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DISCORDANT VOTING PROCESSES ON FINITE GRAPHS[*]

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Abstract. We consider an asynchronous voting process on graphs called discordant voting, which can be described as follows. Initially each vertex holds one of two opinions, red or blue. Neighboring vertices with different opinions interact pairwise along an edge. After an interaction both vertices have the same color. The quantity of interest is the time to reach consensus, i.e., the number of steps needed for all vertices have the same color. We show that for a given initial coloring of the vertices, the expected time to reach consensus depends strongly on the underlying graph and the update rule (i.e., push, pull, oblivious).

Key words. distributed consensus, voter model, discordant voting, interacting particles, discrete random process

AMS subject classifications. C.2.4, G.2, G.3

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1. Introduction. The process of reaching consensus in a graph by means of local interactions is known as voting. It is an abstraction of human behavior and can be implemented in distributed computer networks. As a consequence voting processes have been widely studied.

In the simplest case each vertex has a color or opinion (e.g., red, blue, etc.), and neighboring vertices interact pairwise in a fixed way to update their colors. After this interaction both vertices have the same color. Three basic ways to make an update are as follows:

>*Push:* Pick a random vertex and push its color to a random neighbor.
>*Pull:* Pick a random vertex and pull the color of a random neighbor.
>*Oblivious:* Pick a random edge and push the color of one randomly chosen endpoint to the other one.

In the case of asynchronous voting, all three methods are well defined. For synchronous voting the push and oblivious processes are not well defined, as more than one color could be pushed to a vertex at a given step.

The performance of randomized voting processes is usually measured by the *consensus time* (the expected time to reach consensus) and the probability a given opinion wins.

A common synchronous model is *randomized pull voting*, where at each step, each vertex adopts the opinion of a randomly chosen neighbor. For connected nonbipartite graphs, Hassin and Peleg [13] and Nakata, Imahayashi, and Yamashita [18] proved the probability opinion (color) A wins is $d(A)/2m$, where $d(A)$ is the degree of the vertices initially holding opinion A, and m is the number of edges of the graph.

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If the colors of the vertices are initially distinct, the pull voting process takes $\Theta(n)$ expected steps to reach consensus on many classes of expander graphs on n vertices. This is proved for the complete graph K_n by Aldous and Fill [1] and for r-regular random graphs by Cooper, Frieze, and Radzik [5]. Results for general connected graphs based on the eigenvalue gap and variance of the degree sequence are given by Cooper et al. in [6]. For n-vertex graphs G, they give an expected consensus time of $O(n/(\nu(1 - \lambda)))$, where λ is the second largest eigenvalue in absolute value of the transition matrix P of a simple random walk on G. Thus $\lambda = \max(|\lambda_2|, \ldots, |\lambda_n|)$, where $\lambda_1 = 1$ is the principal eigenvalue. The parameter ν measures the regularity of the degree sequence and ranges from 1 for regular graphs to $\Theta(n)$ for the star graph. The value is given by $\nu = \sum_{v \in V} d^2(v)/(d^2 n)$, where $d(v)$ is the degree of vertex v, and $d = d_{\mathrm{ave}} = 2m/n$ is the average degree.

For regular graphs, the result of [6] achieves an upper bound of $O(n^3)$ in the worst case. Using a different approach, Berenbrink et al. [4] proved a consensus time of $O((d_{\mathrm{ave}}/d_{\min})(n/\Phi))$. Here d_{ave}, $d_{\min}$ are the average and minimum degrees, respectively. The parameter Φ is the graph conductance, $\Phi = \min_{S \subset V(G)} \frac{E(S:S^c)}{\min\{d(S), d(S^c)\}}$, where $E(S : S^c)$ are the edges between S and S^c, and $S \neq \emptyset, V$.

Much of the analysis of *asynchronous pull voting* has been made in the continuous-time model, where edges or vertices have exponential waiting times between events. An example is the work by Cox [8] for toroidal grids. For detailed coverage see Liggett [16]. More recently Oliveira [19] shows that the expected consensus time is $O(H_{\max})$, where $H_{\max} = \max_{v,u \in V} H(v, u)$ and $H(v, u)$ is the expected hitting time of u by a random walk starting at vertex v. Asynchronous pull voting is less studied in a discrete setting. It was shown in [7] that the expected time to consensus for asynchronous pull voting is

$$(1) \qquad\qquad \mathbf{E}T = O(nm/d_{\min}\Phi),$$

where m is the number of edges, $d_{\min}$ is minimum degree, and Φ is graph conductance. Thus $\mathbf{E}T = O(n^5)$ for any connected graph, and $O(n^2)$ for regular expanders.

This paper considers a different asynchronous voting process, *discordant voting*. An edge is discordant if the colors of its endpoints differ, and a vertex is discordant if any of its incident edges are discordant. In the model we consider, initially each vertex has one of two opinions, red or blue. Neighboring vertices of different colors interact pairwise along a discordant edge. Thus, e.g., in discordant push voting, a discordant vertex is chosen at each step, and then a discordant edge incident with the chosen vertex. Henceforth we refer to the voting processes discussed above (based on choosing edges or vertices randomly) as *ordinary voting* processes to distinguish them from discordant voting.

In discordant voting, the expected time to consensus varies considerably, both with the structure of the underlying graph and with the protocol used, and sometimes in a quite counterintuitive way (see Table 1). This behavior is in contrast with ordinary (i.e., nondiscordant) asynchronous voting. Moreover, only the push and pull protocols exhibit this variation in consensus time with the graph structure. The oblivious protocol has a consensus time of $O(n^2)$ on any connected n-vertex graph (see the paragraph following Remark 1). This consensus time depends only on the initial number of red and blue vertices and is independent of the graph structure, and thus serves as a standard of comparison for the other two processes.

Discordant voting originated in the complex networks community as a model of social evolution (see, e.g., [12], [21]). The general version of the model allows for *rewiring*; the interacting vertices can break the edge joining them and reconnect

elsewhere. This serves as a model of social behavior in which vertices either change their opinion or their friends.

Rewiring in the oblivious discordant voting model has been extensively studied. Holme and Newman [14] investigated discordant voting as a model of a self-organizing network which restructures based on the acceptance or rejection of differing opinions among social groups. At each step, a random discordant edge uv is selected, and an endpoint $x \in \{u, v\}$ chosen with probability $1/2$. With probability $1 - \alpha$ the opinion of x is pushed to the other endpoint y, and with probability α, vertex y breaks the edge and rewires to a random vertex with the same opinion as itself. Simulations suggest the existence of threshold behavior in α. This was investigated further by Durrett et al. [10] for sparse random graphs of constant average degree 4 (i.e., $G(n, 4/n)$). The paper studies two rewiring strategies, rewire-to-random and rewire-to-same, and finds experimental evidence of a phase transition in both cases. Basu and Sly [3] made a formal analysis of rewiring for Erdos–Renyi graphs $G(n, 1/2)$ with $1 - \alpha = \beta/n$, $\beta > 0$ constant. They found that for either strategy, if β is sufficiently small the network quickly disconnects, maintaining the initial proportions. As β increases the minority proportion decreases, and in rewire-to-random a positive fraction of both opinions survive. A subsequent paper by Basak, Durrett, and Zhang [2] examines the rewiring phase transitions for the intermediate case of thick graphs $G(n, 1/n^a)$, where $0 < a < 1$.

Although discordant voting seems a natural model of local interaction, its behavior is not well understood even in the simplest cases. Moreover, the analysis of rewiring is highly problematic. First there is no natural model for the space of random graphs derived from the rewiring. Second the voting and rewiring interactions condition the degree sequence in a way which makes subsequent analysis difficult.

In this paper we assume there is no rewiring and evaluate the performance of discordant voting as a function of the graph structure. As discordant voting always chooses an edge between the opposing red and blue sets, intuitively it should finish faster than ordinary asynchronous voting which ignores this discordancy information.

As previously remarked, for discordant voting using the oblivious protocol, the expected time to consensus is the same for any connected n-vertex graph. It is independent of graph structure and of the number of edges, and depends only on the initial number of vertices of each color (red, blue). Whichever discordant edge is chosen, the number of blue vertices in the graph increases (resp., decreases) by one with probability $1/2$ at each step. This is equivalent to an unbiased random walk on the line $(0, 1, \ldots, n)$ with absorbing barriers (see Feller [11, Chapter XIV, section 3]).

Remark 1 (oblivious protocol). Let T be the time to consensus in the two-party asynchronous discordant voting process starting from any initial coloring with $R(0) = r, B(0) = n - r$ red and blue vertices, respectively. For any connected n vertex graph, $\mathbf{E}T(\text{Oblivious}) = r(n - r)$.

Starting with an equal number of red and blue vertices the oblivious protocol takes $\mathbf{E}T \sim n^2/4$ steps for any connected graph. For ordinary asynchronous voting, the performance of the oblivious protocol can also depend on the number of edges m. In the worst case, the expected wait to hit the last red-blue edge is m. If so, the ordinary asynchronous case could take $\mathbf{E}T = O(mn^2)$ steps.

In contrast to the oblivious case, discordant push and pull protocols can exhibit very different expected times to consensus, which depend strongly on the underlying graph in question.

THEOREM 1. *Let T be the time to consensus of the asynchronous discordant voting process starting from any initial coloring with an equal number of red and blue vertices, i.e., $R = B = n/2$. For the complete graph K_n, $\mathbf{E}T(Push) = \Theta(n \log n)$ and $\mathbf{E}T(Pull) = \Theta(2^n)$.*

For the complete graph K_n the various protocols give very different expected completion times, which vary from $\Theta(n \log n)$ for push, to $\Theta(n^2)$ for oblivious, to $\Theta(2^n)$ for pull. On the basis of this evidence, our initial view was that there should be a meta-theorem of the "push is faster than oblivious, oblivious is faster than pull" type. Intuitively, this is supported by the following argument. Suppose red (R) is the larger color class. Choosing a discordant vertex uniformly at random favors the selection of the larger class. In the push process, red vertices push their opinion more often, which tends to increase the size of R. Conversely, the pull process tends to re-balance the set sizes. If R is larger, it is recolored more often.

For the cycle C_n, we prove that all three protocols have similar expected time to consensus; a result which is consistent with the above meta-theorem.

