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FINITE-TIME AND FIXED-TIME CONSENSUS OF MULTIAGENT NETWORKS WITH PINNING CONTROL AND NOISE PERTURBATION*

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Abstract. In this paper we investigate the finite-time and fixed-time consensus problems of multiagent networks with pinning control and noise perturbation. In order to reach the finite-time and fixed-time consensus, several pinning protocols are proposed. Compared with the consensus protocols without pinning control, the proposed finite-time and fixed-time protocols need to control only a small fraction of agents, which is practical and has advantages from the physical viewpoint of energy consumption. More specifically, the deterministic and stochastic protocols include the graph $(p+1)$-Laplacian, a nonlinear generalization of the standard graph Laplacian. We show that, unlike the protocols with the standard (linear) graph Laplacian, those with the graph $(p+1)$-Laplacian solve the finite-time as well as the fixed-time consensus problems. By using the finite-time and fixed-time stability theory and the algebra graph theory, sufficient conditions are established to ensure the finite-time and fixed-time consensus. Finally, numerical simulations are presented to illustrate the correctness of the theoretical results.

Key words. consensus, collective behavior, multiagent system, noise

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1. Introduction.

When a large number of similar units move and interact, under some conditions there may exist a transition in which individual agents start behaving in a coherent fashion, and individuals no longer move independently. When the states of all individuals reach a sort of agreement, a new “mesoscopic” unit emerges and its coherent motion can be described by a new set of variables [40].

Flocking [29], swarming [12], formation control [27], and opinion dynamics [1] are examples of collective behavior, which has a long history and has been attracting an increasing attention across many fields, including biology, physics, mathematics, and engineering [11, 8, 32, 40]. Due to its broad applications, many authors have been trying to uncover the mechanisms leading to eventual consensus [39, 24, 20, 14]. This latter has been often modeled by agent-based models, in which individuals adjust their behavior according to the states of their neighbors [39].

In a more recent approach, each agent is viewed as a node of a graph and interactions among agents are represented by the edges of the corresponding network. Along

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this line of research and based on algebraic graph theory, Olfati-Saber and Murray [30] investigated the consensus problem of multiagent networks with fixed and switching topologies. It was shown that the average consensus can be achieved if the network topology is connected and balanced.

Interactions among moving agents are not necessarily instantaneous, because the information transmission times are finite. Consensus problems of a multiagent system with constant and time-varying delays have been also analyzed in [24, 20]. More recently, some studies have recognized the importance of noise and, by using stability theory of stochastic differential equations, the consensus problems of multiagent networks with multiplicative and additive noise have been discussed [21, 35].

In some practical applications, it is important to drive the multiagent system to a desirable state as soon as possible. Therefore, the convergence rate is an important indicator in the design of protocols. It was shown that, under the linear consensus protocol, the convergence rate is proportional to the algebraic connectivity of the network topology. Several researchers endeavored to improve the convergence rate by optimizing network topologies according to an increased algebraic connectivity [28]. However, the convergence time for the linear consensus protocols was proved to be unbounded [41]. In addition, the infinite time of such protocols costs more energy than the corresponding finite-time ones, because external control is needed before the consensus is reached.

In many applications, it is often required that the eventual consensus is reached in a finite time, whether or not this depends on the initial configuration of the system. This approach is more appealing because parameters can then be used to achieve faster convergence rates [5]. In [10], a discontinuous consensus protocol was proposed to solve the finite-time consensus problem. Some other continuous non-Lipschitz finite-time consensus protocols have been presented in [41, 19]. Most previous studies added the control to each agent. However, this is impractical when the group size is too large. Thus this leaves open a key question: Can consensus be achieved in finite time by controlling a fraction of the interacting agents? This paper will investigate this problem by combining the finite-time and pinning control techniques. Unlike previous investigations on pinning synchronization of complex networks [44], the control protocols presented in this paper ensure that the time to eventual consensus is finite.

Another important element of disturbance is noise, which is ubiquitous in nature as well as in manmade systems. The collective motion of self-propelled multiunits is inevitably affected by intrinsic or environmental noise, which should be taken into account when investigating more realistic multiagent systems [23, 38]. The convergence analysis of the consensus problem in noise-perturbed multiagent systems has been systematically investigated in the literature [35, 36], but so far, to the best of our knowledge, the finite-time stochastic consensus with pinning control has not been studied. This paper will specifically address this issue based on the finite-time stochastic stability theory.

Our analysis will naturally lead to looking into how the settling time of the finite-time consensus depends on the initial states of the system. On the one hand, the settling time provides us with information about characteristic temporal scales for achieving a final coherent state; on the other, however, it seems to be of limited applicability, because of its dependence on the initial states of the system, which should be given beforehand [7]. Recently, this limitation has been overcome by introducing a new control method, named fixed-time control [31]. Unlike the finite-time control, the settling time of network systems with fixed-time control is independent of the initial states. This new methodology has been successfully used in the stabilization and synchronization of complex network systems [7, 26, 25, 17]. Also, as it has been analytically and numerically validated, the fixed-time control techniques cost less time than the finite-time control methods, which, however, usually need less control terms. Thus, the energy cost of the fixed-time control technique may be very high, especially if we add controllers to all nodes. This therefore suggests controlling only part of the nodes, instead of the whole network system. This paper is aimed at investigating the fixed-time consensus problem by using the pinning control technique, in both the deterministic and the stochastic setting. Analytical validations and numerical illustrations will show the high quality of the proposed control technique.

Algebraic graph theory is a natural tool for analyzing consensus problems and in linear consensus protocols the properties of the graph Laplacian of the network topology play a crucial role [30]. Indeed, a necessary condition for a multiagent network to reach the average consensus state is that its graph Laplacian has a zero eigenvalue with geometric and algebraic multiplicity one. Also, the converging speed is proportional to the second smallest eigenvalue of the graph Laplacian. In order to reach the consensus state in a finite time, we propose a nonlinear (and non-Lipschitz) consensus protocol by introducing a nonlinear generalization of the graph Laplacian, the graph $(p+1)$-Laplacian [2, 3, 6], which has important applications in machine learning [16, 22]. In this paper, we show that the graph $(p+1)$-Laplacian can make the system converge to consensus in finite time.

The rest of this paper is organized as follows. The problem formulation and preliminaries are given in section 2. Analytical arguments for the finite-time consensus with and without noise are investigated in section 3. Sufficient conditions for the fixed-time consensus problem with and without noise are given in section 4. Simulation results are presented in section 5, where we show the usefulness of the proposed protocols. In section 6, concluding remarks and perspective on the future research are presented.

Notation. Throughout this paper, $|\cdot|$ denotes the absolute value. If A is a vector or matrix, its transpose is denoted by A^T. For a symmetric matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its largest and smallest eigenvalues, respectively.

2. Problem formulation and preliminaries.

Consider a group of N moving agents. Let $x_i \in \mathbb{R}$ denote the state of agent i. Suppose that the dynamics of each agent is given by

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}, \tag{2.1}$$

where $\mathcal{I} = \{1, 2, \dots, N\}$, u_i is the protocol to be designed. System (2.1) is said to reach the consensus asymptotically if there exists an asymptotically stable equilibrium x^* such that $x_i(t) \to x^*$ $\forall i$ as $t \to \infty$. More specifically, if $x^* = \frac{1}{N}\sum_{i=1}^N x_i(0)$, system (2.1) is said to reach the average consensus.

