Article:
Bhattacharya, Anindya orcid.org/0000-0002-2853-8078, Brosi, Victoria Katharina Franziska and Ciardiello, Francesco (2018) The Uncovered Set and the Core:Cox's Result Revisited. Journal of mechanism and institution design. pp. 1-15. ISSN 2399-844X

https://doi.org/10.22574/jmid.2018.12.001

Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2018.12.001

THE UNCOVERED SET AND THE CORE: COX'S RESULT REVISITED

Anindya Bhattacharya

University of York, United Kingdom
anindya.bhattacharya@york.ac.uk

Victoria Brosi

Formerly of University of York, United Kingdom
vbrosi@gmx.de

Francesco Ciardiello

University of Sheffield, United Kingdom
f.ciardiello@shef.ac.uk

ABSTRACT

In this work first it is shown, in contradiction to the well-known claim in Cox (1987), that the uncovered set in a multidimensional spatial voting situation (under the usual regularity conditions) does not necessarily coincide with the core even when the core is singleton: in particular, the posited coincidence result, while true for an odd number of voters, may cease to be true when the number of voters is even. Second we provide a characterisation result for the case with an even number of voters: a singleton core is the uncovered set in this case if and only if the unique element in the core is the Condorcet winner.

We are greatly indebted to the Editor and two anonymous referees for several helpful comments and suggestions. We also thank Bhaskar Dutta, Gary Cox, John Duggan, Michel Le Breton and Elizabeth Maggie-Penn for their comments and help. This work was born when Brosi was receiving a scholarship from ESRC, UK and Ciardiello visited the Department of Economics and Related Studies, University of York. Brosi thanks the ESRC. Ciardiello thanks Regione Puglia for funding his International visiting scholarship. He thanks Andrea Genovese, head of the division "Operations Management and decision sciences" at Sheffield Management School, for funding recent short visiting period in York. He thanks the Department of Economics and Related Studies in York, especially Annette Johnson and Peter Simmons, for the kind hospitality always received in York. The errors and shortcomings remaining are ours.

Keywords: Spatial voting games, uncovered set, core.

JEL Classification Numbers: D71, C71.

1. INTRODUCTION

In this work first it is shown, in contradiction to the well-known claim in Cox (1987, 409) (repeated in a number of subsequent works), that the uncovered set in a multidimensional spatial voting situation does not necessarily coincide with the core even when the core is non-empty or even singleton: in particular, the posited coincidence result may cease to be true when the number of voters is even. Then we provide a characterisation result for the case with an even number of voters: a singleton core is the uncovered set in this case if and only if the unique element in the core is the Condorcet winner.

In our framework, the set of outcomes or policies under consideration is some compact and convex subset of some finite dimensional Euclidean space and any majority coalition of voters can enforce any outcome over another. For such an environment Cox (1987, 409) made the claim that if individual preferences satisfy a very innocuous symmetry condition then the uncovered set coincides with the core whenever the latter is non-empty. However, he worked with an odd number of voters *for expositional convenience* (Cox, 1987, 409). But actually his proof used the assumption that the cardinality of the voter set is odd in a non-trivial way. This claim has been repeated in subsequent literature. For example, Austen-Smith & Banks (2005, 274) stated in their much-used textbook that "the uncovered set coincides with the core when the latter is nonempty and singleton" (the definition of the uncovered set they use, in fact, gives a superset of the uncovered set with which we have worked here). A similar remark appears in the relatively recent but well-known paper by Penn (2009, 44).

However, in this paper we show that there is a voting situation for which the number of voters is even, the core is a singleton, but the uncovered set does not coincide with the core. Therefore, the message of this counter-example is that this claim (Cox, 1987), which is a powerful result for majority rule voting situations with an odd number of voters, should be invoked with some degree of caution. The strength of this result in the case with an odd number of voters (for which it is true) may lead to its somewhat careless generalised use. While we show this possibility of non-coincidence of the core and the uncovered set for spatial voting, for majority voting situations with a finite set of outcomes, this possibility was noted by Bordes (1983) several years ago. Quite naturally, the question then arises whether an analogue of Cox (1987)'s result is true when the number of voters is even. Addressing this issue we provide the following characterization result: a singleton core is the uncovered set in a majority spatial voting situation (with the usual regularity conditions) with an even number of voters if and only if the unique element in the core is the Condorcet winner of that situation.