THEOREM 2. *Let T be the time to consensus of the asynchronous discordant voting process. For the cycle C_n, and starting from any initial coloring with an equal number of red and blue vertices, the push and pull protocols have $\mathbf{E}T = O(n^2)$.*

For the initial coloring where the first half of the cycle is colored red and the second half blue, as shown in Figure 4, the push and pull protocols have $\mathbf{E}T = \Theta(n^2)$.

At the time of publication of this paper, the constant in the value of $\mathbf{E}T$ was refined by Pongrácz [20]. By combining the techniques of this paper with results for absorbing Markov chains, it is shown that Theorem 2 can be refined to $|\mathbf{E}T - n^2/4| = O(n^{3/2})$ for all three processes.

At this point we are left with a choice: either to produce evidence for a relationship of the form $\mathbf{E}T(\text{Push}) = O(\mathbf{E}T(\text{Pull}))$ for general graphs or to refute it. Mossel and Roch [17] found slow convergence of the iterated prisoner's dilemma (IPD) problem on caterpillar trees. Intuitively push voting is aggressive, whereas pull voting is altruistic, and thus similar to cooperation in the IPD. Motivated by this, we found simple counterexamples, namely, the star graph S_n and the double star S_n^*.

THEOREM 3. *Let T be the time to consensus of the asynchronous discordant voting process.*

For the star graph S_n, and starting from any initial coloring with an equal number of red and blue vertices, $\mathbf{E}T(Push) = \Theta(n^2 \log n)$ and $\mathbf{E}T(Pull) = O(n^2)$.

For the double star S_n^* with the initial coloring of Figure 1, $\mathbf{E}T(Push) = \Omega(2^{n/5})$ and $\mathbf{E}T(Pull) = O(n^4)$.

At this point little remains of the possibility of a meta-theorem except a vague hope that at least one of the push and pull protocols always has polynomial time to consensus. However, this is disproved by the example of the *barbell graph*, which consists of two cliques of size $n/2$ joined by a single edge.

THEOREM 4. *Let T be the time to consensus of the asynchronous discordant voting process. For the barbell graph, there exist initial colorings with an equal number of red and blue vertices, such that $\mathbf{E}T(Push) = \Omega(2^{n/10})$ and $\mathbf{E}T(Pull) = \Omega(2^{n/2})$.*

We use the term *ordinary* to refer to the standard asynchronous voting model in which the protocol makes no distinction between discordant and nondiscordant neighbors.
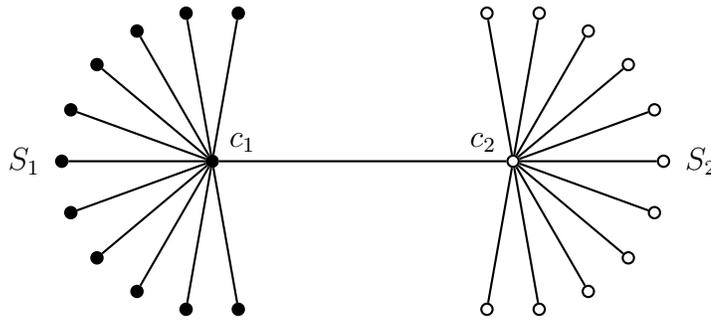
FIG. 1. *Double star S^* with half of the vertices colored red and half colored blue.*

TABLE 1
Worst case expected time to consensus for discordant and ordinary asynchronous voting proto-cols on connected n-vertex graphs.

	Discordant voting			Ordinary voting		
	Push	Pull	Obliv.	Push	Pull	Obliv.
Complete graph K_n	$\Theta(n\log n)$	$\Theta(2^n)$		$O(n^2)$	$O(n^2)$	$O(n^4)$
Cycle C_n	$\Theta(n^2)$	$\Theta(n^2)$		$O(n^2)$	$O(n^2)$	$O(n^3)$
Star graph S_n	$\Theta(n^2\log n)$	$O(n^2)$	$n^2/4$	$O(n^2)$	$O(n^2)$	$O(n^3)$
Double star S_n^*	$\Omega(2^{n/5})$	$O(n^4)$		$O(n^3)$	$O(n^4)$	$O(n^3)$
Barbell graph	$\Omega(2^{n/10})$	$\Omega(2^{n/2})$		$O(n^4)$	$O(n^4)$	$O(n^4)$

A summary of these results is given in the table below. The moral of the story is that there is always some graph and starting configuration for which the chosen push or pull process behaves badly.

The column for ordinary asynchronous pull voting in Table 1 follows from (1). The column for ordinary asynchronous pull voting from $\mathbf{E}T = O(n^2m)$ (see the paragraph following Remark 1). To complete the column for ordinary asynchronous push voting, we used a result of [7]. For any graph $G = (V(G), E(G))$,

$$(2) \qquad \mathbf{E}T(\text{Push}) = O(1/\Psi(G)),$$

where

$$\Psi(G) = \frac{2C(G)}{nd_{\max}} \min_{S\subset V(G)} \frac{1}{\min\{J(S), J(S^c)\}} \sum_{(v,w)\in E(S:S^c)} \frac{1}{d(v)d(w)}.$$

The expression is evaluated over sets $S \neq \emptyset, V(G)$, and $d_{\max}$ is maximum degree, $C(G) = (\sum_{v\in V} 1/d(v))^{-1}$, $E(S : S^c)$ are the edges between S and S^c, and $J(S) = \sum_{v\in S} d(v)^{-1}$. The parameter Ψ does not seem related to the classical graph parameters, but can be directly evaluated for the graphs we consider. For regular graphs,

$$\Psi = \frac{2}{n^2}\Phi,$$

in which case $\mathbf{E}T = O(n^2/\Phi)$, which agrees with the asynchronous pull model in (1).

Asynchronous discordant voting model. We next give a formal definition of the discordant voting process. Given a graph $G = (V, E)$ with $n = |V|$, each vertex $v \in V$ is labeled with an *opinion* $X(v) \in \{0, 1\}$. We call X a *configuration* of opinions.

We can think of the opinions as having colors, e.g., red (0) and blue (1). An edge $e = uv \in E$ is *discordant* if $X(u) \neq X(v)$. Let $K(X)$ denote the set of discordant edges at time t. A vertex v is discordant if it is incident with any discordant edge, and $D(X)$ will denote the set of discordant vertices in X. We consider three random update rules for opinions X_t at time t.

Push: Choose $v_t \in D(X_t)$, uniformly at random, and a discordant neighbor u_t of v_t uniformly at random. Let $X_{t+1}(u_t) \leftarrow X_t(v_t)$, and $X_{t+1}(w) \leftarrow X_t(w)$ otherwise.

Pull: Choose $v_t \in D(X_t)$, uniformly at random, and a discordant neighbor u_t of v_t uniformly at random. Let $X_{t+1}(v_t) \leftarrow X_t(u_t)$, and $X_{t+1}(w) \leftarrow X_t(w)$ otherwise.

Oblivious: Choose $\{u_t, v_t\} \in K(X_t)$ uniformly at random. With probability $1/2$, $X_{t+1}(v_t) \leftarrow X_t(u_t)$ with probability $1/2$, $X_{t+1}(u_t) \leftarrow X_t(v_t)$, and $X_{t+1}(w) \leftarrow X_t(w)$ otherwise.

These three processes are Markov chains on the configurations in G, in which the opinion of exactly one vertex is changed at each step. Assuming G is connected, there are two absorbing states, when $X(v) = 0$ for all $v \in V$ or $X(v) = 1$ for all $v \in V$, where no discordant vertices exist. When the process reaches either of these states, we say that is has converged. Let T be the step at which convergence occurs. Our object of study is $\mathbf{E}T$.

Structure of the paper. A major obstacle in the analysis of discordant voting is that the effect of recoloring a vertex is not always monotone. For each of the graphs studied, the way to bound $\mathbf{E}T$ differs. The proof of the pull voting result for the cycle C_n in particular is somewhat delicate and requires an analysis of the optimum of a linear program based on a potential function.

The general proof methodology is to map the process to a biased random walk on the line $0, \ldots, n$. In section 2 we prove results for a birth-and-death chain which we call the push chain. This chain can be coupled with many aspects of the discordant voting process. We then prove Theorems 1, 2, 3, and 4 in that order.

2. Birth-and-death chains. A Markov chain $(X_t)_{t \geq 0}$ is said to be a birth-and-death chain on state space $S = \{0, \ldots, N\}$ if given $X_t = i$ and then the possible values of X_{t+1} are $i + 1, i$, or $i - 1$ with probability p_i, r_i, and q_i, respectively. Note that $q_0 = p_N = 0$. In this section we assume that $r_i = 0$, $p_0 = 1$, $q_N = 1$, $p_i > 0$ for $i \in \{0, \ldots, N - 1\}$ and $q_i > 0$ for $i \in \{1, \ldots, N\}$. We next summarize some results on birth-and-death chains (see Peres, Levin, and Wilmer [15, Chapter 2, section 5]).

Denote by $\mathbf{E}_i Y$ the expected value of random variable Y when the chain starts in i (i.e., $X_0 = i$). Define the random hitting time of state i as $T_i = \min\{t > 0 : X_t = i\}$. Thus $\mathbf{E}_j T_i$ is the expected hitting time of state i starting from state j or first return time if $j = i$.

A probability distribution π satisfies the detailed balance equations, if

$$(3) \qquad \pi(i)P(i,j) = \pi(j)P(j,i) \text{ for all } i, j \in S.$$

Birth-and-death chains with $p_i = P(i, i+1), q_i = P(i, i-1)$ can be shown to satisfy the detailed balance equations. It follows from this (see, e.g., [15]) that

$$(4) \qquad \mathbf{E}_{i-1} T_i = \frac{1}{q_i \pi(i)} \sum_{k=0}^{i-1} \pi(k).$$

For birth-and-death chains, this equation can be written explicitly in terms of p_i, q_i (see [15]) as $\mathbf{E}_0 T_1 = 1/p_0 = 1$ and in general for $i \in \{1, \ldots, N\}$,

$$(5) \qquad \mathbf{E}_{i-1} T_i = \sum_{k=0}^{i-1} \frac{1}{p_k} \frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}}.$$

In writing this expression we follow the convention that if $k = i-1$, then $\frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}} = 1$ so that the last term is $1/p_{i-1}$. Note also that the final index k on p_k is $k = N - 1$, i.e., we never divide by $p_N = 0$.

