi_n(t,l,\mathbf{d})$ be a flow on $\mathbb{L}_n$ corresponding to the vector field $\sum_{i=1}^N l_i g_{i_n}$, i.e.

$$\begin{cases} \dfrac{d\Phi_n}{dt}(t,l,\mathbf{d}) = \sum_{i=1}^N l_i g_{i_n}(\Phi_n(t,l,\mathbf{d})), \ t \geq 0,\\ \Phi_n(0,l,\mathbf{d}) = \mathbf{d} \in \mathbb{L}_n. \end{cases} \tag{5.2}$$

For $\mathbf{d} \in \mathbb{L}_n$, we denote

$$G_n(l,\mathbf{d}) := \Phi_n(l,\mathbf{d}) - \mathbf{d}, \tag{5.3}$$

$$K_n(l,\mathbf{d}) := \Phi_n(l,\mathbf{d}) - \mathbf{d} - \sum_{i=1}^N l_i g_{i_n}(\mathbf{d}), \tag{5.4}$$

and

$$b_n(\mathbf{d}) := \int_{\mathbb{B}} \left[\Phi_n(l,\mathbf{d}) - \mathbf{d} - \sum_{i=1}^N l_i g_{i_n}(\mathbf{d})\right] \nu_2(dl) := \int_{\mathbb{B}} K_n(l,\mathbf{d})\, \nu_2(dl). \tag{5.5}$$

So the Galerkin approximation of the problem (2.42)-(2.43) is

$$d\mathbf{u}_n(t) + \left[\mathscr{A}\mathbf{u}_n(t) + B_n(\mathbf{u}_n(t)) + M_n(\mathbf{d}_n(t))\right]dt = \int_Y P_n F(t,\mathbf{u}_n(t-);y)\, \tilde{\eta}_1(dt,dy), \quad t \geq 0, \tag{5.6}$$

$$\begin{aligned} d\mathbf{d}_n(t) + &\left[\mathcal{A}\mathbf{d}_n(t) + \tilde{B}_n(\mathbf{u}_n(t),\mathbf{d}_n(t)) + f_n(\mathbf{d}_n(t))\right]dt\\ &= \int_{\mathbb{B}} G_n(l,\mathbf{d}_n(t-))\, \tilde{\eta}_2(dt,dl) + b_n(\mathbf{d}_n(t))\, dt, \quad t \geq 0. \end{aligned} \tag{5.7}$$

The equations (5.6)-(5.7) with initial conditions $\mathbf{u}_n(0) = \mathbf{u}_{0n}$ and $\mathbf{d}_n(0) = \mathbf{d}_{0n}$, form a system of Stochastic Differential Equations with locally Lipschitz coefficients. See [8] for details.

Here we have followed the standard finite dimensional convention (see [3, 26, 42]) of using left limit $t-$ in the stochastic integral with respect to time homogeneous compensated Poisson random measure.

Now consider the following mappings

$$B_n'(\mathbf{u}) := P_n B(\chi_n^1(\mathbf{u}),\mathbf{u}), \quad \mathbf{u}\in \mathrm{H}_n,$$
$$M_n'(\mathbf{d}) := P_n M(\chi_n^2(\mathbf{d}),\mathbf{d}), \quad \mathbf{d}\in \mathbb{L}_n,$$
$$\tilde{B}_n'(\mathbf{u},\mathbf{d}) := P_n \tilde{B}(\chi_n^1(\mathbf{u}),\mathbf{d}), \quad \mathbf{u}\in \mathrm{H}_n, \mathbf{d}\in \mathbb{L}_n,$$

where $\chi_n^1 : \mathrm{H}\to \mathrm{H}$ is defined by $\chi_n^1(\mathbf{u}) = \theta_n(|\mathbf{u}|_{\mathrm{V}'})\mathbf{u}$, and $\chi_n^2 : L^2 \to L^2$ is defined by $\chi_n^2(\mathbf{d}) = \theta_n(|\mathbf{d}|_{L^2})\mathbf{d}$, where $\theta_n : \mathbb{R}\to[0,1]$ of class $\mathscr{C}^\infty$ such that

$$\theta_n(r) = 1 \quad if \quad r\le n \quad and \quad \theta_n(r)=0 \quad if \quad r\ge n+1.$$

The mappings $B_n' : \mathrm{H}_n \to \mathrm{H}_n$, $M_n' : \mathbb{L}_n\to \mathrm{H}_n$, and $\tilde{B}_n' : \mathrm{H}_n\times\mathbb{L}_n \to \mathbb{L}_n$ are well defined and are globally Lipschitz.

Let us consider the Faedo-Galerkin approximation in the space H_n and $\mathbb{L}_n$,

$$d\mathbf{u}_n(t) + \big[\mathscr{A}\mathbf{u}_n(t) + B_n'(\mathbf{u}_n(t)) + M_n'(\mathbf{d}_n(t))\big]dt = \int_Y P_n F(t,\mathbf{u}_n(t-);y)\tilde{\eta}(dt,dy), \tag{5.8}$$

$$\begin{aligned} d\mathbf{d}_n(t) + \big[\mathcal{A}\mathbf{d}_n(t) + \tilde{B}_n'(\mathbf{u}_n(t),\mathbf{d}_n(t)) + f_n(\mathbf{d}_n(t))\big]dt \\ = \int_{\mathbb{B}} G_n(l,\mathbf{d}_n(t-))\,\tilde{\eta}_2(dt,dl) + b_n(\mathbf{d}_n(t))\,dt, \quad t\ge 0. \end{aligned} \tag{5.9}$$

Before we embark on studying the properties of solutions to the Galerkin system, let us list the fundamental properties of the vector fields appearing in it.

Lemma 5.2. *Let*

$$\mathcal{R}_n = \mathcal{R}_n(l) := \sum_{i=1}^N l_i g_{i_n}, \quad n\in\mathbb{N}, \quad l\in\mathbb{R}^N.$$

Then, we have

$$\|\mathcal{R}_n\|_{\mathcal{L}(L^2)} \le |l|\|g\|_{\mathcal{L}(L^2)}, \quad \|\mathcal{R}_n\|_{\mathcal{L}(H^1)} \le |l|\|g\|_{\mathcal{L}(H^1)}, \quad \|\mathcal{R}_n\|_{\mathcal{L}(L^p)} \le |l|\|g\|_{\mathcal{L}(L^p)} \sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^p)}.$$

Moreover,

$$\|e^{t\mathcal{R}_n}\|_{\mathcal{L}(H^1)} \le e^{|t||l|\|g\|_{\mathcal{L}(H^1)}}, \qquad \|e^{\mathcal{R}_n}\|_{\mathcal{L}(L^p)} \le e^{|t||l|\|g\|_{\mathcal{L}(L^p)}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^p)}}, \qquad t\in\mathbb{R}.$$

Lemma 5.3. *(1) There exists $M_1>0$ such that for any $l\in\mathbb{B}$ and $\mathbf{d}\in\mathbb{L}_n$,*

$$|G_n(l,\mathbf{d})|_{\mathbb{L}_n} \le M_1|l|_{\mathbb{R}^N}(1+|\mathbf{d}|_{\mathbb{L}_n}). \tag{5.10}$$

(2) There exists $M_2>0$ such that for any $l\in\mathbb{B}$ and $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{L}_n$,

$$|G_n(l,\mathbf{d}_2) - G_n(l,\mathbf{d}_1)|_{\mathbb{L}_n} \le M_2|l|_{\mathbb{R}^N}|\mathbf{d}_2-\mathbf{d}_1|_{\mathbb{L}_n}. \tag{5.11}$$

(3) There exists $M_3>0$ such that for any $l\in\mathbb{B}$ and $\mathbf{d}\in\mathbb{L}_n$,

$$|K_n(l,\mathbf{d})|_{\mathbb{L}_n} \le M_3|l|^2_{\mathbb{R}^N}(1+|\mathbf{d}|_{\mathbb{L}_n}). \tag{5.12}$$

(4) There exists $M_4>0$ such that for any $l\in\mathbb{B}$ and $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{L}_n$,

$$|K_n(l,\mathbf{d}_2) - K_n(l,\mathbf{d}_1)|_{\mathbb{L}_n} \le M_4|l|^2_{\mathbb{R}^N}|\mathbf{d}_2-\mathbf{d}_1|_{\mathbb{L}_n}. \tag{5.13}$$

Proof. For details of the proof see Lemma 3.3 in [13]. Here the constants $M_i, i=1,2,3,4$ depend upon the bound of $\|g\|_{\mathcal{L}(L^2)}$. □

Lemma 5.4. *There exists a constant $C_1>0$ such that for any $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{L}_n$,*

$$\big|b_n(\mathbf{d}_2) - b_n(\mathbf{d}_1)\big|^2_{\mathbb{L}_n} + \int_{\mathbb{B}}\big|G_n(l,\mathbf{d}_2) - G_n(l,\mathbf{d}_1)\big|^2_{\mathbb{L}_n}\,\nu_2(dl) \le C_1\big|\mathbf{d}_2-\mathbf{d}_1\big|^2_{\mathbb{L}_n}. \tag{5.14}$$

Proof. We refer to Lemma 3.4 in [13] for the proof. □

Lemma 5.5. *There exists a constant $C_2>0$ such that for any $\mathbf{d}\in\mathbb{L}_n$,*

$$\big|b_n(\mathbf{d})\big|^2_{\mathbb{L}_n}+\int_{\mathbb{B}}\big|G_n(l,\mathbf{d})\big|^2_{\mathbb{L}_n}\,\nu_2(dl)\le C_2|\mathbf{d}|^2_{\mathbb{L}_n}. \tag{5.15}$$

Proof. For the proof see Lemma 3.5 in [13]. □

Since all relevant maps are globally Lipschitz, we have the following standard result, see e.g. [2] for references.

Lemma 5.6. *For each $n\in\mathbb{N}$, there exists a unique global, $\mathbb{F}$-progressively measurable, $\mathrm{H}_n\times\mathbb{L}_n$ -valued càdlàg processes $(\mathbf{u}_n,\mathbf{d}_n)$ satisfying the Galerkin approximation equations* (5.8)-(5.9).

It follows easily, see for instance [2], that for each $n\in\mathbb{N}$, the equations (5.6)-(5.7) has a unique local maximal solution. However, by a combination of the proof of [2, Theorem 3.1] with the proofs of Propositions 5.9 and 5.12 below (in the case $p=2$), we infer the following important result.

Corollary 5.7. *For each $n\in\mathbb{N}$, the problem* (5.6)-(5.7) *has a unique global solution.*

5.2. **A Priori Estimates.** The processes $\big(\mathbf{u}_n\big)_{n\in\mathbb{N}}$ and $\big(\mathbf{d}_n\big)_{n\in\mathbb{N}}$ satisfy the following estimates.

Lemma 5.8. *Assume that $\mathbf{h}_i\in L^\infty$ and $T\in(0,\infty)$. Then for each $n=1,2,\dots$ and every $t\in[0,T]$,*

$$\big|\Phi_n(t,l,\mathbf{d})\big|_{L^2}=|\mathbf{d}|_{\mathbb{L}_n}. \tag{5.16}$$

The proof is straightforward and depends upon the fact that $\mathcal{R}_n:=S_n\mathcal{R}S_n$ is self-adjoint, and therefore omitted.

Proposition 5.9. *Assume that $\mathbf{h}_i\in L^\infty$ and $T\in(0,\infty)$. Then for any $p\ge 2$, there exists a positive constant C, depending on p such that*

$$\sup_{n\in\mathbb{N}}\left(\mathbb{E}\left[\sup_{s\in[0,T]}\big|\mathbf{d}_n(s)\big|^p_{L^2}+\int_0^T\big|\mathbf{d}_n(s)\big|^{p-2}_{L^2}\big(|\nabla\mathbf{d}_n(s)|^2_{L^2}+|\mathbf{d}_n(s)|^{2N+2}_{L^{2N+2}}\big)\,ds\right]\right)\le\mathbb{E}\ \mathfrak{C}_0(p,T).$$

where $\mathfrak{C}_0(p,T):=|\mathbf{d}_{0n}|^p\big(1+CTe^{CT}\big)$, which is independent of $n\in\mathbb{N}$ and $s\in[0,T]$.

Proof. For all $n\in\mathbb{N}$ and all $R>0$ let us define a random variable

$$\tau_n^R:=\inf\{t\ge 0:|\mathbf{d}_n(t)|_{L^2}\ge R\}. \tag{5.17}$$

It is a stopping time, since the processes $(\mathbf{d}_n(t))_{t\in[0,T]}$ is $\mathbb{F}$-adapted and right-continuous. Moreover $(\tau_n^R\wedge T)\uparrow T$, as $R\uparrow\infty$, $\mathbb{P}$-a.s.

Let us fix $T>0$ and $p\ge 2$. Let $\psi:\mathbb{L}_n\to\mathbb{R}$ be the mapping defined by

$$\psi(\mathbf{d}):=\frac{1}{p}|\mathbf{d}|^p,\qquad \mathbf{d}\in\mathbb{L}_n. \tag{5.18}$$

The first Fréchet derivative of ψ is

$$\psi'(\mathbf{d})[\mathbf{g}]=|\mathbf{d}|^{p-2}\langle\mathbf{d},\mathbf{g}\rangle,\ \ \mathbf{d}\in\mathbb{L}_n.$$

So applying this to the sequence $\mathbf{d}_n(t)$ we get,

$$d\psi(\mathbf{d}_n(t))=\psi'(\mathbf{d}_n)[d\mathbf{d}_n(t)]=|\mathbf{d}_n|^{p-2}\langle d\mathbf{d}_n(t),\mathbf{d}_n(t)\rangle,\ t\ge 0. \tag{5.19}$$

Applying the Itô formula to the process $\psi(\mathbf{d}_n(t))$, we obtain

$$\begin{aligned}&\psi(\mathbf{d}_n(t))-\psi(\mathbf{d}_n(0))\\&=-\int_0^t\psi'(\mathbf{d}_n(s))\big[\mathcal{A}\mathbf{d}_n(s)+\tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s))+f_n(\mathbf{d}_n(s))\big]\,ds\\&\quad+\int_0^t\int_{\mathbb{B}}\left[\psi(\Phi_n(l,\mathbf{d}_n(s-)))-\psi(\mathbf{d}_n(s-))\right]\tilde{\eta}_2(ds,dl)\\&\quad+\int_0^t\int_{\mathbb{B}}\left[\psi(\Phi_n(l,\mathbf{d}_n(s)))-\psi(\mathbf{d}_n(s))-\sum_{i=1}^N l_i\langle\psi'(\mathbf{d}_n(s)),g_{i_n}(\mathbf{d}_n(s))\rangle_{L^2}\right]\nu_2(dl)ds,\ t\geq 0.\end{aligned}\tag{5.20}$$

From (5.19) we have

$$\begin{aligned}&\psi'(\mathbf{d}_n(s))\big[\mathcal{A}\mathbf{d}_n(s)+\tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s))+f_n(\mathbf{d}_n(s))\big]\\&=|\mathbf{d}_n|^{p-2}\big\langle-\mathcal{A}\mathbf{d}_n(s)-\tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s))-f_n(\mathbf{d}_n(s)),\mathbf{d}_n(s)\big\rangle\\&=|\mathbf{d}_n|^{p-2}\Big(|\nabla\mathbf{d}_n(s)|^2_{L^2}-\big\langle f_n(\mathbf{d}_n(s)),\mathbf{d}_n(s)\big\rangle\Big),\ s\geq 0.\end{aligned}\tag{5.21}$$

From Lemma 5.8, taking the power p of both sides of (5.16) we get,

$$\psi(\Phi_n(l,\mathbf{d}_n(s-)))-\psi(\mathbf{d}_n(s-))=0.\tag{5.22}$$

The above equality is important for our further analysis. We observe this equality (5.22), implies that the martingale part of $\psi(\mathbf{d}_n(t))$ in (5.20) is zero, which is due to our Marcus force. Hence by Lemma 2.8 we obtain from (5.20) that for $s\geq 0$,

$$\big|\psi(\Phi_n(l,\mathbf{d}_n(s)))-\psi(\mathbf{d}_n(s))-\psi'(\mathbf{d}_n(s))\mathcal{R}\mathbf{d}_n(s)\big|\leq\mathscr{N}p^2\,|l|^2_{\mathbb{R}^N}\,\|g\|_{\mathcal{L}(L^2)}|\mathbf{d}_n(s)|^p_{L^2}.\tag{5.23}$$

Let us observe that by (5.21), (5.22) and (5.23), we further write inequality (5.20) as

$$\begin{aligned}\psi(\mathbf{d}_n(t))\leq\psi(\mathbf{d}_0)&-\int_0^t|\mathbf{d}_n|^{p-2}|\nabla\mathbf{d}_n|^2_{L^2}\,ds-\int_0^t|\mathbf{d}_n|^{p-2}\big\langle f_n(\mathbf{d}_n(s)),\mathbf{d}_n(s)\big\rangle\,ds\\&+C\int_0^t\int_{\mathbb{B}}|\mathbf{d}_n(s)|^p\ |l|^2_{\mathbb{R}^N}\,\nu_2(dl)ds,\quad t\geq 0.\end{aligned}\tag{5.24}$$

Thus by (5.18), (5.24) and taking integration over all $t\in[0,T]$,

$$\begin{aligned}&|\mathbf{d}_n(t)|^p+\int_0^t|\mathbf{d}_n(s)|^{p-2}|\nabla\mathbf{d}_n(s)|^2_{L^2}\,ds+\int_0^t|\mathbf{d}_n(s)|^{p-2}\big\langle f_n(\mathbf{d}_n(s)),\mathbf{d}_n(s)\big\rangle\,ds\\&\leq|\mathbf{d}_0|^p+C\Big(\int_0^t|\mathbf{d}_n(s)|^p\,ds\Big)\Big(\int_{\mathbb{B}}|l|^2_{\mathbb{R}^N}\,\nu_2(dl)\Big),\quad t\geq 0.\end{aligned}\tag{5.25}$$

Since by Assumption 2.3 we have

$$\begin{aligned}\big\langle f_n(\mathbf{d}_n),\mathbf{d}_n\big\rangle=\big\langle\tilde{f}(|\mathbf{d}_n|^2)\mathbf{d}_n,\mathbf{d}_n\big\rangle&=\int_{\mathbb{O}}\tilde{f}(|\mathbf{d}_n(x)|^2)|\mathbf{d}_n(x)|^2\,dx\\&=\int_{\mathbb{O}}\sum_{k=0}^N a_k(|\mathbf{d}_n(x)|^2)^{k+1}\,dx=\sum_{l=1}^{N+1}a_{l-1}\int_{\mathbb{O}}|\mathbf{d}_n(x)|^{2l}\,dx,\end{aligned}\tag{5.26}$$

and by Lemma 8.7 of [14] we infer

$$\int_{\mathbb{O}}|\mathbf{d}_n(x)|^{2N+2}\,dx-C\int_{\mathbb{O}}|\mathbf{d}_n(x)|^2\,dx\leq\big\langle f(\mathbf{d}_n),\mathbf{d}_n\big\rangle\text{ for }\mathbf{d}_n\in L^{2N+2}(\mathbb{O}),\tag{5.27}$$

we infer by (5.25) that

$$\begin{aligned}&|\mathbf{d}_n(t)|^p+\int_0^t|\mathbf{d}_n(s)|^{p-2}|\nabla\mathbf{d}_n(s)|^2_{L^2}\,ds+\int_0^t|\mathbf{d}_n(s)|^{p-2}\big(|\mathbf{d}_n(s)|^{2N+2}_{L^{2N+2}}-C|\mathbf{d}_n(s)|^2\big)\,ds\\&\leq|\mathbf{d}_{0n}|^p+C\int_0^t|\mathbf{d}_n(s)|^p\,ds,\ t\geq 0.\end{aligned}\tag{5.28}$$

The last inequality further implies that

$$|\mathbf{d}_n(t)|^p + \int_0^t |\mathbf{d}_n(s)|^{p-2} |\nabla \mathbf{d}_n(s)|_{L^2}^2 \, ds + \int_0^t |\mathbf{d}_n(s)|^{p-2} |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \, ds \\ \leq |\mathbf{d}_{0n}|^p + C \int_0^t |\mathbf{d}_n(s)|^p \, ds, \;\; t \geq 0. \tag{5.29}$$

Therefore we deduce that,

$$\sup_{0 \leq s \leq t} |\mathbf{d}_n(s)|^p + \int_0^t |\mathbf{d}_n(s)|^{p-2} |\nabla \mathbf{d}_n(s)|_{L^2}^2 \, ds + \int_0^t |\mathbf{d}_n(s)|^{p-2} |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \, ds \\ \leq |\mathbf{d}_{0n}|^p + C \int_0^t |\mathbf{d}_n(s)|^p \, ds, \quad t \geq 0. \tag{5.30}$$

Now let us fix $t \geq 0$. Then by (5.29),

$$|\mathbf{d}_n(r)|^p + \int_0^r |\mathbf{d}_n(s)|^{p-2} |\nabla \mathbf{d}_n(s)|_{L^2}^2 \, ds + \int_0^r |\mathbf{d}_n(s)|^{p-2} |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \, ds \\ \leq |\mathbf{d}_{0n}|^p + C \int_0^r |\mathbf{d}_n(s)|^p \, ds \leq |\mathbf{d}_{0n}|^p + C \int_0^t |\mathbf{d}_n(s)|^p \, ds, \quad \text{for any } r \in [0, t]. \tag{5.31}$$

In particular we infer that

$$|\mathbf{d}_n(r)|^p \leq |\mathbf{d}_{0n}|^p + C \int_0^t |\mathbf{d}_n(s)|^p \, ds, \quad r \in [0, t]. \tag{5.32}$$

Next, in the above we take supremum over $r \in [0, t]$ and then taking the expectation we get,

$$\mathbb{E}\left[\sup_{r \in [0,t]} |\mathbf{d}_n(r)|^p \right] \leq \mathbb{E}|\mathbf{d}_{0n}|^p + C \int_0^t \mathbb{E}\big[|\mathbf{d}_n(s)|^p \big] \, ds. \tag{5.33}$$

The Gronwall Lemma thus yields

$$\mathbb{E}\left[\sup_{r \in [0,t]} |\mathbf{d}_n(r)|^p \right] \leq \mathbb{E}|\mathbf{d}_{0n}|^p e^{Ct}, \quad t \geq 0. \tag{5.34}$$

From (5.31) and (5.34) we also get

$$\mathbb{E}\left[\int_0^T |\mathbf{d}_n(s)|^{p-2} \big(|\nabla \mathbf{d}_n(s)|_{L^2}^2 + |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \big) \, ds \right] \\ \leq \mathbb{E}\left[|\mathbf{d}_{0n}|^p + C \int_0^T |\mathbf{d}_n(s)|^p \, ds \right] \leq \mathbb{E}|\mathbf{d}_{0n}|^p + C \, \mathbb{E}|\mathbf{d}_{0n}|^p \int_0^T e^{Ct} \, dt. \tag{5.35}$$

So we infer

$$\sup_{n \in \mathbb{N}} \left(\mathbb{E}\left[\sup_{t \in [0,T]} \big| \mathbf{d}_n(t) \big|_{L^2}^p + \int_0^T \big| \mathbf{d}_n(s) \big|_{L^2}^{p-2} \big(|\nabla \mathbf{d}_n(s)|_{L^2}^2 + |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \big) \, ds \right] \right) \leq \mathbb{E} \; \mathfrak{C}_0(p, T), \tag{5.36}$$

where $\mathfrak{C}_0(p, T) := |\mathbf{d}_{0n}|^p \big(1 + CTe^{CT} \big)$.

Note that in equations (5.34) and (5.36), we could argue in different order by first applying Gronwall lemma and then taking expectation.

□

Now we prove a Lemma which we require in the next a'priori estimate. Basically the term, we estimate in this Lemma, comes from the the Itô formula which we derive later.

Lemma 5.10. *For $z \in H^1$, let $\Psi(\cdot)$ be the mapping defined by $\Psi(z) = \frac{1}{2} |\nabla z|^2 + \frac{1}{2} \int_{\mathbb{O}} \tilde{F}(|z(x)|^2) \, dx$. Then there exists a generic constant $C > 0$, depending on N, $|\mathbb{O}|$, $\|g\|_{\mathcal{L}(H^1)}$, $\sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{2N+2})}$,*

$\|g\|_{\mathcal{L}(L^{2N+2})}$, *such that*

$$\left|\int_0^t \int_{\mathbb{B}} |\Psi(\Phi_n(l,\mathbf{d}_n(s))) - \Psi(\mathbf{d}_n(s))|^2 \nu_2(dl)ds\right| \le C\int_0^t \left[\Psi(\mathbf{d}_n(s)) + |\mathbf{d}_n(s)|^2_{L^2}\right]^2 ds, \tag{5.37}$$

and

$$\left|\int_0^t \int_{\mathbb{B}} \left[\Psi(\Phi_n(l,\mathbf{d}_n(s))) - \Psi(\mathbf{d}_n(s)) - \sum_{i=1}^N l_i \langle \Psi'(\mathbf{d}_n(s)), g_{i_n}(\mathbf{d}_n(s))\rangle_{\mathbb{L}_n}\right] \nu_2(dl)ds\right|$$
$$\le C\int_0^t \left[\Psi(\mathbf{d}_n(s)) + |\mathbf{d}_n(s)|^2_{L^2}\right] ds. \tag{5.38}$$

Proof. Observe that for $v, h, h_1, h_2 \in H_n$

$$\Psi'(v)(h) = \langle -\Delta v + \tilde{f}(|v|^2)v, h\rangle = \langle \nabla v, \nabla h\rangle + \langle f(v), h\rangle, \tag{5.39}$$
$$\Psi''(v)(h_1,h_2) = \langle \nabla h_1, \nabla h_2\rangle + \langle f'(v)(h_1), h_2\rangle. \tag{5.40}$$

For a given $z \in H^1$ and $l \in \mathbb{R}^N$, let us denote $y_n(t) := \Phi_n(t,l,z)$ and $y_n(1) := \Phi_n(l,z)$. In other words, y_n is the unique solution of

$$\frac{dy_n}{dt} = \mathcal{R}_n y_n, \quad y_n(0) = z,$$

where $\mathcal{R}_n := \sum\limits_{i=1}^N l_i g_{i_n}$.