The next section gives the preliminary definitions and notation. Section 3 gives the results and some discussions about the results.

2. PRELIMINARY DEFINITIONS AND NOTATION

Let $Z \subseteq \mathbb{R}^k$ be a compact convex subset of some finite (k-)dimensional Euclidean space. This set, Z, is identified to be the feasible set of policies or outcomes on which a voter votes. Let $N = \{1, 2, \dots, n\}$ be the finite set of players or voters. Suppose that the preferences of a player i on Z is represented by a real-valued continuous and strictly concave pay-off function $u_i \in C^0(Z, \mathbb{R})$. The spatial voting situation we consider below is obtained by introducing the method of majority rule voting.

Definition 2.1 (Domination by Majority Rule)**.** *Given $x, y \in Z$, the policy x beats (or dominates) policy y via coalition $S \subseteq N$, if $|S| > |N|/2$ and $u_i(x) > u_i(y)$ for each $i \in S$. We denote this as $x \succ_S y$. If there exists a majority coalition S via which x dominates y, we denote that as $x \succ y$.*

The collection $G = (Z, N, (u_i)_{i \in N})$ is a spatial voting situation with majority rule. For any $x \in Z$ and $i \in N$, by $D^i(x)$ we denote the set $\{y \in Z : u_i(y) > u_i(x)\}$. Further, $D(x) = \{y \in Z : y \succ x\}$. For any set $A \subseteq Z$, by $cl(A)$ we denote the closure of A. Also, for any two points $x, y \in Z$, by $\rho(x, y)$ we denote the (Euclidean) distance between these two points. Recall the two well-known solution concepts for such situations that we shall discuss: the core and the uncovered set.

Definition 2.2 (The Core of a Voting Situation)**.** *The core of such a voting situation is the subset $K = \{y \in Z : \nexists z \in Z \text{ such that } z \succ y\}$.*

Recall that a point $x \in Z$ is said to be the Condorcet winner of the voting situation if for any other outcome $y \neq x$, $x \succ y$. Recall that if a voting situation admits a Condorcet winner, then it is the unique element in the core.

Definition 2.3 (The (Gillies) Uncovered Set)**.** *Let $x, y \in Z$. We say that x covers y, denoted as $y \prec_c x$ if the following hold:*

$$x \succ y;$$
$$z \in Z,\ z \succ x \Longrightarrow z \succ y.$$

The uncovered set is given by $UC = \{y \in Z : \nexists z \text{ such that } y \prec_c z\}$

As this work is primarily motivated by Cox (1987) we are using the definition he has used. Although the notion of covering (and that of uncovered sets) was introduced explicitly first by Miller, the solution defined here is the set of the maximal elements of the Gillies' covering subrelation (following Gillies, 1959) rather than Miller's subrelation (for clarification, we refer to Bordes et al. 1992; see also Miller 2015). At the end we remark on the analogue of our non-coincidence result for the (Miller) uncovered set. For any $y \in Z$, by $C(y)$ we denote the set of elements in Z, which cover y.

Next recall that the preferences are said to be Euclidean or circular if for every voter $i \in N$, there exists $\bar{x}_i \in Z$ such that for any policy $x \in Z$, $u_i(x) = -(\rho(x, \bar{x}_i))^2$. Cox (1987, 416) introduces a notion of *limited* asymmetry of preferences. For completeness, we reproduce the definition here.