For any $M \leq N$, we have that $\mathbf{E}_0 T_M = \sum_{i=1}^{M} \mathbf{E}_{i-1} T_i$. For example, $\mathbf{E}_0 T_1 = \frac{1}{p_0} = 1$ and $\mathbf{E}_0 T_2 = 1 + \frac{1}{p_1} + \frac{q_1}{p_0 p_1}$ etc. Thus, for $M \geq 1$

$$(6) \qquad \mathbf{E}_0 T_M = \sum_{i=1}^{M} \mathbf{E}_{i-1} T_i = \sum_{i=1}^{M} \sum_{k=0}^{i-1} \frac{1}{p_k} \prod_{j=k+1}^{i-1} \frac{q_j}{p_j}.$$

We define two birth-and-death chains which feature in our analysis. The chains have states $\{0, 1, \ldots, i, \ldots, N\}$, where $N = n/2$ (assume $n \geq 2$ even). We refer to these chains as the push chain and pull chain, respectively.

Push chain. Let Z_t be the state occupied by the push chain at step $t \geq 0$. Let $\delta \in \{-1, 0, +1\}$ be fixed. When applying results for the push chain in our proofs, we will state the value of δ we use. The transition probability $p_i = P(i, i+1)$ from $Z_t = i$ is given by

$$(7) \qquad p_i = \begin{cases} 1, & \text{if } i = 0, \\ 1/2 + i/n + \delta/n, & \text{if } i \in \{1, \ldots, n/2 - 1\}, \\ 0, & \text{if } i = n/2. \end{cases}$$

Pull chain. Let $\overline{Z}_t$ be the state occupied by the pull chain at step $t \geq 0$. Given that $\overline{Z}_t = i$, the transition probability $\overline{p}_i = \overline{P}(i, i+1)$ is given by

$$(8) \qquad \overline{p}_i = \begin{cases} 1, & \text{if } i = 0, \\ 1/2 - i/n - \delta/n, & \text{if } i \in \{1, \ldots, n/2 - 1\}, \\ 0, & \text{if } i = n/2. \end{cases}$$

For $1 \leq i \leq N - 1$ the pull chain is the push chain with the probabilities reversed, i.e., $\overline{p}_i = 1 - p_i$.

Push chain: Bounds on hitting time.

Push chain: Upper bound on hitting time.

LEMMA 5. *For any $M \leq N$, let $\mathbf{E}_0 T_M$ be the expected hitting time of M in the push chain Z_t starting from state 0. Then*

$$\mathbf{E}_0 T_M \leq 2N \log M + O(1).$$

Proof. Using (6) and recalling the notational convention given below (5) we can change the order of summation to give

$$(9) \qquad E_0 T_M = \sum_{k=0}^{M-1} \sum_{i=k+1}^{M} \frac{1}{p_k} \frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}} = \frac{1}{p_{M-1}} + \sum_{k=0}^{M-2} \sum_{i=k+1}^{M-1} \frac{1}{p_k} \frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}}.$$

Using (7), we see for $1 \leq k \leq N-2$ that $q_k/p_k \geq q_{k+1}/p_{k+1}$, $q_1/p_1 \leq 1$, and for $2 \leq k \leq N-1$ that $q_k/p_k < 1$. As $p_0 = 1$, we upper bound $E_0 T_M$ by

$$(10) \qquad E_0 T_M \leq M + \frac{1}{p_{M-1}} + \sum_{k=1}^{M-2} \frac{1}{p_k} \sum_{i=k+1}^{M-1} \left(\frac{q_{k+1}}{p_{k+1}}\right)^{i-k-1}$$

and

$$(11) \qquad \sum_{k=1}^{M-2} \frac{1}{p_k} \sum_{\ell=0}^{\infty} \left(\frac{q_{k+1}}{p_{k+1}}\right)^{\ell} = \sum_{k=1}^{M-2} \frac{1}{p_k} \frac{1}{1 - \frac{q_{k+1}}{p_{k+1}}} = \sum_{k=1}^{M-2} \frac{p_{k+1}}{p_k} \frac{1}{p_{k+1} - q_{k+1}}.$$

As $q_k = 1 - p_k$, $p_k - q_k = 2p_k - 1 > 0$ for all $k \in \{2, \ldots, N-1\}$, then $\frac{1}{p_k - q_k} = \frac{N}{k+\delta}$. For all $k \in \{1, \ldots, N-2\}$ we have $\frac{p_{k+1}}{p_k} \leq 2$. Using (10) with the upper bounds given in (11), we obtain the required conclusion. $\square$

Push chain: Lower bound on hitting time.

LEMMA 6. *Let $\delta = 0$ in (7). Let $\mathbf{E}_0 T_M$ be the expected hitting time of M in the push chain Z_t starting from state 0. There exists a constant C such that, for any $\sqrt{N} \leq M = o(N^{3/4})$,*

$$\mathbf{E}_0 T_M \geq C(N \log M/\sqrt{N} + \sqrt{N}).$$

Proof. For $0 < x < 1$,

$$\frac{1-x}{1+x} = \exp\left\{-2\left(x + \frac{x^3}{3} + \cdots + \frac{x^{2\ell+1}}{2\ell+1} + \cdots\right)\right\}.$$

Thus with $N = n/2$

$$(12) \qquad \prod_{j=k+1}^{i-1} \frac{q_j}{p_j} = \prod_{j=k+1}^{i-1} \frac{1 - j/N}{1 + j/N}$$

$$= \exp\left\{-2\left(\sum_{j=k+1}^{i-1} \frac{j}{N} + \sum \frac{(j/N)^3}{3} + \cdots + \sum \frac{(j/N)^{2\ell+1}}{2\ell+1} + \cdots\right)\right\}$$

$$(13) \qquad = \exp\{-2\Psi\},$$

say. If $f(s)$ is nonnegative and monotone increasing, then $\sum_{s=k+1}^{i-1} f(s) \leq \int_k^i f(s)\, ds$. Thus, the sum of terms in $(j/N)^3$ and above in Ψ can be bounded above by

$$\sum_{\ell \geq 1} \sum_{j=k+1}^{i-1} \frac{(j/N)^{2\ell+1}}{2\ell+1} \leq \sum_{\ell \geq 1} \frac{1}{(2\ell+1)N^{2\ell+1}} \int_k^i x^{2\ell+1} dx$$

$$\leq \sum_{\ell \geq 1} \frac{1}{(2\ell+1)N^{2\ell+1}} \cdot \frac{i^{2\ell+2}}{2\ell+2}$$

$$= O\left(\frac{i^4}{N^3}\right) \sum \frac{1}{(2\ell+1)(2\ell+2)} = O\left(\frac{i^4}{N^3}\right).$$

Thus, using our assumption that $M = o(N^{3/4})$,

$$\Psi = \frac{i(i-1)}{2N} - \frac{k(k+1)}{2N} + O\left(\frac{i^4}{N^3}\right) = \frac{i^2}{2N} - \frac{k^2}{2N} - \frac{i+k}{2N} - o(1).$$

Replacing Ψ in (13) with the upper bound given above gives a lower bound on the term (13) in (6). Thus

$$(14) \qquad \mathbf{E}_0 T_M \geq (1 - o(1)) \sum_{i=0}^{M} \sum_{k=0}^{i-1} \frac{1}{p_k} \exp\left(-\frac{i^2}{N}\right) \exp\left(\frac{k^2}{N}\right).$$

For $i \leq M$ the last term on the right-hand side of (14) is bounded below by a positive constant. Let

$$(15) \qquad \sigma(i) = \sum_{k=0}^{i-1} \exp\left(\frac{k^2}{N}\right).$$

Let $\beta = (1/2)\log 2 \approx 0.34$. We claim that if $i \geq \sqrt{N}$, then

$$(16) \qquad \sigma(i) \geq \frac{\beta N}{2i} e^{i^2/N}.$$

Let $a = \beta N/i$ and then for $i \geq \sqrt{N}$, $i - a > 0$. For $k \geq i - a$

$$\frac{k^2}{N} \geq \frac{i^2}{N} - \frac{2ia}{N} + \frac{a^2}{N} = \frac{i^2}{N} - \frac{2i}{N}\beta\frac{N}{i} + \beta\frac{N}{i^2} \geq \frac{i^2}{N} - 2\beta.$$

If $k \geq i - a$, then $\exp\{k^2/N\} \geq \frac{1}{2}\exp\{i^2/N\}$. As there are at least a such values of k, it follows that $\sigma(i) \geq \beta N/2i\, e^{i^2/N}$.

Let $\sqrt{N} \leq i \leq M = o(N^{3/4})$. Replace (15) in (14) with (16). Noting that $p_0 = 1$ and for $1 \leq k \leq M$, $p_k \sim 1/2$, we can assume $(1 - o(1))/p_k \geq 1/2$ to give

$$\mathbf{E}_0 T_M \geq \sum_{i<\sqrt{N}} \frac{e^{-1}}{2} + \sum_{i=\sqrt{N}}^{M} \frac{\beta N}{2i} \geq \frac{\sqrt{N}}{6} + \frac{\beta N}{3}\log\frac{M}{\sqrt{N}}. \qquad \square$$

3. Voting on the complete graph K_n. For the complete graph K_n, the probability that the number of blue vertices B increases at a given step is $B(t)/n$, whereas in the pull process it is $R(t)/n = 1 - B(t)/n$. The chain defined by $Y_t = \max\{R(t), B(t)\} - n/2$ is a birth-and-death chain. We study the time that it takes Y_t to reach $N = n/2$ starting from 0.

Theorem 1: Push process. For the push model, the process Y_t is identical to the push chain Z_t with transitions given by (7) with $\delta = 0$. This was analyzed in section 2.