In the following, we introduce some basic concepts and notation of algebraic graph theory. Let an $N \times N$ nonnegative matrix A be the adjacency matrix of the graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_N\}$ is the vertex set and $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$ is the edge set of the graph. The adjacency matrix A accounts for the influence of agent j on agent i in such a way that $(v_i, v_j) \in \mathcal{E}$ if and only if $a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ $\forall i \in \mathcal{I}$. Let an $N \times N$ diagonal matrix $D = \mathrm{diag}\{d_1, \dots, d_N\}$ be the degree matrix of $\mathcal{G}(A)$, whose diagonal elements are $d_i = \sum_{j=1}^N a_{ij}$ $\forall i \in \mathcal{I}$. Then the (standard or linear) graph Laplacian of the digraph $\mathcal{G}(A)$ is defined as $L_A = D - A$.

The following linear protocol was proposed by Olfati-Saber and Murray in [30] to solve the average consensus problem:

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \quad i \in \mathcal{I}, \tag{2.2}$$

where $\mathcal{N}_i = \{j : (v_i, v_j) \in \mathcal{E}\}$ is the set of neighbors of agent i. Using the graph Laplacian L_A, the multiagent system (2.1) with the linear consensus protocol (2.2) can be rewritten as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -L_A \mathbf{x},$$

where $\mathbf{x}(t) = (x_1, x_2, \ldots, x_N)^T$. The graph Laplacian L_A always has a zero eigenvalue corresponding to the right eigenvector $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$. If the network topology $\mathcal{G}(A)$ is strongly connected and balanced, the linear consensus protocol (2.2) can solve the average consensus problem and the converging rate is proportional to the second smallest eigenvalue of graph Laplacian [30]. A drawback of linear protocol is that the decay rate is exponential and it costs infinite time to reach the consensus. More realistically, it is interesting to investigate when the agreement can be achieved in finite time. So it makes sense to introduce the following definitions.

Definition 2.1. *System* (2.1) *is said to reach the finite-time (or fixed-time) consensus if there exists a time function T_1 depending on (or independent of) the initial values, such that*

$$\lim_{t \to T_1} |x_i(t) - x^*| = 0, \quad \text{and} \quad x_i(t) = x^*, \forall t \geq T_1, \quad \forall i \in \mathcal{I},$$

where $T_1 = \inf\{T : x_i(t) = x^* \ \forall t \geq T\}$ is called the settling time.

3. Finite-time consensus with pinning control.

As explained in the introduction, the linear consensus protocol can only guarantee the asymptotic consensus. In order to reach the finite-time consensus, some continuous as well as discontinuous consensus protocols have been proposed [41, 10, 19]. However, most of them can only solve the average consensus problem, where the final state of each individual is the mean value of the initial states of all agents. More generally, it is desirable to drive the system toward some specific state, rather than the average state. Therefore, some suitable controllers should be added to achieve this goal. Since the agents have usually stronger connections with their neighbors, it is more practical to control part of the agents rather than adding controllers to all of them. An effective way to reduce the number of controlled agents is to use the pinning control technology [44]. In the following we are going to investigate the finite-time consensus problem of multiagent networks—with or without noise perturbation—by combining the finite-time control and the pinning control technologies. In this paper, the pinning strategy is applied on a small fraction $\delta(1/N < \delta < 1)$ of the agents in system (2.1). Suppose that the nodes $i_1, i_2, \ldots, i_l$ are selected, where $l = [\delta N]$ represents the least integer that is greater than or equal to δN. Without loss of generality, rearrange the nodes and let the first l nodes be controlled.

3.1. Finite-time consensus without noise perturbation.

It is well known that the linear protocol with standard graph Laplacian can only solve the asymptotical consensus problem [30]. To tackle the finite-time consensus problem, we propose a nonlinear protocol with the use of the graph $(p+1)$-Laplacian. Suppose we are given a weighted graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E})$. Let $i \in \mathcal{V}$; then the graph p-Laplacian is defined as

$$(\Delta_p \zeta)_i = \sum_{j \in \mathcal{N}_i} a_{ij} \phi_p(\zeta_i - \zeta_j), \tag{3.1}$$

where ζ is a function on $\mathcal{V}$ and $\phi_p : \mathbb{R} \to \mathbb{R}$ is defined for $z \in \mathbb{R}$ as $\phi_p(z) \doteq |z|^{p-1}\text{sign}(z)$. Note that $\phi_2(z) = z$, and it is easy to see that $L_A \zeta = \Delta_2 \zeta$, i.e., the graph p-Laplacian becomes the standard (linear) graph Laplacian when $p = 2$.

In this paper, we consider the following finite-time pinning protocol:

$$\begin{aligned} u_i &= -(\Delta_{p+1}\mathbf{x})_i - \alpha \phi_{p+1}(x_i - x^*), \quad i \in \mathcal{I}_c, \\ u_i &= -(\Delta_{p+1}\mathbf{x})_i, \quad i \in \overline{\mathcal{I}}_c, \end{aligned} \tag{3.2}$$

where $\mathcal{I}_c = \{1, 2, \ldots, l\}$ represents the index set of controlled nodes, $\overline{\mathcal{I}}_c = \{l+1, l+2, \ldots, N\}$, $0 < p < 1, \alpha > 0$, $(\Delta_{p+1}\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} a_{ij} \phi_{p+1}(x_i - x_j)$, and $\phi_{p+1}(z) = |z|^p \text{sign}(z)$.

Remark 3.1. According to the theory of differential equations, there exists a unique solution to the controlled system (2.1) if the protocol is Lipschitz continuous. Thus, the Lipschitz continuity of the control protocol can only guarantee the asymptotic consensus. Notice that the function ϕ_{p+1} is continuous, but it does not satisfy the Lipschitz condition at some points, which is the least requirement for finite-time consensus protocols. Our main motivation for the use of the graph $(p+1)$-Laplacian in (3.2) is that the graph p-Laplacian is a natural nonlinear generalization of the standard graph Laplacian [6]. From the physical standpoint, although linear interactions between pairs of agents are appealing for their simplicity, it is also interesting to study what kind of new features emerge when similar—but nonlinear—interactions are considered. These may be important in collective behavior of flocking, where we do not know the specific form of interaction. We will show that the non-Lipschitz properties of the graph $(p+1)$-Laplacian are crucial to reach consensus in finite time.