Definition 2.4 (Preferences that are limited in asymmetry)**.** *Preferences are said to be limited in asymmetry by α if for every line L intersecting Z, every $i \in N$, and every $r \in \mathbb{R}$,*

$$V_L^i(r) \neq \emptyset \Longrightarrow f(L, i.r) \leq \alpha$$

where $V_L^i(r) = \{x \in L : u_i(x) = r\}$; $f(L, i, r) = \frac{max_{x \in V_L^i(r)} \rho(x, b_L^i)}{min_{x \in V_L^i(r)} \rho(x, b_L^i)}$; the induced ideal point of $i \in N$ on the line L being denoted by b_L^i.

We merely note here that Euclidean preferences obviously satisfy this condition with $\alpha = 1$.

3. THE RESULTS AND DISCUSSIONS

Cox's result is as follows.

Proposition 3.1. *Take a voting situation G. Suppose there exist a finite α such that the condition of limited asymmetry of preferences by α holds for every voter. Now suppose $K \neq \emptyset$. Then $K = UC$.*

As we mentioned in the introduction, one primary motivation behind this work is that this result has been repeated in a number of subsequent works but not with, in our view, sufficient care. However, we find the following.

Proposition 3.2. *There is a voting situation G for which $|N|$ is even, the core K is singleton, the preference of each of the voters is Euclidean and the uncovered set does not coincide with the core.*

To prove this proposition we shall use an intermediate result. First recall the definition of a von-Neumann-Morgenstern stable set for the voting situations we consider here.

Definition 3.1 (von-Neumann-Morgenstern Stable Sets)**.** *A set $V \subseteq Z$ is a (von-Neumann-Morgenstern) stable set for G if it satisfies*

- *(internal stability:) there do not exist $x, y \in V$ such that $x \succ y$;*
- *(external stability) if $x \in Z \setminus V$ it must be the case that there exists $y \in V$ such that $y \succ x$.*

The following Proposition 3.3 is useful to prove Proposition 3.2. Although variants of this result are well-known (see, e.g., McKelvey (1986)), for completeness we provide a short proof, within our framework, below.

Proposition 3.3. *If a stable set V exists then $K \subseteq V \subseteq UC$.*

Proof. If $K \not\subseteq V$ then that violates the external stability of V.

Let V be a stable set and take, if possible, and $x \in V \setminus UC$. That is, there exists $y \in Z$ such that $x \prec_c y$. This implies that $y \succ x$. Since, $y \notin V$ (otherwise, the internal stability of V is violated), by external stability of V, there exists $z \in V$ such that $z \succ y$. But, then, by the definition of the covering relation, $z \succ x$ which again violates the internal stability of V. □

Proof of the Proposition 3.2. Below we give an example of a situation where the core is singleton, a stable set exists and the stable set does not coincide with the core. Then, by Proposition 3.3 we are done. Let $N = \{1,2,3,4\}$. The set of outcomes, $Z = \{x \in \mathbb{R}^2 | x_1 \in [-1,1]; x_2 \in [-1,1]\}$. Each player i has an ideal point $\bar{x}_i$ whose coordinates are given as follows. The point $\bar{x}_1$ (labelled by A')$= (-1,-1)$; $\bar{x}_2$ (labelled by B')$= (1,-1)$; $\bar{x}_3$ (labelled by C')$= (1,1)$ and $\bar{x}_4$ (labelled by D')$= (-1,1)$ (please see Figure 1 below). The players' preferences are Euclidean, i.e., for any $i \in N$, and $x \in Z$, $u_i(x) = -(\rho(x,\bar{x}_i))^2$. We show below that the core of this situation is the singleton set containing the point $(0,0)$ (labelled as point O) while the set $V = \{x \in Z | x_1 = 0 \text{ or } x_2 = 0\}$ is a stable set. For convenience later call the set $\{x \in Z | x_1 = 0\}$ as V_1 and the set $\{x \in Z | x_2 = 0\}$ as V_2.