Hence,

$$y_n(t) = \Phi_n(t,l,z) = \exp(t\mathcal{R}_n)z, \quad t \ge 0.$$

We begin with the observation that

$$\begin{aligned}
&\Psi(\Phi_n(l,z)) - \Psi(z) - \Psi'(z)\mathcal{R}_n z = \Psi(y_n(1)) - \Psi(y_n(0)) - \Psi'(y_n(0))\mathcal{R}_n(y_n(0))\\
&= \int_0^1 \frac{d}{ds}[\Psi\circ y_n](s)ds - \frac{d}{ds}[\Psi\circ y_n](0) := \int_0^1 (\Psi\circ y_n)'(s)ds - (\Psi\circ y_n)'(0)\\
&= \int_0^1\int_0^s (\Psi\circ y_n)''(r)drds = \int_0^1\int_0^s \left[\Psi''(y_n(r))(y_n'(r), y_n'(r)) + \Psi'(y_n(r))(y_n''(r))\right] drds\\
&= \int_0^1\int_0^s \Psi''(y_n(r))(\mathcal{R}_n y_n(r), \mathcal{R}_n y_n(r))drds + \int_0^1\int_0^s \Psi'(y_n(r))(\mathcal{R}_n^2 y_n(r))drds\\
&:= I_1 + I_2.
\end{aligned} \tag{5.41}$$

By the notation C, we will denote a generic positive constant, whose value and dependencies might differ from place to place.

We first estimate I_2. By (5.39) and (2.27) we find

$$\begin{aligned}
|I_2| &\le \int_0^1\int_0^s \left[\left|\langle \nabla y_n(r), \nabla \mathcal{R}_n^2 y_n(r)\rangle\right| + \left|\langle f(y_n(r)), \mathcal{R}_n^2 y_n(r)\rangle\right|\right] drds\\
&\le \int_0^1\int_0^s \left[\|\mathcal{R}_n\|^2_{\mathcal{L}(H^1)} e^{2r\|\mathcal{R}_n\|_{\mathcal{L}(H^1)}} |\nabla z|^2_{L^2} + |f(y_n(r))|_{L^{\frac{2N+2}{2N+1}}} |\mathcal{R}_n^2 y_n(r)|_{L^{2N+2}}\right] drds\\
&\le C\int_0^1\int_0^s \Big[\|\mathcal{R}_n\|^2_{\mathcal{L}(H^1)} e^{2r\|\mathcal{R}_n\|_{\mathcal{L}(H^1)}} |\nabla z|^2_{L^2} + \|\mathcal{R}_n\|^2_{\mathcal{L}(L^{2N+2})} e^{(2N+2)r\|\mathcal{R}_n\|_{\mathcal{L}(L^{2N+2})}} |z|^{2N+2}_{L^{2N+2}}\\
&\quad + \|\mathcal{R}_n\|^2_{\mathcal{L}(L^{2N+2})} e^{r\|\mathcal{R}_n\|_{\mathcal{L}(L^{2N+2})}} |z|_{L^{2N+2}}\Big] drds.
\end{aligned} \tag{5.42}$$

We now employ Young's inequality

$$|z|_{L^{2N+2}} \le \frac{|z|^{2N+2}_{L^{2N+2}}}{2N+2} + \frac{2N+1}{2N+2},$$

perform the integration and make use of the estimates in Lemma 5.2 to obtain

$$|I_2| \le C|l|^2\left(|\nabla z|^2_{L^2} + |z|^{2N+2}_{L^{2N+2}}\right). \tag{5.43}$$

Using (5.40), we split $I_1 = I_{1,1} + I_{1,2}$, where

$$I_{1,1} = \int_0^1 \int_0^s |\nabla \mathcal{R}_n y_n(r)|_{L^2}^2 \, drds,$$

$$I_{1,2} = \int_0^1 \int_0^s \langle f'(y_n(r))\mathcal{R}_n y_n(r), \mathcal{R}_n y_n(r)\rangle drds.$$

By Lemma 5.2

$$|I_{1,1}| \leq \int_0^1 \int_0^s \|\mathcal{R}_n\|_{\mathcal{L}(H^1)}^2 e^{2r\|\mathcal{R}_n\|_{\mathcal{L}(H^1)}} |\nabla z|_{L^2}^2 drds \leq C|l|^2 |\nabla z|_{L^2}^2. \tag{5.44}$$

To estimate $I_{1,2}$, we proceed similar to estimate of the second term on the right hand side of I_2. In particular, making use of (2.28) and Young's inequality, then integrating and finally employing Lemma 5.2, we obtain

$$|I_{1,2}| \leq C|l|^2 |z|_{L^{2N+2}}^{2N+2}. \tag{5.45}$$

Therefore, combining (5.43), (5.44) and (5.45), we get from (5.41)

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{B}} \left[\Psi(\Phi_n(l, \mathbf{d}_n(s))) - \Psi(\mathbf{d}_n(s)) - \sum_{i=1}^N l_i \langle \Psi'(\mathbf{d}_n(s)), g_{i_n}(\mathbf{d}_n(s))\rangle_{\mathbb{L}_n} \right] \nu_2(dl) ds \right| \\
&\leq C \int_0^t \int_{\mathbb{B}} |l|^2 \left[|\nabla \mathbf{d}_n(s)|_{L^2}^2 + |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \right] \nu_2(dl) ds \\
&\leq C \int_0^t \left[|\nabla \mathbf{d}_n(s)|_{L^2}^2 + |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2} \right] ds \leq C \int_0^t \left[\Psi(\mathbf{d}_n(s)) + |\mathbf{d}_n(s)|_{L^2}^2 \right] ds,
\end{aligned}$$

where the last step followed from the definition of Ψ and (2.29).

The proof is now complete. ☐

Remark 5.11. *Since $H^1 \hookrightarrow L^{2N+2}$ for both two and three dimensions, $\|\mathcal{R}_n\|_{\mathcal{L}(L^{2N+2})} \leq C\|\mathcal{R}_n\|_{H^1}$ and hence one need not necessarily invoke Proposition 5.1. In other words, one could avoid the techniques of modified Faedo-Galerkin approximations based on the Littlewood-Paley-decomposition for the noise term and instead apply the classical Faedo-Galerkin approximations. However, we introduce such notions of approximations keeping a bigger picture in mind, where the polynomial $\tilde{F}$ (or f) might be considered as more general and without any restriction on the degree. We plan to address such issues in our forthcoming work.*

Now we derive estimates for the processes $\mathbf{u}_n$ and $\nabla \mathbf{d}_n$.

Proposition 5.12. *For $z \in H^1$, let $\Psi(\cdot)$ be the mapping defined by $\Psi(z) = \frac{1}{2}|\nabla z|^2 + \frac{1}{2}\int_{\mathbb{O}} \tilde{F}(|z(x)|^2)\, dx$. Then for every $p \geq 1$ and $T > 0$, there exists a positive constant $C = C(p, T)$ such that*

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \Bigg(&\mathbb{E}\left[\sup_{s\in[0,T]} \left(\Psi(\mathbf{d}_n(s)) + \left|\mathbf{u}_n(s)\right|_{\mathrm{H}}^2 \right)^p \right] \\
&+ \mathbb{E}\left[\int_0^T \left(\|\mathbf{u}_n(s)\|^2 + \left|\Delta \mathbf{d}_n(s) - f_n(\mathbf{d}_n(s))\right|_{L^2}^2 \right) ds \right]^p \Bigg) \leq C.
\end{aligned}$$

Proof. For all $n \in \mathbb{N}$ and all $R > 0$ let us define

$$\tau_R^n := \inf\{t \geq 0 : |\mathbf{u}_n(t)|_{\mathrm{H}} + |\mathbf{d}_n(t)|_{L^2} \geq R\}. \tag{5.46}$$

This function is a stopping time, see the Dèbut Theorem [43, Theorem 76.1], since the processes $(\mathbf{u}_n(t))_{t\in[0,T]}$ and $(\mathbf{d}_n(t))_{t\in[0,T]}$ are $\mathbb{F}$-adapted and right-continuous. Moreover $\tau_R^n \uparrow \infty$, $\mathbb{P}$-a.s., as $R \uparrow \infty$.

Step -1 : Define a mapping ϕ as follows

$$\phi(\mathbf{u}) : \mathrm{H}_n \ni \mathbf{u} \mapsto \frac{1}{2}|\mathbf{u}|^2 := \frac{1}{2}|\mathbf{u}|_{\mathrm{H}_n}^2 \in \mathbb{R}.$$

Since the first Fréchet derivative of ϕ is given by

$$\phi'(\mathbf{u})[\mathbf{v}] = \langle \mathbf{u}, \mathbf{v}\rangle, \quad \mathbf{u}, \mathbf{v} \in \mathrm{H}_n,$$

by applying the Itô formula to the process $\phi(\mathbf{u}_n(t))$, where $\mathbf{u}_n = \mathbf{u}_n(t),\ t \geq 0$, is the solution to equation (5.6), we infer that for $t \geq 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}
&\phi(\mathbf{u}_n(t)) - \phi(\mathbf{u}_n(0)) \\
&= -\int_0^t \big\langle \mathscr{A}\mathbf{u}_n(s) + B_n(\mathbf{u}_n(s)) + M_n(\mathbf{d}_n(s)), \mathbf{u}_n(s)\big\rangle\, ds \\
&\quad + \frac{1}{2}\int_0^t \int_Y \left\{|\mathbf{u}_n(s-) + P_nF(s, \mathbf{u}_n(s-); y)|^2 - |\mathbf{u}_n(s-)|^2\right\} \tilde{\eta}_1(ds, dy) \\
&\quad + \frac{1}{2}\int_0^t \int_Y \Big\{|\mathbf{u}_n(s) + P_nF(s, \mathbf{u}_n(s); y)|^2 - |\mathbf{u}_n(s)|^2 - 2\big\langle \mathbf{u}_n(s), P_nF(s, \mathbf{u}_n(s); y)\big\rangle\Big\}\, d\nu_1(y)ds.
\end{aligned} \tag{5.47}$$

By applying the Itô formula to $\Psi(\mathbf{d}_n(t))$, where $\mathbf{d}_n = \mathbf{d}_n(t),\ t \geq 0$, is the solution to equation (5.7), we deduce

$$\begin{aligned}
&\Psi(\mathbf{d}_n(t)) - \Psi(\mathbf{d}_n(0)) \\
&= -\int_0^t \Psi'(\mathbf{d}_n(s))\big[\mathcal{A}\mathbf{d}_n(s) + \tilde{B}_n(\mathbf{u}_n(s), \mathbf{d}_n(s)) + f_n(\mathbf{d}_n(s))\big]\, ds \\
&\quad + \int_0^t \int_{\mathbb{B}} \left[\Psi(\Phi_n(l, \mathbf{d}_n(s-))) - \Psi(\mathbf{d}_n(s-))\right] \tilde{\eta}_2(ds, dl) \\
&\quad + \int_0^t \int_{\mathbb{B}} \left[\Psi(\Phi_n(l, \mathbf{d}_n(s))) - \Psi(\mathbf{d}_n(s)) - \sum_{i=1}^N l_i \langle \Psi'(\mathbf{d}_n(s)), g_{i_n}(\mathbf{d}_n(s))\rangle_{\mathbb{L}_n}\right] \nu_2(dl)ds.
\end{aligned} \tag{5.48}$$

From (5.39) we have,

$$\begin{aligned}
&-\int_0^t \Psi'(\mathbf{d}_n(s))\big[\mathcal{A}\mathbf{d}_n(s) + \tilde{B}_n(\mathbf{u}_n(s), \mathbf{d}_n(s)) + f_n(\mathbf{d}_n(s))\big]\, ds \\
&= \int_0^t -|f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|^2\, ds - \big\langle \tilde{B}_n(\mathbf{u}_n(s), \mathbf{d}_n(s)), f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)\big\rangle\, ds,\ t \geq 0.
\end{aligned} \tag{5.49}$$

From (5.48), (5.49), Lemma 5.10 and Lemma A.7 we obtain,

$$\begin{aligned}
&\Psi(\mathbf{d}_n(t)) - \Psi(\mathbf{d}_n(0)) \\
&\leq -\int_0^t |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\, ds + \int_0^t \langle M_n(\mathbf{d}_n(s)), \mathbf{u}_n(s)\rangle\, ds \\
&\quad + \int_0^t \int_{\mathbb{B}} \left[\Psi(\Phi_n(l, \mathbf{d}_n(s-))) - \Psi(\mathbf{d}_n(s-))\right] \tilde{\eta}_2(ds, dl) + C\int_0^t \left[\Psi(\mathbf{d}_n(s) + |\mathbf{d}_n(s)|_{L^2}^2\right] ds, \quad t \geq 0.
\end{aligned} \tag{5.50}$$

Since $\langle B_n(\mathbf{u}_n(t)), \mathbf{u}_n(t)\rangle = 0$, adding equations (5.47) and (5.48) and using (5.50) we get after rearrngement

$$\begin{aligned}
&\Psi(\mathbf{d}_n(t)) + \phi(\mathbf{u}_n(t)) + \int_0^t \left(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\right) ds \\
&\leq \Psi(\mathbf{d}_{0n}) + \phi(\mathbf{u}_{0n}) + C\int_0^t \left[\Psi(\mathbf{d}_n(s) + |\mathbf{d}_n(s)|_{L^2}^2\right] ds + I_n^1(t) + I_n^2(t) + I_n^3(t), \quad t \geq 0.
\end{aligned} \tag{5.51}$$

Here we use the following shortcut notation, for $t \geq 0$,

$$I_n^1(t) := \frac{1}{2}\int_0^t \int_Y \left\{|\mathbf{u}_n(s-) + P_nF(s,\mathbf{u}_n(s-);y)|^2 - |\mathbf{u}_n(s-)|^2\right\} \tilde{\eta}_1(ds,dy), \tag{5.52}$$

$$I_n^2(t) := \frac{1}{2}\int_0^t \int_Y \left\{|\mathbf{u}_n(s) + P_nF(s,\mathbf{u}_n(s);y)|^2 - |\mathbf{u}_n(s)|^2 - 2\big\langle \mathbf{u}_n(s), P_nF(s,\mathbf{u}_n(s);y)\big\rangle\right\} d\nu_1(y)ds, \tag{5.53}$$

$$I_n^3(t) := \int_0^t \int_{\mathbb{B}} \left\{\Psi(\Phi_n(l,\mathbf{d}_n(s-))) - \Psi(\mathbf{d}_n(s-))\right\} \tilde{\eta}_2(ds,dl). \tag{5.54}$$

Step -2 : Now for all $t \in [0,T]$, taking expectation both sides we can rewrite the above equation as

$$\begin{aligned}
&\mathbb{E}\big[\Psi(\mathbf{d}_n(t\wedge\tau_R^n)) + \big|\mathbf{u}_n(t\wedge\tau_R^n)\big|_{\mathrm{H}}^2\big] + \mathbb{E}\left[\int_0^{t\wedge\tau_R^n} \big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\big)\, ds\right] \\
&\quad \leq \mathbb{E}\big[\Psi(\mathbf{d}_{0n}) + |\mathbf{u}_{0n}|^2\big] + C\int_0^{t\wedge\tau_R^n} \big[\Psi(\mathbf{d}_n(s) + |\mathbf{d}_n(s)|_{L^2}^2\big]\, ds \\
&\quad\quad + \mathbb{E}\big[I_n^1(t\wedge\tau_R^n) + I_n^2(t\wedge\tau_R^n) + I_n^3(t\wedge\tau_R^n)\big].
\end{aligned} \tag{5.55}$$

In order to estimate the RHS of (5.55), let us first observe that by part (D) of Assumption 2.3 we have,

$$\begin{aligned}
|I_n^2(t\wedge\tau_n^R)| &= \int_0^{t\wedge\tau_n^R} \int_Y \bigg\{|\mathbf{u}_n(s) + P_nF(s,\mathbf{u}_n(s);y)|^2 - |\mathbf{u}_n(s)|^2 \\
&\qquad\qquad - 2\big\langle \mathbf{u}_n(s), P_nF(s,\mathbf{u}_n(s);y)\big\rangle\bigg\} d\nu_1(y)ds \\
&\leq C\int_0^{t\wedge\tau_n^R}\int_Y |P_nF(s,\mathbf{u}_n(s);y)|^2\, d\nu_1(y)ds \leq C\int_0^{t\wedge\tau_n^R} \left\{1 + |\mathbf{u}_n(s)|^2\right\} ds, \quad t \geq 0.
\end{aligned} \tag{5.56}$$

Thus, by the Fubini Theorem, we infer that

$$\mathbb{E}\big[|I_n^2(t\wedge\tau_n^R)|\big] \leq C(t\wedge\tau_n^R) + \int_0^{t\wedge\tau_n^R} \mathbb{E}\big[|\mathbf{u}_n(s)|^2\big]\, ds, \quad t \geq 0. \tag{5.57}$$

Next by definition of τ_R^n and from (2.21), we observe that the process $I_n^1(t\wedge\tau_R^n)$ and $I_n^3(t\wedge\tau_R^n)$ are square integrable martingales and hence

$$\mathbb{E}\big[I_n^1(t\wedge\tau_R^n)\big] = \mathbb{E}\big[I_n^3(t\wedge\tau_R^n)\big] = 0, \qquad t \in [0,T]. \tag{5.58}$$

Using (5.57), (5.58) and Proposition 5.9 we can further deduce from (5.55) that

$$\begin{aligned}
&\mathbb{E}\big[\Psi(\mathbf{d}_n(t\wedge\tau_R^n)) + |\mathbf{u}_n(t\wedge\tau_R^n)|^2\big] + \mathbb{E}\left[\int_0^{t\wedge\tau_R^n} \big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\big)\, ds\right] \\
&\quad \leq \bar{C} + \mathbb{E}\big[\Psi(\mathbf{d}_{0n}) + |\mathbf{u}_{0n}|^2\big] + C\int_0^t \mathbb{E}\big[\Psi(\mathbf{d}_n(s\wedge\tau_R^n)) + |\mathbf{u}_n(s\wedge\tau_R^n)|^2\big]\, ds, \ t \geq 0.
\end{aligned} \tag{5.59}$$

Applying the Gronwall Lemma we get,

$$\begin{aligned}
&\mathbb{E}\big[\Psi(\mathbf{d}_n(t\wedge\tau_R^n)) + |\mathbf{u}_n(t\wedge\tau_R^n)|^2\big] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_R^n} \big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\big)\, ds\right] \leq \mathfrak{C}(T), \quad t \geq 0,
\end{aligned} \tag{5.60}$$

where $\mathfrak{C}(T) := \big(\bar{C} + \mathbb{E}\big[\Psi(\mathbf{d}_{0n}) + |\mathbf{u}_{0n}|^2_{\mathrm{H}}\big]\big)e^{CT}$.
In particular, we have

$$\sup_{n\in\mathbb{N}}\Bigg(\mathbb{E}\Big[\Psi(\mathbf{d}_n(T\wedge\tau_R^n)) + \big|\mathbf{u}_n(T\wedge\tau_R^n)\big|^2_{\mathrm{H}}\Big] + \mathbb{E}\bigg[\int_0^{T\wedge\tau_R^n}\big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|^2_{L^2}\big)\,ds\bigg]\Bigg) \le \mathfrak{C}(T). \tag{5.61}$$

Step -3 : If we put $\psi(s) := \Psi(\mathbf{d}_n(s)) + \frac{1}{2}\big|\mathbf{u}_n(s)\big|^2_{\mathrm{H}}$, then by inequality (5.51) we infer that

$$\psi(t) + \int_0^t\big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|^2_{L^2}\big)ds \le \psi(0) + \int_0^t\big[\Psi(\mathbf{d}_n(s)) + |\mathbf{d}_n(s)|^2_{L^2}\big]\,ds + I_n^1(t) + I_n^2(t) + I_n^3(t),\ t\in[0,T].$$

Let C be a generic positive constant whose value and dependencies may differ from place to place. Now raising both side to the power $p\ge 1$, taking supremum over $s\in[0,T\wedge\tau_R^n]$ and finally taking the expectation we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}[\psi(s)]^p\bigg] + \mathbb{E}\bigg[\int_0^{T\wedge\tau_R^n}\big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|^2_{L^2}\big)\,ds\bigg]^p \\
&\le \mathbb{E}[\psi(0)]^p + C\mathbb{E}\int_0^{T\wedge\tau_R^n}\Big[[\Psi(\mathbf{d}_n(s))]^p + |\mathbf{d}_n(s)|^{2p}_{L^2}\Big]ds \\
&\quad + C\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^1(s)|^p\bigg] + C\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^2(s)|^p\bigg] + C\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^3(s)|^p\bigg].
\end{aligned} \tag{5.62}$$

We will first find a suitable estimate for the third term on RHS of (5.62). We observe by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^1(s)|^p\bigg] \\
&\le C\mathbb{E}\Bigg[\bigg(\int_0^{T\wedge\tau_R^n}\int_Y\big\{|\mathbf{u}_n(s) + P_nF(s,\mathbf{u}_n(s);y)|^2 - |\mathbf{u}_n(s)|^2\big\}^2\,\nu_1(dy)ds\bigg)^{\frac{p}{2}}\Bigg], \quad s\ge 0.
\end{aligned} \tag{5.63}$$

Using the Taylor formula, it follows that for each $r\ge 2$, there exists a positive constant $c_r>0$ such that

$$\big||a+b|^r_{\mathrm{H}} - |a|^r_{\mathrm{H}} - r|a|^{r-2}_{\mathrm{H}}\langle a,b\rangle\big| \le c_r\big(|a|^{r-2}_{\mathrm{H}} + |b|^{r-2}_{\mathrm{H}}\big)|b|^2_{\mathrm{H}}, \quad a,b\in\mathrm{H}.$$

Using the Hölder inequality we further have

$$\big(|a+b|^r_{\mathrm{H}} - |a|^r_{\mathrm{H}}\big)^2 \le 2r^2|a|^{2r-2}_{\mathrm{H}}|b|^2_{\mathrm{H}} + 4c_r^2|a|^{2r-4}_{\mathrm{H}}|b|^4_{\mathrm{H}} + 4c_r^2|b|^{2r}_{\mathrm{H}}, \quad a,b\in\mathrm{H}. \tag{5.64}$$

Now for $r=2$ in (5.64), using (2.21) and the Young inequality

$$\begin{aligned}
&\int_Y\big\{|\mathbf{u}_n(s) + P_nF(s,\mathbf{u}_n(s);y)|^2 - |\mathbf{u}_n(s)|^2\big\}^2\,\nu_1(dy) \\
&\le c\ |\mathbf{u}_n(s)|^2\int_Y|F(s,\mathbf{u}_n(s);y)|^2\,\nu_1(dy) + c\int_Y|F(s,\mathbf{u}_n(s);y)|^4\,\nu_1(dy) \\
&\le c + c_1|\mathbf{u}_n(s)|^2 + c_2|\mathbf{u}_n(s)|^4 \le k_1 + k_2\ |\mathbf{u}_n(s)|^4, \quad s\ge 0.
\end{aligned} \tag{5.65}$$

Hence using the fact that $T\wedge\tau_R^n\le T$ we have,

$$\begin{aligned}
&\bigg(\int_0^{T\wedge\tau_R^n}\int_Y\big\{|\mathbf{u}_n(s) + P_nF(s,\mathbf{u}_n(s);y)|^2 - |\mathbf{u}_n(s)|^2\big\}^2\,\nu_1(dy)ds\bigg)^{\frac{p}{2}} \\
&\le c(k_1T)^{\frac{p}{2}} + c(k_2)^{\frac{p}{2}}\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^4\,ds\bigg)^{\frac{p}{2}}.
\end{aligned}$$

So we have

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^1(s)|^p\bigg] \leq C + C\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^4\,ds\bigg)^{\frac{p}{2}}\bigg]$$
$$\leq C + \frac{1}{2}\mathbb{E}\bigg[\bigg(\sup_{s\in[0,T\wedge\tau_R^n]}|\mathbf{u}_n(s)|^2\bigg)^p\bigg] + \frac{C^2}{2}\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^2\,ds\bigg)^p\bigg], \quad s\geq 0. \tag{5.66}$$

On the other hand, using the Hölder inequality and the fact that $T\wedge\tau_R^n \leq T$, we have

$$\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^2\,ds \leq T^{\frac{p-1}{p}}\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^{2p}\,ds\bigg)^{\frac{1}{p}}.$$

Now taking power p in both sides, taking expectation, then using the Fubini Theorem we get

$$\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^2\,ds\bigg)^p\bigg] \leq T^{p-1}\int_0^{T\wedge\tau_R^n}\mathbb{E}\big[|\mathbf{u}_n(s)|^{2p}\big]\,ds, \quad s\geq 0. \tag{5.67}$$