Theorem 1: Pull process. For the pull model, the process Y_t is identical to the pull chain $\overline{Z}_t$ with transitions given by (8) with $\delta = 0$

For the pull model, the process Y_t is identical to the pull chain $\overline{Z}_t$ with transitions given by (8). To begin with, observe that $w_k = \binom{n}{N+k}, k = 0, 1, \ldots, N$ satisfies the detailed balance equation (3). Hence we have $\pi(k) = w_k/W$, where $W = w_0 + w_1 + \cdots + w_N$.

It follows from (4) that

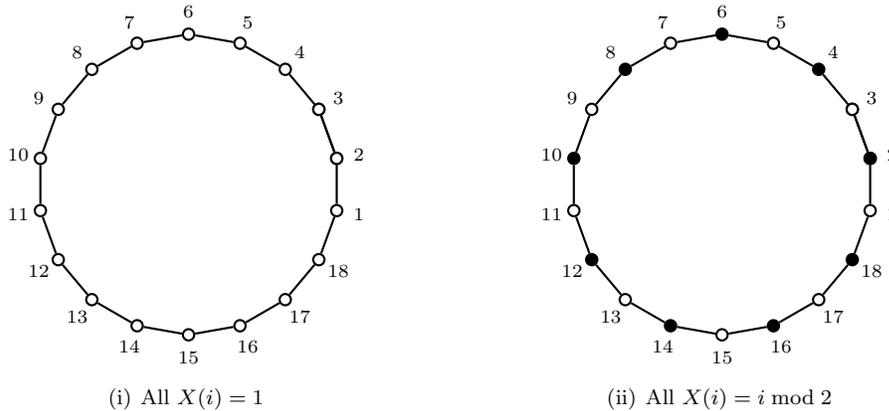$$\mathbf{E}_{i-1} T_i = \frac{2n}{n+2i} \cdot \frac{1}{\binom{n}{N+i}} \cdot \sum_{k=0}^{i-1} \binom{n}{N+k}.$$

(i) All $X(i) = 1$ (ii) All $X(i) = i \bmod 2$

FIG. 2. *Cycle with $n = 18$.*

Putting $i = N$ we have

$$(17) \qquad \mathbf{E}_{N-1}T_N = \sum_{k=0}^{N-1} \binom{n}{N+k} = \frac{1}{2}\left(2^n - 2 + \binom{n}{N}\right) = \Omega(2^n).$$

On the other hand, an upper bound

$$\sum_{i=1}^{N} \mathbf{E}_{i-1}T_i \le 2 \cdot 2^n \cdot \sum_{i=1}^{N} \frac{1}{\binom{n}{N+i}} = O(2^n)$$

follows from a result of Sury [22] that

$$\sum_{i=1}^{N} \frac{1}{\binom{n}{N+i}} = \frac{n+1}{2^n} \sum_{i=0}^{n} \frac{2^i}{i+1} = O(1).$$

4. Voting on the cycle. An n-cycle G with $V = [n]$ has $E = \{(i, i+1) : i \in [n]\}$, where we identify vertex $n + i$ with vertex i. See Figure 2.

Let $X = X(t)$ denote the (configuration of opinions) of the voting process at time t. Let $K(X)$ denote the set of discordant edges of X and let $k(X) = |K|$. Let $D(X)$ denote the set of discordant vertices in X. We note that, during voting, the number of discordant edges can never increase from its current value.

We say $i + 1, i + 2, \dots, j$ is a *run* of length $(j - i)$ $(1 \le j - i < n)$ if $X(i) \ne X(i+1) = X(i+2) = \cdots = X(j) \ne X(j+1)$. A *singleton* is a run of length 1, a single vertex. These vertices require special treatment, since they lie in two discordant edges. If a singleton is recolored, the number of discordant edges decreases by two, and indeed recoloring a singleton is the only way to decrease the number of discordant edges.

Note that the number of runs, $k(X)$, in X is equal to the number of discordant edges. Also k is even, since red and blue runs must alternate, so we will write $r(X) = \frac{1}{2}k(X)$, and $k_0 = 2r_0 = k(X_0)$. Thus $r(X)$ is the number of paths of a given color. Then T is the first t for which $k(X_t) = r(X_t) = 0$ (a cycle is not a path).

Let the k runs in X have lengths $\ell_1, \ell_2, \dots, \ell_k$, respectively, and let $s(X)$ denote the number of singletons. Clearly $\sum_{i=1}^{k} \ell_i = n$, and there are $\kappa = 2k - s$ discordant vertices, so $k \le \kappa \le 2k$.

We wish to determine the convergence time T for an arbitrary configuration X_0 of the push or pull process to reach an absorbing state X_T with $X_T(i) = X_T(1)$ ($i \in [n]$). In these processes, the run lengths behave rather like symmetric random walks on the line. However, an analysis using classical random walk techniques [11] seems problematic. There are two main difficulties. First, the k "walks" (run lengths) are correlated. If a run is long, the adjacent runs are likely to be shorter, and vice versa. Second, when the change vertex is a singleton, the lengths of three adjacent runs are combined, so three walks suddenly merge into one. One of the three runs is a singleton, but the other two may have arbitrary lengths.

Therefore, we will use the random walk view only to give a lower bound on the convergence time. For the upper bound, we use a different approach. We will define a *potential function*

$$\psi(X) = \sum_{i=1}^{k} \sqrt{\ell_i}\,,$$

where $\psi(X) = 0$ if and only if $k(X) = 0$. The important feature of ψ is that it is a separable and strictly concave function of the ℓ_i ($i \in [k]$). Almost any other function with these properties would give similar results.

LEMMA 7. *For any configuration X on the n-cycle with k runs, $\psi(X) \leq \sqrt{kn}$.*

Proof. If $k = 0$, this is clearly true. Otherwise, if $k \geq 2$, by concavity we have $\psi(X)/k = \frac{1}{k} \sum_{i=1}^{k} \sqrt{\ell_i} \leq \sqrt{\frac{1}{k} \sum_{i=1}^{k} \ell_i} = \sqrt{n/k}$, so $\psi(X) \leq \sqrt{kn}$. $\qquad\square$

Observe that $k(X_{t+1}) = k(X_t)$ at step t of either the push or pull process, unless the change vertex is a singleton, in which case we may have $k(X_{t+1}) = k(X_t) - 2$. Thus $\{t : k(X_t) = 2r\}$ is an interval $[t_r, t_{r-1})$, which we will call *phase r* of the process.

Let $v_t = v \in D(X_t)$ be the active vertex, i.e., the vertex selected to push in the push rule, or pull in the pull rule. Let δ_v be the expected change in ψ, i.e.,

$$\delta_v = \mathbf{E}[\,\psi(X_{t+1}) - \psi(X_t) \mid v_t = v\,].$$

If there are $\kappa = 2k - s$ discordant vertices, the total expected change δ in ψ is

$$\Delta = \mathbf{E}[\,\psi(X_{t+1}) - \psi(X_t)] = \frac{1}{\kappa} \sum_{v \in D} \delta_v.$$

We will show that Δ is negative, so $\psi(X_t)$ is monotonically decreasing with t, in expectation. Unfortunately we cannot simply bound δ_v for each $v \in D$, since it is possible to have $\delta_v > 0$. Thus we will consider discordant *edges*. We partition the set K of discordant edges uv into three subsets:
(A) $A = \{uv : u \text{ and } v \text{ not singleton}\}$;
(B) $B = \{uv : u \text{ not singleton}, v \text{ singleton}\}$;
(C) $C = \{uv : u \text{ and } v \text{ both singleton}\}$.
See Figure 3, where ℓ_z is the length of the run containing discordant vertex z for $z \in \{u, v, w, q\}$.

Note that k can change only if $uv \in B \cup C$. Now let

$$\lambda_{uv} = \begin{cases} \sqrt{\ell_u} + \sqrt{\ell_v}, & uv \in A\,; \\ \sqrt{\ell_u} + \frac{1}{2}\sqrt{\ell_v}, & uv \in B\,; \\ \frac{1}{2}\sqrt{\ell_u} + \frac{1}{2}\sqrt{\ell_v}, & uv \in C\,; \end{cases} \qquad \delta_{uv} = \begin{cases} \delta_u + \delta_v, & uv \in A\,; \\ \delta_u + \frac{1}{2}\delta_v, & uv \in B\,; \\ \frac{1}{2}\delta_u + \frac{1}{2}\delta_v, & uv \in C\,. \end{cases}$$

(A) u and v not singleton

(B) u not singleton, v singleton
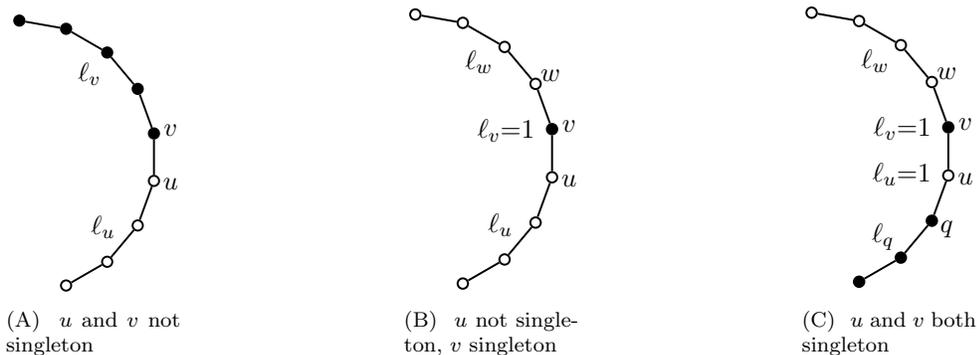
(C) u and v both singleton

FIG. 3. *Cases for discordant edge uv.*

Each singleton is in two discordant edges, all other discordant vertices in one, and each run is bounded by two discordant vertices. Therefore

$$\psi = \frac{1}{2} \sum_{v \in D} \sqrt{\ell_v} = \sum_{uv \in K} \lambda_{uv}, \qquad \delta = \frac{1}{\kappa} \sum_{v \in D} \delta_v = \frac{1}{\kappa} \sum_{uv \in K} \delta_{uv}.$$

We will show that $\delta_{uv} < 0$ for all $uv \in K$. We consider cases (A), (B), and (C) separately. So far, the analysis is identical for pull and push voting. Now we must distinguish them. First we consider the push process.

Push voting.