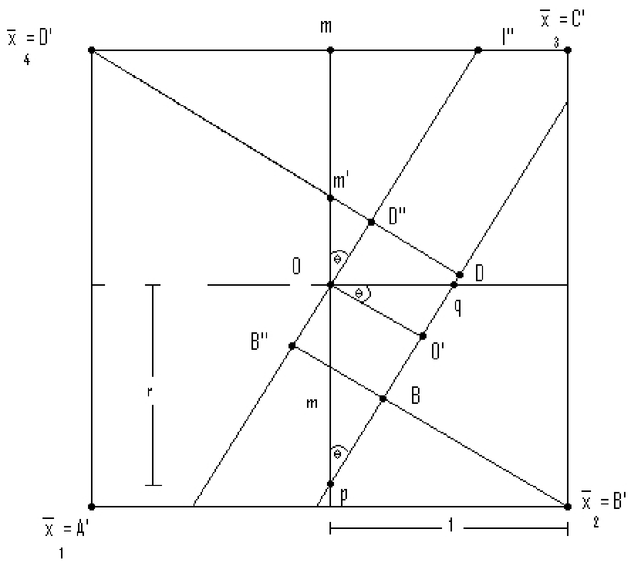

Figure 1: The voting situation for which the singleton core is not a stable set.

We take the following steps.

Step 1 : Notice that the point $O = (0,0)$ satisfies the Plott condition (Austen-Smith & Banks, 1999, 142) and so, it is in the core of this voting situation (see Schofield, 2008, 90), too.

Step 2 : Next we show that no point other than O is in the core. We start with the subset $\Delta_1 = \{x \in Z \setminus \{O\} | x_1 \leq 0, x_2 < 0, x_1 \geq x_2\}$. Note that this is a triangle (without the point O. Choose a point $x \in \Delta_1$ such that $x_1 < 0, x_2 < 0$ and $x_1 > x_2$. Draw a line of slope 1, L_x, passing through this x and let the line intersect the line V_1 at the point y. It is obvious that y dominates x via coalition $\{2,3,4\}$. Next, take a point $x \in \Delta_1$ such that $x_1 = x_2$. Then it is obvious that $(0,0)$ dominates such a point via $\{2,3,4\}$. Finally, choose a point $x \in \Delta_1$ such that $x_1 = 0$ and $x_2 < 0$, i.e., the point is in V_1. Again, draw a line of slope 1, L_x, passing through x and let that intersect the line given by $\{x \in Z | x_1 = -x_2\}$ at a point y. Again, it is obvious that y dominates x via coalition $\{2,3,4\}$.
Thus, no point of Δ_1 is in the core.
Note that $Z \setminus \{(0,0)\}$ is the union of 8 such triangles like Δ_1. Therefore, using the symmetry between these triangles we can show that no point other than O is in the core.

Next we show that V is a stable set.

Step 3 (External stability of V): From Step 2 itself we see that for any $x \in Z \setminus V$, there exists an $y \in V$ such that $y \succ x$.

Step 4 (Internal stability of V): It is obvious that a point in V_1 cannot be dominated by another point in V_1 and a point in V_2 cannot be dominated by another point in V_2. Next we show that a point in V_1 cannot dominate, nor can be dominated by a point in V_2. Let p be a point in V_1 such that $p_2 < 0$. Let q be a point in V_2 such that $q_1 > 0$. Let $\overline{Op}$, the length of the line segment Op, be $0 < r \leq 1$ and let the angle $\widehat{Opq}$ be θ. (Please refer to Figure 1.) Let, without loss of generality, $0 < \theta \leq \pi/4$. Call the line passing through p and q, L. Let the perpendiculars from A', B', C' and D' on L (or, to put more rigorously, the orthogonal projections of these points on L) be denoted respectively by A, B, C and D. Note that if $\theta = \pi/4$, then the points B and D coincide and then it is obvious that, neither p can dominate q, nor q can dominate p. Now suppose

$0 < \theta < \pi/4$. It is easy to see that $\overline{qD} < \overline{pD}$. Since $\overline{Cq} < \overline{Cp}$, p cannot dominate q. Next we show that the line segment $\overline{B'p} \leq \overline{B'q}$. Note that this is true if and only if $\underline{(1+p_2)} \leq (1-q_1)$. But this follows from the fact that $\theta < \pi/4$. Since $\overline{A'p} \leq \overline{A'q}$, (obviously) q cannot dominate p. From this, using the symmetry of this example, we can show that no point in V_1 can dominate another point in V_2 and vice versa. □

One point of curiosity is to identify the uncovered set in the example we used in proving Proposition 3.2. [1] Indeed, we find that for that example, the uncovered set is a strict superset of the stable set we identified, a feature which might be somewhat interesting. We summarize the finding as an additional result below.