From inequalities (5.66) and (5.67) we get,

$$c\,\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^1(s)|^p\bigg] \leq \frac{1}{2}\mathbb{E}\bigg[\bigg(\sup_{s\in[0,T\wedge\tau_R^n]}|\mathbf{u}_n(s)|^2\bigg)^p\bigg] + C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\big[|\mathbf{u}_n(s)|^{2p}\big]\,ds, s\geq 0. \tag{5.68}$$

Now we will find a suitable estimate for the fourth term on the RHS of (5.62). From (5.56) and using the fact that $T\wedge\tau_R^n \leq T$, we have

$$\mathbb{E}|I_n^2(T\wedge\tau_R^n)|^p \leq C\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}\big\{1+|\mathbf{u}_n(s)|^2\big\}\,ds\bigg)^p\bigg]$$
$$= C\,\mathbb{E}\bigg[\bigg((T\wedge\tau_R^n) + \int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^2\,ds\bigg)^p\bigg] \leq C_p\bigg[C_{p,T} + \mathbb{E}\bigg(\int_0^{T\wedge\tau_R^n}|\mathbf{u}_n(s)|^2\,ds\bigg)^p\bigg].$$

From(5.67) we obtain

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^2(s)|^p\bigg] \leq C_{p,T} + C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\big[|\mathbf{u}_n(s)|^{2p}\big]\,ds, \quad s\geq 0. \tag{5.69}$$

Finally we deal with the fifth term on the RHS of (5.62). Again, by using the Burkholder-Davis-Gundy inequality, and (5.37), we obtain, for $s\geq 0$,

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|I_n^3(s)|^p\bigg] \leq C\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}\int_{\mathbb{B}}|\Psi(\Phi_n(l,\mathbf{d}_n(s))) - \Psi(\mathbf{d}_n(s))|^2\,\nu_2(dl)ds\bigg)^{\frac{p}{2}}\bigg]$$
$$\leq C\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\tau_R^n}\big[\Psi(\mathbf{d}_n(s) + |\mathbf{d}_n(s)|^2_{L^2}\big]^2\,ds\bigg)^{\frac{p}{2}}\bigg]$$
$$\leq C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\big[\Psi(\mathbf{d}_n(s))]^p + |\mathbf{d}_n(s)|^{2p}\big]\,ds. \tag{5.70}$$

Now from (5.62), (5.68), (5.69) and (5.70) and applying Proposition 5.9 we obtain,

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}[\psi(s)]^p\bigg] \leq C_{p,T} + \frac{1}{2}\mathbb{E}\bigg[\bigg(\sup_{s\in[0,T\wedge\tau_R^n]}|\mathbf{u}_n(s)|^2\bigg)^p\bigg]$$
$$+ C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\big[\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\big]\,ds. \tag{5.71}$$

As $[\psi(s)]^p \geq [\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}$, we further observe

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}\Big([\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big)\bigg] \leq \mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}[\psi(s)]^p\bigg]\\
&\leq C_{p,T} + \frac{1}{2}\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}|\mathbf{u}_n(s)|^{2p}\bigg] + C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\Big[[\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big]\,ds. \qquad (5.72)
\end{aligned}$$

Now taking second term of RHS to the LHS and multiplying both side by 2 we have,

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}\Big([\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big)\bigg] \leq C_{p,T} + C_{p,T}\int_0^{T\wedge\tau_R^n}\mathbb{E}\Big[[\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big]\,ds. \tag{5.73}$$

Now Gronwall's Lemma yields

$$\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}\Big([\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big)\bigg] \leq C_{p,T}\ \exp\big(C_{p,T}\cdot T\wedge\tau_R^n\big). \tag{5.74}$$

We can write further using the fact that $T\wedge\tau_R^n \leq T$,

$$\sup_{n\in\mathbb{N}}\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}\Big([\Psi(\mathbf{d}_n(s))]^p + |\mathbf{u}_n(s)|^{2p}\Big)\bigg] \leq C(p,T). \tag{5.75}$$

Finally from (5.62) for $p \geq 1$ we get,

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\Bigg(\mathbb{E}\bigg[\sup_{s\in[0,T\wedge\tau_R^n]}\big(\Psi(\mathbf{d}_n(s)) + \big|\mathbf{u}_n(s)\big|_{\mathrm{H}}^2\big)^p\bigg]\\
&\qquad + \mathbb{E}\bigg[\int_0^{T\wedge\tau_R^n}\big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\big)\,ds\bigg]^p\Bigg) \leq C_{p,T}. \qquad (5.76)
\end{aligned}$$

Since the constant in the RHS does not depend on m or R, so letting $m, R \to \infty$ we get $T\wedge\tau_R^n \to T$. We finally have

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\Bigg(\mathbb{E}\bigg[\sup_{s\in[0,T]}\big(\Psi(\mathbf{d}_n(s)) + \big|\mathbf{u}_n(s)\big|_{\mathrm{H}}^2\big)^p\bigg]\\
&\qquad + \mathbb{E}\bigg[\int_0^{T}\big(\|\mathbf{u}_n(s)\|^2 + |f_n(\mathbf{d}_n(s)) - \Delta\mathbf{d}_n(s)|_{L^2}^2\big)\,ds\bigg]^p\Bigg) \leq C_{p,T}, \qquad (5.77)
\end{aligned}$$

So the proof is complete. $\square$

Using the previous results we can deduce that

Proposition 5.13. *For every $q \geq 1$, there exists a positive constant C, depending on q such that*

$$\mathbb{E}\bigg[\int_0^T |\mathbf{d}_n(s)|_{D(\mathcal{A})}^2\,ds\bigg]^q \leq C(q).$$

Proof. Let us choose and fix $q \geq 1$. Since $\mathbf{d}_n$ take values in $D(\mathcal{A})$, it is sufficient to prove that

$$\mathbb{E}\bigg[\int_0^T |\Delta\mathbf{d}_n(s)|^2\,ds\bigg]^q \leq C(q).$$

Since $N \in I_{\mathbf{n}}$, we have $H^1 \hookrightarrow L^{4N+2}$ and using Remark 2.4

$$\begin{aligned}
|\Delta\mathbf{d}_n(s)|^2 &\leq 2|\Delta\mathbf{d}_n(s) - f_n(\mathbf{d}_n(s))|^2 + 2|f_n(\mathbf{d}_n(s))|^2\\
&\leq 2|\Delta\mathbf{d}_n(s) - f_n(\mathbf{d}_n(s))|^2 + c|\mathbf{d}_n(s)|_{L^{\tilde{q}}}^{4N+2} + C. \qquad (5.78)
\end{aligned}$$

Therefore, we infer that

$$\mathbb{E}\bigg[\int_0^T |\Delta\mathbf{d}_n(s)|^2\,ds\bigg]^q \leq c\mathbb{E}\bigg[\int_0^T |\Delta\mathbf{d}_n(s) - f_n(\mathbf{d}_n(s))|^2\,ds\bigg]^q + c\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathbf{d}_n(s)|^{(4N+2)q}\bigg] + C. \tag{5.79}$$

Using (5.76) we obtain the desired result.

□

Corollary 5.14. *For every $q \geq 2$, there exists a constant $C > 0$, depending on q such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbf{d}_n(t)\big|_{H^1}^q\Big] \leq C_q.$$

Proof. Follows from Proposition 5.9 and Proposition 5.12. □

Remark 5.15. *Since a stochastic integral with respect to the time homogeneous compensated Poisson random measure is defined for all progressively measurable processes, each $\mathbf{u}_n$ satisfies a version of* (5.6) *with $t-$ replaced by t, i.e. $\mathbf{u}_n$ satisfies*

$$d\mathbf{u}_n(t) + \big[\mathscr{A}\mathbf{u}_n(t) + B_n(\mathbf{u}_n(t)) + M_n(\mathbf{d}_n(t))\big]dt = \int_Y P_n F(t, \mathbf{u}_n(t); y)\, \tilde{\eta}_1(dt, dy). \tag{5.80}$$

Similar argument says $\mathbf{d}_n$ satisfies

$$d\mathbf{d}_n(t) + \big[\mathcal{A}\mathbf{d}_n(t) + \tilde{B}_n(\mathbf{u}_n(t), \mathbf{d}_n(t)) + f_n(\mathbf{d}_n(t))\big]dt = \int_{\mathbb{B}} G_n(l, \mathbf{d}_n(t))\, \tilde{\eta}_2(dt, dl) + b_n(\mathbf{d}_n(t))\, dt. \tag{5.81}$$

6. Tightness of the Laws of Approximating Sequences

In this section we will show that all the conditons of Corollary 4.10 and Corollary 4.11 satisfiy for $p = 2$. This will yield tightness of laws of $\mathbf{u}_n$ and $\mathbf{d}_n$. Let us consider the space $\mathcal{Z}_T = \mathcal{Z}_{T,1} \times \mathcal{Z}_{T,2}$, where

$$\mathcal{Z}_{T,1} = L^2_w(0, T; \mathrm{V}) \cap L^2(0, T; \mathrm{H}) \cap \mathbb{D}([0, T]; \mathrm{V}') \cap \mathbb{D}([0, T]; \mathrm{H}_w) \tag{6.1}$$

and

$$\mathcal{Z}_{T,2} = L^2_w(0, T; D(\mathcal{A})) \cap L^2(0, T; H^1) \cap \mathbb{D}([0, T]; L^2) \cap \mathbb{D}([0, T]; H^1_w). \tag{6.2}$$

For each $n \in \mathbb{N}$, the solution $(\mathbf{u}_n, \mathbf{d}_n)$ of the Galerkin approximation equations defines a measure $\mathscr{L}(\mathbf{u}_n, \mathbf{d}_n)$ on $(\mathcal{Z}_T, \mathscr{T})$, where $\mathscr{T}$ is the supremum of $\mathscr{T}^1$ and $\mathscr{T}^2$. We will show that the set of measures $\{\mathscr{L}(\mathbf{u}_n, \mathbf{d}_n), n \in \mathbb{N}\}$ is tight on $(\mathcal{Z}_T, \mathscr{T})$ using Corollaries 4.10 and 4.11.

Before embarking on the proof of the main result of this section let us write-down some important estimates we derived in Section 5. In what follows by $C_1, C_2, \cdots, C_6$, we will denote generic constants independent of n (but possibly dependent on p, q and $T > 0$).

From Proposition 5.9 we observe that for $p \geq 2, T > 0$, there exists $C_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbf{d}_n(t)\big|_{L^2}^p\Big] \leq C_1. \tag{6.3}$$

Similarly, from (5.36) there exists $C_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T |\mathbf{d}_n(s)|_{L^2}^{p-2} |\nabla \mathbf{d}_n(s)|_{L^2}^2\, ds\Big] \leq C_2. \tag{6.4}$$

From Corollary 5.14, there exists $C_3 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \big|\mathbf{d}_n(s)\big|_{H^1}^2\, ds\Big] \leq C_3. \tag{6.5}$$

From Proposition 5.13 for $q = 1$, there exists $C_4 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \big|\mathbf{d}_n(s)\big|_{D(\mathcal{A})}^2\, ds\Big] \leq C_4. \tag{6.6}$$

From Proposition 5.12, there exists $C_5 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbf{u}_n(t)\big|_{\mathrm{H}}^{2p}\Big] \leq C_5. \tag{6.7}$$

Again from Proposition 5.12 for $p = 1$ we get,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|\mathbf{u}_n(t)\big|_{\mathrm{H}}^2\Big] \le C_5, \tag{6.8}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \|\mathbf{u}_n(s)\|^2\, ds\Big] \le C_5. \tag{6.9}$$

So from (6.9), there exists $C_6 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \big|\mathbf{u}_n(s)\big|_{\mathrm{V}}^2\, ds\Big] \le C_6. \tag{6.10}$$

Proposition 6.1. *There exists a positive constant $\mathcal{C}$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \Big[\big|B(\mathbf{u}_n(s))\big|_{\mathrm{V}'}^p + \big|M(\mathbf{d}_n(s))\big|_{\mathrm{V}'}^p + \big|\tilde{B}(\mathbf{u}_n(s),\mathbf{d}_n(s))\big|_{L^2}^p + \big|f(\mathbf{d}_n(s))\big|_{L^2}^2\Big]\, ds \le \mathcal{C}, \tag{6.11}$$

where $p = 2$ if dimension $\mathbf{n} = 2$ and $p = \frac{4}{3}$ if dimension $\mathbf{n} = 3$.

Proof. In what follows by $C_1, C_2, \cdots, C_7$, we will denote generic constants independent of n (but possibly dependent on T). Let us fix $T > 0$. We will prove inequality (6.11) in four steps considering separately for the cases $\mathbf{n} = 2$ and $\mathbf{n} = 3$.
Consider first the case $\mathbf{n} = 2$. From (2.11) and using Proposition 5.12 we infer

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|B(\mathbf{u}_n(s))\big|_{\mathrm{V}'}^2\, ds\Big] \le c\,\mathbb{E}\Big[\int_0^T \big|\mathbf{u}_n(s)\big|^2 \cdot \big\|\mathbf{u}_n(s)\big\|^2\, ds\Big] \\
&\le c\Big\{\mathbb{E}\Big[\sup_{s\in[0,T]} |\mathbf{u}_n(s)|^4\Big]\Big\}^{\frac{1}{2}} \cdot \Big\{\mathbb{E}\Big[\int_0^T \|\mathbf{u}_n(s)\|^2\, ds\Big]^2\Big\}^{\frac{1}{2}} \le C_1.
\end{aligned} \tag{6.12}$$

Now consider $\mathbf{n} = 3$. From (2.11) and using Proposition 5.12 we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|B(\mathbf{u}_n(s))\big|_{\mathrm{V}'}^{\frac{4}{3}}\, ds\Big] \le c\,\mathbb{E}\Big[\int_0^T |\mathbf{u}_n(s)|^{\frac{2}{3}} \cdot \|\mathbf{u}_n(s)\|^2\, ds\Big] \\
&\le \Big\{\mathbb{E}\Big[\sup_{s\in[0,T]} |\mathbf{u}_n(s)|^{\frac{8}{3}}\Big]\Big\}^{\frac{1}{4}} \cdot \Big\{\mathbb{E}\Big[\int_0^T \|\mathbf{u}_n(s)\|^2\, ds\Big]^{\frac{4}{3}}\Big\}^{\frac{3}{4}} \le C_2.
\end{aligned} \tag{6.13}$$

Consider $\mathbf{n} = 2$. From (2.15) and using Proposition 5.12 and Proposition 5.13 we infer

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|M(\mathbf{d}_n(s))\big|_{\mathrm{V}'}^2\, ds\Big] \le c\,\mathbb{E}\Big[\int_0^T \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^2 \cdot \big|\Delta\mathbf{d}_n(s)\big|^2\, ds\Big] \\
&\le \Big\{\mathbb{E}\Big[\sup_{s\in[0,T]} \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^4\Big]\Big\}^{\frac{1}{2}} \cdot \Big\{\mathbb{E}\Big[\int_0^T \big|\Delta\mathbf{d}_n(s)\big|^2\, ds\Big]^2\Big\}^{\frac{1}{2}} \le C_3.
\end{aligned} \tag{6.14}$$

Now consider $\mathbf{n} = 3$. From (2.15) and using Proposition 5.12 and Proposition 5.13 we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|M(\mathbf{d}_n(s))\big|_{\mathrm{V}'}^{\frac{4}{3}}\, ds\Big] \le c\,\mathbb{E}\Big[\int_0^T \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^{\frac{2}{3}} \cdot \big|\Delta\mathbf{d}_n(s)\big|^2\, ds\Big] \\
&\le \Big\{\mathbb{E}\Big[\sup_{s\in[0,T]} \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^{\frac{8}{3}}\Big]\Big\}^{\frac{1}{4}} \cdot \Big\{\mathbb{E}\Big[\int_0^T \big|\Delta\mathbf{d}_n(s)\big|^2\, ds\Big]^{\frac{4}{3}}\Big\}^{\frac{3}{4}} \le C_4.
\end{aligned} \tag{6.15}$$

Again consider $\mathbf{n} = 2$. From (2.12) and referring to inequalities (6.12) and (6.14) we infer,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|\tilde{B}(\mathbf{u}_n(s),\mathbf{d}_n(s))\big|_{L^2}^2\, ds\Big] \le c\,\mathbb{E}\Big[\int_0^T \big|\mathbf{u}_n(s)\big| \cdot \big\|\mathbf{u}_n(s)\big\| \cdot \big|\nabla\mathbf{d}_n(s)\big|_{L^2} \cdot \big|\Delta\mathbf{d}_n(s)\big|\, ds\Big] \\
&\le \frac{c}{2}\,\mathbb{E}\Big[\int_0^T \big|\mathbf{u}_n(s)\big|^2 \cdot \big\|\mathbf{u}_n(s)\big\|^2\, ds\Big] + \frac{c}{2}\,\mathbb{E}\Big[\int_0^T \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^2 \cdot \big|\Delta\mathbf{d}_n(s)\big|^2\, ds\Big] \le C_5.
\end{aligned} \tag{6.16}$$

Now consider $\mathbf{n}=3$. From (2.12) and referring to inequalities (6.13) and (6.15) we get,

$$\mathbb{E}\bigg[\int_0^T \big|\tilde{B}(\mathbf{u}_n(s),\mathbf{d}_n(s))\big|_{L^2}^{\frac{4}{3}}\,ds\bigg] \le c\,\mathbb{E}\bigg[\int_0^T \big|\mathbf{u}_n(s)\big|^{\frac{1}{3}}\cdot\big\|\mathbf{u}_n(s)\big\|\cdot\big|\nabla\mathbf{d}_n(s)\big|_{L^2}^{\frac{1}{3}}\cdot\big|\Delta\mathbf{d}_n(s)\big|\,ds\bigg]$$

$$\le \frac{c}{2}\,\mathbb{E}\bigg[\int_0^T \big|\mathbf{u}_n(s)\big|^{\frac{2}{3}}\cdot\big\|\mathbf{u}_n(s)\big\|^2\,ds\bigg] + \frac{c}{2}\,\mathbb{E}\bigg[\int_0^T \big|\nabla\mathbf{d}_n(s)\big|_{L^2}^{\frac{2}{3}}\cdot\big|\Delta\mathbf{d}_n(s)\big|^2\,ds\bigg] \le C_6. \tag{6.17}$$

Now consider $\mathbf{n}=2,3$. From Assumption 2.3, we have $N\in\{1,2,\cdots\}$ for $\mathbf{n}=2$ and $N=1$ for $\mathbf{n}=3$. We know $H^1\hookrightarrow L^{\tilde{q}}$ for $\tilde{q}=4N+2$. From Remark 2.4 and Proposition 5.12 we infer,

$$\mathbb{E}\bigg[\int_0^T \big|f(\mathbf{d}_n(s))\big|_{L^2}^2\,ds\bigg] \le C_T + c\mathbb{E}\bigg[\int_0^T \big|\mathbf{d}_n(s)\big|_{L^{\tilde{q}}}^{\tilde{q}}\,ds\bigg] \le C_T + C_T\mathbb{E}\bigg[\sup_{s\in[0,T]}\big|\mathbf{d}_n(s)\big|_{H^1}^{\tilde{q}}\,ds\bigg] \le C_7. \tag{6.18}$$

where C_7 is a positive constant independent of n.

From inequalities (6.12) to (6.18) we conclude our desired inequality (6.11) for some positive constant $\mathcal{C}$ which is independent of n. $\square$

Now we can state (and prove) the tightness Lemma.

Lemma 6.2. *The set of measures $\{\mathscr{L}(\mathbf{u}_n,\mathbf{d}_n), n\in\mathbb{N}\}$ is tight on $(\mathcal{Z}_T,\mathscr{T})$.*

Proof. From (6.8), (6.10), (6.6) and Corollary 6.11, we obtain the first two conditions of Corollaries 4.10 and 4.11 for $\mathbf{u}_n$ and $\mathbf{d}_n$ respectively.

Hence, it is sufficient to prove that the sequences $(\mathbf{u}_n)_{n\in\mathbb{N}}$ and $(\mathbf{d}_n)_{n\in\mathbb{N}}$ satisfy the Aldous condtion in the spaces V' and L^2 respectively. We begin with the former sequence. We will use Lemma 4.9. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times such that $0\le\tau_n\le T$. From (5.80) we have

$$\mathbf{u}_n(t) = \mathbf{u}_{0n} - \int_0^t \mathscr{A}\mathbf{u}_n(s)\,ds - \int_0^t B_n(\mathbf{u}_n(s))\,ds - \int_0^t M_n(\mathbf{d}_n(s))\,ds$$
$$+\int_0^t\int_Y P_nF(s,\mathbf{u}_n(s),y)\,\tilde{\eta}_1(ds,dy) \qquad =: k_1^n + \sum_{j=2}^5 k_j^n(t), \qquad t\in[0,T].$$

Let $\theta>0$. We will check that each term $k_j^n, j=1,\cdots,5$, satisfies condition (4.4) in Lemma 4.9. It is easy to see that k_1^n satifies condition (4.4) with $\alpha=1$ and $\beta=1$.

Now consider $k_2^n(t)$. Since $\mathscr{A}:\mathrm{V}\to\mathrm{V}'$ and $|\mathscr{A}(\mathbf{u})|_{\mathrm{V}'}\le\|\mathbf{u}\|$, by the Hölder inequality and (6.9) we have

$$\mathbb{E}\big[\big|k_2^n(\tau_n+\theta)-k_2^n(\tau_n)\big|_{\mathrm{V}'}\big] = \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}\mathscr{A}\mathbf{u}_n(s)\,ds\bigg|_{\mathrm{V}'}\bigg] \le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|\mathscr{A}\mathbf{u}_n(s)\big|_{\mathrm{V}'}\,ds\bigg]$$
$$\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\|\mathbf{u}_n(s)\|\,ds\bigg] \le c\,\mathbb{E}\bigg[\theta^{\frac{1}{2}}\bigg(\int_0^T\|\mathbf{u}_n(s)\|^2\,ds\bigg)^{\frac{1}{2}}\bigg] \le c\cdot\bigg(\mathbb{E}\bigg[\int_0^T\|\mathbf{u}_n(s)\|^2\,ds\bigg]\bigg)^{\frac{1}{2}}\cdot\theta^{\frac{1}{2}}$$
$$\le c\sqrt{C_5}\cdot\theta^{\frac{1}{2}} = c_2\cdot\theta^{\frac{1}{2}}.$$

Thus k_2^n satisfies condition (4.4) with $\alpha=1$ and $\beta=\frac{1}{2}$.

In the following calculations we take p and q to be Hölder conjugates i.e. $\frac{1}{p}+\frac{1}{q}=1$. Let us consider the term k_3^n. From Proposition 6.1 for some Hölder conjugates p and q we infer

$$\mathbb{E}\big[\big|k_3^n(\tau_n+\theta)-k_3^n(\tau_n)\big|_{\mathrm{V}'}\big] = \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}B_n(\mathbf{u}_n(s))\,ds\bigg|_{\mathrm{V}'}\bigg] \le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|B(\mathbf{u}_n(s))\big|_{\mathrm{V}'}\,ds\bigg]$$
$$\le c\bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|B(\mathbf{u}_n(s))\big|_{\mathrm{V}'}^p\,ds\bigg]\bigg\}^{\frac{1}{p}}\cdot\bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}1\,ds\bigg]\bigg\}^{\frac{1}{q}} \le c\,\mathcal{C}^{\frac{1}{p}}\cdot\theta^{\frac{1}{q}} =: c_3\cdot\theta^{\frac{1}{q}},$$

where $p=2$ for $\mathbf{n}=2$ and $p=\frac{4}{3}$ for $\mathbf{n}=3$.

Thus k_3^n satisfies condition (4.4) with $\alpha=1$, $\beta=\frac{1}{2}$ for 2-D and $\alpha=1$, $\beta=\frac{1}{4}$ for 3-D.

Now consider k_4^n. From Proposition 6.1 for some Hölder conjugates p and q we infer

$$\mathbb{E}\big[\big|k_4^n(\tau_n+\theta)-k_4^n(\tau_n)\big|_{\mathrm{V}'}\big] = \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} M_n(\mathbf{d}_n(s))\,ds\bigg|_{\mathrm{V}'}\bigg] \le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|M(\mathbf{d}_n(s))\big|_{\mathrm{V}'}\,ds\bigg]$$
$$\le c\bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|M(\mathbf{d}_n(s))\big|_{\mathrm{V}'}^p\,ds\bigg]\bigg\}^{\frac{1}{p}} \cdot \bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} 1\,ds\bigg]\bigg\}^{\frac{1}{q}} \le c\,\mathcal{C}^{\frac{1}{p}}\cdot\theta^{\frac{1}{q}} =: c_4\cdot\theta^{\frac{1}{q}},$$

where $p=2$ for $\mathbf{n}=2$ and $p=\frac{4}{3}$ for $\mathbf{n}=3$.

Thus k_4^n satisfies condition (4.4) with $\alpha=1$, $\beta=\frac{1}{2}$ for 2-D and $\alpha=1$, $\beta=\frac{1}{4}$ for 3-D.

Now consider k_5^n. Since the embedding $\mathrm{H}\hookrightarrow\mathrm{V}'$ is continuous, from (2.21), (6.8) and using Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}\big[\big|k_5^n(\tau_n+\theta)-k_5^n(\tau_n)\big|_{\mathrm{V}'}^2\big] &= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}\int_Y P_nF(s,\mathbf{u}_n(s),y)\,\tilde{\eta}_1(ds,dy)\bigg|_{\mathrm{V}'}^2\bigg]\\
&\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\int_Y \big|P_nF(s,\mathbf{u}_n(s),y)\big|_{\mathrm{H}}^2\,\nu_1(dy)ds\bigg] \le c\cdot\theta + c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}|\mathbf{u}_n(s)|_{\mathrm{H}}^2\,ds\bigg]\\
&\le c\cdot\theta + c\cdot\theta\,\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathbf{u}_n(s)|_{\mathrm{H}}^2\bigg] \le c\cdot\theta(1+C_5) =: c_5\cdot\theta. \qquad (6.19)\end{aligned}$$

Where the constant $C_{1,T}$ is used in (6.8). Thus k_5^n satisfies condition (4.4) with $\alpha=2$ and $\beta=1$.