(A)

$$\delta_v = \sqrt{\ell_v + 1} - \sqrt{\ell_v} + \sqrt{\ell_u - 1} - \sqrt{\ell_u},$$
$$\delta_u = \sqrt{\ell_v - 1} - \sqrt{\ell_v} + \sqrt{\ell_u + 1} - \sqrt{\ell_u}.$$

Hence $\delta_{uv} = (\sqrt{\ell_v + 1} + \sqrt{\ell_v - 1} - 2\sqrt{\ell_v}) + (\sqrt{\ell_u + 1} + \sqrt{\ell_u - 1} - 2\sqrt{\ell_u}) \leq -\frac{1}{4}(\ell_v^{-3/2} + \ell_u^{-3/2})$, using Lemma 8.

LEMMA 8. *For all $\ell \geq 1$, $\sqrt{\ell + 1} + \sqrt{\ell - 1} \leq 2\sqrt{\ell} - \frac{1}{4}\ell^{-3/2}$.*

Proof. First, we prove the inequality $\sqrt{1 + x} + \sqrt{1 - x} \leq 2 - \frac{1}{4}x^2$ for all $x \leq 1$. By squaring both sides, the inequality is true if $2 + 2\sqrt{1 - x^2} \leq 4 - x^2 + \frac{1}{16}x^4$. This is true if $\sqrt{1 - y} \leq 1 - \frac{1}{2}y$ with $y = x^2$. Squaring both sides, this is $1 - y^2 \leq 1 - y^2 + \frac{1}{4}y^4$, which is clearly true. Now, letting $x = 1/\ell$, $\sqrt{\ell + 1} + \sqrt{\ell - 1} \leq 2\sqrt{\ell} - \frac{1}{4}\ell^{-3/2}$ is equivalent to $\sqrt{1 + x} + \sqrt{1 - x} \leq 2 - \frac{1}{4}x^2$ with $x \leq 1$. ∎

(B) Let u, w be the discordant neighbors of v. Then

$$\delta_v = \frac{1}{2}(\sqrt{\ell_u - 1} - \sqrt{\ell_u} + \sqrt{2} - 1 + \sqrt{\ell_w - 1} - \sqrt{\ell_w} + \sqrt{2} - 1).$$

Since $\sqrt{\ell - 1} \leq \sqrt{\ell}$, $\delta_v \leq \sqrt{2} - 1$. Also

$$\delta_u = \sqrt{\ell_w + \ell_u + 1} - \sqrt{\ell_w} - \sqrt{\ell_u} - 1 \leq \sqrt{3} - 3,$$

using Lemma 9. Thus

$$\delta_{uv} \leq \frac{1}{2}(\sqrt{2} - 1) + \sqrt{3} - 3 < -1 \leq -\frac{1}{2}(\ell_v^{-3/2} + \ell_u^{-3/2}).$$

LEMMA 9. *For all $\ell_1, \ell_2 \geq 1$, $\sqrt{\ell_1} + \sqrt{\ell_2} + 1 \geq \sqrt{\ell_1 + \ell_2 + 1} + (3 - \sqrt{3})$.*

Proof. Consider $f(\ell_1, \ell_2) = \sqrt{\ell_1} + \sqrt{\ell_2} + 1 - \sqrt{\ell_1 + \ell_2 + 1} + (\sqrt{3} - 3)$. Then, for all $\ell_1, \ell_2 > 0$,

$$\frac{\partial f}{\partial \ell_i} = \frac{1}{2\sqrt{\ell_i}} - \frac{1}{2\sqrt{\ell_1 + \ell_2 + 1}} > 0 \qquad (i = 1, 2).$$

Hence $f(\ell_1, \ell_2) \geq f(1, 1) = 0$ for all $\ell_1, \ell_2 \geq 1$. $\qquad \qquad \square$

(C) Let u, w be the discordant neighbors of v, and v, q the discordant neighbors of u. Then

$$\delta_v = \frac{1}{2}(\sqrt{\ell_w - 1} - \sqrt{\ell_w} + \sqrt{2} - 1 + \sqrt{\ell_q + 2} - \sqrt{\ell_q} - 2).$$

Now $\sqrt{\ell - 1} \leq \sqrt{\ell}$ and $\sqrt{\ell + 2} - \sqrt{\ell} - 2 \leq \sqrt{3} - 3$, using Lemma 9 with $\ell_1 = 1$. Thus $\delta_v \leq \frac{1}{2}(\sqrt{2} - 1 + \sqrt{3} - 3) < -0.425$. Similarly $\delta_u < -0.425$, so $\delta_{uv} < -0.425 \leq -\frac{1}{5}(\ell_v^{-3/2} + \ell_u^{-3/2})$.

Hence we have $\delta_{uv} < -\frac{1}{5}(\ell_v^{-3/2} + \ell_u^{-3/2})$ for all $uv \in K$, so

$$\delta = \frac{1}{\kappa} \sum_{v \in D} \delta_v = \frac{1}{\kappa} \sum_{uv \in K} \delta_{uv} \leq -\frac{1}{5\kappa} \sum_{uv \in K} (\ell_v^{-3/2} + \ell_u^{-3/2}) < -\frac{1}{5\kappa} \sum_{v \in D} \ell_v^{-3/2}.$$

Thus

$$\mathbf{E}[\psi(X_{t+1})] < \psi(X_t) - \frac{1}{5\kappa} \sum_{v \in D} \ell_v^{-3/2}.$$

Since $f(x) = x^{-3}$ is a convex function, $\mathbf{E}[f(X)] \geq f(\mathbf{E}[X])$ by Jensen's inequality [23, 6.6], so

$$\frac{1}{\kappa} \sum_{v \in D} \ell_v^{-3/2} \geq \left(\frac{1}{\kappa} \sum_{v \in D} \sqrt{\ell_v}\right)^{-3} = \left(\frac{\kappa}{2\psi(X_t)}\right)^3 \geq \left(\frac{k}{2\psi(X_t)}\right)^3,$$

as $\kappa = 2k - s$, and we assume t is in the interval $[t_r, t_{r-1})$ in which there are $k(X_t) = 2r$ runs. Therefore,

$$(18) \qquad \mathbf{E}[\psi(X_{t+1})] < \psi(X_t) - \frac{1}{5}\left(\frac{k}{2\psi(X_t)}\right)^3 = \psi(X_t) - \frac{k^3}{40\psi(X_t)^3}.$$

Hence, using Lemma 7 we have that $\psi(X_t) \leq \sqrt{kn}$ and so

$$(19) \qquad \mathbf{E}[\psi(X_{t+1})] - \mathbf{E}[\psi(X_t)] \leq -\tfrac{1}{40}k^3/(kn)^{3/2} = -\tfrac{1}{40}(k/n)^{3/2}.$$

During phase r of the process (the interval $[t_r, t_{r-1})$) the number of runs is $k = 2r$ for $r \in [r_0]$, where $r_0 = 2k(X_0)$ is the initial number of runs. Let $\varphi_r = \mathbf{E}[\psi(X_{t_r})]$. Since $r_0 = \frac{1}{2}k(X_0)$, $t_{r_0} = 0$, and since $r(X_T) = k(X_T) = 0$, $t_0 = T$ and $\varphi_0 = 0$. Let $m_r = \mathbf{E}[t_{r-1} - t_r]$ for $r \in [r_0]$ and $\gamma_r = \frac{1}{40}(2r/n)^{3/2}$. Then (7) implies that $\psi(X_t) + (t - t_{r-1})\gamma_r$ is a supermartingale [23, Chapter 10, section 3] during phase r, and t_{r-1} is a stopping time. Then the optional stopping theorem [23, Chapter 10, section 10] implies that

$$\varphi_{r-1} + \gamma_r m_r = \mathbf{E}[\psi(X_{t_{r-1}}) + \gamma_r(t_r - t_{r-1})] \leq \mathbf{E}[\psi(X_{t_r})] = \varphi_r,$$

which implies

$$(20) \qquad \varphi_r - \varphi_{r-1} \geq \gamma_r m_r = \tfrac{1}{40} m_r (2r/n)^{3/2} \qquad (r \in [r_0]).$$

Note, in particular, that $\varphi_r \geq \varphi_{r-1}$ for all $r \in [r_0]$.

From Lemma 7, $\varphi_r \leq \sqrt{2rn}$. Then, from (20), we have $m_r \leq 40\sqrt{2rn}(2r/n)^{-3/2} = 20n^2/r$. Thus

$$\mathbf{E}[T] = \sum_{j=1}^{r_0} m_j \leq 20n^2 \sum_{j=1}^{r_0} 1/j < 20n^2 (\log r_0 + 1).$$

Since $r_0 \leq n/2$, this gives an absolute bound of $20n^2 \log(en/2) = O(n^2 \log n)$. However, we can improve this with a more careful analysis.

Let $x_r = \varphi_r - \varphi_{r-1} \geq 0$ for $r \in [r_0]$. By Lemma 7, $\sum_{i=1}^{r} x_j = \varphi_r \leq \sqrt{2rn}$. Also, from (20), we have $m_r \leq 40x_r(n/2r)^{3/2} = 10\sqrt{2}\,n^{3/2}x_r/r^{3/2}$, so $\mathbf{E}[T] = \sum_{j=1}^{r_0} m_j < 10\sqrt{2}\,n^{3/2} \sum_{j=1}^{r_0} x_r/r^{3/2}$.

Thus $E[T]$ is bounded above by $T^\star$, the optimal value of the following linear program.

$$T^\star = \max\ 10\sqrt{2}\,n^{3/2} \sum_{r=1}^{r_0} x_r/r^{3/2}$$

$$(21) \qquad \text{such that } \sum_{j=1}^{r} x_j \leq \sqrt{2rn} \qquad\qquad (r \in [r_0]),$$

$$x_j \geq 0 \qquad\qquad (j \in [r_0]).$$

This linear program can be solved easily by a greedy procedure. In fact, it is a polymatroidal linear program [9], but we will give a self-contained proof for this simple case, using linear programming duality.