In the following, we shall derive sufficient conditions for the finite-time consensus. Let $e_i(t) = x_i(t) - x^*$ be the state error of agent i and define $\mathbf{e}(t) = (e_1, \ldots, e_N)^T$. From (2.1) and (3.2) we have the following error systems:

$$\begin{aligned} \dot{e}_i &= -(\Delta_{p+1}\mathbf{e})_i - \alpha \phi_{p+1}(e_i), \quad i \in \mathcal{I}_c, \\ \dot{e}_i &= -(\Delta_{p+1}\mathbf{e})_i, \quad i \in \overline{\mathcal{I}}_c. \end{aligned} \tag{3.3}$$

Consider the following Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^2(t). \tag{3.4}$$

Differentiating the function $V(t)$ along the trajectory of (3.3) gives

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{N} e_i \sum_{j=1}^{N} a_{ij} \phi_{p+1}(e_j - e_i) - \alpha \sum_{i=1}^{l} e_i \phi_{p+1}(e_i) \\ &\doteq V_1 + V_2. \end{aligned} \tag{3.5}$$

From Lemma A.2, we have

$$\begin{aligned} V_1 &= -\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(e_j - e_i)\phi_{p+1}(e_j - e_i) \\ &= -\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}|e_j - e_i|^{p+1}. \end{aligned} \tag{3.6}$$

Let $\alpha_i = \alpha$ when $i \in \mathcal{I}_c$ and $\alpha_i = 0$ while $i \in \overline{\mathcal{I}}_c$, thus

$$V_2 = -\sum_{i=1}^{N} \alpha_i |e_i|^{p+1}. \tag{3.7}$$

Therefore, from Lemma A.3, we have

$$\begin{aligned} V_1 + V_2 &= -\frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N} a_{ij}|e_j - e_i|^{p+1} + 2\alpha_i|e_i|^{p+1}\right] \\ &= -\frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N}(a_{ij}^{\frac{1}{p+1}}|e_j - e_i|)^{p+1} + ((2\alpha_i)^{\frac{1}{p+1}}|e_i|)^{p+1}\right] \\ &\le -\frac{1}{2}\left\{\sum_{i=1}^{N}\left[\sum_{j=1}^{N} a_{ij}^{\frac{2}{p+1}}|e_j - e_i|^2 + (2\alpha_i)^{\frac{2}{p+1}}|e_i|^2\right]\right\}^{\frac{p+1}{2}}. \end{aligned} \tag{3.8}$$

Define a new matrix $B = (b_{ij})$ and $b_{ij} = a_{ij}^{\frac{2}{p+1}}$, then the matrix B can be regarded as the adjacency matrix of the graph $\mathcal{G}(B)$. Let L_B denote the Laplacian matrix of the graph $\mathcal{G}(B)$. If the graph $\mathcal{G}(A)$ is strongly connected and undirected, then the graph $\mathcal{G}(B)$ is strongly connected and undirected as well. Then, according to Lemma A.1,

$$\sum_{i,j=1}^{N} a_{ij}^{\frac{2}{p+1}}|e_j - e_i|^2 = 2\mathbf{e}^T L_B \mathbf{e}. \tag{3.9}$$

Let $D_\alpha = \mathrm{diag}(\underbrace{(2\alpha)^{\frac{2}{p+1}}, \ldots, (2\alpha)^{\frac{2}{p+1}}}_{l}, \underbrace{0, \ldots, 0}_{N-l})$; then we have

$$\sum_{j=1}^{N}(2\alpha_i)^{\frac{2}{p+1}}|e_i|^2 = \mathbf{e}^T D_\alpha \mathbf{e}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$V_1 + V_2 \le -\frac{1}{2}\left[\mathbf{e}^T(2L_B + D_\alpha)\mathbf{e}\right]^{\frac{p+1}{2}}. \tag{3.11}$$

If the graph $\mathcal{G}(A)$ is strongly connected, then the graph $\mathcal{G}(B)$ is also strongly connected and L_B is irreducible. According to Definition A.4, the matrix $2L_B + D_\alpha$ is irreducibly diagonally dominant if $l \ge 1$. If the graph $\mathcal{G}(A)$ is undirected, then the matrix

$2L_B + D_\alpha$ is symmetric with strictly positive entries in its diagonal. Thus, from Lemma A.5 we obtain that all eigenvalues of $2L_B+D_\alpha$ are strictly positive. Therefore,

$$V_1 + V_2 \le -\frac{1}{2}(\lambda_1 \mathbf{e}^T\mathbf{e})^{\frac{p+1}{2}} = -\frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}, \tag{3.12}$$

where $\lambda_1 \doteq \lambda_{\min}(2L_B + D_\alpha)$.

Combining (3.5) and (3.12), one has

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le -\frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}. \tag{3.13}$$

By Lemma A.6, $V(t)$ converges to zero in a finite time. And the settling time is estimated by

$$T_1 \le \frac{2V^{\frac{1-p}{2}}(0)}{\lambda_1^{\frac{p+1}{2}}2^{\frac{p-1}{2}}(1-p)}, \tag{3.14}$$

where $V(0) = \frac{1}{2}\sum_{i=1}^N e_i^2(0)$, $e_i(0) = x_i(0) - x^*$ is the initial value of $e_i(t)$.

From the above analysis, we have the following theorem.

THEOREM 3.2. *Consider the multiagent network* (1) *with topology $\mathcal{G}(A)$. If the graph $\mathcal{G}(A)$ is undirected and strongly connected, then under the pinning protocol* (3.2)*, the system* (2.1) *can reach the finite-time consensus and the settling time satisfies*

$$T_1 \le \frac{2V^{\frac{1-p}{2}}(0)}{\lambda_1^{\frac{p+1}{2}}2^{\frac{p-1}{2}}(1-p)}.$$

Remark 3.3. Theorem 3.2 shows that the non-Lipschitz form of the graph $(p+1)$-Laplacian is responsible for the finite-time consensus. The inequality (3.14) gives an upper bound estimation of the settling time. We expect a dependence of the settling time for the finite-time consensus on the parameters p, λ_1 and the initial values of the multiagent system. The upper bound in (3.14) indicates that a steeper behavior of the function ϕ_{p+1} close to the origin allows the system to converge faster to the final state, for a given initial error. Although the estimate in (3.14) does not explicitly indicate the effect of the parameters l and α on the settling time, the eigenvalue λ_1 does depend on the number of pinned nodes and the control parameter α. Thus, the settling time also depends on the density of pinned nodes and α as will be further illustrated in section 5.

3.2. Finite-time stochastic consensus. In noisy environments, the agent cannot measure its neighbor's state exactly. To investigate the impact of environmental noise on the finite-time consensus, we consider the following stochastic consensus protocol:

$$\begin{aligned} u_i &= -(L_A\mathbf{x})_i - (\Delta_{p+1}\mathbf{x})_i - k(x_i - x_i^*) - \alpha\phi_{p+1}(x_i - x^*) \\ &\quad + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(x_j - x_i)\xi_i(t), \quad i \in \mathcal{I}_c, \\ u_i &= -(L_A\mathbf{x})_i - (\Delta_{p+1}\mathbf{x})_i + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(x_j - x_i)\xi_i(t), \quad i \in \overline{\mathcal{I}}_c, \end{aligned} \tag{3.15}$$

where $(L_A\mathbf{x})_i = (\Delta_2\mathbf{x})_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j)$ is the ith component of $L_A\mathbf{x}$, $0 < p < 1, k > 0, \gamma > 0$, $\xi_i(t)$ are independent white noises with statistical properties

$\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t), \xi_j(t') \rangle = \delta_{ij}\delta(t-t')(i,j=1,2,\ldots,N)$, and δ is the Dirac delta function. The parameter $\sigma_{ij} \geq 0$ represents the strength of noise. The matrix $\Xi = [\sigma_{ij}] \in \mathbb{R}^{N\times N}$ is called the noise intensity matrix. To highlight the presence of noise, it is natural to assume that $\sigma_{ij} > 0$ if and only if $a_{ij} > 0$.

Definition 3.4. *Under the control protocol* (3.15)*, system* (2.1) *is said to reach the finite-time (or fixed-time) consensus with probability one if there exists a time function T_1 depending on (or independent of) the initial values, such that*

$$P\{|x_i(t) - x^*| = 0\} = 1, \quad \forall t \geq T_1 \quad \forall i \in \mathcal{I},$$

where $T_1 = \inf\{T : x_i(t) = x^* \ \forall t \geq T\}$ is called the stochastic settling time.