Hence by Lemma 4.9 the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition in the space V'. Now we will consider the sequence $(\mathbf{d}_n)$. We begin by rewriting (5.81) as

$$\begin{aligned}\mathbf{d}_n(t) &= \mathbf{d}_{0n} - \int_0^t \mathcal{A}\mathbf{d}_n(s)\,ds - \int_0^t \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s))\,ds - \int_0^t f_n(\mathbf{d}_n(s))\,ds\\
&\quad + \int_0^t\int_{\mathbb{B}} G_n(l,\mathbf{d}_n(s))\,\tilde{\eta}_2(ds,dl) + \int_0^t b_n(\mathbf{d}_n(s))\,ds =: j_1^n + \sum_{k=2}^{6} j_k^n(t), \quad t\in[0,T].\end{aligned}$$

It is obvious that j_1^n satifies condition (4.4) with $\alpha=1$ and $\beta=1$.

Now consider $j_2^n(t)$. By (6.6) and the Hölder inequality we get,

$$\mathbb{E}\big[\big|j_2^n(\tau_n+\theta)-j_2^n(\tau_n)\big|_{L^2}\big] = \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}\mathcal{A}\mathbf{d}_n(s)\,ds\bigg|_{L^2}\bigg]$$
$$\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|\mathbf{d}_n(s)\big|_{D(\mathcal{A})}\,ds\bigg] \le c\,\theta^{\frac{1}{2}}\left(\mathbb{E}\bigg[\int_0^T\big|\mathbf{d}_n(s)\big|_{D(\mathcal{A})}^2\,ds\bigg]\right)^{\frac{1}{2}} \le c\,\theta^{\frac{1}{2}}\sqrt{C_4} =: \bar{c}_2\cdot\theta^{\frac{1}{2}}.$$

Thus j_2^n satisfies condition (4.4) with $\alpha=1$ and $\beta=\frac{1}{2}$.

Now consider $j_3^n(t)$. Using Proposition 6.1, with $p=2$ for $\mathbf{n}=2$ and $p=\frac{4}{3}$ for $\mathbf{n}=3$, and where q is the Hölder conjugates of p, we infer we infer

$$\mathbb{E}\big[\big|j_3^n(\tau_n+\theta)-j_3^n(\tau_n)\big|_{L^2}\big] \le \mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}|\tilde{B}(\mathbf{u}_n(s),\mathbf{d}_n(s))|_{L^2}\,ds\bigg] \qquad (6.20)$$
$$\le \bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|\tilde{B}(\mathbf{u}_n(s),\mathbf{d}_n(s))\big|_{L^2}^p\,ds\bigg]\bigg\}^{\frac{1}{p}} \cdot \bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}1\,ds\bigg]\bigg\}^{\frac{1}{q}} \le c\,\mathcal{C}^{\frac{1}{p}}\cdot\theta^{\frac{1}{q}} =: \bar{c}_3\cdot\theta^{\frac{1}{q}},$$

Thus $j_3^n(t)$ satisfies condition (4.4) with $\alpha=1$ and $\beta=\frac{1}{2}$ for $\mathbf{n}=2$ and $\alpha=1$ and $\beta=\frac{1}{4}$ for $\mathbf{n}=3$.

At last consider $j_4^n(t)$. Using Proposition 6.1 we obtain,

$$\mathbb{E}\big[\big|j_4^n(\tau_n+\theta)-j_4^n(\tau_n)\big|_{L^2}\big] \le \mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}|f(\mathbf{d}_n(s))|_{L^2}\,ds\bigg] \qquad (6.21)$$
$$\le \bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}|f(\mathbf{d}_n(s))|_{L^2}^2\,ds\bigg]\bigg\}^{\frac{1}{2}} \cdot \bigg\{\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}1\,ds\bigg]\bigg\}^{\frac{1}{2}} \le c\,\mathcal{C}^{\frac{1}{2}}\cdot\theta^{\frac{1}{2}} := \bar{c}_4\cdot\theta^{\frac{1}{2}}. \qquad (6.22)$$

Thus $j_4^n(t)$ satisfies condition (4.4) with $\alpha=1$ and $\beta=\frac{1}{2}$.

Using Itô isometry, from Lemma 5.5 we obtain,

$$\mathbb{E}\big[\big|j_5^n(\tau_n+\theta)-j_5^n(\tau_n)\big|_{L^2}\big]=\mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}\int_{\mathbb{B}}G_n(l,\mathbf{d}_n(s))\,\tilde{\eta}_2(ds,dl)\bigg|_{L^2}\bigg]$$
$$\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta}\int_{\mathbb{B}}\big|G_n(l,\mathbf{d}_n(s))\big|_{L^2}^2\nu_2(dl)ds\bigg]\le\tilde{c}\cdot\theta\,\mathbb{E}\bigg[\sup_{s\in[0,T]}\big|\mathbf{d}_n(s)\big|_{L^2}^2\bigg]\le\bar{c}_5\cdot\theta. \tag{6.23}$$

Thus $j_5^n(t)$ satisfies the condition (4.4) with $\alpha=2$ and $\beta=1$.

At last consider $j_6^n(t)$. Using Schwarz's inequality, from Lemma 5.5 we infer,

$$\mathbb{E}\big[\big|j_6^n(\tau_n+\theta)-j_5^n(\tau_n)\big|_{L^2}\big]=\mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}b_n(\mathbf{d}_n(s))\,ds\bigg|_{L^2}\bigg]$$
$$\le c\,\theta^{\frac{1}{2}}\,\mathbb{E}\bigg[\bigg(\int_{\tau_n}^{\tau_n+\theta}|b_n(\mathbf{d}_n(s))|_{L^2}^2\,ds\bigg)^{\frac{1}{2}}\bigg]\le\tilde{c}\,\theta^{\frac{1}{2}}\,\mathbb{E}\bigg[\bigg(\int_{\tau_n}^{\tau_n+\theta}|\mathbf{d}_n(s)|_{L^2}^2\,ds\bigg)^{\frac{1}{2}}\bigg]$$
$$\le\tilde{c}\,\theta\,\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathbf{d}_n(s)|_{L^2}^2\bigg]^{\frac{1}{2}}\le\tilde{c}\,\theta\bigg\{\mathbb{E}\bigg[\sup_{s\in[0,T]}|\mathbf{d}_n(s)|_{L^2}^2\bigg]\bigg\}^{\frac{1}{2}}\le\bar{c}_6\cdot\theta. \tag{6.24}$$

Thus $j_6^n(t)$ satisfies the condition (4.4) with $\alpha=1$ and $\beta=1$.
So this is enough to gurantee that the laws are tight.

□

7. Existence of a Martingale Solution

We will now prove the existence of a martingale solution. The main difficulties lie in the terms containing the nonlinearity of B,M and the noise terms F and G in both the equations. The Skorokhod Theorem for nonmetric spaces helps us constructing a martingale solution. Let us define $M_{\bar{\mathbb{N}}}([0,T]\times Y)$ as the set of all $\bar{\mathbb{N}}$-valued measures on the measurable space $\big([0,T]\times Y,\mathscr{B}([0,T])\otimes\mathscr{B}(Y)\big)$. Similarly, we define $M_{\bar{\mathbb{N}}}([0,T]\times\mathbb{B})$.

7.1. **Construction of a new probability space and processes.** By Lemma 6.2 we have shown the set of measures $\{\mathscr{L}(\mathbf{u}_n,\mathbf{d}_n),n\in\mathbb{N}\}$ is tight on $(\mathcal{Z}_{T,1}\times\mathcal{Z}_{T,2},\mathscr{T})$.

Let us denote $(\eta_n)_{n\in\mathbb{N}}:=\big(\eta_{1n},\eta_{2n}\big)_{n\in\mathbb{N}}$ and $\eta_*=\big(\eta_{1*},\eta_{2*}\big)$. Similarly, we define $\bar{\eta}_n$ and $\tilde{\eta}_n$. Let $\big(\eta_{1n},\eta_{2n}\big):=(\eta_1,\eta_2)$, for $n\in\mathbb{N}$. Then the set of measures $\{\mathscr{L}(\eta_{1n},\eta_{2n}),n\in\mathbb{N}\}$ is tight on the space $M_{\bar{\mathbb{N}}}([0,T]\times Y)\times M_{\bar{\mathbb{N}}}([0,T]\times\mathbb{B})$. Thus the set $\{\mathscr{L}(\mathbf{u}_n,\mathbf{d}_n,\eta_n),\ n\in\mathbb{N}\}$ is tight on $\mathcal{Z}_T\times M_{\bar{\mathbb{N}}}([0,T]\times Y)\times M_{\bar{\mathbb{N}}}([0,T]\times\mathbb{B})$.

By Theorem 4.13, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ and on this space, $\mathcal{Z}_T\times M_{\bar{\mathbb{N}}}([0,T]\times Y)\times M_{\bar{\mathbb{N}}}([0,T]\times\mathbb{B})$-valued random variables $(\mathbf{u}_*,\mathbf{d}_*,\eta_*)$, $(\bar{\mathbf{u}}_k,\bar{\mathbf{d}}_k,\bar{\eta}_k)$, $k\in\mathbb{N}$ such that

- (a) $\mathscr{L}\big((\bar{\mathbf{u}}_k,\bar{\mathbf{d}}_k,\bar{\eta}_k)\big)=\mathscr{L}\big((\mathbf{u}_{n_k},\mathbf{d}_{n_k},\eta_{n_k})\big)$ for all $k\in\mathbb{N}$;
- (b) $(\bar{\mathbf{u}}_k,\bar{\mathbf{d}}_k,\bar{\eta}_k)\to(\mathbf{u}_*,\mathbf{d}_*,\eta_*)$ in $\mathcal{Z}_T\times M_{\bar{\mathbb{N}}}([0,T]\times Y)\times M_{\bar{\mathbb{N}}}([0,T]\times\mathbb{B})$ with probability 1 on $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ as $k\to\infty$;
- (c) $\bar{\eta}_k(\bar{\omega})=\eta_*(\bar{\omega})$ for all $\bar{\omega}\in\bar{\Omega}$.

We will continue to denote these sequences by $\big((\mathbf{u}_n,\mathbf{d}_n,\eta_n)\big)_{n\in\mathbb{N}}$ and $\big((\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_n)\big)_{n\in\mathbb{N}}$.

Using the definiton of the space $\mathcal{Z}_T$, we deduce that $\bar{\mathbb{P}}$-a.s.

$$\bar{\mathbf{u}}_n\to\mathbf{u}_*\ in\ L_w^2(0,T;\mathrm{V})\cap L^2(0,T;\mathrm{H})\cap\mathbb{D}([0,T];\mathrm{V}')\cap\mathbb{D}([0,T];\mathrm{H}_w) \tag{7.1}$$

and

$$\bar{\mathbf{d}}_n\to\mathbf{d}_*\ \text{in}\ L_w^2(0,T;D(\mathcal{A}))\cap L^2(0,T;H^1)\cap\mathbb{D}([0,T];L^2)\cap\mathbb{D}([0,T];H_w^1). \tag{7.2}$$

7.2. **Properties of the new processes and the limiting processes.** It is easy to verify that the spaces $\mathcal{Z}_{T,1}$ and $\mathcal{Z}_{T,2}$ (defined in Section 6) are not Polish spaces. So the following result cannot be deduced from the Kuratowski theorem [31] directly, see Lemma 4.2 in [16].

Proposition 7.1. *The sets $\mathbb{D}([0,T],\mathrm{H}_n)\cap\mathcal{Z}_{T,1}$ and $\mathbb{D}([0,T],\mathbb{L}_n)\cap\mathcal{Z}_{T,2}$ are Borel subsets of $\mathcal{Z}_{T,1}$ and $\mathcal{Z}_{T,2}$ respectively and the corresponding embeddings transforms Borel sets into Borel subsets.*

Proof. The space $\mathbb{D}([0,T],\mathrm{V}')\cap L^2(0,T;\mathrm{H})$ is a Polish space. Then by Kuratowski theorem, $\mathbb{D}([0,T],\mathrm{H}_n)$ is a Borel subset of $\mathbb{D}([0,T],\mathrm{V}')\cap L^2(0,T;\mathrm{H})$. Hence $\mathbb{D}([0,T],\mathrm{H}_n)\cap\mathcal{Z}_{T,1}$ is a Borel subset of $\mathbb{D}([0,T],\mathrm{V}')\cap L^2(0,T;\mathrm{H})\cap\mathcal{Z}_{T,1}=\mathcal{Z}_{T,1}$. Similarly one can show $\mathbb{D}([0,T],\mathbb{L}_n)\cap\mathcal{Z}_{T,2}$ is a Borel subset of $\mathcal{Z}_{T,2}$. □

Since the laws of $\mathbf{u}_n$ and $\bar{\mathbf{u}}_n$ are same in the space $\mathcal{Z}_{T,1}$, from equations (6.8), (6.10) and Proposition 7.1 we have,

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}\big|\bar{\mathbf{u}}_n(s)\big|_{\mathrm{H}}^2\Big]\le C_5, \tag{7.3}$$

and

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\int_0^T\big|\bar{\mathbf{u}}_n(s)\big|_{\mathrm{V}}^2\,ds\Big]\le C_6. \tag{7.4}$$

From (7.3) and Banach-Alaoglu Theorem we conclude there exists a subsequence of $(\bar{\mathbf{u}}_n)$ convergent weak star in $L^2(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$. So from (7.1) we infer $\mathbf{u}_*\in L^2(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$, i.e.,

$$\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\mathbf{u}_*(t)\big|_{\mathrm{H}}^2\Big]<\infty, \tag{7.5}$$

Similarly by (7.1), (7.4) and Banach-Alaoglu Theorem, there exists a subsequence of $(\bar{\mathbf{u}}_n)$, weakly convergent in $L^2([0,T]\times\bar{\Omega};\mathrm{V})$, i.e.,

$$\bar{\mathbb{E}}\Big[\int_0^T\big|\mathbf{u}_*(t)\big|_{\mathrm{V}}^2\,ds\Big]<\infty. \tag{7.6}$$

Also from (6.7) we get,

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}\big|\bar{\mathbf{u}}_n(s)\big|_{\mathrm{H}}^{2p}\Big]\le C_5, \tag{7.7}$$

From Proposition A.5 in Appendix we obtain,

$$\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}\big|\mathbf{u}_*(s)\big|_{\mathrm{H}}^{2p}\Big]<C_p. \tag{7.8}$$

From Proposition 5.12, for $p\ge 2$ we observe,

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\int_0^T\|\bar{\mathbf{u}}_n(s)\|^2\,ds\Big]^p\le C. \tag{7.9}$$

Since the laws of $\mathbf{d}_n$ and $\bar{\mathbf{d}}_n$ are same in the space $\mathcal{Z}_{T,2}$, from (6.3) and (6.5), we have for $p=2$,

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}\big|\bar{\mathbf{d}}_n(s)\big|_{L^2}^2\Big]\le C_1, \tag{7.10}$$

and

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\Big[\int_0^T\big|\bar{\mathbf{d}}_n(s)\big|_{H^1}^2\,ds\Big]\le C_3. \tag{7.11}$$

From (7.10) and Banach-Alaoglu Theorem we conclude there exists a subsequence of $(\bar{\mathbf{d}}_n)$ convergent weak star in $L^2(\bar{\Omega};L^\infty(0,T;L^2))$. So from (7.2) we infer $\mathbf{d}_*\in L^2(\bar{\Omega};L^\infty(0,T;L^2))$.

Similarly for $p=2$, from (7.2), (7.11) and Banach-Alaoglu Theorem, there exists a subsequence of $(\bar{\mathbf{d}}_n)$, weakly convergent in $L^2([0,T]\times\bar{\Omega};H^1)$.

From (6.3) we also have for $p \geq 2$,

$$\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]} \big|\bar{\mathbf{d}}_n(s)\big|^p_{L^2}\Big] < C_1, \tag{7.12}$$

Again from Corollary 5.14 we obtain,

$$\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]} \big|\bar{\mathbf{d}}_n(s)\big|^q_{H^1}\Big] \leq C_q. \tag{7.13}$$

From Proposition A.6 in Appendix we obtain for $q > 2$,

$$\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]} \big|\mathbf{d}_*(s)\big|^q_{H^1}\Big] \leq C_q. \tag{7.14}$$

Similarly, from Proposition 5.13 we have,

$$\bar{\mathbb{E}}\Big[\int_0^T \big|\bar{\mathbf{d}}_n(s)\big|^2_{D(\mathcal{A})}\, ds\Big]^q \leq C(q). \tag{7.15}$$

Using Banach-Alaoglu Theorem, we have a subsequence of $\mathbf{d}_n$, convergent weakly in $L^2([0,T]\times\bar{\Omega}; D(\mathcal{A}))$. As from (7.2), $\mathbf{d}_n \to \mathbf{d}_*$ in $L^2_w([0,T]; D(\mathcal{A}))$, we obtain for $q = 1$,

$$\bar{\mathbb{E}}\Big[\int_0^T \big|\mathbf{d}_*(s)\big|^2_{D(\mathcal{A})}\, ds\Big] \leq C. \tag{7.16}$$

7.3. **Convergence of the New Processes to the Corresponding Limiting Processes.** Let us fix $v \in \mathrm{V}$ and denote for $t \in [0,T]$

$$\begin{aligned}\mathscr{K}_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{1n}, v)(t) := \big(\bar{\mathbf{u}}_n(0), v\big)_{\mathrm{H}} - \int_0^t \big\langle \bar{\mathbf{u}}_n(s), \mathscr{A} v\big\rangle\, ds - \int_0^t \big\langle B_n(\bar{\mathbf{u}}_n(s)), v\big\rangle\, ds \\ - \int_0^t \big\langle M_n(\bar{\mathbf{d}}_n(s)), v\big\rangle\, ds + \int_0^t \int_Y \big(P_n F(s, \bar{\mathbf{u}}_n(s); y), v\big)_{\mathrm{H}}\, \tilde{\bar{\eta}}_{1n}(ds, dy),\end{aligned} \tag{7.17}$$

and fixing $v \in D(\mathcal{A})$, we denote

$$\begin{aligned}\Lambda_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{2n}, v)(t) := \big(\bar{\mathbf{d}}_n(0), v\big)_{L^2} - \int_0^t \big(\bar{\mathbf{d}}_n(s), \mathcal{A} v\big)_{L^2}\, ds - \int_0^t \big(\tilde{B}_n(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s), v\big)_{L^2}\, ds \\ - \int_0^t \big(f_n(\bar{\mathbf{d}}_n(s)), v\big)_{L^2}\, ds + \int_0^t \int_{\mathbb{B}} \big(G_n(l, \bar{\mathbf{d}}_n(s)), v\big)_{L^2}\, \tilde{\bar{\eta}}_{2n}(ds, dl) \\ + \int_0^t \big(b_n(\bar{\mathbf{d}}_n(s)), v\big)_{L^2}\, ds, \qquad t \in [0,T].\end{aligned} \tag{7.18}$$

Now for the limiting processes we denote for $v \in \mathrm{V}$,

$$\begin{aligned}\mathscr{K}(\mathbf{u}_*, \mathbf{d}_*, \eta_{1*}, v)(t) := \big(\mathbf{u}_*(0), v\big)_{\mathrm{H}} - \int_0^t \big\langle \mathbf{u}_*(s), \mathscr{A} v\big\rangle\, ds - \int_0^t \big\langle B(\mathbf{u}_*(s)), v\big\rangle\, ds \\ - \int_0^t \big\langle M(\mathbf{d}_*(s)), v\big\rangle\, ds + \int_0^t \int_Y \big(F(s, \mathbf{u}_*(s); y), v\big)_{\mathrm{H}}\, \tilde{\eta}_{1*}(ds, dy),\ \ t \in [0,T]\end{aligned} \tag{7.19}$$

and for $v \in D(\mathcal{A})$,

$$\begin{aligned}\Lambda(\mathbf{u}_*, \mathbf{d}_*, \eta_{2*}, v)(t) := \big(\mathbf{d}_*(0), v\big)_{L^2} - \int_0^t \big(\mathbf{d}_*(s), \mathcal{A}v\big)_{L^2}\, ds - \int_0^t \big(\tilde{B}(\mathbf{u}_*(s), \mathbf{d}_*(s)), v\big)_{L^2}\, ds \\ - \int_0^t \big(f(\mathbf{d}_*(s)), v\big)_{L^2}\, ds + \int_0^t \int_{\mathbb{B}} \big(G(l, \mathbf{d}_*(s)), v\big)_{L^2}\, \tilde{\eta}_{2*}(ds, dl) \\ + \int_0^t \big(b(\mathbf{d}_*(s)), v\big)_{L^2}\, ds, \qquad t \in [0,T]. \end{aligned} \tag{7.20}$$

We will show that

$$\lim_{n\to\infty} |\mathscr{K}_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{1n}, v) - \mathscr{K}(\mathbf{u}_*, \mathbf{d}_*, \eta_{1*}, v)|_{L^2([0,T]\times\bar{\Omega})} = 0. \tag{7.21}$$

and

$$\lim_{n\to\infty} |\Lambda_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{2n}, v) - \Lambda(\mathbf{u}_*, \mathbf{d}_*, \eta_{2*}, v)|_{L^2([0,T]\times\bar{\Omega})} = 0. \tag{7.22}$$

Now for proving (7.21), using Fubini's Theorem, we have

$$\begin{aligned} &|\mathscr{K}_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{1n}, v) - \mathscr{K}(\mathbf{u}_*, \mathbf{d}_*, \eta_{1*}, v)|^2_{L^2([0,T]\times\bar{\Omega})} \\ &= \int_0^T \int_{\bar{\Omega}} |\mathscr{K}_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{1n}, v)(t) - \mathscr{K}(\mathbf{u}_*, \mathbf{d}_*, \eta_{1*}, v)(t)|^2\, d\bar{\mathbb{P}}(\omega)\; dt \\ &= \int_0^T \bar{\mathbb{E}}\big[|\mathscr{K}_n(\bar{\mathbf{u}}_n, \bar{\mathbf{d}}_n, \bar{\eta}_{1n}, v)(t) - \mathscr{K}(\mathbf{u}_*, \mathbf{d}_*, \eta_{1*}, v)(t)|^2\big]\, dt. \end{aligned} \tag{7.23}$$

So we will show each term of right hand side of (7.17) tends to the corresponding terms in (7.19) in $L^2([0,T]\times\bar{\Omega})$. Similarly for proving (7.22), we will show each term of right hand side of (7.18) tends to the corresponding terms in (7.20) in $L^2([0,T]\times\bar{\Omega})$. So we need to prove the following Lemmas.

Lemma 7.2. *For all $v \in \mathrm{V}$*

(a) $\lim_{n\to\infty} \bar{\mathbb{E}}\big[\int_0^T \big|(\bar{\mathbf{u}}_n(t) - \mathbf{u}_*(t), v)_{\mathrm{H}}\big|^2\, dt\big] = 0$,
(b) $\lim_{n\to\infty} \big|(\bar{\mathbf{u}}_n(0) - \mathbf{u}_*(0), v)_{\mathrm{H}}\big|^2_{L^2([0,T]\times\bar{\Omega})} = 0$,
(c) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \langle \bar{\mathbf{u}}_n(s) - \mathbf{u}_*(s), \mathscr{A}v\rangle\, ds\big|^2\big]\, dt = 0$,
(d) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big\langle B_n\big(\bar{\mathbf{u}}_n(s)\big) - B\big(\mathbf{u}_*(s)\big), v\big\rangle\, ds\big|^2\big]\, dt = 0$,
(e) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big\langle M_n\big(\bar{\mathbf{d}}_n(s)\big) - M\big(\mathbf{d}_*(s)\big), v\big\rangle\, ds\big|^2\big]\, dt = 0$,
(f) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \int_Y \big(P_n F(s, \bar{\mathbf{u}}_n(s), y) - F(s, \mathbf{u}_*(s), y), v\big)_{\mathrm{H}}\, \tilde{\eta}_{1*}(ds, dy)\big|^2\big]\, dt = 0$.

Lemma 7.3. *For all $v \in D(\mathcal{A})$*

(a) $\lim_{n\to\infty} \bar{\mathbb{E}}\big[\int_0^T \big|(\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t), v)_{L^2}\big|^2\, dt\big] = 0$,
(b) $\lim_{n\to\infty} \big|\big(\bar{\mathbf{d}}_n(0) - \mathbf{d}_*(0), v\big)_{L^2}\big|^2_{L^2([0,T]\times\bar{\Omega})} = 0$,
(c) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big(\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s), \mathcal{A}v\big)_{L^2}\, ds\big|^2\big]\, dt = 0$,
(d) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big(\tilde{B}_n\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big) - \tilde{B}\big(\mathbf{u}_*(s), \mathbf{d}_*(s)\big), v\big)_{L^2}\, ds\big|^2\big]\, dt = 0$,
(e) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big(f_n\big(\bar{\mathbf{d}}_n(s)\big) - f\big(\mathbf{d}_*(s)\big), v\big)_{L^2}\, ds\big|^2\big]\, dt = 0$.
(f) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \int_{\mathbb{B}} \big(G_n\big(l, \bar{\mathbf{d}}_n(s)\big) - G\big(l, \mathbf{d}_*(s)\big), v\big)_{L^2}\, \tilde{\eta}_{2*}(ds, dy)\big|^2\, dt\big] = 0$;
(g) $\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\big[\big|\int_0^t \big(b_n\big(\bar{\mathbf{d}}_n(s)\big) - b\big(\mathbf{d}_*(s)\big), v\big)_{L^2}\, ds\big|^2\big]\, dt = 0$.