LEMMA 10. *Let $0 < b_1 < b_2 < \cdots < b_\nu$ and $c_1 > c_2 > \cdots > c_\nu > 0$. Then the linear program $\max \sum_{j=1}^{\nu} c_j x_j$ subject to $\sum_{j=1}^{r} x_j \leq b_r$, $x_r \geq 0$ $(r \in [\nu])$ has optimal solution $x_1 = b_1$, $x_j = b_j - b_{j-1}$ $(j = 2, 3, \ldots, \nu)$.*

Proof. This solution has objective function value $c_1 b_1 + c_2(b_2 - b_1) + \cdots + c_\nu(b_\nu - b_{\nu-1})$. The dual linear program is $\min \sum_{i=1}^{\nu} b_i y_i$ subject to $\sum_{i=j}^{\nu} y_i \geq c_j$, $y_j \geq 0$ $(j \in [\nu])$, and has feasible solution $y_\nu = c_\nu$, $y_j = c_j - c_{j+1}$ $(j \in [\nu - 1])$. Then the dual objective function has value $b_\nu c_\nu + b_{\nu-1}(c_{\nu-1} - c_\nu) + \cdots + b_1(c_1 - c_2)$. However,

$$c_1 b_1 + c_2(b_2 - b_1) + \cdots + c_\nu(b_\nu - b_{\nu-1}) = b_\nu c_\nu + b_{\nu-1}(c_{\nu-1} - c_\nu) + \cdots + b_1(c_1 - c_2).$$

Since the objective function values are equal, it follows that the two solutions are optimal in the primal and dual, respectively. $\qquad\square$

Thus, the optimal solution to (21) is $x_r = \sqrt{2nr} - \sqrt{2n(r-1)} = \sqrt{2nr}(1 - \sqrt{1-1/r}) \leq \sqrt{2n/r}$ for $r \in [r_0]$, since $1 - y \leq \sqrt{1-y}$ for $0 \leq y \leq 1$. Thus

$$T^\star \leq 10\sqrt{2}\,n^{3/2} \sum_{j=1}^{r_0} x_r/r^{3/2} \leq 10\sqrt{2}\,n^2 \sum_{j=1}^{r_0} \sqrt{2}/(\sqrt{r}\,r^{3/2})$$

$$= 20n^2 \sum_{r=1}^{r_0} 1/r^2 < (10\pi^2/3)n^2,$$

since $\sum_{r=1}^{\infty} 1/r^2 = \pi^2/6$. Thus we have an absolute bound of $\mathbf{E}[T] = O(n^2)$.
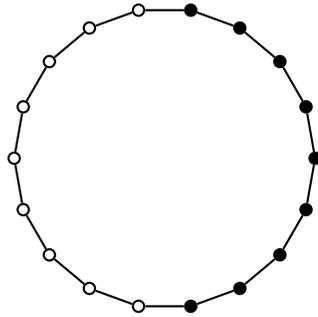
FIG. 4. *Lower bound configuration.*

Pull voting. The case of pull voting is similar, but the calculations for cases (A)–(C) are changed as follows.

(A′) The analysis for this case is identical to (A), except that δ_u and δ_v are interchanged. Hence $\delta_{uv} \leq -\frac{1}{4}(\ell_v^{-3/2} + \ell_u^{-3/2})$, as before.

(B′) $\delta_v = \sqrt{\ell_u + \ell_w + 1} - \sqrt{\ell_u} - \sqrt{\ell_w} - 1 \leq \sqrt{3} - 3$, using Lemma 9. Also $\delta_u = \sqrt{2} + \sqrt{\ell_u - 1} - \sqrt{\ell_u} - 1 \leq \sqrt{2} - 1$. Thus $\delta_{uv} \leq \sqrt{2} - 1 + \frac{1}{2}(\sqrt{3} - 3) < -0.22 \leq -\frac{1}{10}(\ell_v^{-3/2} + \ell_u^{-3/2})$.

(C′) $\delta_v = \sqrt{\ell_w + 2} - \sqrt{\ell_w} - 2 < \sqrt{3} - 3$, from Lemma 9 with $\ell = 1$. Similarly $\delta_u < \sqrt{3} - 3$, so $\delta_{uv} \leq \sqrt{3} - 3 < -1.25 < -\frac{1}{2}(\ell_v^{-3/2} + \ell_u^{-3/2})$.

Hence we have $\delta_{uv} < -\frac{1}{10}(\ell_v^{-3/2} + \ell_u^{-3/2})$ for all $uv \in K$, whereas we had $\delta_{uv} < -\frac{1}{5}(\ell_v^{-3/2} + \ell_u^{-3/2})$ for push voting. Thus the estimated rate of convergence is only half that for push voting. The rest of the analysis follows the same lines as before, except that the convergence time estimates are doubled. However, we may still conclude that $\mathbf{E}[T] = O(n^2)$.

Lower bound. Suppose G is an n-cycle, with $n = 2\nu$ even, and the push or pull process starts with $X_0(i) = 0$ $(i = 1, \ldots, \nu)$, $X_0(i) = 1$ $(i = \nu + 1, \ldots, n)$. Thus $k = 2$ and $\ell_1 = \ell_2 = \nu$. See Figure 4. At each step before convergence, there are two discordant edges and four discordant vertices and the push and pull processes proceed identically. Let L_t be the length of (say) the red run at step t, so $L_0 = \nu$, $L_T \in \{0, n\}$. At each step before convergence, we have $k(X_t) = 2$, $L_{t+1} \leftarrow L_t - 1$ with probability $1/2$ and $L_{t+1} \leftarrow L_t + 1$ with probability $1/2$. Thus L_t is a symmetric simple random walk. The number of runs $k(X_t)$ can only be reduced from two to zero if either $L_t = 1$ or $L_t = n - 1$, when one of the runs is a singleton. Thus $\mathbf{E}[T]$ is bounded below by the expected time for a symmetric simple random walk started at ν to reach either 1 or $(n - 1)$. This is well known [11, Chapter XIV, section 3] and is exactly $(\nu - 1)^2 = \Omega(n^2)$. Therefore the expected convergence time for either the push or pull process is $\Theta(n^2)$.

5. Voting on the star graph S_n. Let (r, b, X) denote the coloring of the star graph S_n on n vertices in which there are r red vertices $b = n - r$ blue vertices. The central vertex has color $X \in \{R, B\}$.

Push voting on the star.

THEOREM 11. *Starting from $|R| = n/2$, the push process on the star with n vertices has expected consensus time $\mathbf{E}T = \Theta(n^2 \log n)$.*
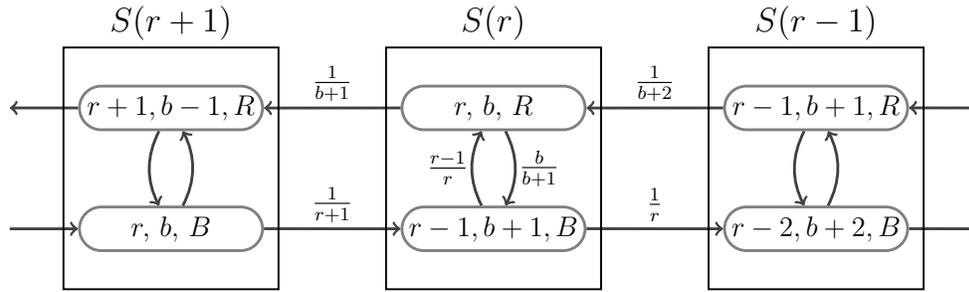
FIG. 5. *Pseudostates for the push process.*

In the case of the push process, the transitions from state (r, b, R) are to state $(r+1, b-1, R)$ with probability $1/(b+1)$ and to state $(r-1, b+1, B)$ with probability $b/(b+1)$. The transitions from state $(r-1, b+1, B)$ are to (r, b, R) with probability $(r-1)/r$ and to $(r-2, b+2, B)$ with probability $1/r$. For the purposes of discussion we group the states $(r, R) = (r, b, R)$ and $(r-1, B) = (r-1, b+1, B)$ into a single pseudostate $S(r)$. The transitions probabilities within or between $S(r+1)$ or $S(r-1)$ are shown in Figure 5, and are derived as follows:

Let $X, Y \in \{R, B\}$. For a particle occupying a state (of color) X in $S(r)$ let $P_X(Y, r)$ be the probability of exit from $S(r)$ via state Y. For example, $P_R(R, r)$ is the probability that a particle starting at (r, R) eventually exits from $S(r)$ via state (r, R) to state $(r+1, R)$ in $S(r+1)$. Thus

$$P_R(R, r) = \frac{1}{b+1} \left(1 + \frac{b}{b+1}\frac{r-1}{r} + \cdots + \left(\frac{b}{b+1}\frac{r-1}{r} \right)^k + \cdots \right),$$

so that

$$P_R(R, r) = \frac{1}{b+1}\frac{1}{1 - [b(r-1)/(b+1)r]} = \frac{r}{n}.$$

Similarly let $P_B(R, r)$ be the probability that a particle currently at $(r-1, B)$ in $S(r)$ moves from $S(r)$ to $(r+1, R)$ in $S(r+1)$. Then

$$P_B(R, r) = \frac{r-1}{r} P_R(R, r) = \frac{r-1}{n}.$$

In summary, starting from state $X \in \{R, B\}$ of $S(r)$, for $1 \le r \le n-1$ the transition probability $p_X(r)$ from $S(r)$ to $S(r+1)$ (resp., transition probability $p_X(b)$ from $S(r)$ to $S(r-1)$) is given by

(22) $$p_X(r) = \frac{r - 1_{(X=B)}}{n}, \qquad p_X(b) = \frac{b + 1_{(X=B)}}{n}.$$

States $(0, B)$ (i.e., $S(0)$) and (n, R) (i.e., $S(n)$) are absorbing.