Result 3.1 Consider the set U specified as:

$$\begin{gathered}(\{z \in Z \,|\, \rho(A',z) \leq \rho(A',O)\} \cap \{z \in Z \,|\, \rho(B',z) \leq \rho(B',O)\}) \cup \\ (\{z \in Z \,|\, \rho(B',z) \leq \rho(B',O)\} \cap \{z \in Z \,|\, \rho(C',z) \leq \rho(C',O)\}) \cup \\ (\{z \in Z \,|\, \rho(C',z) \leq \rho(C',O)\} \cap \{z \in Z \,|\, \rho(D',z) \leq \rho(D',O)\}) \cup \\ (\{z \in Z \,|\, \rho(D',z) \leq \rho(D',O)\} \cap \{z \in Z \,|\, \rho(A',z) \leq \rho(A',O)\}).\end{gathered}$$

The set U (shown as the shaded area in the Figure 2 below) is the uncovered set in the example used in Proposition 3.2.

The proof of Result 3.1 involves repeated (and somewhat tedious) use of similar arguments from elementary Euclidean geometry. We provide a sketch proof below.

[1] We are indebted to one of the referees for inducing us to explore this issue.

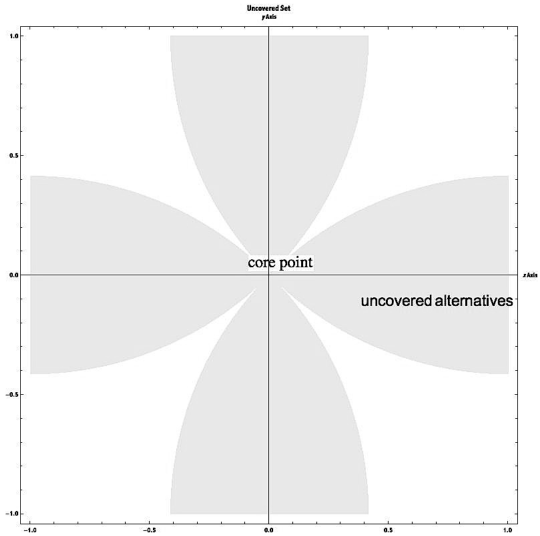

Figure 2: The uncovered set for the example used in Proposition 3.2.

Proof of Result 3.1. We start with defining the following subsets of Z :

$$\begin{aligned}
\Delta_1 &= \{x \in Z | x_1 < 0, x_2 < 0, x_1 > x_2\}, \\
\Delta_2 &= \{x \in Z | x_1 < 0, x_2 < 0, x_1 < x_2\}, \\
\Delta_3 &= \{x \in Z | x_1 > 0, x_2 > 0, x_1 > x_2\}, \\
\Delta_4 &= \{x \in Z | x_1 > 0, x_2 > 0, x_1 < x_2\}, \\
\Delta_5 &= \{x \in Z | x_1 > 0, x_2 < 0, x_1 > -x_2\}, \\
\Delta_6 &= \{x \in Z | x_1 > 0, x_2 < 0, x_1 < -x_2\}, \\
\Delta_7 &= \{x \in Z | x_1 < 0, x_2 > 0, -x_1 < x_2\}, \\
\Delta_8 &= \{x \in Z | x_1 < 0, x_2 > 0, -x_1 > x_2\}.
\end{aligned}$$

Then the proof proceeds along the following steps.