Proof. First we establish the proof of Lemma 7.2.

(a) Let us consider

$$|(\bar{\mathbf{u}}_n(\cdot), v)_{\mathrm{H}} - (\mathbf{u}_*(\cdot), v)_{\mathrm{H}}|^2_{L^2([0,T]\times\bar{\Omega})} = \bar{\mathbb{E}}\bigg[\int_0^T \big|(\bar{\mathbf{u}}_n(t) - \mathbf{u}_*(t), v)_{\mathrm{H}}\big|^2\, dt\bigg] \tag{7.24}$$

Moreover,

$$\int_0^T \left|(\bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t),v)_{\mathrm{H}}\right|^2 dt = \int_0^T \left|{}_{\mathrm{V}'}\langle \bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t),v\rangle_{\mathrm{V}}\right|^2 dt \le |v|_{\mathrm{V}}^2 \int_0^T \left|\bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t)\right|_{\mathrm{V}'}^2 dt \tag{7.25}$$

By (7.1), $\bar{\mathbf{u}}_n \to \mathbf{u}_*$ in $\mathbb{D}([0,T];\mathrm{V}')$ and from (7.3), $\sup_{t\in[0,T]}|\bar{\mathbf{u}}_n(t)|_{\mathrm{H}}^2<\infty$, $\bar{\mathbb{P}}$-a.s.. The embedding $\mathrm{H}\hookrightarrow\mathrm{V}'$ is continuous. Then by Dominated Convergence Theorem we observe that $\bar{\mathbf{u}}_n\to\mathbf{u}_*$ in $L^2(0,T;\mathrm{V}')$. So from (7.25),

$$\lim_{n\to\infty}\int_0^T \left|(\bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t),v)_{\mathrm{H}}\right|^2 dt = 0. \tag{7.26}$$

Moreover, from (7.7), Proposition A.5 in Appendix and using the Hölder inequality, for every $n\in\mathbb{N}$ and every $r>1$ we obtain

$$\begin{aligned}\bar{\mathbb{E}}\left[\left|\int_0^T \left|\bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t)\right|_{\mathrm{H}}^2 dt\right|^{2+r}\right] &\le c\,\bar{\mathbb{E}}\left[\int_0^T \left(\left|\bar{\mathbf{u}}_n(t)\right|_{\mathrm{H}}^{2(2+r)}+\left|\mathbf{u}_*(t)\right|_{\mathrm{H}}^{2(2+r)} dt\right)\right]\\ &\le \tilde{c}\,\bar{\mathbb{E}}\left[\sup_{t\in[0,T]}\left|\bar{\mathbf{u}}_n(t)\right|_{\mathrm{H}}^{2(2+r)}\right] \le \tilde{c}\cdot C(4+2r,T)<\infty.\end{aligned} \tag{7.27}$$

for some constant $\tilde{c}>0$. Then by (7.26), (7.27) and Vitali's Theorem we obtain

$$\lim_{n\to\infty}\bar{\mathbb{E}}\left[\int_0^T \left|(\bar{\mathbf{u}}_n(t)-\mathbf{u}_*(t),v)_{\mathrm{H}}\right|^2 dt\right]=0, \quad \text{which proves (a).}$$

(b) From (7.1), $\bar{\mathbf{u}}_n\to\mathbf{u}_*$ in $\mathbb{D}([0,T];\mathrm{H}_w)$ $\bar{\mathbb{P}}$-a.s. and $\mathbf{u}_*$ is right continuous at $t=0$. So we obtain, by Remark 4.1, $(\bar{\mathbf{u}}_n(0),v)_{\mathrm{H}}\to(\mathbf{u}_*(0),v)_{\mathrm{H}},\bar{\mathbb{P}}$-a.s. From (7.3) and applying Vitali's Theorem, we get

$$\lim_{n\to\infty}\left|(\bar{\mathbf{u}}_n(0)-\mathbf{u}_*(0),v)_{\mathrm{H}}\right|_{L^2([0,T]\times\bar{\Omega})}^2=0. \tag{7.28}$$

(c) Now from (7.1), $\bar{\mathbf{u}}_n\to\mathbf{u}_*$ in $L_w^2(0,T;\mathrm{V}),\bar{\mathbb{P}}$-a.s., then from (2.16) and for all $v\in\mathrm{V}$ we obtain $\bar{\mathbb{P}}$-a.s.,

$$\lim_{n\to\infty}\int_0^t\langle\bar{\mathbf{u}}_n(s),\mathscr{A}v\rangle\,ds=\lim_{n\to\infty}\int_0^t((\bar{\mathbf{u}}_n(s),v))\,ds=\int_0^t((\mathbf{u}_*(s),v))\,ds=\int_0^t\langle\mathbf{u}_*(s),\mathscr{A}v\rangle\,ds. \tag{7.29}$$

By (7.9) and using the Hölder inequality we obtain for all $t\in[0,T], r>2$ and $n\in\mathbb{N}$,

$$\begin{aligned}\bar{\mathbb{E}}\left[\left|\int_0^t\langle\bar{\mathbf{u}}_n(s),\mathscr{A}v\rangle\,ds\right|^{2+r}\right] &= \bar{\mathbb{E}}\left[\left|\int_0^t((\bar{\mathbf{u}}_n(s),v))\,ds\right|^{2+r}\right]\\ &\le \bar{\mathbb{E}}\left[\left(\int_0^t\|\bar{\mathbf{u}}_n(s)\||v|_{\mathrm{V}}\,ds\right)^{2+r}\right]\le\tilde{c}\,\bar{\mathbb{E}}\left[\left(\int_0^T\|\bar{\mathbf{u}}_n(s)\|^2\,ds\right)^{1+\frac{r}{2}}\right]\le C\end{aligned} \tag{7.30}$$

for some constant $C>0$. Then by (7.29), (7.30) and using Vitali's Theorem we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\left[\left|\int_0^t\langle\bar{\mathbf{u}}_n(s)-\mathbf{u}_*(s),\mathscr{A}v\rangle\,ds\right|^2\right]=0 \tag{7.31}$$

Now from (7.4), using Dominated Convergence Theorem, for all $t\in[0,T]$ and all $n\in\mathbb{N}$ we get,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\left[\left|\int_0^t\langle\bar{\mathbf{u}}_n(s)-\mathbf{u}_*(s),\mathscr{A}v\rangle\,ds\right|^2\right]dt=0. \tag{7.32}$$

Now we advance to the nonlinear term.

(d) From Lemma A.4 of Appendix, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t \big\langle B_n\big(\bar{\mathbf{u}}_n(s),\bar{\mathbf{u}}_n(s)\big)-B\big(\mathbf{u}_*(s),\mathbf{u}_*(s)\big),v\big\rangle\,ds\\&=\lim_{n\to\infty}\int_0^t \big\langle B\big(\bar{\mathbf{u}}_n(s),\bar{\mathbf{u}}_n(s)\big)-B\big(\mathbf{u}_*(s),\mathbf{u}_*(s)\big),P_nv\big\rangle\,ds=0\quad \bar{\mathbb{P}}\text{-a.s.}\end{aligned}\tag{7.33}$$

Now from (2.11) and using the Hölder inequality, we obtain for all $t\in[0,T], r>1$ and $n\in\mathbb{N}$

$$\begin{aligned}&\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big\langle B_n\big(\bar{\mathbf{u}}_n(s)\big),v\big\rangle\,ds\bigg|^r\bigg]\le\bar{\mathbb{E}}\bigg[\bigg(\int_0^t|B_n\big(\bar{\mathbf{u}}_n(s)\big)|_{\mathrm{V}'}|v|_{\mathrm{V}}\,ds\bigg)^r\bigg]\\&\le|v|_{\mathrm{V}}^r\,t^{r-1}\,\bar{\mathbb{E}}\bigg[\int_0^t|B_n\big(\bar{\mathbf{u}}_n(s)\big)|_{\mathrm{V}'}^r\,ds\bigg]\le c_t\,\bar{\mathbb{E}}\bigg[\int_0^t|\bar{\mathbf{u}}_n(s)|^{2r-\frac{r\mathbf{n}}{2}}\|\bar{\mathbf{u}}_n(s)\|^{\frac{r\mathbf{n}}{2}}\,ds\bigg]\end{aligned}\tag{7.34}$$

Now we will estimate separately for $\mathbf{n}=2$ and 3. First consider the case for $\mathbf{n}=2$. From (7.7) and (7.9), for $r\in(1,2]$, we have

$$\bar{\mathbb{E}}\bigg[\int_0^t|\bar{\mathbf{u}}_n(s)|^r\|\bar{\mathbf{u}}_n(s)\|^r\,ds\bigg]\le\bigg\{\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\bar{\mathbf{u}}_n|^{2r}\bigg]\bigg\}^{\frac12}\bigg\{\bar{\mathbb{E}}\bigg[\int_0^t\|\bar{\mathbf{u}}_n\|^r\,ds\bigg]^2\bigg\}^{\frac12}\le C(T,r).\tag{7.35}$$

Now for $\mathbf{n}=3$, from (7.7) and (7.9), for $r\in[1,\frac43)$, we obtain

$$\begin{aligned}&\bar{\mathbb{E}}\bigg[\int_0^t|\bar{\mathbf{u}}_n|^{\frac r2}\|\bar{\mathbf{u}}_n\|^{\frac{3r}{2}}\,ds\bigg]\le\bar{\mathbb{E}}\bigg[\bigg(\int_0^t(|\bar{\mathbf{u}}_n|^{\frac r2})^{\frac{4}{4-3r}}\,ds\bigg)^{\frac{4-3r}{4}}\bigg(\int_0^t(\|\bar{\mathbf{u}}_n\|^{\frac{3r}{2}})^{\frac{4}{3r}}\,ds\bigg)^{\frac{3r}{4}}\bigg]\\&\le c\,\bigg\{\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\bar{\mathbf{u}}_n|^{\frac{2r}{4-3r}}\bigg]\bigg\}^{\frac{4-3r}{4}}\bigg\{\bar{\mathbb{E}}\bigg[\int_0^T\|\bar{\mathbf{u}}_n\|^2\,ds\bigg]\bigg\}^{\frac{3r}{4}}\le\bar{C}(T,r).\end{aligned}\tag{7.36}$$

So from (7.33), (7.34), (7.35), (7.36) and using Vitali's Theorem we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big\langle B_n\big(\bar{\mathbf{u}}_n(s)\big)-B\big(\mathbf{u}_*(s)\big),v\big\rangle\,ds\bigg|^2\bigg]=0.\tag{7.37}$$

Then from (7.37) and Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big\langle B_n\big(\bar{\mathbf{u}}_n(s)\big)-B\big(\mathbf{u}_*(s)\big),v\big\rangle\,ds\bigg|^2\bigg]\,dt=0.\tag{7.38}$$

(e) Now we come to the second nonlinear term. From Lemma A.2 we have,

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\big\langle M_n\big(\bar{\mathbf{d}}_n(s)\big)-M\big(\mathbf{d}_*(s)\big),v\big\rangle\,ds\\&=\lim_{n\to\infty}\int_0^t\big\langle M\big(\bar{\mathbf{d}}_n(s)\big)-M\big(\mathbf{d}_*(s)\big),P_nv\big\rangle\,ds=0\quad\bar{\mathbb{P}}\text{-a.s.}\end{aligned}\tag{7.39}$$

Now from (2.15) and using the Hölder inequality, we obtain for all $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned}&\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big\langle M_n\big(\bar{\mathbf{d}}_n(s)\big),v\big\rangle\,ds\bigg|^r\bigg]\le\bar{\mathbb{E}}\bigg[\bigg(\int_0^t|M_n\big(\bar{\mathbf{d}}_n(s)\big)|_{\mathrm{V}'}|v|_{\mathrm{V}}\,ds\bigg)^r\bigg]\\&\le|v|_{\mathrm{V}}^r\,t^{r-1}\,\bar{\mathbb{E}}\bigg[\int_0^t|M_n\big(\bar{\mathbf{d}}_n(s)\big)|_{\mathrm{V}'}^r\,ds\bigg]\le c_t\,\bar{\mathbb{E}}\bigg[\int_0^t|\nabla\bar{\mathbf{d}}_n(s)|_{L^2}^{2r-\frac{r\mathbf{n}}{2}}|\Delta\bar{\mathbf{d}}_n(s)|_{L^2}^{\frac{r\mathbf{n}}{2}}\,ds\bigg]\end{aligned}\tag{7.40}$$

Now we will estimate separately for $\mathbf{n}=2$ and 3. First consider the case for $\mathbf{n}=2$. From (7.13) and (7.15), for $r\in(1,2]$, we have

$$\bar{\mathbb{E}}\bigg[\int_0^t|\nabla\bar{\mathbf{d}}_n(s)|_{L^2}^r|\Delta\bar{\mathbf{d}}_n(s)|_{L^2}^r\,ds\bigg]\le\bigg\{\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\nabla\bar{\mathbf{d}}_n(s)|_{L^2}^{2r}\bigg]\bigg\}^{\frac12}\bigg\{\bar{\mathbb{E}}\bigg[\int_0^t|\Delta\bar{\mathbf{d}}_n|^r\,ds\bigg]^2\bigg\}^{\frac12}\le C_{r,T}.\tag{7.41}$$

Now for $\mathbf{n}=3$, from (7.13) and (7.15), for $r\in[1,\frac{4}{3})$, we obtain

$$\bar{\mathbb{E}}\bigg[\int_0^t |\nabla \bar{\mathbf{d}}_n(s)|_{L^2}^{\frac{r}{2}}|\Delta\bar{\mathbf{d}}_n|^{\frac{3r}{2}}\,ds\bigg] \le \bar{\mathbb{E}}\bigg[\bigg(\int_0^t (|\nabla\bar{\mathbf{d}}_n(s)|_{L^2}^{\frac{r}{2}})^{\frac{4}{4-3r}}\,ds\bigg)^{\frac{4-3r}{4}}\bigg(\int_0^t (|\Delta\bar{\mathbf{d}}_n|^{\frac{3r}{2}})^{\frac{4}{3r}}\,ds\bigg)^{\frac{3r}{4}}\bigg]$$
$$\le c\left\{\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\nabla\bar{\mathbf{d}}_n(s)|_{L^2}^{\frac{2r}{4-3r}}\bigg]\right\}^{\frac{4-3r}{4}}\left\{\bar{\mathbb{E}}\bigg[\int_0^T |\Delta\bar{\mathbf{d}}_n|^2\,ds\bigg]\right\}^{\frac{3r}{4}} \le C(r,T). \tag{7.42}$$

So from (7.39), (7.40), (7.41), (7.42) and using Vitali's Theorem we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t \big\langle M_n\big(\bar{\mathbf{d}}_n(s)\big)-M\big(\mathbf{d}_*(s)\big),v\big\rangle\,ds\bigg|^2\bigg]=0. \tag{7.43}$$

From Proposition 6.1 and (7.43), then using Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\bigg[\bigg|\int_0^t \big\langle M_n\big(\bar{\mathbf{d}}_n(s)\big)-M\big(\mathbf{d}_*(s)\big),v\big\rangle\,ds\bigg|^2\bigg]=0. \tag{7.44}$$

(f) Let us proceed to the noise terms. Assume that $v\in\mathrm{H}$. Using Lipschitz property of F, for all $t\in[0,T]$ we have,

$$\int_0^t\int_Y \big|\big(F(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big)_{\mathrm{H}}\big|^2\,d\nu_1(y)\,ds$$
$$\le\int_0^t\int_Y \big|F(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y)\big|_{\mathrm{H}}^2\cdot\big|v\big|_{\mathrm{H}}^2\,d\nu_1(y)\,ds\le C\int_0^T\big|\bar{\mathbf{u}}_n(s)-\mathbf{u}_*(s)\big|_{\mathrm{H}}^2\,ds. \tag{7.45}$$

From (7.1) we have, $\bar{\mathbf{u}}_n\to\mathbf{u}_*$ in $L^2(0,T;\mathrm{H})$, $\bar{\mathbb{P}}$-a.s. Then we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\int_0^t\int_Y\big|\big(F(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big)_{\mathrm{H}}\big|^2\,d\nu_1(y)\,ds=0. \tag{7.46}$$

Moreover, from (2.21), (7.7) and Proposition A.5, for every $t\in[0,T]$, every $r\ge1$ and every $n\in\mathbb{N}$,

$$\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\int_Y\big|\big(F(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big)_{\mathrm{H}}\big|^2\,d\nu_1(y)\,ds\bigg|^r\bigg]$$
$$\le C\,|v|_{\mathrm{H}}^{2r}\;\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\int_Y\big\{|F(s,\bar{\mathbf{u}}_n(s),y)|_{\mathrm{H}}^2+|F(s,\mathbf{u}_*(s),y)|_{\mathrm{H}}^2\big\}\,d\nu_1(y)\,ds\bigg|^r\bigg]$$
$$\le C(r,T)\bigg(1+\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\bar{\mathbf{u}}_n(s)|_{\mathrm{H}}^{2r}\bigg]+\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\mathbf{u}_*(s)|_{\mathrm{H}}^{2r}\bigg]\bigg)\le C. \tag{7.47}$$

Where $C>0$ is a constant. Then by (7.46), (7.47) and by Vitali's Theorem, for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\int_0^t\int_Y\big|\big(F(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big)_{\mathrm{H}}\big|^2\,d\nu_1(y)\,ds\bigg]=0,\quad v\in\mathrm{H}. \tag{7.48}$$

Since the restriction of P_n to the space H is the $(\cdot,\cdot)_{\mathrm{H}}$-projection onto H_n, we obtain

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\int_0^t\int_Y\big|\big(P_nF(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big)_{\mathrm{H}}\big|^2\,d\nu_1(y)\,ds\bigg]=0,\quad v\in\mathrm{H}. \tag{7.49}$$

Since $\mathrm{V}\subset\mathrm{H}$, (7.49) holds for all $v\in\mathrm{V}$. As $\bar{\eta}_{1n}=\eta_{1*}$, for all $n\in\mathbb{N}$. From (7.49) andusing Itô isometry we have,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\int_Y\big\langle P_nF(s,\bar{\mathbf{u}}_n(s),y)-F(s,\mathbf{u}_*(s),y),v\big\rangle\,\tilde{\eta}_{1*}(ds,dy)\bigg|^2\bigg]=0. \tag{7.50}$$

Moreover, from (7.47) and using Itô isometry with $r = 1$ we obtain,

$$\begin{aligned}\bar{\mathbb{E}}\Big[\Big|\int_0^t \int_Y \big\langle P_n F(s, \bar{\mathbf{u}}_n(s), y) - F(s, \mathbf{u}_*(s), y), v\big\rangle\, \tilde{\eta}_{1*}(ds, dy)\Big|^2\Big] \\ = \bar{\mathbb{E}}\Big[\int_0^t \int_Y \big|\big(F(s, \bar{\mathbf{u}}_n(s), y) - F(s, \mathbf{u}_*(s), y), v\big)_{\mathrm{H}}\big|^2\, d\nu_1(y)\, ds\Big] \le C. \end{aligned} \tag{7.51}$$

Finally, from (7.50), (7.51) and using Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\Big[\Big|\int_0^t \int_Y \big\langle P_n F(s, \bar{\mathbf{u}}_n(s), y) - F(s, \mathbf{u}_*(s), y), v\big\rangle\, \tilde{\eta}_{1*}(ds, dy)\Big|^2 dt\Big] = 0. \tag{7.52}$$

□

Now we will give the proof of Lemma 7.3. Let us fix $v \in D(\mathcal{A})$.

Proof. (a) Let us consider

$$|(\bar{\mathbf{d}}_n(\cdot), v)_{L^2} - (\mathbf{d}_*(\cdot), v)_{L^2}|^2_{L^2([0,T]\times\bar{\Omega})} = \bar{\mathbb{E}}\Big[\int_0^T \big|(\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t), v)_{L^2}\big|^2\, dt\Big] \tag{7.53}$$

Moreover,

$$\int_0^T \big|(\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t), v)_{L^2}\big|^2\, dt \le |v|^2_{L^2} \int_0^T \big|\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t)\big|^2_{L^2}\, dt \tag{7.54}$$

By (7.2), $\bar{\mathbf{d}}_n \to \mathbf{d}_*$ in $\mathbb{D}([0,T]; L^2)$ and from (7.10), $\sup_{t\in[0,T]} |\bar{\mathbf{d}}_n(t)|^2_{L^2} < \infty$, $\bar{\mathbb{P}}$-a.s.. Then by Dominated Convergence Theorem, from (7.54), we deduce

$$\lim_{n\to\infty} \int_0^T \big|(\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t), v)_{L^2}\big|^2\, dt = 0. \tag{7.55}$$

Moreover, from (7.12), Proposition A.6 and using the Hölder inequality, for every $n \in \mathbb{N}$ and every $r > 1$ we obtain

$$\bar{\mathbb{E}}\Big[\Big|\int_0^T \big|\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t)\big|^2_{L^2}\, dt\Big|^r\Big] \le c\, \bar{\mathbb{E}}\Big[\int_0^T \big(\big|\bar{\mathbf{d}}_n(t)\big|^{2r}_{L^2} + \big|\mathbf{d}_*(t)\big|^{2r}_{L^2}\big)\, dt\Big] < \infty. \tag{7.56}$$

for some constant $c > 0$. Then by (7.55), (7.56) and Vitali's Theorem we obtain

$$\lim_{n\to\infty} \bar{\mathbb{E}}\Big[\int_0^T \big|(\bar{\mathbf{d}}_n(t) - \mathbf{d}_*(t), v)_{L^2}\big|^2\, dt\Big] = 0, \quad \text{which proves (a).}$$

(b) From (7.2), $\bar{\mathbf{d}}_n \to \mathbf{d}_*$ in $\mathbb{D}([0,T]; L^2)$ $\bar{\mathbb{P}}$-a.s. and $\mathbf{d}_*$ is right continuous at $t = 0$. So we obtain, by Remark 4.1, $(\bar{\mathbf{d}}_n(0), v)_{L^2} \to (\mathbf{d}_*(0), v)_{L^2}, \bar{\mathbb{P}}$-a.s. Applying Vitali's Theorem, we get

$$\lim_{n\to\infty} \bar{\mathbb{E}}\Big[\big|(\bar{\mathbf{d}}_n(0) - \mathbf{d}_*(0), v)_{L^2}\big|^2\Big] = 0$$

Hence,

$$\lim_{n\to\infty} \big|(\bar{\mathbf{d}}_n(0) - \mathbf{d}_*(0), v)_{L^2}\big|^2_{L^2([0,T]\times\bar{\Omega})} = 0. \tag{7.57}$$

which proves (b).

(c) Now from (7.2), $\mathbf{d}_n \to \mathbf{d}_*$ in $L^2(0,T; H^1), \bar{\mathbb{P}}$-a.s., then for all $v \in D(\mathcal{A})$ we obtain $\bar{\mathbb{P}}$-a.s.,

$$\begin{aligned}\lim_{n\to\infty} \int_0^t \big(\bar{\mathbf{d}}_n(s), \mathcal{A}v\big)_{L^2}\, ds &= \lim_{n\to\infty} \int_0^t ((\bar{\mathbf{d}}_n(s), v))\, ds \\ &= \int_0^t ((\mathbf{d}_*(s), v))\, ds = \int_0^t \big(\mathbf{d}_*(s), \mathcal{A}v\big)_{L^2}\, ds. \end{aligned} \tag{7.58}$$

By (7.12) and using the Hölder inequality we obtain for all $t \in [0,T], r > 2$ and $n \in \mathbb{N}$,

$$\bar{\mathbb{E}}\left[\left|\int_0^t \big(\bar{\mathbf{d}}_n(s), \mathcal{A}v\big)_{L^2}\, ds\right|^{2+r}\right] \le \bar{\mathbb{E}}\left[\left(\int_0^t |\bar{\mathbf{d}}_n(s)|_{L^2}|v|_{D(\mathcal{A})}\, ds\right)^{2+r}\right]$$
$$\le\le c\,|v|^{2+r}_{D(\mathcal{A})}\ \bar{\mathbb{E}}\left[\left(\int_0^T |\bar{\mathbf{d}}_n(s)|_{L^2}\, ds\right)^{2+r}\right] \le \tilde{c}\ \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} \big|\bar{\mathbf{d}}_n(s)\big|^{2+r}_{L^2}\, ds\right] \le C, \tag{7.59}$$

for some constant $C > 0$. Then by (7.58), (7.59) and using Vitali's Theorem we obtain for all $t \in [0,T]$,

$$\lim_{n\to\infty} \bar{\mathbb{E}}\left[\left|\int_0^t \big(\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s), \mathcal{A}v\big)_{L^2}\, ds\right|^2\right] = 0 \tag{7.60}$$

Using Dominated Convergence Theorem, for all $t \in [0,T]$ and all $n \in \mathbb{N}$ we get,

$$\lim_{n\to\infty} \int_0^T \bar{\mathbb{E}}\left[\left|\int_0^t \big(\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s), \mathcal{A}v\big)_{L^2}\, ds\right|^2\right] dt = 0. \tag{7.61}$$

(d) From Lemma A.3 we have,

$$\lim_{n\to\infty} \int_0^t \big(\tilde{B}_n\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big) - \tilde{B}\big(\mathbf{u}_*(s), \mathbf{d}_*(s)\big), v\big)_{L^2}\, ds$$
$$= \lim_{n\to\infty} \int_0^t \big(\tilde{B}\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big) - \tilde{B}\big(\mathbf{u}_*(s), \mathbf{d}_*(s)\big), \tilde{P}_n v\big)_{L^2}\, ds = 0 \quad \bar{\mathbb{P}}\text{-a.s.} \tag{7.62}$$

Now from (2.12) and the Hölder inequality, we obtain for all $t \in [0,T], n \in \mathbb{N}$ and $r > 1$,

$$\bar{\mathbb{E}}\left[\left|\int_0^t \big(\tilde{B}_n\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big), v\big)_{L^2}\, ds\right|^r\right] \le\le |v|^r_{L^2}\ t^{r-1}\ \bar{\mathbb{E}}\left[\int_0^t |\tilde{B}_n\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big)|^r_{L^2}\, ds\right]$$
$$\le c\,\bar{\mathbb{E}}\left[\int_0^t |\bar{\mathbf{u}}_n|_{\mathrm{H}}^{r-\frac{r\mathbf{n}}{4}} \|\bar{\mathbf{u}}_n\|^{\frac{r\mathbf{n}}{4}} |\nabla\bar{\mathbf{d}}_n|_{L^2}^{r-\frac{r\mathbf{n}}{4}} |\Delta\bar{\mathbf{d}}_n|_{L^2}^{\frac{r\mathbf{n}}{4}}\, ds\right], \tag{7.63}$$

for $\mathbf{n} = 2, 3$.