Remark 3.5. To investigate the influence of environmental noise on the collective behavior, additive or multiplicative noises have been added to the deterministic model [21, 35]. Here, we introduce a multiplicative noise to protocol (3.2). The multiplicative structure of the noise term in (3.15) implies that the measurement error increases when the group alignment is low. This assumption on the noise term is satisfied in most biological systems, including swarming locusts [34] and flocking birds [36, 13]. Therefore the noise term in (3.15) may hinder the emergence of consensus. Unlike the consensus protocol in (3.2), the stochastic consensus protocol in (3.15) contains the standard graph Laplacian, the graph $(p+1)$-Laplacian, and the linear feedback control. This is because the linear graph Laplacian and the linear feedback control facilitate suppressing the negative effect of noise and guarantee that the system moves in a neighborhood of the desired state x^*.

In the following, we will derive sufficient conditions that guarantee the multiagent system will reach the finite-time stochastic consensus. From (2.1) and (3.15) we have the following error system:

$$\begin{aligned} \dot{e}_i &= -(L_A\mathbf{e})_i - (\Delta_{p+1}\mathbf{e})_i - ke_i - \alpha\text{sig}^p(e_i) + \gamma \sum_{j\in\mathcal{N}_i} \sigma_{ij}(e_j - e_i)\xi_i(t), \ i \in \mathcal{I}_c, \\ \dot{e}_i &= -(L_A\mathbf{e})_i - (\Delta_{p+1}\mathbf{e})_i + \gamma \sum_{j\in\mathcal{N}_i} \sigma_{ij}(e_j - e_i)\xi_i(t), \quad i \in \overline{\mathcal{I}}_c. \end{aligned} \tag{3.16}$$

Let $\sigma_i = \sum_{j\in\mathcal{N}_i} \sigma_{ij}$, $L_\Xi = \text{diag}(\sigma_1, \ldots, \sigma_N) - \Xi$, which can be regarded as the Laplacian matrix of graph $\mathcal{G}(\Xi)$. Then,

$$\sum_{j\in\mathcal{N}_i} \sigma_{ij}(e_j - e_i)\xi_i(t) = -(L_\Xi)_i\mathbf{e}\xi_i(t),$$

where $(L_\Xi)_i$ denotes the ith row of matrix L_Ξ.

With the same Lyapunov function as (3.4), the diffusion operator $\mathcal{L}$ defined in (A.3) on to the function $V(t)$ along the error system (3.16) gives

$$\begin{aligned} \mathcal{L}V(t) = &-\sum_{i=1}^N e_i(L_A\mathbf{e})_i + \sum_{i=1}^N e_i \sum_{j=1}^N a_{ij}\phi_{p+1}(e_j - e_i) \\ &- k\sum_{i=1}^l e_i^2 - \alpha \sum_{i=1}^l e_i\phi_{p+1}(e_i) + \frac{\gamma^2}{2}\sum_{i=1}^N \left[(L_\Xi)_i\mathbf{e}\right]^2 \\ \doteq\ &\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 + \tilde{V}_4 + \tilde{V}_5. \end{aligned} \tag{3.17}$$

It is easy to see that

$$\tilde{V}_1 = -\mathbf{e}^T L_A \mathbf{e}. \tag{3.18}$$

Let $D_k = \operatorname{diag}(\underbrace{k, \ldots, k}_{l}, \underbrace{0, \ldots, 0}_{N-l})$. Then,

$$\tilde{V}_1 + \tilde{V}_3 = -\mathbf{e}^T \big(L_A + D_k)\mathbf{e}. \tag{3.19}$$

Note that the matrix $L_A + D_k$ is irreducibly diagonally dominant when the graph $\mathcal{G}(A)$ is strongly connected and $l \geq 1$. If graph $\mathcal{G}(A)$ is undirected, then the matrix $L_A + D_k$ is symmetric with strictly positive entries in its diagonal. From Lemma A.5, all the eigenvalues of $L_A + D_k$ are strictly positive. Thus,

$$\tilde{V}_1 + \tilde{V}_3 \leq -\lambda_{\min}(L_A + D_k)\mathbf{e}^T \mathbf{e}. \tag{3.20}$$

Similar to (3.12), we have

$$\tilde{V}_2 + \tilde{V}_4 \leq -\frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}. \tag{3.21}$$

For the last term in the inequality (3.17), we have

$$\begin{aligned} \tilde{V}_5 &= \frac{\gamma^2}{2}\sum_{i=1}^{N}[(L_\Xi)_i \mathbf{e}]^2 \\ &= \frac{\gamma^2}{2}\mathbf{e}^T L_\Xi^T L_\Xi \mathbf{e} \\ &\leq \frac{\gamma^2}{2}\lambda_{\max}^2(L_\Xi)\mathbf{e}^T\mathbf{e}. \end{aligned} \tag{3.22}$$

Combining (3.20)–(3.22), one has

$$\mathcal{L}V \leq -2\lambda_{\min}(L_A + D_k)V(t) + \gamma^2\lambda_{\max}^2(L_\Xi)V(t) - \frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}. \tag{3.23}$$

If $\gamma^2\lambda_{\max}^2(L_\Xi) \leq 2\lambda_{\min}(L_A + D_k)$, then we have

$$\mathcal{L}V \leq -\frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}. \tag{3.24}$$

By Lemma A.10, $V(t)$ converges to zero in a finite time with probability one, i.e., the multiagent system (2.1) reaches the finite-time stochastic consensus. And the settling time is estimated by

$$\mathbb{E}(T_1) \leq \frac{2V^{\frac{1-p}{2}}(0)}{\lambda_1^{\frac{p+1}{2}} 2^{\frac{p-1}{2}}(1-p)}. \tag{3.25}$$

Thus, from above analysis, we get the following result on the finite-time stochastic consensus.

THEOREM 3.6. *Consider the multiagent system* (2.1) *with topology* $\mathcal{G}(A)$. *Suppose that the graph* $\mathcal{G}(A)$ *is undirected and strongly connected. Suppose that* $\Xi = \Xi^T$ *and* $\gamma^2\lambda_{\max}^2(L_\Xi) \leq 2\lambda_{\min}(L_A + D_k)$. *Then, under the pinning protocol* (3.15), *system* (2.1) *can reach the finite-time stochastic consensus. And the stochastic settling time satisfies the inequality* (3.25).

Remark 3.7. The analytical results in Theorem 3.6 show that the multiagent systems can reach the finite-time stochastic consensus if the strength of noise is small enough. Moreover, it can be seen from (3.14) and (3.25) that the settling time of the finite-time consensus depends on the initial states of the system. If the initial states are not given, we cannot get the estimation of the settling time. And the settling time may become very large when the initial state grows. In order to make the settling time independent of the initial values, we propose a fixed-time consensus protocol in next section.

4. Fixed-time consensus with pinning control. The converging time of finite-time consensus depends on the initial states of agents. By using the fixed-time control technology, we can obtain the estimation of the converging time which is independent of the initial states of agents. This section investigates the fixed-time consensus problem with or without noise.