Step 1 First note the rather obvious fact that no point $x \in Z$ is dominated by another point y, via the grand coalition $\{1,2,3,4\}$. For completeness we give a brief proof. Consider, without loss of generality, a point x in Δ_1. Again, without loss of generality, consider a point y in the "north-east" of x. Then $u_1(x) \geq u_1(y)$. By using similar arguments, and by the symmetry of the voting situation in this example, this fact can be verified.
Take a point $X \in \Delta_1 \cap \{y \in Z : \rho(y, B') < \rho(O, B')\}$ (refer to Figure 3

below as well). Below, for $y \in Z$ and a positive real number r, by $L(y,r)$ we denote the set $\{z \in Z : \rho(y,z) < r\}$.

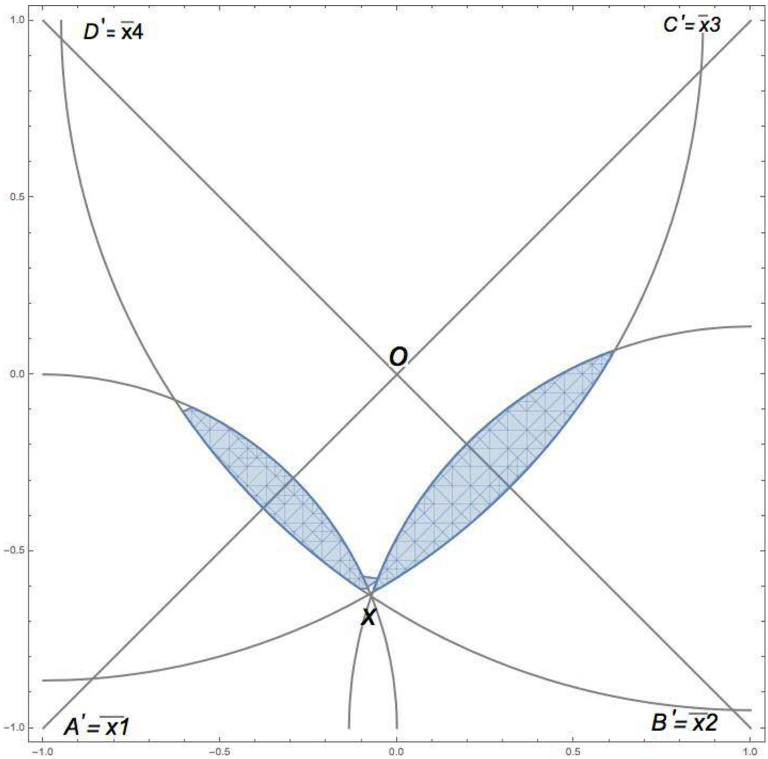

Figure 3: The set of points dominating the point X.

Step 2 Step 1 implies that for any $x, y \in Z$, if y dominates x, then that must be via a coalition of cardinality 3. Now take the point X without loss of generality. Note that if some $y \succ_S X$ via some coalition S, then $\{1,2\} \not\subset S$. To see this, assume the contrary. Then, there exists $y \in L(A', \rho(A', X) \cap L(B', \rho(B', X)$ which dominates X via some coalition S which contains voters 1 and 2. Let, without loss of generality, the third member of S be 3. Consider the angles $A'C'X = \phi$ and $A'C'y = \phi'$. It is easy to see that $\pi/4 > \phi' > \phi$. Then, from the fact that $\cos\phi' < \cos\phi$, it is straightforward that $\rho(C', y) > \rho(C', X)$. But this leads to a contradiction.

Step 3 It is easy to see that, then, $D(X) = (D^1(X) \cap D^3(X)) \cup (D^2(X) \cap D^4(X))$ (this is depicted as the shaded area, without the boundaries, in Firgure 3).