First we consider the case $\mathbf{n} = 2$. Using Young's inequality we have,

$$\bar{\mathbb{E}}\left[\int_0^t (|\bar{\mathbf{u}}_n|^{\frac{r}{2}}\|\bar{\mathbf{u}}_n\|^{\frac{r}{2}})(|\nabla\bar{\mathbf{d}}_n|^{\frac{r}{2}}_{L^2}|\Delta\bar{\mathbf{d}}_n|^{\frac{r}{2}})\, ds\right]$$
$$\le \bar{\mathbb{E}}\left[\int_0^t |\bar{\mathbf{u}}_n|^r\|\bar{\mathbf{u}}_n\|^r\, ds\right] + \bar{\mathbb{E}}\left[\int_0^t |\nabla\bar{\mathbf{d}}_n|^r_{L^2}|\Delta\bar{\mathbf{d}}_n|^r\, ds\right]. \tag{7.64}$$

Now the estimate for the second term of Right Hand Side follows from (7.41). So let us estimate the first term. From (7.7) and (7.4), using the Hölder inequality, for $r \in [1,2]$ and for all $t \in [0,T], n \in \mathbb{N}$ we obtain,

$$\bar{\mathbb{E}}\left[\int_0^t |\bar{\mathbf{u}}_n|^r\|\bar{\mathbf{u}}_n\|^r\, ds\right] \le \left\{\bar{\mathbb{E}}\left[\sup_{s\in[0,T]} |\bar{\mathbf{u}}_n|^{2r}\right]\right\}^{\frac12}\left\{\bar{\mathbb{E}}\left[\int_0^t \|\bar{\mathbf{u}}_n\|^r\, ds\right]^2\right\}^{\frac12} \le C(r,T). \tag{7.65}$$

Similarly, from (7.63) for $\mathbf{n} = 3$, using Young's inequality we obtain,

$$\bar{\mathbb{E}}\left[\int_0^t (|\bar{\mathbf{u}}_n|^{\frac{r}{4}}\|\bar{\mathbf{u}}_n\|^{\frac{3r}{4}})(|\nabla\bar{\mathbf{d}}_n|^{\frac{r}{4}}_{L^2}|\Delta\bar{\mathbf{d}}_n|^{\frac{3r}{4}})\, ds\right]$$
$$\le \frac12\bar{\mathbb{E}}\left[\int_0^t |\bar{\mathbf{u}}_n|^{\frac{r}{2}}\|\bar{\mathbf{u}}_n\|^{\frac{3r}{2}}\, ds\right] + \frac12\bar{\mathbb{E}}\left[\int_0^t |\nabla\bar{\mathbf{d}}_n|^{\frac{r}{2}}_{L^2}|\Delta\bar{\mathbf{d}}_n|^{\frac{3r}{2}}\, ds\right] \tag{7.66}$$

Similarly, the estimate for the second term of Right Hand Side follows from (7.42). So we handle only the first term. From (7.3) and (7.4) for $r \in [1, \frac43)$ and for all $t \in [0,T], n \in \mathbb{N}$, using the Hölder inequality we obtain,

$$\bar{\mathbb{E}}\bigg[\int_0^t |\bar{\mathbf{u}}_n|^{\frac{r}{2}}\|\bar{\mathbf{u}}_n\|^{\frac{3r}{2}}\,ds\bigg] \le \bar{\mathbb{E}}\bigg[\bigg(\int_0^t (|\bar{\mathbf{u}}_n|^{\frac{r}{2}})^{\frac{4}{4-3r}}\,ds\bigg)^{\frac{4-3r}{4}}\bigg(\int_0^t (\|\bar{\mathbf{u}}_n\|^{\frac{3r}{2}})^{\frac{4}{3r}}\,ds\bigg)^{\frac{3r}{4}}\bigg]$$

$$\le c\left\{\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}|\bar{\mathbf{u}}_n|^{\frac{2r}{4-3r}}\bigg]\right\}^{\frac{4-3r}{4}}\left\{\bar{\mathbb{E}}\bigg[\int_0^T \|\bar{\mathbf{u}}_n\|^2\,ds\bigg]\right\}^{\frac{3r}{4}} \le C(r,T). \tag{7.67}$$

So from (7.62), (7.63), (7.64), (7.65), (7.66), (7.67) and using Vitali's Theorem we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t \big(\tilde{B}_n\big(\bar{\mathbf{u}}_n(s),\bar{\mathbf{d}}_n(s)\big)-\tilde{B}\big(\mathbf{u}_*(s),\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\bigg|^2\bigg]=0. \tag{7.68}$$

From (7.3), (7.4), (7.11), (7.15) and (7.68), using Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\bigg[\bigg|\int_0^t \big(\tilde{B}_n\big(\bar{\mathbf{u}}_n(s),\bar{\mathbf{d}}_n(s)\big)-\tilde{B}\big(\mathbf{u}_*(s),\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\bigg|^2\bigg]\,dt=0. \tag{7.69}$$

(e) $\bar{\mathbf{d}}_n\to\mathbf{d}_*$ in $\mathcal{Z}_{T,2}$. Since f is a polynomial function of order $2N+1$, we have $\bar{\mathbb{P}}$-a.s.,

$$\lim_{n\to\infty}\int_0^t\big(f_n\big(\bar{\mathbf{d}}_n(s)\big)-f\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds=\lim_{n\to\infty}\int_0^t\big(f\big(\bar{\mathbf{d}}_n(s)\big)-f\big(\mathbf{d}_*(s)\big),\tilde{P}_nv\big)_{L^2}\,ds=0 \tag{7.70}$$

Since $H^1\hookrightarrow L^{\bar{q}}$, for $\bar{q}=4N+2$, where $N\in I_{\mathbf{n}}$. From Remark 2.4 and (7.13), we obtain for all $t\in[0,T], r>1$ and $n\in\mathbb{N}$,

$$\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big(f_n\big(\bar{\mathbf{d}}_n(s)\big),v\big)_{L^2}\,ds\bigg|^r\bigg]\le\le C+|v|^r_{L^2}\;t^{r-1}\;\bar{\mathbb{E}}\bigg[\int_0^t\big|\bar{\mathbf{d}}_n(s)\big|^{\frac{r\bar{q}}{2}}_{L^{\bar{q}}}\,ds\bigg]$$

$$\le C+C_{r,t}\;\bar{\mathbb{E}}\bigg[\int_0^t\big|\bar{\mathbf{d}}_n(s)\big|^{\frac{r\bar{q}}{2}}_{H^1}\,ds\bigg]\le C+\tilde{C}_{r,t}\;\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}\big|\bar{\mathbf{d}}_n(s)\big|^{\frac{r\bar{q}}{2}}_{H^1}\bigg]\le C(r,N,T). \tag{7.71}$$

So from (7.70), (7.71) and using Vitali's Theorem we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big(f_n\big(\bar{\mathbf{d}}_n(s)\big)-f\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\bigg|^2\bigg]=0. \tag{7.72}$$

Again from (7.13), Remark 2.4 and using Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big(f_n\big(\bar{\mathbf{d}}_n(s)\big)-f\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\bigg|^2\bigg]\,dt=0. \tag{7.73}$$

(f) Since, $\bar{\mathbf{d}}_n\to\mathbf{d}_*$ in $L^2(0,T;H^1)$, using the Lipschitz property of b, owing to the similar calculations previously we obtain,

$$\lim_{n\to\infty}\int_0^t\big(b_n\big(\bar{\mathbf{d}}_n(s)\big)-b\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds=\lim_{n\to\infty}\int_0^t\big(b\big(\bar{\mathbf{d}}_n(s)\big)-b\big(\mathbf{d}_*(s)\big),\tilde{P}_nv\big)_{L^2}\,ds=0 \tag{7.74}$$

Now using Lemma 5.15 and Proposition 5.9, for $r\ge1$ we get,

$$\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big(b_n\big(\bar{\mathbf{d}}_n(s)\big),v\big)_{L^2}\,ds\bigg|^r\bigg]\le c_t\;|v|^r_{L^2}\,\bar{\mathbb{E}}\bigg[\int_0^t\Big(\big|b_n\big(\bar{\mathbf{d}}_n(s)\big)\big|^2_{L^2}\Big)^{\frac{r}{2}}\,ds\bigg]$$

$$\le c_t\,\bar{\mathbb{E}}\bigg[\int_0^t\Big(\big|\bar{\mathbf{d}}_n(s)\big|^2_{L^2}\Big)^{\frac{r}{2}}\,ds\bigg]\le\bar{c}_t\,\bar{\mathbb{E}}\bigg[\sup_{s\in[0,T]}\big|\bar{\mathbf{d}}_n(s)\big|^r_{L^2}\bigg]\le C\le\infty. \tag{7.75}$$

So from (7.74), (7.75) and using Vitali's theorem we get,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\bigg[\bigg|\int_0^t\big(b_n\big(\bar{\mathbf{d}}_n(s)\big)-b\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\bigg|^2\bigg]=0. \tag{7.76}$$

Finally from (7.76) and Dominated convergence theorem we get

$$\lim_{n\to\infty}\int_0^T \bar{\mathbb{E}}\Big[\Big|\int_0^t \big(b_n\big(\bar{\mathbf{d}}_n(s)\big)-b\big(\mathbf{d}_*(s)\big),v\big)_{L^2}\,ds\Big|^2\Big]\,dt=0. \tag{7.77}$$

(g) Now we will prove the convergence of the stochastic integral. Using the fact that $H^1\hookrightarrow L^2$ is continuous and Lipschitz property of G, we obtain for all $v\in D(\mathcal{A})$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{B}}\big|\big(G(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds\\ &\le C|v|_{L^2}^2\int_0^t\big|\bar{\mathbf{d}}_n(s)-\mathbf{d}_*(s)\big|_{L^2}^2\,ds\le \tilde{C}\int_0^T\big|\bar{\mathbf{d}}_n(s)-\mathbf{d}_*(s)\big|_{H^1}^2\,ds.\end{aligned} \tag{7.78}$$

From (7.2) we have, $\bar{\mathbf{d}}_n\to\mathbf{d}_*$ in $L^2(0,T;H^1)$, $\bar{\mathbb{P}}$-a.s. Then we obtain for all $t\in[0,T]$,

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{B}}\big|\big(G(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds=0. \tag{7.79}$$

Moreover, from Proposition A.6, for every $t\in[0,T]$, every $r\ge 1$ and every $n\in\mathbb{N}$,

$$\begin{aligned}&\bar{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathbb{B}}\big|\big(G(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds\Big|^r\Big]\\ &\le C\ |v|_{L^2}^{2r}\ \bar{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathbb{B}}\big\{|G(l,\bar{\mathbf{d}}_n(s))|_{L^2}^2+|G(l,\mathbf{d}_*(s))|_{L^2}^2\big\}\,\nu_2(dl)ds\Big|^r\Big]\\ &\le \tilde{C}\ \bar{\mathbb{E}}\Big[\Big|\int_0^t\big\{|\bar{\mathbf{d}}_n(s)|_{L^2}^2+|\mathbf{d}_*(s)|_{L^2}^2\big\}\,ds\Big|^r\Big]\le \bar{C}\Big(\bar{\mathbb{E}}\Big[\int_0^t|\bar{\mathbf{d}}_n(s)|_{L^2}^{2r}\,ds\Big]+\bar{\mathbb{E}}\Big[\int_0^t|\mathbf{d}_*(s)|_{L^2}^{2r}\,ds\Big]\Big)\\ &\le C(r,T)\Big(\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\bar{\mathbf{d}}_n(s)|_{L^2}^{2r}\Big]+\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\mathbf{d}_*(s)|_{L^2}^{2r}\Big]\Big)\le C.\end{aligned} \tag{7.80}$$

Then by (7.79), (7.80) and by Vitali's Theorem, for all $t\in[0,T]$,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\Big[\int_0^t\int_{\mathbb{B}}\big|\big(G(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds\Big]=0,\quad v\in D(\mathcal{A}). \tag{7.81}$$

Since the restriction of S_n to the space L^2 is the $(\cdot,\cdot)_{L^2}$-projection onto $\mathbb{L}_n$ (see proposition 5.1), we obtain

$$\lim_{n\to\infty}\bar{\mathbb{E}}\Big[\int_0^t\int_{\mathbb{B}}\big|\big(G_n(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds\Big]=0,\quad v\in D(\mathcal{A}). \tag{7.82}$$

As $\bar{\eta}_{2n}=\eta_{2*}$, for all $n\in\mathbb{N}$. From (7.82) and the Itô isometry we have,

$$\lim_{n\to\infty}\bar{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathbb{B}}\big(G_n(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\,\tilde{\eta}_{2*}(ds,dl)\Big|^2\Big]=0. \tag{7.83}$$

Moreover, from (7.80) and the Itô isometry, with $r=1$, we obtain,

$$\begin{aligned}&\bar{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathbb{B}}\big(G_n(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\,\tilde{\eta}_{2*}(ds,dl)\Big|^2\Big]\\ &=\bar{\mathbb{E}}\Big[\int_0^t\int_{\mathbb{B}}\big|\big(G_n(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\big|^2\,\nu_2(dl)ds\Big]\le C.\end{aligned} \tag{7.84}$$

Finally, from (7.83), (7.84) and using Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_0^T\bar{\mathbb{E}}\Big[\Big|\int_0^t\int_{\mathbb{B}}\big(G_n(l,\bar{\mathbf{d}}_n(s))-G(l,\mathbf{d}_*(s)),v\big)_{L^2}\,\tilde{\eta}_{2*}(ds,dl)\Big|^2\,dt\Big]=0. \tag{7.85}$$

□

Now we are ready to prove the existence of martingale solution.

Theorem 7.4. *Let Assumption 2.3 holds. Then there exists a martingale solution* $\big(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{F}},\bar{\mathbb{P}},\bar{\mathbf{u}},\bar{\mathbf{d}},\bar{\eta}_1,\bar{\eta}_2\big)$ *of the problem* (2.42)-(2.43).

Proof. From Lemma 7.2, we have

$$\lim_{n\to\infty}\big|(\bar{\mathbf{u}}_n(\cdot),v)_{\mathbb{H}}-(\mathbf{u}_*(\cdot),v)_{\mathbb{H}}\big|_{L^2([0,T]\times\bar{\Omega})}=0 \tag{7.86}$$

and

$$\lim_{n\to\infty}\big|\mathscr{K}_n(\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_{1n},v)-\mathscr{K}(\mathbf{u}_*,\mathbf{d}_*,\eta_{1*},v)\big|_{L^2([0,T]\times\bar{\Omega})}=0. \tag{7.87}$$

From Lemma 7.3, we obtain

$$\lim_{n\to\infty}\big|(\bar{\mathbf{d}}_n(\cdot),v)_{L^2}-(\mathbf{d}_*(\cdot),v)_{L^2}\big|_{L^2([0,T]\times\bar{\Omega})}=0 \tag{7.88}$$

and

$$\lim_{n\to\infty}\big|\Lambda_n(\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_{2n},v)-\Lambda(\mathbf{u}_*,\mathbf{d}_*,\eta_{2*},v)\big|_{L^2([0,T]\times\bar{\Omega})}=0 \tag{7.89}$$

Since $(\mathbf{u}_n,\mathbf{d}_n)$ is a solution of the Galerkin approximation equations (5.80)-(5.81) for all $t\in[0,T]$, we have for $\mathbb{P}$-a.s.

$$(\mathbf{u}_n(t),v)_{\mathbb{H}}=\mathscr{K}_n(\mathbf{u}_n,\mathbf{d}_n,\eta_{1n},v)(t)$$

and

$$(\mathbf{d}_n(t),v)_{L^2}=\Lambda_n(\mathbf{u}_n,\mathbf{d}_n,\eta_{2n},v)(t).$$

In particular,

$$\int_0^T\mathbb{E}\big[\big|(\mathbf{u}_n(t),v)_{\mathbb{H}}-\mathscr{K}_n(\mathbf{u}_n,\mathbf{d}_n,\eta_{1n},v)(t)\big|^2\big]\,dt=0$$

and

$$\int_0^T\mathbb{E}\big[\big|(\mathbf{d}_n(t),v)_{L^2}-\Lambda_n(\mathbf{u}_n,\mathbf{d}_n,\eta_{2n},v)(t)\big|^2\big]\,dt=0.$$

Since $\mathscr{L}(\mathbf{u}_n,\mathbf{d}_n,\eta_n)=\mathscr{L}(\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_n)$, we conclude

$$\int_0^T\bar{\mathbb{E}}\big[\big|(\bar{\mathbf{u}}_n(t),v)_{\mathbb{H}}-\mathscr{K}_n(\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_{1n},v)(t)\big|^2\big]\,dt=0$$

and

$$\int_0^T\bar{\mathbb{E}}\big[\big|(\bar{\mathbf{d}}_n(t),v)_{L^2}-\Lambda_n(\bar{\mathbf{u}}_n,\bar{\mathbf{d}}_n,\bar{\eta}_{2n},v)(t)\big|^2\big]\,dt=0.$$

From (7.86), (7.87), (7.88) and (7.89), we have

$$\int_0^T\bar{\mathbb{E}}\big[\big|(\mathbf{u}_*(t),v)_{\mathbb{H}}-\mathscr{K}(\mathbf{u}_*,\mathbf{d}_*,\eta_{1*},v)(t)\big|^2\big]\,dt=0$$

and

$$\int_0^T\bar{\mathbb{E}}\big[\big|(\mathbf{d}_*(t),v)_{L^2}-\Lambda(\mathbf{u}_*,\mathbf{d}_*,\eta_{2*},v)(t)\big|^2\big]\,dt=0.$$

Hence for l-almost all $t\in[0,T]$ and $\bar{\mathbb{P}}$-almost all $\omega\in\bar{\Omega}$, we obtain

$$(\mathbf{u}_*(t),v)_{\mathbb{H}}-\mathscr{K}(\mathbf{u}_*,\mathbf{d}_*,\eta_{1*},v)(t)=0$$

and

$$(\mathbf{d}_*(t),v)_{L^2}-\Lambda(\mathbf{u}_*,\mathbf{d}_*,\eta_{2*},v)(t)=0.$$

In particular, for every $v \in \mathbb{V}$,

$$(\mathbf{u}_*(t), v)_{\mathrm{H}} + \int_0^t \big\langle \mathbf{u}_*(s), \mathscr{A} v \big\rangle \, ds + \int_0^t \big\langle B(\mathbf{u}_*(s)), v \big\rangle \, ds + \int_0^t \big\langle M(\mathbf{d}_*(s)), v \big\rangle \, ds$$
$$= \big(\mathbf{u}_0, v\big)_{\mathrm{H}} + \int_0^t \int_Y \big(F(s, \mathbf{u}_*(s); y), v\big)_{\mathrm{H}} \, \tilde{\eta}_*(ds, dy) \tag{7.90}$$

and for every $v \in D(\mathcal{A})$,

$$(\mathbf{d}_*(t), v)_{L^2} + \int_0^t \big(\mathbf{d}_*(s), \mathcal{A} v\big)_{L^2} \, ds + \int_0^t \big(\tilde{B}(\mathbf{u}_*(s), \mathbf{d}_*(s)), v\big)_{L^2} \, ds,$$
$$= \big(\mathbf{d}_0, v\big)_{L^2} - \int_0^t \big(f(\mathbf{d}_*(s)), v\big)_{L^2} \, ds + \int_0^t \int_{\mathbb{B}} \big(G(l, \mathbf{d}_*(s)), v\big)_{L^2} \, \tilde{\eta}_{2*}(ds, dl) + \int_0^t \big(b(\mathbf{d}_*(s)), v\big)_{L^2} \, ds. \tag{7.91}$$

Since $(\mathbf{u}_*, \mathbf{d}_*)$ is $\mathcal{Z}_{T,1} \times \mathcal{Z}_{T,2}$-valued random variable, and $\mathbf{u}_*, \mathbf{d}_*$ are weakly càdlàg, we obtain that the equalities (7.90)-(7.91) hold for all $t \in [0,T]$ and all $v \in \mathrm{V}$ and $v \in D(\mathcal{A})$ respectively. Putting $\bar{\mathbf{u}} := \mathbf{u}_*$, $\bar{\mathbf{d}} := \mathbf{d}_*$ and $\bar{\eta}_1 := \eta_{1*}$ and $\bar{\eta}_2 := \eta_{2*}$, we infer that the system $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{u}}, \bar{\mathbf{d}}, \bar{\eta}_1, \bar{\eta}_2\big)$ is a martingale solution of (2.42)-(2.43).

□

Remark 7.5. *The limiting process $\mathbf{u}_*$ satisfies* (7.5) *and* (7.6) *and the process $\mathbf{d}_*$ satisfies* (7.14) *and* (7.16). *In Theorem* (7.4)*, we proved the system $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{u}}, \bar{\mathbf{d}}, \bar{\eta}\big)$ is a martingale solution by equating $\bar{\mathbf{u}} := \mathbf{u}_*$ and $\bar{\mathbf{d}} := \mathbf{d}_*$. So we infer that $\bar{\mathbf{u}}$ and $\bar{\mathbf{d}}$ satisfy* (3.9) *and* (3.10) *respectively.*

8. Pathwise Uniqueness and Existence of Strong Solution in 2-D

In this section we will prove the pathwise uniqueness of the solutions of (2.42)-(2.43). Then we will use results from [41], to deduce the existence of a strong solution of (2.42)-(2.43) as well. We consider these cases only in two dimensions. Our results from this section could be seen as a double generalization of the results from section 7 in [15] to the Ericksen-Leslie Equations driven by a Poisson random measure.

In the following Lemma we will show that almost all trajectories of the solution $(\mathbf{u}, \mathbf{d})$ are almost everywhere equal to a $\mathrm{H} \times H^1$-valued function defined on $[0,T]$.

Lemma 8.1. *Let Assumption 2.3 holds. Let $\mathbf{n} = 2$. Assume that $(\mathbf{u}_0, \mathbf{d}_0) \in \mathrm{H} \times H^1$. Let $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\mathbf{u}}, \bar{\mathbf{d}}, \bar{\eta}\big)$ be a martingale solution of* (2.42)-(2.43)*. Then for $\bar{\mathbb{P}}$-almost all $\omega \in \bar{\Omega}$, the trajectory $\bar{\mathbf{u}}(\cdot, \omega)$ is almost everywhere equal to a càdlàg* H*-valued function and the trajectory $\bar{\mathbf{d}}(\cdot, \omega)$ is almost everywhere equal to a càdlàg H^1-valued function defined on* $[0,T]$.

Proof. From previous results we have for $t \in [0,T]$,

$$\bar{\mathbf{u}}(t) - \bar{\mathbf{u}}_0 = \int_0^t \mathscr{A} \bar{\mathbf{u}}(s) \, ds - \int_0^t B(\bar{\mathbf{u}}(s)) \, ds - \int_0^t M(\bar{\mathbf{d}}(s)) \, ds + \int_0^t \int_Y F(s, \bar{\mathbf{u}}(s), y) \, \tilde{\eta}_1(ds, dy) \tag{8.1}$$

and

$$\bar{\mathbf{d}}(t) - \bar{\mathbf{d}}_0 = \int_0^t \mathcal{A} \bar{\mathbf{d}}(s) \, ds - \int_0^t \tilde{B}(\bar{\mathbf{u}}(s), \bar{\mathbf{d}}(s)) \, ds - \int_0^t f(\bar{\mathbf{d}}(s)) \, ds + \int_0^t \int_{\mathbb{B}} G(l, \bar{\mathbf{d}}(s)) \, \tilde{\eta}_2(ds, dl)$$
$$+ \int_0^t b(\bar{\mathbf{d}}(s)) \, ds. \tag{8.2}$$

By Gyöngy and Krylov [23], we need to verify the first three terms on the RHS of (8.1) are V'-valued and that the $L^2([0,T] \times \Omega; \mathrm{V}')$ -norm of each of them is finite.