4.1. Fixed-time consensus without noise. In order to realize the fixed-time consensus, we consider the following pinning protocol:

$$\begin{aligned} &u_i = -(\Delta_{p+1}\mathbf{x})_i - (\Delta_{q+1}\mathbf{x})_i - \alpha\phi_{p+1}(x_i - x^*) - \beta\phi_{q+1}(x_i - x^*), \quad i \in \mathcal{I}_c, \\ &u_i = -(\Delta_{p+1}\mathbf{x})_i - (\Delta_{q+1}\mathbf{x})_i, \quad i \in \overline{\mathcal{I}}_c, \end{aligned} \tag{4.1}$$

where $0 < p < 1, q > 1$ and α, β are positive constants.

Remark 4.1. Unlike the finite-time protocol in (3.2), the fixed-time protocol in (4.1) contains both control inputs $(\Delta^{\alpha}_{p+1})_i - (\Delta_{p+1}\mathbf{x})_i - \alpha\phi_{p+1}(x_i - x^*)$ and $(\Delta^{\beta}_{q+1})_i \doteq -(\Delta_{q+1}\mathbf{x})_i - \beta\phi_{q+1}(x_i - x^*)$. It is easy to see that $|x|^p < |x| < |x|^q$ when $|x| > 1$ and $0 < p < 1 < q$. Thus, the strength of control input $(\Delta^{\alpha}_{p+1})_i$ is weaker (stronger) than the control input $(\Delta^{\beta}_{q+1})_i$ when $\|\mathbf{x}\| > 1$ ($\|\mathbf{x}\| < 1$). Thus, the settling time for the finite-time protocol may become very large when the initial state grows. In order to make the settling time independent of the initial values, we introduce both $(\Delta^{\alpha}_{p+1})_i$ and $(\Delta^{\beta}_{q+1})_i$ to the fixed-time protocol. Based on the fixed-time Lyapunov theorem in [31], we will show that the settling time of system (2.1) with protocol (4.1) is independent of the initial states.

From (2.1) and (4.1) we have the following error system:

$$\begin{aligned} &\dot{e}_i = -(\Delta_{p+1}\mathbf{e})_i - (\Delta_{q+1}\mathbf{e})_i - \alpha\phi_{p+1}(e_i) - \beta\phi_{q+1}(e_i), \quad i \in \mathcal{I}_c, \\ &\dot{e}_i = -(\Delta_{p+1}\mathbf{e})_i - (\Delta_{q+1}\mathbf{e})_i \quad i \in \overline{\mathcal{I}}_c. \end{aligned} \tag{4.2}$$

Define the Lyapunov function as in (3.4); differentiating it along (4.2), one has

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{N} e_i \sum_{j=1}^{N} a_{ij}\phi_{p+1}(e_j - e_i) + \sum_{i=1}^{N} e_i \sum_{j=1}^{N} a_{ij}\phi_{q+1}(e_j - e_i) \\ &\quad - \alpha \sum_{i=1}^{l} e_i\phi_{p+1}(e_i) - \beta \sum_{i=1}^{l} e_i\phi_{q+1}(e_i) \\ &\doteq \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \hat{V}_4. \end{aligned} \tag{4.3}$$

Similar to the analysis in the above section, if the graph $\mathcal{G}(A)$ is strongly connected and undirected we have

$$\hat{V}_1 + \hat{V}_3 \leq -\frac{1}{2}\lambda_1^{\frac{p+1}{2}}(2V)^{\frac{p+1}{2}}. \tag{4.4}$$

Let $\beta_i = \beta$ when $i \in \mathcal{I}_c$ and $\beta_i = 0$ while $i \in \overline{\mathcal{I}}_c$, thus

$$
\begin{aligned}
\hat{V}_2 + \hat{V}_4 &= \sum_{i=1}^N e_i \sum_{j=1}^N a_{ij}\phi_{q+1}(e_j - e_i) - \sum_{i=1}^N \beta_i e_i \phi_{q+1}(e_i) \\
&= -\frac{1}{2}\sum_{i=1}^N \left[\sum_{j=1}^N a_{ij}|e_j - e_i|^{q+1} + 2\beta_i |e_i|^{q+1}\right] \\
&= -\frac{1}{2}\sum_{i=1}^N \left[\sum_{j=1}^N (a_{ij}^{\frac{1}{q+1}}|e_j - e_i|)^{q+1} + ((2\beta_i)^{\frac{1}{q+1}}|e_i|)^{q+1}\right].
\end{aligned}
\tag{4.5}
$$

From Lemma A.3, we obtain

$$
\hat{V}_2 + \hat{V}_4 \le -\frac{1}{2}N^{\frac{1-q}{2}}\left(\sum_{i,j=1}^N a_{ij}^{\frac{2}{q+1}}|e_j - e_i|^2 + \sum_{i=1}^N (2\beta_i)^{\frac{2}{q+1}}|e_i|^2\right)^{\frac{q+1}{2}}. \tag{4.6}
$$

Let $c_{ij} = a_{ij}^{\frac{2}{q+1}}$; then the matrix $C = (c_{ij})$ can be regarded as the adjacency matrix of a graph $\mathcal{G}(C)$. Let L_C denote the Laplacian matrix of the graph $\mathcal{G}(C)$. Then, according to Lemma A.1,

$$
\sum_{i=1}^N \sum_{j=1}^N a_{ij}^{\frac{2}{q+1}}|e_j - e_i|^2 = 2\mathbf{e}^T L_C \mathbf{e}. \tag{4.7}
$$

Define $D_\beta = \text{diag}\big(\underbrace{(2\beta)^{\frac{2}{q+1}}, \dots, (2\beta)^{\frac{2}{q+1}}}_{l}, \underbrace{0, \dots, 0}_{N-l}\big)$; then we have

$$
\sum_{j=1}^N (2\beta_i)^{\frac{2}{q+1}}|e_i|^2 = \mathbf{e}^T D_\beta \mathbf{e}. \tag{4.8}
$$

From (4.6) and (4.8), we have

$$
\hat{V}_2 + \hat{V}_4 \le -\frac{1}{2}N^{\frac{1-q}{2}}\left[\mathbf{e}^T(2L_C + D_\beta)\mathbf{e}\right]^{\frac{q+1}{2}}. \tag{4.9}
$$

If the graph $\mathcal{G}(A)$ is undirected and strongly connected, then the graph $\mathcal{G}(C)$ is undirected and strongly connected as well. Then, from Lemma A.5, the matrix $2L_C + D_\beta$ is positive definite. Hence,

$$
\hat{V}_2 + \hat{V}_4 \le -\frac{1}{2}N^{\frac{1-q}{2}}(\mu_1 e^T e)^{\frac{q+1}{2}} = -\frac{1}{2}N^{\frac{1-q}{2}}\mu_1^{\frac{q+1}{2}}(2V)^{\frac{q+1}{2}}, \tag{4.10}
$$

where $\mu_1 = \lambda_{\min}(2L_C + D_\beta)$.