Step 4 Now consider, if possible, $z \in Z$, such that z can cover X. Suppose $z \in D^2(X) \cap D^4(X) \cap \Delta_1$. Then $D(z)$ must contain a point in $\Delta_1 \setminus (D^1(X) \cap D^3(X))$. Suppose $z \in D^2(X) \cap D^4(X) \cap \Delta_6$ such that $\rho(A', z) < \rho(A', O)$. Then, again, $D(z)$ must contain a point in $\Delta_1 \setminus (D^1(X) \cap D^3(X))$. Suppose $z \in D^2(X) \cap D^4(X) \cap \Delta_6$ such that $\rho(A', z) > \rho(A', O)$. Then, again, $D(z)$ must contain O, the unique point in the core, but $O \notin D(X)$. By replicating the arguments above in this fashion, we can verify that X cannot be covered.

Step 5 Steps like the above can be replicated for every point in the set U, which would verify that U is indeed the uncovered set. Note that a point of Z which is neither in the core and nor in U is covered by O, the single point in the core. □

Given Proposition 3.2 (and Result 3.1), the immediate question which obviously arises is what might be the conditions under which an analogue of Cox's result–coincidence of the core and the uncovered set–holds when the number of voters is even? We find the following.[2]

Proposition 3.4. *Consider a voting situation G for which $|N|$ is even and for which the core $K = \{x_0\}$ is singleton. Then K is the uncovered set if and only if x_0 is the Condorcet winner for the voting situation.*

The following elementary lemma would be useful for proving this proposition.

Lemma 3.1. *Suppose $x_0 \in Z$ is a point in the core and that it does not dominate a point $y \in Z; y \neq x_0$. Then there exists a coalition T of exactly $n/2$ voters such that for each $i \in T$, $u_i(x_0) > u_i(y)$.*

Proof. Suppose not. Then there exists a coalition S containing at least $n/2+1$ voters such that for every $i \in S$, $u_i(x_0) \leq u_i(y)$. Consider a point $z \neq x_0 \neq y$ such that z is a convex combination of x_0 and y. But then, since u_i is strictly

[2] We are especially indebted to the Editor for encouraging us to explore a result like Proposition 3.4.

concave for each $i \in N$, for each $i \in S$, $u_i(z) > u_i(x_0)$ which contradicts the supposition that x_0 is in the core. The proof is completed by noting the fact that if x_0 is strictly preferred to y by more than $n/2$ voters, then x_0 dominates y which, once again, leads to a contradiction. □

Proof of Proposition 3.4. The proof of the "if" part is obvious. The proof of the "only if" part proceeds along the following steps.

Step 1 Recall that the covering relation, $\prec_c$, is transitive: i.e., $z \prec_c y$ and $y \prec_c x$ implies that $z \prec_c x$ (see, e.g., Bordes et al. (1992)).

Step 2 Now pick, if possible, $y \in Z \setminus K$, such that x_0 does not dominate y. Then, setting $y^0 = y$, construct an infinite sequence of sets $(G^q)_{q=1}^{\infty}$ in the following manner:
$G^1 = C(y^0)$;[3]
then choose some member $y^1 \in G^1$, and
$G^2 = C(y^1)$;
and proceeding in this manner, given G^k and choosing some $y^k \in G^k$,
$G^{k+1} = C(y^k)$.
By the transitivity of $\prec_c$, for each k, $G^k \subset G^{k-1}$. Observe also, that for each k, $G^k \subseteq D(y^{k-1})$. Note that by the construction of the sequence, if for any k, $x_0 \in G^k$, then, by the transitivity of $\prec_c$, $x_0 \succ y$ and we reach an immediate contradiction. Therefore, for every k, $x_0 \notin G^k$. Further, by the same reasoning, by the construction of the sequence $(G^q)_{q=1}^{\infty}$,

$$\cap_{j=1}^{\infty} G^j = \emptyset.$$

Step 3 Then, further consider the closure of each of these sets in the sequence. Then the sequence of sets $(cl(G^j))_{j=1}^{\infty}$ is a sequence of non-empty compact subsets of Z such that for every j,

$$cl(G^{j+1}) \subseteq cl(G^j).$$

Then, by the Cantor intersection lemma (see, if necessary, e.g. (Rudin, 1976, 38)),

$$\cap_{j=1}^{\infty} cl(G^j) \neq \emptyset.$$

[3] Recall from Section 2 that for any $z \in Z$, by $C(z)$ we denote the set of points which cover z.