For this aim, let us first observe that from 2.17 and Proposition 5.12 we get,

$$\bar{\mathbb{E}} \int_0^T \big|\mathscr{A} \bar{\mathbf{u}}(s)\big|_{\mathrm{V}'}^2 \, ds \le \bar{\mathbb{E}} \int_0^T \|\bar{\mathbf{u}}(s)\|^2 \, ds < \infty, \tag{8.3}$$

where $\| \cdot \| := |\nabla \cdot|_{L^2}$.

From 2.11 we have,

$$\bar{\mathbb{E}}\int_0^T \left|B(\bar{\mathbf{u}}(s))\right|^2_{\mathrm{V}'}\,ds \le \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{u}}(s)|^{4-\mathbf{n}}\|\bar{\mathbf{u}}(s)\|^{\mathbf{n}}\,ds. \tag{8.4}$$

As $\mathbf{n}=2$, applying Young's inequality and then in view of Remark 7.5 we get,

$$\bar{\mathbb{E}}\int_0^T |\bar{\mathbf{u}}(s)|^2\|\bar{\mathbf{u}}(s)\|^2\,ds \le c\,\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\bar{\mathbf{u}}(s)|^2\Big]^2 + c\,\bar{\mathbb{E}}\Big[\int_0^T \|\bar{\mathbf{u}}(s)\|^2\,ds\Big]^2 < \infty. \tag{8.5}$$

From (2.15) we obtain,

$$\bar{\mathbb{E}}\int_0^T \left|M(\bar{\mathbf{d}}(s))\right|^2_{\mathrm{V}'}\,ds \le \bar{\mathbb{E}}\int_0^T |\nabla\bar{\mathbf{d}}(s)|^{4-\mathbf{n}}_{L^2}|\Delta\bar{\mathbf{d}}(s)|^{\mathbf{n}}\,ds. \tag{8.6}$$

As $\mathbf{n}=2$, applying Young's inequality and in view of Remark 7.5 we get,

$$\bar{\mathbb{E}}\int_0^T |\nabla\bar{\mathbf{d}}(s)|^2_{L^2}|\Delta\bar{\mathbf{d}}(s)|^2\,ds \le c\,\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\nabla\bar{\mathbf{d}}(s)|^2_{L^2}\Big]^2 + c\,\bar{\mathbb{E}}\Big[\int_0^T |\Delta\bar{\mathbf{d}}(s)|^2\,ds\Big]^2 < \infty. \tag{8.7}$$

Finally, by using Itô isometry, (2.21) and from Remark 7.5 we obtain,

$$\begin{aligned}\bar{\mathbb{E}}\int_0^T \left|\int_Y F(s,\bar{\mathbf{u}}(s),y)\,\tilde{\eta}_1(ds,dy)\right|^2_{\mathrm{H}} &\le \bar{\mathbb{E}}\int_0^T\int_Y \left|F(s,\bar{\mathbf{u}}(s),y)\right|^2_{\mathrm{H}}\,\nu_1(dy)ds\\ \le \bar{\mathbb{E}}\int_0^T \left(1+|\bar{\mathbf{u}}(s)|^2_{\mathrm{H}}\right)ds &\le C_T + \bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\bar{\mathbf{u}}(s)|^2_{\mathrm{H}}\Big] < \infty. \end{aligned}\tag{8.8}$$

Now we consider the second equation (8.2). By Gyöngy and Krylov [23], it is sufficient to show the each term on the RHS of (8.2) is L^2-valued and that the $L^2([0,T]\times\Omega;L^2)$-norm of each of them is finite.

We begin by observing that in view of Remark 7.5 we have,

$$\bar{\mathbb{E}}\int_0^T \left|\mathcal{A}\bar{\mathbf{d}}(s)\right|^2_{L^2}\,ds \le \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{d}}(s)|^2_{D(\mathcal{A})}\,ds < \infty. \tag{8.9}$$

From (2.12) we get,

$$\bar{\mathbb{E}}\int_0^T \left|\tilde{B}(\bar{\mathbf{u}}(s),\bar{\mathbf{d}}(s))\right|^2_{L^2}\,ds \le c\,\bar{\mathbb{E}}\int_0^T \left(|\bar{\mathbf{u}}(s)|^{2-\frac{\mathbf{n}}{2}}\|\bar{\mathbf{u}}(s)\|^{\frac{\mathbf{n}}{2}}|\nabla\bar{\mathbf{d}}(s)|^{2-\frac{\mathbf{n}}{2}}_{L^2}|\Delta\bar{\mathbf{d}}(s)|^{\frac{\mathbf{n}}{2}}\right)ds \tag{8.10}$$

As $\mathbf{n}=2$, using the Remark 7.5 we infer,

$$\begin{aligned}\bar{\mathbb{E}}\int_0^T &\left(|\bar{\mathbf{u}}(s)|\ \|\bar{\mathbf{u}}(s)\|\ |\nabla\bar{\mathbf{d}}(s)|_{L^2}\ |\Delta\bar{\mathbf{d}}(s)|\right)ds \le \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{u}}(s)|^2\|\bar{\mathbf{u}}(s)\|^2 + \bar{\mathbb{E}}\int_0^T |\nabla\bar{\mathbf{d}}(s)|^2_{L^2}|\Delta\bar{\mathbf{d}}(s)|^2\,ds\\ &\le c\,\bar{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{\mathbf{u}}(s)|^2\Big)^2 + c\,\bar{\mathbb{E}}\Big(\int_0^T\|\bar{\mathbf{u}}(s)\|^2\,ds\Big)^2\\ &\quad + c\,\bar{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\nabla\bar{\mathbf{d}}(s)|^2_{L^2}\Big)^2 + c\,\bar{\mathbb{E}}\Big(\int_0^T|\Delta\bar{\mathbf{d}}(s)|^2\,ds\Big)^2 < \infty. \end{aligned}\tag{8.11}$$

Finally, still for $\mathbf{n}=2$, in the view of Remark 7.5 and Proposition A.6 in the Appendix, for $\bar{q}=4N+2$ we obtain

$$\bar{\mathbb{E}}\int_0^T |f(\bar{\mathbf{d}}(s))|^2_{L^2}\,ds \le C_T + \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{d}}(s)|^{\bar{q}}_{L^{\bar{q}}}\,ds \le C_T + C(T)\,\bar{\mathbb{E}}\sup_{s\in[0,T]}|\bar{\mathbf{d}}(s)|^{\bar{q}}_{H^1} < \infty. \tag{8.12}$$

Now we need to show G is L^2-valued. So using Itô isometry, Lemma 5.5 we obtain,

$$\begin{aligned}\bar{\mathbb{E}}\int_0^T \left|\int_{\mathbb{B}} G(l,\bar{\mathbf{d}}(s))\,\tilde{\eta}_2(ds,dl)\right|^2_{L^2} &= \bar{\mathbb{E}}\int_0^T\int_{\mathbb{B}} \left|G(l,\bar{\mathbf{d}}(s))\right|^2_{L^2}\,\nu_2(dl)ds\\ \le \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{d}}(s)|^2_{L^2}\,ds &\le c_T\,\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\bar{\mathbf{d}}(s)|^2_{L^2}\Big] < \infty. \end{aligned}\tag{8.13}$$

Again from Lemma 5.5 we have,

$$\bar{\mathbb{E}}\int_0^T |b(\bar{\mathbf{d}}(s))|_{L^2}^2\,ds \le \bar{\mathbb{E}}\int_0^T |\bar{\mathbf{d}}(s)|_{L^2}^2\,ds \le c_T\,\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]}|\bar{\mathbf{d}}(s)|_{L^2}^2\Big] < \infty. \tag{8.14}$$

Thus the proof is complete. □

Remark 8.2. *It is important to point out that in the case $\mathbf{n}=3$, the same proof as above shows that if $p=\frac{4}{3}$, then*

$$\bar{\mathbb{E}}\int_0^T \big|B(\bar{\mathbf{u}}(s))\big|_{\mathrm{V}'}^p\,ds<\infty,\quad \bar{\mathbb{E}}\int_0^T \big|M(\bar{\mathbf{d}}(s))\big|_{\mathrm{V}'}^p\,ds<\infty,\quad \bar{\mathbb{E}}\int_0^T \big|\tilde{B}(\bar{\mathbf{u}}(s),\bar{\mathbf{d}}(s))\big|_{L^2}^p\,ds<\infty. \tag{8.15}$$

This is the reason behind our Remark 3.4.

We first formulate a definition of the pathwise uniqueness and then in the following lemma we show that the solutions of (2.42)-(2.43) are pathwise unique. We use Gyöngy and Krylov's version of the Itô formula (see [23]) for a suitable function in this proof.

Definition 8.3. *It is said that the solutions to problem* (2.42)-(2.43) *are **pathwise unique** iff for any two solutions $\mathbf{u}_i:[0,T]\times\Omega\to\mathrm{H}$ and $\mathbf{d}_i:[0,T]\times\Omega\to H^1$, $i=1,2$, to problem to* (2.42)-(2.43) *defined on the same stochastic basis $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$ with the same time homogeneous Poisson random measures η_1 on $(Y,\mathscr{B}(Y))$ and η_2 on $(\mathbb{B},\mathscr{B}(\mathbb{B}))$ over the above stochastic basis with intensity measure ν_1 and ν_2 respectively, if $\big(\mathbf{u}_1(0),\mathbf{d}_1(0)\big)=\big(\mathbf{u}_2(0),\mathbf{d}_2(0)\big)$, $\mathbb{P}-a.s.$, then*

$$(\mathbf{u}_1(t),\mathbf{d}_1(t))=(\mathbf{u}_2(t),\mathbf{d}_2(t))\quad \mathbb{P}\text{-}a.s.\quad \text{for all } t\in(0,T].$$

Lemma 8.4. *Let us assume that $(\mathbf{u}_1,\mathbf{d}_1)$ and $(\mathbf{u}_2,\mathbf{d}_2)$ be two solutions of problem* (2.42)-(2.43) *defined on the same stochastic basis $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\tilde{\eta}_1,\tilde{\eta}_2\big)$, where $\mathbb{F}=\big(\mathcal{F}_t\big)$ with the same initial data $(\mathbf{u}_0,\mathbf{d}_0)\in\mathrm{H}\times H^1$. Then*

$$(\mathbf{u}_1(t),\mathbf{d}_1(t))=(\mathbf{u}_2(t),\mathbf{d}_2(t))\quad \mathbb{P}\text{-}a.s.\quad \text{for all } t\in(0,T]. \tag{8.16}$$

Proof. Let us denote the norms as per section 2.1 in this proof. Let $\mathbf{u}(t)=\mathbf{u}_1(t)-\mathbf{u}_2(t)$ and $\mathbf{d}(t)=\mathbf{d}_1(t)-\mathbf{d}_2(t)$, with $(\mathbf{u}(0),\mathbf{d}(0))=(0,0)$. Let us denote $F_d(t,y):=\big(F(t,\mathbf{u}_1(t);y)-F(t,\mathbf{u}_2(t);y)\big)$ and $G_d(l):=\big(G(l,\mathbf{d}_1(t))-G(l,\mathbf{d}_2(t))\big)$. These processes satisfy

$$\begin{aligned}
&d\mathbf{u}(t)+\Big(\mathscr{A}\mathbf{u}(t)+B(\mathbf{u}(t),\mathbf{u}_1(t))+B(\mathbf{u}_2(t),\mathbf{u}(t))\Big)\,dt\\
&\qquad=-\Big(M(\mathbf{d}(t),\mathbf{d}_1(t))+M(\mathbf{d}_2(t),\mathbf{d}(t))\Big)\,dt+\int_Y F_d(t,y)\,\tilde{\eta}_1(dt,dy),
\end{aligned}$$

and

$$\begin{aligned}
&d\mathbf{d}(t)+\Big(\mathcal{A}\mathbf{d}(t)+\tilde{B}(\mathbf{u}(t),\mathbf{d}_1(t))+\tilde{B}(\mathbf{u}_2(t),\mathbf{d}(t))\Big)\,dt\\
&\qquad=-\Big(f(\mathbf{d}_2(t))-f(\mathbf{d}_1(t))\Big)\,dt+\int_{\mathbb{B}} G_d(l)\,\tilde{\eta}_2(dt,dl)+\Big(b(\mathbf{d}_1(t))-b(\mathbf{d}_2(t))\Big)\,dt.
\end{aligned}$$

The estimates in the following equation (8.17) can be found in [9], but for the sake of the readers we provide the details. From (2.11), (2.15), (2.12) and using Poincaré and Young's inequalities, we obtain for any $\kappa_3>0,\kappa_4>0,\kappa_5>0,\kappa_6>0,\kappa_7>0$ and $\kappa_8>0$, there exist $C(\kappa_3)>0, C(\kappa_4,\kappa_5)>0, C(\kappa_6,\kappa_8)>0$ and $C(\kappa_7)>0$ such that

$$\begin{aligned}
&|\langle B(\mathbf{u},\mathbf{u}_1),\mathbf{u}\rangle|\le\kappa_3\|\mathbf{u}\|^2+C(\kappa_3)|\mathbf{u}_1|^2\|\mathbf{u}_1\|^2|\mathbf{u}|^2,\\
&|\langle M(\mathbf{d}_2,\mathbf{d}),\mathbf{u}\rangle|\le\kappa_4\|\mathbf{u}\|^2+\kappa_5|\Delta\mathbf{d}|^2+C(\kappa_4,\kappa_5)|\nabla\mathbf{d}_2|_{L^2}^2|\Delta\mathbf{d}_2|^2|\nabla\mathbf{d}|^2,\\
&|\langle M(\mathbf{d},\mathbf{d}_1),\mathbf{u}\rangle|\le\kappa_8\|\mathbf{u}\|^2+\kappa_6|\Delta\mathbf{d}|^2+C(\kappa_6,\kappa_8)|\nabla\mathbf{d}_1|_{L^2}^2|\Delta\mathbf{d}_1|^2|\nabla\mathbf{d}|_{L^2}^2,\\
&|\langle\tilde{B}(\mathbf{u}_2,\mathbf{d}),\Delta\mathbf{d}\rangle|\le\kappa_7|\Delta\mathbf{d}|^2+C(\kappa_7)|\mathbf{u}_2|^2\|\mathbf{u}_2\|^2|\nabla\mathbf{d}|_{L^2}^2.
\end{aligned}\tag{8.17}$$

From Gagliardo-Nirenberg inequality and from the Sobolev embedding $H^2 \subset L^\infty$, we obtain for any $\kappa_9 > 0$ there exists $C(\kappa_9) > 0$ such that

$$|\langle \tilde{B}(\mathbf{u}, \mathbf{d}_1), \mathbf{d}\rangle| \le |\mathbf{u}|\ |\nabla \mathbf{d}_1|_{L^2}\ |\mathbf{d}|_{L^\infty} \le \kappa_9 |\Delta \mathbf{d}|^2 + C(\kappa_9)|\mathbf{u}|^2 |\nabla \mathbf{d}_1|^2_{L^2}.$$

Now let us define

$$\Upsilon(t) := \exp\bigg(-2\int_0^t (\xi_1(s) + \xi_2(s) + \xi_3(s))\, ds\bigg), \quad \text{for any } t > 0.$$

where

$$\begin{aligned}
\xi_1(s) &:= C(\kappa_3)|\mathbf{u}_1(s)|^2 \|\mathbf{u}_1(s)\|^2 + C(\kappa_9)|\nabla \mathbf{d}_1(s)|^2_{L^2},\\
\xi_2(s) &:= C(\kappa_4,\kappa_5)|\nabla \mathbf{d}_2(s)|^2_{L^2}|\Delta \mathbf{d}_2(s)|^2 + C(\kappa_6,\kappa_8)|\nabla \mathbf{d}_1(s)|^2_{L^2}|\Delta \mathbf{d}_1(s)|^2\\
&\qquad + C(\kappa_7)|\mathbf{u}_2(s)|^2\|\mathbf{u}_2(s)\|^2 + C_1(\kappa_2)\beta(\mathbf{d}_1,\mathbf{d}_2),\\
\xi_3(s) &:= \big(C(\kappa_1) + C_2(\kappa_2)\big)\beta(\mathbf{d}_1,\mathbf{d}_2),
\end{aligned}$$

where $\beta(\mathbf{d}_1, \mathbf{d}_2)$ is defined in (A.15) of Appendix.

Now applying the Itô formula to $\Upsilon(t)|\mathbf{d}(t)|^2$, we obtain

$$\begin{aligned}
d\big[\Upsilon(t)|\mathbf{d}(t)|^2\big] &= -2\Upsilon(t)\Big[|\nabla \mathbf{d}(t)|^2_{L^2} + \big\langle \tilde{B}(\mathbf{u}(t), \mathbf{d}_1(t)), \mathbf{d}(t)\big\rangle + \big\langle f(\mathbf{d}_2(t)) - f(\mathbf{d}_1(t)), \mathbf{d}(t)\big\rangle\\
&\qquad + \big\langle b(\mathbf{d}_1(t)) - b(\mathbf{d}_2(t)), \mathbf{d}(t)\big\rangle\Big]\, dt\\
&\qquad + 2\Upsilon(t)\Big[\int_{\mathbb{B}} \big(G_d(l), \mathbf{d}(t)\big)_{L^2}\, \tilde{\eta}_2(dt, dl) + \int_{\mathbb{B}} \big|G_d(l)\big|^2_{L^2}\, \nu_2(dl)dt\Big] + \Upsilon'(t)|\mathbf{d}(t)|^2\, dt.
\end{aligned} \tag{8.18}$$

In this proof we will use the Itô formula due to [23]. So applying the Itô formula to $\Upsilon(t)|\nabla \mathbf{d}(t)|^2_{L^2}$ and $\Upsilon(t)|\mathbf{u}(t)|^2$ we get,

$$\begin{aligned}
d\big[\Upsilon(t)|\nabla \mathbf{d}(t)|^2_{L^2}\big] &= 2\Upsilon(t)\Big[- |\Delta \mathbf{d}(t)|^2 + \big\langle \tilde{B}(\mathbf{u}(t), \mathbf{d}_1(t)) + \tilde{B}(\mathbf{u}_2(t), \mathbf{d}(t)), \Delta \mathbf{d}(t)\big\rangle\\
&\qquad + \big\langle f(\mathbf{d}_2(t)) - f(\mathbf{d}_1(t)), \Delta \mathbf{d}(t)\big\rangle + \big\langle b(\mathbf{d}_1(t)) - b(\mathbf{d}_2(t)), \Delta \mathbf{d}(t)\big\rangle\Big]\, dt\\
&\qquad + 2\Upsilon(t)\Big[\int_{\mathbb{B}} \big(G_d(l), \Delta \mathbf{d}(t)\big)_{L^2}\, \tilde{\eta}_2(dt, dl) + \int_{\mathbb{B}} \big|G_d(l)\big|^2_{L^2}\, \nu_2(dl)dt\Big]\\
&\qquad + \Upsilon'(t)|\nabla \mathbf{d}(t)|^2_{L^2}\, dt,
\end{aligned} \tag{8.19}$$

and

$$\begin{aligned}
d\big[\Upsilon(t)|\mathbf{u}(t)|^2\big] &= -2\Upsilon(t)\Big[\|\mathbf{u}(t)\|^2 + \big\langle B(\mathbf{u}(t), \mathbf{u}_1(t)) + M(\mathbf{d}(t), \mathbf{d}_1(t)), \mathbf{u}(t)\big\rangle\\
&\qquad + \big\langle M(\mathbf{d}_2(t), \mathbf{d}(t)), \mathbf{u}(t)\big\rangle\Big]\, dt\\
&\qquad + 2\Upsilon(t)\Big[\int_Y \big(F_d(t,y), \mathbf{u}(t)\big)_{\mathrm{H}}\, \tilde{\eta}_1(dt, dy) + \int_Y \big|F_d(t,y)\big|^2_{\mathrm{H}}\, \nu_1(dy)ds\Big]\\
&\qquad + \Upsilon'(t)|\mathbf{u}(t)|^2\, dt.
\end{aligned} \tag{8.20}$$

Now we estimate few terms from above equations. Using Cauchy-Schwartz and Young's inequalities and Lipschitz property of b

$$\begin{aligned}
&\big|\big\langle b(\mathbf{d}_1(t)) - b(\mathbf{d}_2(t)), \Delta \mathbf{d}(t)\big\rangle\big| \le |b(\mathbf{d}_1(t)) - b(\mathbf{d}_2(t))|_{L^2}|\Delta \mathbf{d}(t)|_{L^2}\\
&\le \frac{1}{2}\big(C\,\big|\mathbf{d}_1(t) - \mathbf{d}_2(t)\big|^2_{L^2} + \big|\Delta \mathbf{d}(t)\big|^2_{L^2}\big) = \frac{1}{2}\big(\kappa_{10}\,\big|\mathbf{d}\big|^2_{L^2} + \big|\Delta \mathbf{d}(t)\big|^2_{L^2}\big).
\end{aligned} \tag{8.21}$$

Similarly, we can show

$$\big|\big\langle b(\mathbf{d}_1(t)) - b(\mathbf{d}_2(t)), \mathbf{d}(t)\big\rangle\big| \le \kappa_{10}\big|\mathbf{d}\big|^2_{L^2}. \tag{8.22}$$

Using Lipschitz property of G we have,

$$\int_{\mathbb{B}} \big|G_d(l)\big|^2_{L^2}\, \nu_2(dl) = \int_{\mathbb{B}} \big|G(l, \mathbf{d}_1) - G(l, \mathbf{d}_2)\big|^2_{L^2}\, \nu_2(dl) \le \kappa_{11}\big|\mathbf{d}_1 - \mathbf{d}_2\big|^2_{L^2} := \kappa_{11}\,\big|\mathbf{d}\big|^2_{L^2}. \tag{8.23}$$

Now adding (8.18), (8.19) and (8.20), using the inequalities (8.17), (8.21), (8.22), (8.23) and Lemma A.9 we get,

$$
\begin{aligned}
& d\Big[\Upsilon(t)\big(|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2_{L^2}\big)\Big]+2\Upsilon(t)\Big[\|\mathbf{u}(t)\|^2+|\nabla \mathbf{d}(t)|^2_{L^2}+|\Delta \mathbf{d}(t)|^2\Big]\,dt \\
& \leq 2\Upsilon(t)\Big[\xi_1(t)|\mathbf{u}(t)|^2+\xi_2(t)|\mathbf{d}(t)|^2+\xi_3(t)|\nabla \mathbf{d}(t)|^2\Big]\,dt \\
& \quad +2\Upsilon(t)\Big[\Big(\kappa_2+\kappa_9+\sum_{i=5}^{7}\kappa_i+\frac{1}{2}\Big)|\Delta \mathbf{d}(t)|^2\,dt\Big]+\Big[\int_Y\big(F_d(t,y),\mathbf{u}(t)\big)_{\mathbb{H}}\,\tilde{\eta}_1(dt,dy) \\
& \quad +\int_{\mathbb{B}}\big(G_d(l),\Delta \mathbf{d}(t)\big)_{L^2}\,\tilde{\eta}_2(dt,dl)+\int_{\mathbb{B}}\big(G_d(l),\mathbf{d}(t)\big)_{L^2}\,\tilde{\eta}_2(dt,dl)\Big] \\
& \quad +2\Upsilon(t)\Big[\Big(\frac{3}{2}\kappa_{10}+2\kappa_{11}\Big)|\mathbf{d}(t)|^2+L|\mathbf{u}(t)|^2+(\kappa_3+\kappa_4+\kappa_8)\|\mathbf{u}(t)\|^2+\kappa_1|\nabla \mathbf{d}(t)|^2\Big]\,dt \\
& \quad +\Upsilon'(t)\Big[|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2\Big]\,dt. \qquad (8.24)
\end{aligned}
$$

By the choice of Υ we have

$$
2\Upsilon(t)\Big[\xi_1(t)|\mathbf{u}(t)|^2+\xi_2(t)|\mathbf{d}(t)|^2+\xi_3(t)|\nabla \mathbf{d}(t)|^2_{L^2}\Big]+\Upsilon'(t)\Big[|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2_{L^2}\Big]\leq 0.
$$

So dropping the above term from the right hand side of (8.24), then choosing $\kappa_2+\kappa_9+\sum_{i=5}^{7}\kappa_i=0$, $\kappa_3=\kappa_4=\kappa_8=\frac{1}{6},\kappa_1=\frac{1}{2}$ and rearranging we obtain,

$$
\begin{aligned}
& d\Big[\Upsilon(t)\big(|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2\big)\Big]+\Upsilon(t)\Big[\|\mathbf{u}(t)\|^2+|\nabla \mathbf{d}(t)|^2+|\Delta \mathbf{d}(t)|^2\Big]\,dt \\
& \leq 2\Upsilon(t)\Big[C\big(|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2\big)\Big]\,dt+2\Upsilon(t)\Big[\int_Y\big(F_d(t,y),\mathbf{u}(t)\big)_{\mathbb{H}}\,\tilde{\eta}_1(dt,dy) \\
& \qquad +\int_{\mathbb{B}}\big(G_d(l),\Delta \mathbf{d}(t)\big)_{L^2}\,\tilde{\eta}_2(dt,dl)+\int_{\mathbb{B}}\big(G_d(l),\mathbf{d}(t)\big)_{L^2}\,\tilde{\eta}_2(dt,dl)\Big] \qquad (8.25)
\end{aligned}
$$

Now integrating both side and taking mathematical expectation we get

$$
\begin{aligned}
& \mathbb{E}\Big[\Upsilon(t)\big(|\mathbf{u}(t)|^2+|\mathbf{d}(t)|^2+|\nabla \mathbf{d}(t)|^2_{L^2}\big)\Big]+\mathbb{E}\Big[\int_0^t\Upsilon(s)\big(\|\mathbf{u}(s)\|^2+|\nabla \mathbf{d}(s)|^2_{L^2}+|\Delta \mathbf{d}(s)|^2\big)\,ds\Big] \\
& \leq C\int_0^t\mathbb{E}\Big[\Upsilon(s)\big(|\mathbf{u}(s)|^2+|\mathbf{d}(s)|^2+|\nabla \mathbf{d}(s)|^2_{L^2}\big)\Big]\,ds. \qquad (8.26)
\end{aligned}
$$

Now applying Gronwall's inequality we obtain (8.16).