Let $i = \max(r, b) - n/2$. To obtain lower and upper bounds on the number of transitions *between pseudostates $S(r)$ before absorption*, we can couple the process with a biased random walk on the line $L = \{0, 1, \ldots, n/2\}$ with a reflecting barrier at 0 and an absorbing barrier at $n/2$. We assume n is even here. For $0 < i < n/2$, let p_i be the probability of a transition from i to $i+1$ on L, and let $q_i = 1 - p_i$ be the probability of a transition from i to $i-1$. It follows from (22) that to obtain bounds

on the number of transitions between pseudostates $S(r)$ before absorption we can use a value of p_i given by

$$(23) \quad p_i = 1/2 + (i+1)/n \quad \text{lower bound}, \qquad p_i = 1/2 + (i-1)/n \quad \text{upper bound}.$$

We next consider the number of loops, for example, $(r, R) \to (r-1, B) \to (r, R)$, made within $S(r)$ before exit. For a particle starting from state X of $S(r)$ let $C_{XY} = C_{XY}(r)$ be the number of loops before exit at state Y. Let $\lambda = \frac{b}{b+1}\frac{r-1}{r}$ and $\rho = \lambda/(1-\lambda)^2$, then

$$\mathbf{E}C_{RR} = \sum_{k \geq 0} \frac{1}{b+1} k\lambda^k = \frac{1}{b+1}\frac{\lambda}{(1-\lambda)^2} = \rho\frac{1}{b+1}.$$

Similarly,

$$\mathbf{E}C_{BR} = \rho\frac{r-1}{r(b+1)}, \qquad \mathbf{E}C_{RB} = \rho\frac{b}{r(b+1)}, \qquad \mathbf{E}C_{BB} = \rho\frac{1}{r}.$$

The conditional expectations $\mu_{XY}(r) = \mathbf{E}C_{XY}(r)/P_X(Y, r)$ are given by

$$(24) \qquad \mu_{XY}(r) = \begin{cases} \rho\frac{n}{r}\frac{1}{b+1}, & XY = RR, \\ \rho\frac{n}{r}\frac{1}{b+1}, & XY = BR, \\ \rho\frac{n}{n-r}\frac{b}{r(b+1)}, & XY = RB, \\ \rho\frac{n}{n-r+1}\frac{1}{r}, & XY = BB. \end{cases}$$

The value of $\rho = (rb(r-1)(b+1))/n^2$. In particular if $b, r = (1+o(1))n/2$, then whatever colors X, Y

$$(25) \qquad \mu_{XY}(r) = (1+o(1))\frac{n}{4}.$$

Let $N = n/2$. Starting from $r = b = n/2$ let T'_N be the number of transitions between states $S(r)$ to reach $\max(r, b) = N + n/2$. Referring to (23), we consider a biassed random walk with transition probabilities of $Z = \max\{r, b\} - n/2$ given by

$$(26) \qquad p_i = \begin{cases} 1, & \text{if } i = 0, \\ 1/2 + i/n + \delta/n, & \text{if } i \in \{1, \ldots, n/2 - 1\}, \\ 0, & \text{if } i = n/2, \end{cases}$$

where we set $\delta = 1$ for a lower bound on the number of steps T' to absorption, and $\delta = -1$ for an upper bound.

The walk in (26) is the push chain Z_t with transitions given by (7) as analyzed in section 2. Referring to (7) and (6) we set $\delta = 0$ for a lower bound on $\mathbf{E}_0 T_M$. For $M = N^{3/4}$, from Lemma 6,

$$\mathbf{E}_0 T_M \geq \Theta(1) \sum_{i=\sqrt{N}}^{M} \frac{N}{i} \geq \Theta(N)\log\frac{M}{\sqrt{N}} = \Theta(n \log n).$$

For all states $i = \sqrt{N}, \ldots, N^{3/4}$, the corresponding value of $r = (1+o(1))n/2$. Referring to (25), whatever the type of transition XY between $S(r)$ and neighboring states, $\mu_{XY}(r) = (1+o(1))n/4$. Let $\mu = \min_{X,Y}(\mu_{XY}(r) : n/2 \leq r \leq M)$, then $\mu \geq n/5$. As $\mathbf{E}_0 T_N \geq \mathbf{E}_0 T_M = \Theta(n \log n)$ we have that

$$\mathbf{E}T(Push) \geq \mu \, \mathbf{E}_0 T_M = \Omega(n^2 \log n).$$

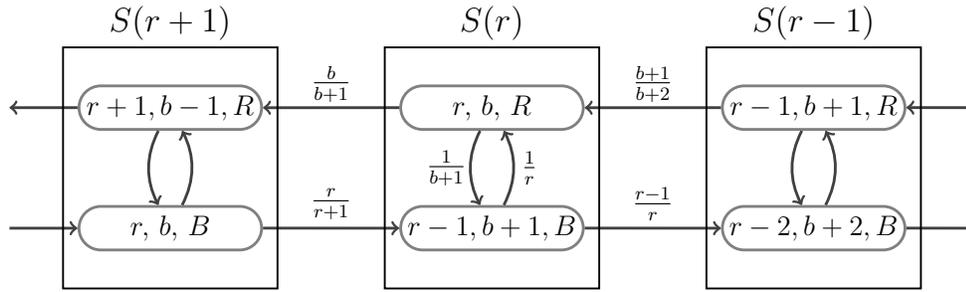The upper bound follows by a similar argument. Put $\delta = -1$ in (7), and use Lemma 5.

FIG. 6. *Pseudostates for the pull process.*

Pull voting on the star.

THEOREM 12. *The pull process on the star with n vertices has expected consensus time* $\mathbf{E}T = O(n^2)$.

As before, we group the states $(r, R) = (r, b, R)$ and $(r - 1, B) = (r - 1, b + 1, B)$ into a single pseudostate $S(r)$. The transitions probabilities within or between $S(r+1)$ or $S(r-1)$ are shown in Figure 6 and are obtained by calculations similar to the push case. In the final pseudostate $S(n)$ on the left, the state $(n, 0, R)$ is absorbing, and so the state $(n - 1, 1, B)$ cannot be reached. As an initial state, $(n - 1, 1, B)$ goes to $(n - 2, 2, B)$ with probability 1.

The pull process seems much easier to analyze. Suppose the star currently has a red central vertex, and we are in state (r, b, R) of $S(r)$. The probability of a direct transition from (r, b, R) to $(r + 1, b - 1, R)$ is $b/(b + 1)$. This occurs when a blue leaf vertex is chosen and pulls the color of the red central vertex. We say a *run* is a sequence of transitions which leave the color of the central vertex unchanged. Let $\rho(r, x, R)$ be run given by the sequence of transitions

$$(r, b, R) \to (r + 1, b - 1, R) \to \cdots \to (x - 1, n - x + 1, R) \to (x, n - x, R).$$

Then
$$\mathbf{Pr}(\rho(r, x, R)) = \frac{n - r}{n - r + 1} \frac{n - r - 1}{n - r} \cdots \frac{n - x + 1}{n - x + 2} = \frac{n - x + 1}{n - r + 1}.$$

The probability a run starting at $(r, n - r, R)$ finishes by absorption at $(n, 0, R)$ is

$$\mathbf{Pr}(\rho(r, n, R)) = \frac{1}{n - r + 1} \geq \frac{1}{n}.$$

Each run is terminated by absorption, or by a change of color of the central vertex, say, from R to B. In the latter case, this marks the start of a new run (possibly of length zero) in the opposite direction. Starting from $(r, n - r, R)$, let X be the number of changes of color of the central vertex from R to B, or vice versa, before absorbtion at $(n, 0, R)$ or $(0, n, B)$. Let Y be the winning step for a sequence of independent trials with success probability $p = 1/n$. Then $\mathbf{E}X \leq \mathbf{E}Y = n$. Each run has a length between zero and n, so $\mathbf{E}T(Pull) = O(n^2)$.

6. Voting on the double star.

Push voting on the double star. A *double star* $S^{\star}_{2n+2}$ comprises two stars S_1, S_2, each with n leaves, and their central vertices c_1, c_2 joined by an edge. See Figure 1. Let $X_t : V \to \{R, B\}$ identify the colors of the vertices $v \in V$ at time t. We
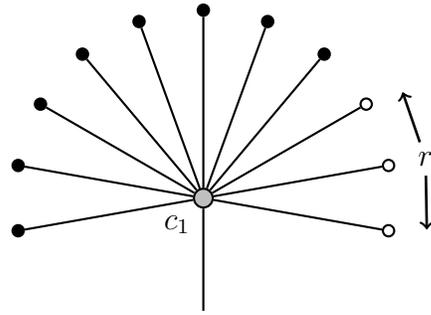
FIG. 7. S_1 with r leaves colored R.

will show that the convergence time for the push process on $S_{2n+2}^{\star}$ can be exponential in n.

THEOREM 13. *The push process on the double star with $2n+2$ vertices has worst case convergence time $\Omega(2^{2n/5})$.*

Proof. We will assume that the initial configuration for the process has $X_0(v) = B$ ($v \in S_1$), and $X_0(v) = R$ ($v \in S_2$). Then, for convergence to occur, we must have either $X(v) = R$ (for all $v \in S_1$) or $X(v) = B$ (for all $v \in S_2$). Without loss of generality, we suppose S_1 that must be recolored R, and temporarily restrict attention to S_1.

Let $r_t = |\{v \in S_1 \setminus c_1 : X_t(v) = R\}|$ be the number of leaves in S_1 which are colored R, and hence $(n - r_t)$ leaves are colored B. We make no assumption about $X_t(c_1)$ or $X_t(c_2)$. See Figure 7.

Now, if $r_{t-1} = r$, at step t either $r_t \leftarrow r + 1$, $r_t \leftarrow r - 1$, c_1 changes color, or the step involves S_2. We discard all steps which involve S_2 or c_1 and consider the time t as changing only when either $r_{t+1} \leftarrow r_t + 1$ or $r_{t+1} \leftarrow r_t - 1$. Thus t is a lower bound on the duration of the process.