Therefore, there exists an $x \in Z$, such that x is in the closure of every G^j. Recall, however, from Step 2 above that $\cap_{j=1}^{\infty} G^j = \emptyset$. Therefore, for every $j \geq k$ (where k is some positive integer) there exists a sequence (y_j^r), with each element from the sequence in G^j, which converges to x.

Step 4 Pick such a G^j, i.e., for which $j \geq k$. By the definition of G^j, for every element $z \in G^j$, $D(z) \subset D(y^{j-1})$. Therefore, by Lemma 7.2 in Austen-Smith & Banks (2005, 272), $D(x) \subseteq D(y^{j-1})$.[4] Therefore,

$$D(x) \subseteq \cap_{j=1}^{\infty} D(y^{j-1}).$$

Note that by the construction of the sequence in Step 2 above,

$$\cap_{j=1}^{\infty} D(y^{j-1}) = \emptyset.$$

Therefore, $D(x) = \emptyset$, i.e., $x = x_0$.

Step 5 Therefore, $x_0 \in cl(G^1)$. Since $G^1 \subset D(y^0)$, $x_0 \in cl(D(y^0))$. If $x_0 \in D(y^0)$ (i.e., $D(y)$), then again we get an immediate contradiction. So, suppose otherwise.

Step 6 Then $x_0 \in cl(D(y)) \setminus D(y)$. Therefore, there exists a sequence (y^r), with each element from the sequence in $D(y)$, which converges to x_0. Further, since there are only finitely many coalitions, there exists a fixed majority coalition S and a sequence $(z^l)_{l=1}^{\infty}$ such that for each element z of the sequence, $z \in D(y)$, z dominates y via the coalition S, and the sequence (z^l) converges to x_0. Since x_0 does not dominate y via S, there exists a voter $i \in S$ such that for each z in the sequence, $u_i(z) > u_i(y) \geq u_i(x_0)$. Moreover, since z^l converges to x_0, by continuity of u_i, $u_i(y) = u_i(x_0)$.

Step 7 Recall that by Lemma 3.1, there exists a coalition T of exactly $n/2$ voters such that for each $i \in T$, $u_i(x_0) > u_i(y)$. Moreover, conversely, for each voter j in the $n/2$-voter coalition $N \setminus T$, $u_j(y) \geq u_j(x_0)$. Then, by strict concavity of the pay-off functions, for each $w \in Z$ which is a convex combination of x_0 and y, (and different from either x_0 or y) $u_j(w) > u_j(x_0)$ for every $j \in N \setminus T$. But then, since x_0 is in the core,

[4] Recall that the lemma, in the present notation-style, states: let (z^j) be an infinite sequence converging to $x \in Z$ such that for every j, $D(z^j) \subseteq D(y)$ for some $y \in Z$. Then $D(x) \subseteq D(y)$.

for each $j \in T$, $u_j(x_0) > u_j(w)$ for each such w on the linear segment x_0y (otherwise x_0 gets dominated). In particular, none of the w's on the segment x_0y is dominated by x_0. Pick one such w. But then, replicating the argument with respect to y above, given in Steps 2 to 6, there must exist some voter $k \in N$ for whom $u_k(x_0) = u_k(w)$. But this contradicts the fact that for every $i \in N$, either $u_i(x_0) > u_i(w)$ or $u_i(x_0) < u_i(w)$.

□

4. CONCLUDING REMARKS

Here, at the end, we provide a few remarks related to our results. First, in view of Proposition 3.4, notice that in the example used in Proposition 3.2, the single point in the core is not the Condorcet winner. Next, the fact that this non-coincidence is still true with, e.g., the Miller definition of the uncovered set, follows in a straightforward manner from Proposition 30 in Duggan (2013). And finally, there remain several open questions: e.g., how far a result like Proposition 3.4 can be generalised, under what primitive conditions a singleton core can never be a Condorcet winner, etc. These are topics of ongoing and further research.

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