□

Definition 8.5. *It is said that the solutions to problem* (2.42)-(2.43) *are unique in law if for any two martingale solutions* $\big(\Omega_i,\mathcal{F}_i,\mathbb{F}_i,\mathbb{P}_i,\{\mathbf{u}_i(t)\}_{t\geq 0},\{\mathbf{d}_i(t)\}_{t\geq 0},\{\tilde{\eta}_1^i(t,\cdot)\}_{t\geq 0},\tilde{\eta}_2^i(t,\cdot)\}_{t\geq 0}\big)$, $i=1,2$, *of problem* (2.42)-(2.43) *with*

$$\mathscr{L}_{\mathbb{P}_1}(\mathbf{u}_1(0))=\mathscr{L}_{\mathbb{P}_2}(\mathbf{u}_2(0)) \quad on \quad \mathrm{H}$$

and

$$\mathscr{L}_{\mathbb{P}_1}(\mathbf{d}_1(0))=\mathscr{L}_{\mathbb{P}_2}(\mathbf{d}_2(0)) \quad on \quad H^1,$$

the laws of the solutions are also equal, i.e.

$$\mathscr{L}_{\mathbb{P}_1}(\mathbf{u}_1)=\mathscr{L}_{\mathbb{P}_2}(\mathbf{u}_2) \quad on \quad L^2(0,T;\mathrm{V})\cap\mathbb{D}([0,T];\mathrm{H})$$

and

$$\mathscr{L}_{\mathbb{P}_1}(\mathbf{d}_1)=\mathscr{L}_{\mathbb{P}_2}(\mathbf{d}_2) \quad on \quad L^2(0,T;D(\mathcal{A}))\cap\mathbb{D}([0,T];H^1),$$

where $\mathscr{L}_{\mathbb{P}_i}(\mathbf{u}_i)$ *and* $\mathscr{L}_{\mathbb{P}_i}(\mathbf{d}_i)$ *for* $i=1,2$ *are probability measures on* $L^2(0,T;\mathrm{V})\cap\mathbb{D}([0,T];\mathrm{H})$ *and* $L^2(0,T;D(\mathcal{A}))\cap\mathbb{D}([0,T];H^1)$ *respectively.*

Corollary 8.6. *Let* $\mathbf{n}=2$. *Let Assumption 2.3 holds. Then*

(1) there exists a pathwise unique strong solution to the problem (2.42)-(2.43).

(2) *Moreover, if* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \mathbf{u}, \mathbf{d}, \tilde{\eta}_1, \tilde{\eta}_2)$ *is a strong solution of* (2.42)-(2.43) *then for* $\mathbb{P}$*-almost all* $\omega \in \Omega$ *the trajectories* $\mathbf{u}(\cdot, \omega)$ *is almost everywhere equal to a càdlàg* H*-valued function and* $\mathbf{d}(\cdot, \omega)$ *is almost everywhere equal to a càdlàg* H^1*-valued function defined on* $[0, T]$.

(3) *The martingale solution of* (2.42)-(2.43) *is unique in law.*

Proof. The existence of a martingale solution is shown in Theorem 7.4. From Lemma 8.4 we obtained the solutions are pathwise unique. Thus the first assertion follows from the Yamada-Watanabe Theorem in the version proved in [41, Theorem 2], see also discussion before Theorem 4.10 in [5]. The second assertion is a direct consequence of Lemma 8.1. The third assertion follows from [41, Theorems 2, 11]. □

9. Proof of Theorem 3.3

The existence of a martingale solution has been proved in Section 7. From Remark 7.5 we infer that the solutions $\bar{\mathbf{u}}$ and $\bar{\mathbf{d}}$ satisfy (3.9) and (3.10) respectively. The pathwise uniqueness of solutions has been proved in Section 8.3 (see Lemma 8.4). The existence of a strong solution has been done in Corollary 8.6.

Appendix A. Some important results

In this section we recall some important results which are needed in our proof of main result.

Remark A.1. *We will show the existence of the countable family of real valued continuous functions which are defined on* $\mathcal{Z}_T$ *and separate points of this space.*

(1) We know $L^2(0, T; \mathrm{H})$ and $\mathbb{D}([0, T]; \mathrm{V}')$ are completely metrizable and separable spaces, we deduce that there exists a countable family of continuous real valued functions on each of these spaces which separate points. For example see [4], exposé 8.

(2) For the space $L^2_w(0, T; \mathrm{V})$ we define

$$g_m(\mathbf{u}) := \int_0^T ((\mathbf{u}(t), v_m(t)))\, dt \in \mathbb{R}, \qquad \mathbf{u} \in L^2(0, T; \mathrm{V}), \quad m \in \mathbb{N},$$

where $\{v_m, m \in \mathbb{N}\}$ is a dense subset of $L^2(0, T; \mathrm{V})$. then $(g_m)_{m\in\mathbb{N}}$ is a sequence of continuous real valued functions separating points of the space $L^2_w(0, T; \mathrm{V})$.

(3) Let $\mathrm{H}_0 \subset \mathrm{H}$ be a countable and dense subset of H. Then for each $h \in \mathrm{H}_0$ the mapping

$$\mathbb{D}([0, T]; \mathrm{H}_w) \ni \mathbf{u} \mapsto (\mathbf{u}(\cdot), h)_{\mathrm{H}} \in \mathbb{D}([0, T]; \mathbb{R})$$

is continuous. Since $\mathbb{D}([0, T]; \mathbb{R})$ is a separable complete metric space, there exists a sequence $(f_l)_{l\in\mathbb{N}}$ of real valued continuous functions defined on $\mathbb{D}([0, T]; \mathbb{R})$ separating points of this space. Then the mappings $m_{h,l}$, where $h \in H_0, l \in \mathbb{N}$ defined by

$$m_{h,l}(\mathbf{u}) := f_l((\mathbf{u}(\cdot), h)_{\mathrm{H}}, \qquad \mathbf{u} \in \mathbb{D}([0, T]; \mathrm{H}_w),$$

form a countable family of continuous functions on $\mathbb{D}([0, T]; \mathrm{H}_w)$ which separates points of this space.

Similarly we can define for the space $\mathcal{Z}_{T,2}$.

Lemma A.2. *Let* $(\bar{\mathbf{d}}_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^\infty(0, T; H^1) \cap L^2(0, T; D(\mathcal{A}))$ *such that* $\bar{\mathbf{d}}_n \to \mathbf{d}_*$ *in* $L^2(0, T; H^1)$*. Then for all* $v \in \mathrm{V}$ *and all* $t \in [0, T]$,

$$\lim_{n\to\infty} \int_0^t \left\langle M\big(\bar{\mathbf{d}}_n(s)\big), v\right\rangle ds = \int_0^t \left\langle M\big(\mathbf{d}_*(s)\big), v\right\rangle ds. \tag{A.1}$$

Proof. Recall the definition of $\mathcal{V}$ from Section 2. Assume that $v \in \mathcal{V}$. Then for $\mathbf{d}_1, \mathbf{d}_2 \in H^2$, using integration byparts we get

$$\begin{aligned} \left|{}_{\mathrm{V}'}\langle M(\mathbf{d}_1, \mathbf{d}_2), v\rangle_{\mathrm{V}}\right| &= \left|\big(M(\mathbf{d}_1, \mathbf{d}_2), v\big)_{L^2}\right| = \left|\int_{\mathbb{O}} \nabla \cdot (\nabla \mathbf{d}_1 \odot \nabla \mathbf{d}_2) \cdot v\, dx\right| \\ &= \left|\int_{\mathbb{O}} (\nabla \mathbf{d}_1 \odot \nabla \mathbf{d}_2) \cdot \nabla v\, dx\right| \le c\, |\nabla \mathbf{d}_1|_{L^2}\, |\nabla \mathbf{d}_2|\, |\nabla v|_{L^\infty} \le c\, |\mathbf{d}_1|_{H^1}\, |\mathbf{d}_2|_{H^1}\, |v|_{H^3}. \end{aligned} \tag{A.2}$$

Moreover,

$$M\big(\bar{\mathbf{d}}_n, \bar{\mathbf{d}}_n\big) - M\big(\mathbf{d}_*, \mathbf{d}_*\big) = M\big(\bar{\mathbf{d}}_n - \mathbf{d}_*, \bar{\mathbf{d}}_n\big) + M\big(\mathbf{d}_*, \bar{\mathbf{d}}_n - \mathbf{d}_*\big). \tag{A.3}$$

Then from (A.2), (A.3) and using the Hölder inequality, we obtain

$$\begin{aligned}
&\Big| \int_0^t \big\langle M\big(\bar{\mathbf{d}}_n(s), \bar{\mathbf{d}}_n(s)\big), v\big\rangle\, ds - \int_0^t \big\langle M\big(\mathbf{d}_*(s), \mathbf{d}_*(s)\big), v\big\rangle\, ds \Big| \\
&\le \Big| \int_0^t \big\langle M\big(\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s), \bar{\mathbf{d}}_n(s)\big), v\big\rangle\, ds \Big| + \Big| \int_0^t \big\langle M\big(\mathbf{d}_*(s), \bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s)\big), v\big\rangle\, ds \Big| \\
&\le \Big(\int_0^t |\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s)|_{H^1} |\bar{\mathbf{d}}_n(s)|_{H^1}\, ds + \int_0^t |\mathbf{d}_*(s)|_{H^1} |\bar{\mathbf{d}}_n(s) - \mathbf{d}_*(s)|_{H^1}\, ds \Big) |v|_{H^3} \\
&\le c\, |\bar{\mathbf{d}}_n - \mathbf{d}_*|_{L^2(0,T;H^1)} \big(|\bar{\mathbf{d}}_n|_{L^2(0,T;H^1)} + |\mathbf{d}_*|_{L^2(0,T;H^1)} \big) |v|_{H^3},
\end{aligned} \tag{A.4}$$

where the constant $c > 0$. Using the fact that $\bar{\mathbf{d}}_n \to \mathbf{d}_*$ in $L^2(0,T;H^1)$ we infer that (A.1) holds for all $v \in \mathcal{V}$.

If $v \in \mathrm{V}$, then for every $\epsilon > 0$ there exists $v_\epsilon \in \mathcal{V}$ such that $|v - v_\epsilon|_{\mathrm{V}} \le \epsilon$. Therefore we get

$$\begin{aligned}
&\big|\big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v\big\rangle\big| \\
&\quad \le \quad \big|\big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v - v_\epsilon\big\rangle\big| \\
&\quad\ \ + \quad \big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v_\epsilon\big\rangle\big| \\
&\quad \le \quad \big(|M(\mathbf{d}_n(r), \mathbf{d}_n(r))|_{\mathrm{V}'} + |M(\mathbf{d}_*(r), \mathbf{d}_*(r))|_{\mathrm{V}'} \big) |v - v_\epsilon|_{\mathrm{V}} \\
&\quad\ \ + \quad \big|\big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v_\epsilon\big\rangle\big|
\end{aligned} \tag{A.5}$$

For $\mathbf{n} = 2$, following the previous calculations in (A.5) we get,

$$\begin{aligned}
&\Big| \int_s^t \big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v\big\rangle\, dr \Big| \\
&\le \epsilon \int_0^t \big(|\nabla \mathbf{d}_n|_{L^2} |\Delta \mathbf{d}_n| + |\nabla \mathbf{d}_*| |\Delta \mathbf{d}_*| \big)\, dr + \Big| \int_s^t \big\langle M(\mathbf{d}_n, \mathbf{d}_n) - M(\mathbf{d}_*, \mathbf{d}_*), v_\epsilon\big\rangle\, dr \Big| \\
&\le \epsilon \Big[\sup_{n \ge 1} \big(|\mathbf{d}_n|_{L^\infty(0,T;H^1)} \cdot |\mathbf{d}_n|_{L^2(0,T;D(\mathcal{A}))} \big) + |\mathbf{d}_*|_{L^\infty(0,T;H^1)} \cdot |\mathbf{d}_*|_{L^2(0,T;D(\mathcal{A}))} \Big] \\
&\quad + \Big| \int_s^t \big\langle M(\mathbf{d}_n, \mathbf{d}_n) - M(\mathbf{d}_*, \mathbf{d}_*), v_\epsilon\big\rangle\, dr \Big|
\end{aligned} \tag{A.6}$$

Passing to the limit as $n \to \infty$,

$$\limsup_{n \to \infty} \Big| \int_s^t \big\langle M(\mathbf{d}_n(r), \mathbf{d}_n(r)) - M(\mathbf{d}_*(r), \mathbf{d}_*(r)), v\big\rangle\, dr \Big| \le \epsilon\, \tilde{C}, \tag{A.7}$$

where $\tilde{C} := |\mathbf{d}_n|_{L^\infty(0,T;H^1)} \cdot |\mathbf{d}_n|_{L^2(0,T;D(\mathcal{A}))} + |\mathbf{d}_*|_{L^\infty(0,T;H^1)} \cdot |\mathbf{d}_*|_{L^2(0,T;D(\mathcal{A}))} < \infty$.

Since $\epsilon > 0$ is arbitrary, we infer that

$$\lim_{n \to \infty} \int_s^t \big\langle M\big(\bar{\mathbf{d}}_n(r), \bar{\mathbf{d}}_n(r)\big), v\big\rangle\, dr = \int_s^t \big\langle M\big(\mathbf{d}_*(r), \mathbf{d}_*(r)\big), v\big\rangle\, dr. \tag{A.8}$$

This completes the proof for the case of dimension $\mathbf{n} = 2$.

For $\mathbf{n} = 3$, we proceed similarly as above. □

Lemma A.3. *Let $(\bar{\mathbf{u}}_n)_{n \in \mathbb{N}}$ and $(\bar{\mathbf{d}}_n)_{n \in \mathbb{N}}$ are bounded sequences in $L^2(0,T;\mathrm{H})$ and $L^2(0,T;H^1)$ respecitvely such that $\bar{\mathbf{u}}_n \to \mathbf{u}_*$ in $L^2(0,T;\mathrm{H})$ and $\bar{\mathbf{d}}_n \to \mathbf{d}_*$ in $L^2(0,T;H^1)$. Then for all $v \in D(\mathcal{A})$, and all $t \in [0,T]$,*

$$\lim_{n \to \infty} \int_0^t \big(\tilde{B}\big(\bar{\mathbf{u}}_n(s), \bar{\mathbf{d}}_n(s)\big), v \big)_{L^2}\, ds = \int_0^t \big(\tilde{B}\big(\mathbf{u}_*(s), \mathbf{d}_*(s)\big), v \big)_{L^2}\, ds. \tag{A.9}$$

Proof. It can be proved similarly as Lemma A.2. □

Lemma A.4. *Let $\mathbf{u}_* \in L^2(0,T;\mathrm{H})$. Let $(\bar{\mathbf{u}}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;\mathrm{H})$ such that $\bar{\mathbf{u}}_n \to \mathbf{u}_*$ in $L^2(0,T;\mathrm{H})$. Then for all $v \in \mathrm{V}$ and all $t \in [0,T]$,*

$$\lim_{n\to\infty} \int_0^t \big\langle B\big(\bar{\mathbf{u}}_n(s)\big), v\big\rangle\, ds = \int_0^t \big\langle B\big(\mathbf{u}_*(s)\big), v\big\rangle\, ds.$$

Proof. The proof is similar to the proof of previous Lemma. □

Proposition A.5. *Let $\mathbf{u}_*$ be the process as defined in* (7.1). *Then for $p \geq 1$, we have*

$$\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]} \big|\mathbf{u}_*(s)\big|_{\mathrm{H}}^{2p}\Big] < C_p.$$

Proof. From (7.3) we have, $\big(\bar{\mathbf{u}}_n\big)_{n\in\mathbb{N}}$ is uniformly bounded in $L^{2p}(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$. Since the dual of $L^{2p}(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$ is $\big(L^{\frac{2p}{2p-1}}(\bar{\Omega};L^1(0,T;\mathrm{H}))\big)'$, by Banach-Alaoglu Theorem, there exists a subsequence of $\bar{\mathbf{u}}_n$, again denoted by the same and there exists $v \in L^{2p}(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$ such that $\bar{\mathbf{u}}_n$ convergent weakly-star to v in $L^{2p}(\bar{\Omega};L^\infty(0,T;\mathrm{H}))$. In particular,

$$\big\langle \bar{\mathbf{u}}_n, \phi\big\rangle \rightharpoonup \big\langle v, \phi\big\rangle, \quad \phi \in L^{\frac{2p}{2p-1}}(\bar{\Omega};L^1(0,T;\mathrm{H})).$$

i.e.,

$$\int_{\bar{\Omega}}\int_0^T \big\langle \bar{\mathbf{u}}_n, \phi\big\rangle\, dt\, d\mathbb{P}(\omega) \to \int_{\bar{\Omega}}\int_0^T \big\langle v, \phi\big\rangle\, dt\, d\mathbb{P}(\omega) \tag{A.10}$$

Again we have $\bar{\mathbf{u}}_n$ convergent weakly to $\mathbf{u}_*$ in $L^2(\bar{\Omega};L^2(0,T;\mathrm{V}))$. Hence by using the compactness of the embedding $\mathrm{V} \hookrightarrow \mathrm{H}$, we infer that for $\phi \in L^2(\bar{\Omega};L^2(0,T;\mathrm{H}))$,

$$\bar{\mathbb{E}}\Big[\int_0^T \big(\bar{\mathbf{u}}_n(t,\omega), \phi(t,\omega)\big)_{\mathrm{H}}\, dt\Big] \to \bar{\mathbb{E}}\Big[\int_0^T \big(\mathbf{u}_*(t,\omega), \phi(t,\omega)\big)_{\mathrm{H}}\, dt\Big]. \tag{A.11}$$

For $p \geq 1$, $L^2(\bar{\Omega};L^2(0,T;\mathrm{H}))$ is dense subspace of $L^{\frac{2p}{2p-1}}(\bar{\Omega};L^1(0,T;\mathrm{H}))$.
From (A.10) and (A.11) we infer that for $\phi \in L^2(\bar{\Omega};L^2(0,T;\mathrm{H}))$,

$$\bar{\mathbb{E}}\Big[\int_0^T (v(t,\omega), \phi(t,\omega))_{\mathrm{H}}\, dt\Big] = \bar{\mathbb{E}}\Big[\int_0^T (\mathbf{u}_*(t,\omega), \phi(t,\omega))_{\mathrm{H}}\, dt\Big].$$

Thus we have, $\mathbf{u}_* = v$ and $\mathbf{u}_* \in L^{2p}(\bar{\Omega};L^2(0,T;\mathrm{H}))$, which is the desired result. □

Proposition A.6. *Let $\mathbf{d}_*$ be the process as defined in* (7.2). *Then for $p \geq 1$ and $q \geq 2$, we have*

$$\bar{\mathbb{E}}\Big[\sup_{s\in[0,T]} \big|\mathbf{d}_*(s)\big|_{L^2}^{2p}\Big] < C(p) \quad \textit{and} \quad \bar{\mathbb{E}}\Big[\sup_{s\in[0,T]} \big|\mathbf{d}_*(s)\big|_{H^1}^{q}\Big] < C_q.$$

Proof. Proof will follow somewhat similar to the proof of Proposition A.5. □

The following three Lemmas can be found in [9]. We provide the proofs for the sake of the readers convenience.

Lemma A.7. *Let $(\mathbf{u}_n, \mathbf{d}_n)$ be the solution of the Galerkin system* (5.6)-(5.7). *Then the following holds.*

$$\langle \tilde{B}_n(\mathbf{u}_n, \mathbf{d}_n), \Delta\mathbf{d}_n + f_n(\mathbf{d}_n)\rangle = \langle M_n(\mathbf{d}_n), \mathbf{u}_n\rangle. \tag{A.12}$$

Proof. Using integration by parts and the divergence free condition of $\mathbf{u}_n$ we get,

$$\begin{aligned}
\langle \tilde{B}_n(\mathbf{u}_n, \mathbf{d}_n), \Delta\mathbf{d}_n\rangle &= \int_{\mathbb{O}} \mathbf{u}_n^{(j)} \frac{\partial \mathbf{d}_n^{(k)}}{\partial x_j} \frac{\partial^2 \mathbf{d}_n^{(k)}}{\partial x_i \partial x_i}\, dx \\
&= -\int_{\mathbb{O}} \frac{\partial \mathbf{u}_n^{(j)}}{\partial x_i} \frac{\partial \mathbf{d}_n^{(k)}}{\partial x_j} \frac{\partial \mathbf{d}_n^{(k)}}{\partial x_i}\, dx - \frac{1}{2}\int_{\mathbb{O}} \mathbf{u}_n^{(j)} \frac{\partial}{\partial x_j}(|\nabla \mathbf{d}_n|^2)\, dx \\
&= \langle M_n(\mathbf{d}_n), \mathbf{u}_n\rangle + \frac{1}{2}\int_{\mathbb{O}} \frac{\partial \mathbf{u}_n^{(j)}}{\partial x_j} |\nabla \mathbf{d}_n|^2\, dx = \langle M_n(\mathbf{d}_n), \mathbf{u}_n\rangle.
\end{aligned} \tag{A.13}$$

And

$$\big\langle \tilde{B}_n(\mathbf{u}_n, \mathbf{d}_n), f_n(\mathbf{d}_n)\big\rangle = \int_{\mathbb{O}} \mathbf{u}_n^{(i)} \frac{\partial \mathbf{d}_n^{(j)}}{\partial x_i} \tilde{f}_n(|\mathbf{d}_n|^2)\mathbf{d}_n^{(j)}\, dx = \frac{1}{2}\int_{\mathbb{O}} \mathbf{u}_n^{(i)} \frac{\partial \tilde{\mathrm{F}}_n(|\mathbf{d}_n|^2)}{\partial x_i}\, dx$$
$$= \frac{1}{2}\big\langle \mathbf{u}_n, \nabla \tilde{\mathrm{F}}_n(|\mathbf{d}_n|^2)\big\rangle = -\frac{1}{2}\big\langle \nabla\cdot \mathbf{u}_n, \tilde{\mathrm{F}}_n(|\mathbf{d}_n|^2)\big\rangle = 0. \tag{A.14}$$

Then adding (A.13) and (A.14) we get the desired result. ☐

Lemma A.8. *Let the Assumption (C) of section 2.3 holds. Then there exists a positive constant C such that for any $\mathbf{d}_1, \mathbf{d}_2 \in H^2$, we obtain*

$$|f(\mathbf{d}_1) - f(\mathbf{d}_2)|_{H^1} \le C\big(1 + \big|\mathbf{d}_1\big|_{H^2}^{2N} + \big|\mathbf{d}_2\big|_{H^2}^{2N}\big)|\mathbf{d}_1 - \mathbf{d}_2|_{H^2}.$$

Proof. We use the fact that H^2 is an algebra. Then it is straight forward to prove for the leading term $b_N|\mathbf{d}|^{2N}\mathbf{d}$. ☐

Lemma A.9. *For any $\kappa_1 > 0$ and $\kappa_2 > 0$, there exist $C(\kappa_1) > 0, C_1(\kappa_2) > 0$ and $C_2(\kappa_2) > 0$ such that*

$$\big|\big\langle f(\mathbf{d}_1) - f(\mathbf{d}_2), \mathbf{d}_1 - \mathbf{d}_2\big\rangle\big| \le \kappa_1|\nabla\mathbf{d}_1 - \nabla\mathbf{d}_2|_{L^2}^2 + C(\kappa_1)|\mathbf{d}_1 - \mathbf{d}_2|_{L^2}^2\beta(\mathbf{d}_1, \mathbf{d}_2),$$

and

$$\big|\big\langle f(\mathbf{d}_1) - f(\mathbf{d}_2), \Delta\mathbf{d}_1 - \Delta\mathbf{d}_2\big\rangle\big|$$
$$\le \kappa_2|\Delta\mathbf{d}_1 - \Delta\mathbf{d}_2|_{L^2}^2 + \big\{C_1(\kappa_2)|\nabla\mathbf{d}_1 - \nabla\mathbf{d}_2|_{L^2}^2 + C_2(\kappa_2)|\mathbf{d}_1 - \mathbf{d}_2|_{L^2}^2\big\}\beta(\mathbf{d}_1, \mathbf{d}_2),$$

where

$$\beta(\mathbf{d}_1, \mathbf{d}_2) := C\big(1 + |\mathbf{d}_1|_{L^{4N+2}}^{2N} + |\mathbf{d}_2|_{L^{4N+2}}^{2N}\big)^2. \tag{A.15}$$

Proof. We use the inequality

$$\big|\big\langle f(\mathbf{d}_1) - f(\mathbf{d}_2), \mathbf{d}_1 - \mathbf{d}_2\big\rangle\big| \le C\int_{\mathbb{O}} \big(1 + |\mathbf{d}_1|^{2N} + |\mathbf{d}_2|^{2N}\big)|\mathbf{d}_1 - \mathbf{d}_2|^2\, dx.$$

Then using the Hölder, the Gagliardo-Nirenberg and the Young inequalities and using the fact that $L^{4N+2} \subset L^{4N}$, we get the desired result. ☐

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