Combining (4.4) and (4.10), we have

$$
\frac{\mathrm{d}V}{dt} \le -2^{\frac{p-1}{2}}\lambda_1^{\frac{p+1}{2}}V^{\frac{p+1}{2}} - 2^{\frac{q-1}{2}}N^{\frac{1-q}{2}}\mu_1^{\frac{q+1}{2}}V^{\frac{q+1}{2}}. \tag{4.11}
$$

Therefore, from Lemma A.7, we have

$$
V(t) \equiv 0 \qquad \forall t \ge \hat{T},
$$

and the settling time is estimated by

$$\hat{T} \le 2\left[\frac{1}{\kappa(1-p)} + \frac{1}{\varrho(q-1)}\right], \tag{4.12}$$

where $\kappa = 2^{\frac{p-1}{2}}\lambda_1^{\frac{p+1}{2}}$ and $\varrho = 2^{\frac{q-1}{2}}N^{\frac{1-q}{2}}\mu_1^{\frac{q+1}{2}}$. Therefore, the above performed argument leads to the following theorem.

THEOREM 4.2. *Consider the multiagent system* (2.1) *with topology* $\mathcal{G}(A)$. *Suppose that the graph* $\mathcal{G}(A)$ *is undirected and strongly connected. Then, under the pinning protocol* (4.1), *the multiagent system* (2.1) *can reach the fixed-time consensus. And the settling time satisfies the inequality* (4.12).

Remark 4.3. The estimation of the settling time for the fixed-time consensus in (4.12) depends on the control parameters p, q and the eigenvalues λ_1, μ_1. As shown in Remark 3.3, both λ_1 and μ_1 depend on the number of pinned nodes. In section 5, we will numerically show that the settling time of the fixed-time consensus decreases with increasing pinned nodes.

4.2. Fixed-time stochastic consensus. To investigate the impact of environmental noise on the fixed-time stochastic consensus, we consider the following stochastic consensus protocol:

$$\begin{aligned} u_i &= -(L_A\mathbf{x})_i - (\Delta_{p+1}\mathbf{x})_i - (\Delta_{q+1}\mathbf{x})_i + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(x_j - x_i)\xi_i(t) \\ &\quad - k(x_i - x^*) - \alpha\phi_{p+1}(x_i - x^*) - \beta\phi_{q+1}(x_i - x^*), \quad i \in \mathcal{I}_c, \\ u_i &= -(L_A\mathbf{x})_i - (\Delta_{p+1}\mathbf{x})_i - (\Delta_{q+1}\mathbf{x})_i + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(x_j - x_i)\xi_i(t), \quad i \in \overline{\mathcal{I}}_c. \end{aligned} \tag{4.13}$$

where $0 < p < 1, q > 1$, $\gamma > 0$. Therefore, we have the following error systems:

$$\begin{aligned} \dot{e}_i &= -(L_A\mathbf{e})_i - (\Delta_{p+1}\mathbf{e})_i - (\Delta_{q+1}\mathbf{e})_i + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(e_j - e_i)\xi_i(t) \\ &\quad - ke_i - \alpha\phi_{p+1}(e_i) - \beta\phi_{q+1}(e_i), \quad i \in \mathcal{I}_c, \\ \dot{e}_i &= -(L_A\mathbf{e})_i - (\Delta_{p+1}\mathbf{e})_i - (\Delta_{q+1}\mathbf{e})_i + \gamma\sum_{j\in\mathcal{N}_i}\sigma_{ij}(e_j - e_i)\xi_i(t), \quad i \in \overline{\mathcal{I}}_c. \end{aligned} \tag{4.14}$$

With the same Lyapunov function as in (3.4), the diffusion operator $\mathcal{L}$ defined in (A.3) on to the function $V(t)$ along the error system (4.14) gives

$$\begin{aligned} \mathcal{L}V(t) = &-\sum_{i=1}^{N} e_i(L_A\mathbf{e})_i + \sum_{i=1}^{N} e_i\sum_{j=1}^{N} a_{ij}\phi_{p+1}(e_j - e_i) \\ &+ \frac{\gamma^2}{2}\sum_{i=1}^{N}\left[(L_\Xi)_i e\right]^2 - k\sum_{i=1}^{l} e_i^2 - \alpha\sum_{i=1}^{l} e_i\phi_{p+1}(e_i) \\ &+ \sum_{i=1}^{N} e_i\sum_{j=1}^{N} a_{ij}\phi_{q+1}(e_j - e_i) - \beta\sum_{i=1}^{l} e_i\phi_{q+1}(e_i). \end{aligned} \tag{4.15}$$

Combining (3.20), (3.21), (3.22), and (4.10), we have

$$\mathcal{L}V \le -2\lambda_{\min}(L_A + D_k)V + \gamma^2\lambda_{\max}^2(L_\Xi)V - 2^{\frac{p-1}{2}}\lambda_1^{\frac{p+1}{2}}V^{\frac{p+1}{2}} - 2^{\frac{q-1}{2}}N^{\frac{1-q}{2}}\mu_1^{\frac{q+1}{2}}V^{\frac{q+1}{2}}. \tag{4.16}$$

If $\gamma^2 \lambda_{\max}^2(L_\Xi) \leq 2\lambda_{\min}(L_A + D_k)$, then we have

$$\mathcal{L}V \leq -2^{\frac{p-1}{2}} \lambda_1^{\frac{p+1}{2}} V^{\frac{p+1}{2}} - 2^{\frac{q-1}{2}} N^{\frac{1-q}{2}} \mu_1^{\frac{q+1}{2}} V^{\frac{q+1}{2}}. \tag{4.17}$$

By Lemma A.11, $V(t)$ converges to zero in the fixed time with probability one, i.e., the multiagent system (2.1) reaches the fixed-time stochastic consensus, and the settling time is estimated by

$$\mathbb{E}(T_1) \leq 2\left[\frac{1}{\kappa(1-p)} + \frac{1}{\varrho(q-1)}\right]. \tag{4.18}$$

Thus, from the above analysis, we can get the following result on the fixed-time stochastic consensus.

THEOREM 4.4. *Consider the multiagent system* (2.1) *with topology* $\mathcal{G}(A)$. *Suppose that the graph* $\mathcal{G}(A)$ *is undirected and strongly connected. Suppose that* $\Xi = \Xi^T$ *and* $\gamma^2 \lambda_{\max}^2(L_\Xi) \leq 2\lambda_{\min}(L_A + D_k)$. *Then, under the pinning protocol* (4.13), *system* (2.1) *can reach the fixed-time stochastic consensus and the stochastic settling time can be estimated by the inequality* (4.18).

5. Simulation results. In this section, some numerical simulations will be given to verify the theoretical results established in previous sections. In all the simulations, we take the undirected and strongly connected scale-free networks as the network topologies with $N = 200$ and mean degree $\langle d \rangle = 10$. All the scale-free networks are generated by using the algorithm in [4]. The strength of noise is $\sigma_{ij} = \sigma_0 = 1.5$ if and only if $a_{ij} > 0$. The desirable state x^* of the system is 0.5. The initial states of multiagent networks are uniformly taken from the interval $[-1, 1]$.

First, we randomly select N_D nodes as pinned nodes. Let $n_D = N_D/N$ be the density of pinned nodes. All edges have weight 0.1. Taking $\alpha = \beta = 0.5, k = 0.1, p = 0.9, q = 1.1, n_D = 0.2, \sigma_0 = 1.0$, Figures 1 and 2 display the evolutions of the states $x(t)$ and the consensus indicator $E(t) = \frac{1}{N}\sum_{i=1}^{N} |x_i(t) - x^*|$ for the finite-time and fixed-time consensus, respectively. It is shown that the finite-time and fixed-time consensus are achieved with the proposed pinning protocols. According to (3.14) and (4.12), the theoretical upper bounds of the settling time for the finite-time and fixed-time consensus are 150.57 and 296.93, respectively. The settling times of Figure 1(a) and (b) are 36.73 and 23.59, respectively. The settling times of Figure 2(a) and (b) are 18.13 and 14.18, respectively. Clearly, the numerical settling times of Figures 1 and 2 are far less than the theoretical upper bounds. Compared with the finite-time protocols, the fixed-time protocols contain extra control terms. Thus, as shown in Figures 1 and 2, systems with fixed-time protocols converge faster than systems with finite-time protocols.