We will upper bound $\mathbf{Pr}(r_{t+1} = r + 1)$, when $r_t = r$. This event occurs only when c_1 is chosen and will be maximized when $X_t(c_1) = R$, since otherwise c_1 must first change color. It is also maximized when $X_t(c_2) = R$, since then $c_1 c_2$ cannot be chosen as a discordant edge. However, c_1 may be recolored B, R any number of times, k, say, between t and $t + 1$. The probability that c_1 is recolored B is at most $(n-r+1)/(n-r+2)$, when c_2 is colored B. Subsequent to this, the probability that c_1 is recolored R is at most $(r+1)/(r+2)$, when c_2 is colored R.

$$
\begin{aligned}
\mathbf{Pr}(r_{t+1} = r + 1 \mid r_t = r) &\leq \frac{1}{n-r+1} \sum_{k=0}^{\infty} \left(\frac{r+1}{r+2} \frac{n-r+1}{n-r+2} \right)^k \\
&= \frac{1}{n-r+1} \left(1 - \frac{r+1}{r+2} \frac{n-r+1}{n-r+2} \right)^{-1} \\
&= \frac{(r+2)(n-r+2)}{(n+3)(n-r+1)} \\
&\leq \frac{r+3}{n+3}, \quad \text{if } r \leq (n-1)/2.
\end{aligned}
$$

Since the only alternative is that $r_{t+1} = r - 1$, when $r \leq (n-1)/2$, we also have

$$
\mathbf{Pr}(r_{t+1} = r - 1 \mid r_t = r) = 1 - \mathbf{Pr}(r_{t+1} = r + 1) \geq 1 - \frac{r+3}{n+3} = \frac{n-r}{n+3}.
$$

Now $\mathbf{Pr}(r_{t+1} = r + 1 \mid r_t = r) \leq (r + 3)/(n + 3) \leq \frac{1}{5}$ if $r \leq (n - 12)/5$. Let $\nu = \lfloor(n - 12)/5\rfloor$. Thus, in the range $0 \leq r_t \leq \nu$, the process r_t is dominated by a random walk Z_t with $\mathbf{Pr}(Z_{t+1} = r+1 \mid Z_t = r) = \frac{1}{5}$, $\mathbf{Pr}(Z_{t+1} = r-1 \mid Z_t = r) = \frac{4}{5}$. Let a *trial* of this process be the sequence of T steps, starting with $Z_0 = 1$, until either of the events $E_0 : Z_T = 0$ or $E_\nu : Z_T = \nu$ occurs. From [11, p. 314], we have

$$\mathbf{Pr}(E_\nu) \;=\; \frac{3}{4^\nu - 1} \;\leq\; 4^{1-\nu} \quad \text{for } \nu > 1.$$

Let $E_\nu^{1,k}$ be the event that E_ν ever occurs in k trials. Thus $\mathbf{Pr}(E_\nu^{1,k}) \leq k4^{1-\nu} = 4k/4^\nu$. The corresponding event $E_\nu^{2,k}$ in S_2 is that $n - r_t = \nu$ occurs in k trials, and so similarly $\mathbf{Pr}(E_\nu^{2,k}) \leq k4^{1-\nu}$. Let $E_\nu^k = E_\nu^{1,k} \vee E_\nu^{2,k}$, so

$$\mathbf{Pr}(E_\nu^k) \;=\; \mathbf{Pr}(E_\nu^{1,k} \vee E_\nu^{2,k}) \;\leq\; \mathbf{Pr}(E_\nu^{1,k}) + \mathbf{Pr}(E_\nu^{2,k}) \;\leq\; 8k/4^\nu \;=\; k/2^{2\nu-3}.$$

Clearly convergence requires E_ν^k to have occurred. However, if $k \leq 2^{2(\nu-5)}$, E_ν^k occurs with probability at most $\frac{1}{4}$. Thus we need at least $\Omega(4^\nu) = \Omega(2^{2n/5})$ trials before there is any appreciable probability of convergence. Hence $\Omega(2^{2n/5})$ is a lower bound on the time for convergence with high probability. □

For a double star S_N^* on $N = 2n + 2$ vertices, it follows that for the push process $\mathbf{E}T = \Omega(2^{N/5})$, as stated in Theorem 3.

Pull voting on the double star.

LEMMA 14. *Let T be the expected time to complete discordant pull voting on the double star of $2n + 2$ vertices. Then for any starting configuration $\mathbf{E}T = O(n^4)$.*

Proof. Our proof mimics that for pull voting on the star graph. If the centers c_1, c_2 are the same color (say, red) we call the central edge monochromatic. If the central vertices are both red, then a run is a sequence of steps in which a blue leaf vertex is chosen at each step and pulls the red color from one of the central vertices.

Let r_1, b_1 be the number of red and blue leaves in S_1 (resp., r_2, b_2 in S_2). Let $b_1 + b_2 = b$. Let $\rho(b, k \mid R)$ be the probability of a run of length at least $k \geq 0$ given the central vertices are red. The probability that a central vertex is recolored at the next step is $\rho(b, 0 \mid R) = 2/(b + 2)$. The required probabilities are

$$\rho(b, k \mid R) = \begin{cases} \frac{b}{b+2}, & k = 1, \\[2mm] \frac{b(b-1)}{(b+2)(b+1)}, & k = 2, \\[2mm] \frac{(b-k+2)(b-k+1)}{(b+2)(b+1)}, & k = 3, \ldots, b-1, \\[2mm] \frac{2}{(b+1)(b+2)}, & k = b. \end{cases}$$

Before cancelation of terms, for $k \geq 3$ the expression for $\rho(b, k \mid R)$ is

$$\frac{b}{b+2}\frac{b-1}{b+1}\frac{b-2}{b} \cdots \quad \cdots \frac{b-(k-3)}{b-(k-3)+2}\frac{b-(k-2)}{b-(k-2)+2}\frac{b-(k-1)}{b-(k-1)+2}.$$

The cases $k = 1, 2$ are given by the first two terms of this expression.

If the central edge is monochromatic, then the probability P to finish voting without recoloring either of c_1, c_2 is $P = \rho(b, b \mid R) \geq 1/n^2$. Let μ' be an upper bound

on the expected number of runs required for an exit (i.e., for the entire coloring to be monochromatic). Then $\mu' \leq 1/P = n^2$.

If the central edge is not monochromatic, e.g., c_1 is red and c_2 is blue, let the probability of becoming monochromatic in a given step be $\phi(r_1, b_1, r_2, b_2)$. Thus

$$\phi(r_1, b_1, r_2, b_2) \geq \min \left\{ \frac{2}{b_1 + r_2 + 2}, \frac{2}{r_1 + b_2 + 2} \right\} \geq \frac{2}{2n + 2} = \frac{1}{n + 1}.$$

Let μ be an upper bound on the expected wait for the central edge to become monochromatic. Then $\mu \leq n + 1$.

The number of steps in any run is at most $s = 2n + 1$. Thus for the pull process

$$\mathbf{E}T \leq \mu\mu's = (n + 1)\, n^2\, (2n + 1) = O(n^4). \qquad \square$$

7. Voting on the barbell graph. The barbell or dumbbell graph of $N = 2n$ vertices, B_{2n}, is formed from two disjoint cliques S_1 and S_2 of size n joined by a single edge e.

Push voting on the barbell. We start with the following configuration: all vertices in S_1 are red, and all vertices in S_2 are blue. Initially e is the only discordant edge. Let T the first time when the whole of S_1 is blue (or S_2 is red). Clearly T is less than or equal to the time to reach consensus. For simplicity, we just look at S_1 and assume the final color of S_1 (and S_2) is blue. Suppose that N_t is the number of blue vertices in S_1, where initially $N_0 = 0$. Let M_t be the number of discordant vertices, where $M_0 = 2$. When $1 \leq N_t \leq n/5 - 9$ then $M_t \geq n$, and

$$\mathbf{Pr}(N_{t+1} = N_t + 1 | N_t) \leq (N_t + 1)/M_t \leq (N_t + 1)/n \leq 1/5,$$
$$\mathbf{Pr}(N_{t+1} = N_t - 1 | N_t) = (n - N_t)/M_t \geq 2/5.$$

In the regime $1 \leq N_t \leq n/5 - 9$, N_t is dominated by a process N'_t with

$$\mathbf{Pr}(N'_{t+1} = N'_t + 1 | N'_t) = 1/5,$$
$$\mathbf{Pr}(N'_{t+1} = N'_t - 1 | N'_t) = 2/5,$$
(27) $$\mathbf{Pr}(N'_{t+1} = N'_t | N'_t) = 2/5.$$

Let Z be N'_t observed when N'_t changes, and thus we ignore the loop steps given by (27), in which case the probability p that Z increases by one is $p = 1/3$, and the probability q that Z decreases by one is $q = 2/3$. We now follow the analysis for push voting on the double star. Let a *trial* of this process be the sequence of T steps, starting with $Z_0 = 1$, until either of the events $E_0 : Z_T = 0$ or $E_\nu : Z_T = \nu$ occurs. From [11, p. 314], we have

$$\mathbf{Pr}(E_\nu) \;=\; \frac{1}{2^\nu - 1} \;\leq\; 2^{1-\nu} \quad \text{for } \nu > 1.$$

From now on, the same argument used for the double star works here. We just repeat the conclusion that $\mathbf{E}T = \Omega(2^\nu) = \Omega(2^{n/5}) = \Omega(2^{N/10})$, where $N = 2n$ is the total number of vertices.

Pull voting on the barbell. We suppose we have reached a configuration in which all vertices except one are red. Suppose the unique blue vertex is in S_1. We modify our process so that the system reaches consensus faster. To do that, in each

round we only select vertices in S_1 and assume the final color will be red. If the final color would be blue, then we must also recolor all of S_2. Even if the vertex c_1 of the bridge edge $e = (c_1, c_2)$ is blue, the interaction between S_1 and S_2 does not affect the outcome. If S_1 is not in consensus, then each vertex in S_1 has at least one discordant neighbor in S_1, so the (red) opinions in S_2 will not affect the outcome.

We use a result from the proof of Theorem 1 for K_n as given in section 3. Inequality (17) shows the expected time for pull voting to reach consensus in K_n, when all but one vertex is red is $\Omega(2^n)$. So, the time to finish in the modified process is $\Omega(2^n) = \Omega(2^{N/2})$ on a graph with N vertices.

8. Concluding remarks. In contrast to ordinary randomized voting which performs consistently on a wide variety of graphs, the consensus time of push or pull discordant voting depends strongly on the structure of the underlying graph (see Figure 1). The exact structural properties which could be used to predict this behavior are unclear. This performance seems counterintuitive, as focusing on the conflicts between vertices rather than ignoring them ought to allow us to reach consensus more quickly.

Unlike ordinary voting which admits a general analysis (see, e.g., [7]), the methods used for discordant voting are somewhat ad hoc. In the case of discordant voting on the cycle, one of the referees suggested an alternative approach to upper bounding the consensus time based on coupling with a simple random walk.

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