The theoretical results in the above sections show that the upper bound of the settling time for the finite-time and fixed-time consensus is inversely proportional to the values of λ_1 and μ_1. In Figure 3, we plot λ_1 and μ_1 as a function of n_D, respectively. It can be seen that both λ_1 and μ_1 are increasing functions of n_D, which means that the settling time may decrease by increasing the number of controlled nodes. This is confirmed by Figure 4, where we plot the settling time estimated from numerical simulations as a function of the density of pinned nodes. We also observe the decrease of the settling time with increasing density of pinned nodes, which implies that controlling more nodes can reduce the settling time significantly. Figure 4 shows that, for given n_D, the fixed-time consensus protocol takes less time than the finite-time consensus protocol.

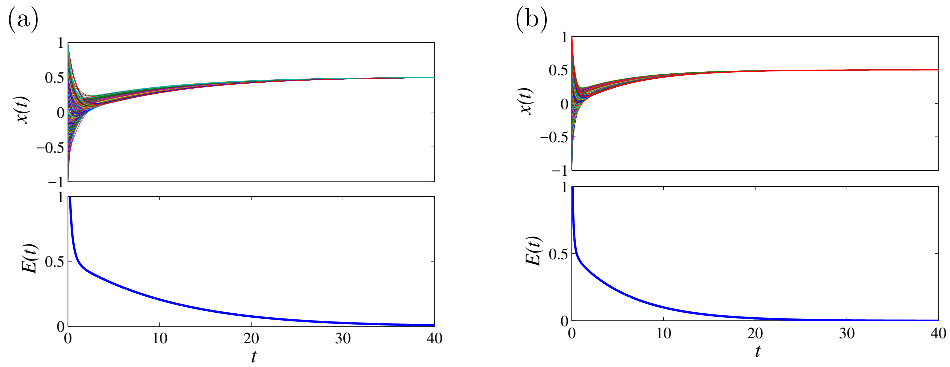

FIG. 1. (a) *Finite-time consensus for multiagent system* (2.1) *with protocol* (3.2). (b) *Fixed-time consensus for multiagent system* (2.1) *with protocol* (4.1). *The parameter values used are* $N = 200, \alpha = \beta = 0.5, p = 0.9, q = 1.1$. *All the edges have weight* 0.1 *and the density of pinned nodes is* $n_D = 0.2$.

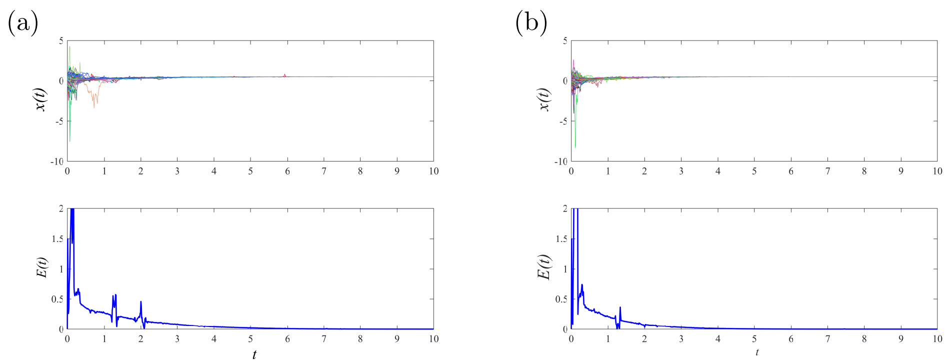

FIG. 2. (a) *Finite-time stochastic consensus for multiagent system* (2.1) *with protocol* (3.15). (b) *Fixed-time stochastic consensus for multiagent system* (2.1) *with protocol* (4.13). *Here the parameters are* $N = 200, \alpha = \beta = 0.5, k = 0.1, p = 0.9, q = 1.1$, *and* $\sigma_0 = 1.0$. *All the edges have weight* 0.1 *and the density of pinned nodes is* $n_D = 0.2$.

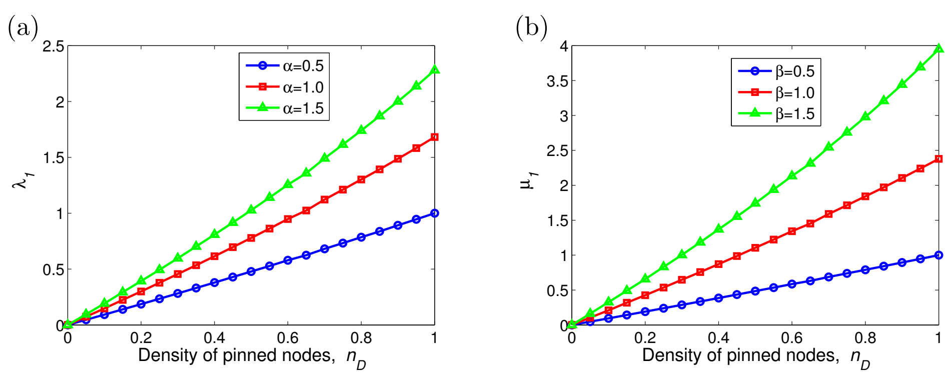

FIG. 3. *The impact of the density of pinned nodes on* λ_1 *and* μ_1. (a) λ_1 *versus* n_D *for scale-free networks with* $N = 200, p = 0.5$, *and* $\alpha = 0.5, 1.0, 1.5$; (b) μ_1 *versus* n_D *for scale-free networks with* $N = 200, q = 1.5$, *and* $\beta = 0.5, 1.0, 1.5$. *All the edges have weight* 0.1 *and the results are averaged over* 100 *realizations.*

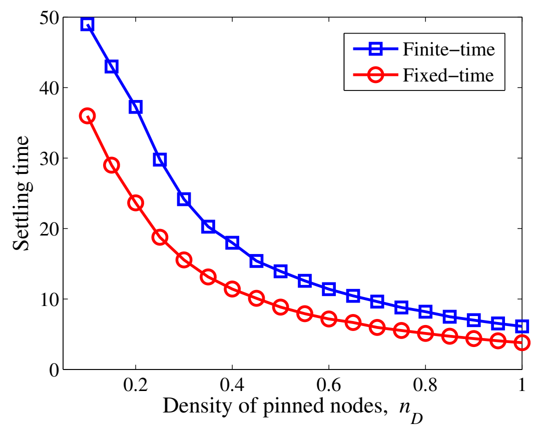

FIG. 4. *Settling time as a function of the density of pinned nodes for multiagent systems with finite-time (squares) and fixed-time (circles) consensus protocols. The other parameter values are the same as those in Figure* 1 *and the results are averaged over* 100 *realizations.*

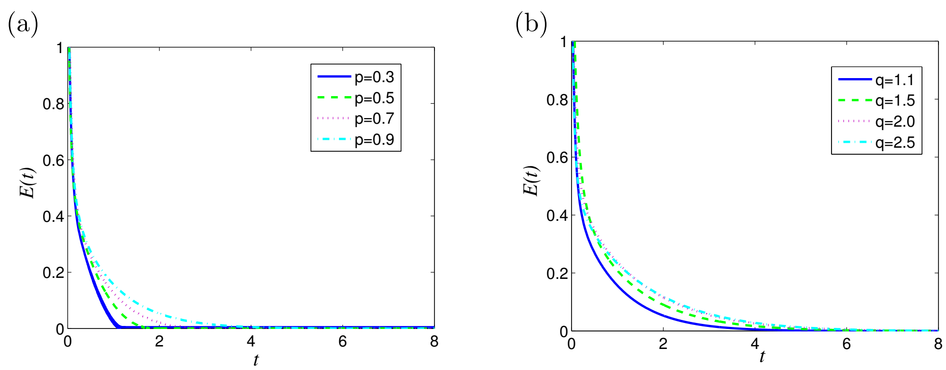

FIG. 5. *The impact of parameters p, q on the convergence rate of fixed-time consensus.* (a) *The variations of consensus indicator $E(t)$ of multiagent networks with $\alpha = \beta = 0.5, q = 1.1$, and* $p = 0.3, 0.5, 0.7, 0.9$. (b) *The same as* (a) *but for networks with $p = 0.9$ and $q = 1.1, 1.5, 2.0, 2.5$. All the edges have weight* 0.1 *and the density of pinned nodes is $n_D = 0.2$.*

To further verify the influence of the control parameters p, q on the settling time, we simulate the multiagent systems with fixed-time protocol (3.15) by taking different values of p, q. Figure 5(a) displays the evolutions of the consensus indicator $E(t)$ with $p = 0.3, 0.5, 0.7, 0.9$ and $q = 1.1$. It is shown that the smaller the parameter p, the faster the fixed-time consensus can be reached. Taking $p = 0.9$ and $q = 1.1, 1.5, 2.0, 2.5$, the simulation results in Figure 5(b) imply that, for given parameter p, the settling times for networks with different values of q are very similar. Thus, the convergence rate for the fixed-time consensus depends weakly on the parameter q.

Finally, we investigate the effect of parameters α, β on the settling time of fixed-time consensus. For given density of pinned nodes, Figure 3 shows that λ_1 is an increasing function of the parameter α, and μ_1 is an increasing function of the parameter β. The simulation result in Figure 6(a) implies that the larger the α, the faster the consensus can be realized. Figure 6(b) shows that the settling time is weakly influenced by the parameter β. In order to make the settling time independent of

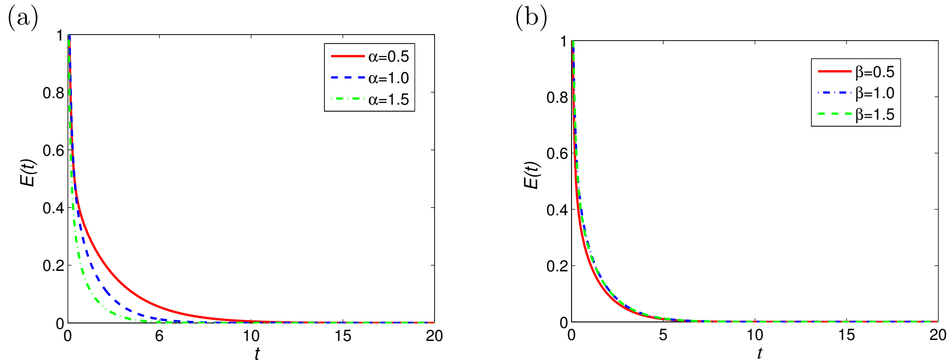

FIG. 6. *The impact of parameters α, β on the convergence rate of fixed-time consensus.* (a) *The variations of consensus indicator $E(t)$ of multiagent networks with $p = 0.9, q = 1.1, \beta = 0.5$, and $\alpha = 0.5, 1.0, 1.5$.* (b) *The same with* (a) *but for networks with $p = 0.9, q = 1.1, \alpha = 0.5$, and $\beta = 0.5, 1.0, 1.5$. All the edges have weight* 0.1 *and the density of pinned nodes is $n_D = 0.2$.*

the initial value, we proposed the fixed-time protocol (4.1) by adding the additional control term on the finite-time protocol. Note that $|x|^q < |x|^p$ when $|x| < 1$ and $0 < p < 1 < q$. Thus, compared with the control input $\phi_{p+1}(x_i - x^*)$, the control input $\phi_{q+1}(x_i - x^*)$ becomes very small when the error approaches zero. Therefore, the simulation results in Figures 4, 5, and 6 show that the settling time depends mainly on the parameters of the finite-time control term.

6. Conclusion. In this paper, we have investigated the finite-time and fixed-time consensus problem of multiagent systems with pinning control and noise perturbation. Both the theoretical and numerical results show that if the network topology is undirected and strongly connected, the multiagent systems can reach the finite-time and fixed-time consensus. The results in this paper complement and extend existing results. Also, we have showed that the graph $(p + 1)$-Laplacian plays a crucial role when considering consensus in network systems. Indeed, unlike linear protocols with the standard graph Laplacian for the asymptotic consensus, protocols with the graph $(p+1)$-Laplacian are able to solve the finite-time and fixed-time consensus problems. Compared with the consensus protocols without pinning control, the proposed finite-time and fixed-time pinning protocols are practical and show some advantages from a physical viewpoint of energy consumption, since it only needs to control a small fraction of agents. We found that, for given density of pinned nodes, the settling time of the fixed-time consensus is less than that of the finite-time consensus.

How to minimize the energy and time cost is an important issue in network control theory. In our previous work [33], we found that, for networks without pinning control, there is a trade-off between time and energy cost. For the multiagent networks with pinning control, it is very important to find the optimal number of nodes to minimize the time and energy cost. In addition, communication time delays due to finite information transmission and processing speed may arise naturally [34, 13]. Therefore, studying finite-time and fixed-time pinning consensus of time-delayed multiagent systems is important. The protocols designed in this paper can make all agents converge to the same state, which is usually called the "complete consensus." However, in more realistic situations, a group of agents may evolve into several subgroups where the consensus is achieved in each subgroup. As a result, agents in different subgroups may evolve toward different states, which is known as the "cluster consensus." The protocols proposed in this paper can also be used to investigate the "cluster consensus." These problems are our future research direction.

Appendix A. Lemmas. The following results are useful in order to prove the theorems outlined in the text.

LEMMA A.1 (see [41, 15]). *If the graph $\mathcal{G}(A)$ is strongly connected, then the eigenvalue 0 of the graph Laplacian L_A is algebraically simple and all other eigenvalues are with positive real parts. If $\mathcal{G}(A)$ is also undirected, then $\xi^T L_A \xi = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(\xi_j - \xi_i)^2$, where $\xi = (\xi_1, \dots, \xi_N)^T \in \mathbb{R}^N$.*

LEMMA A.2 (see [19]). *Suppose function ψ satisfies $\psi(x_i, x_j) = -\psi(x_j, x_i)$ $\forall i, j \in \{1, \dots, N\}, i \neq j$. Then for any undirected graph $\mathcal{G}(A)$ and a group of numbers $y_1, y_2, \dots, y_N$,*

$$\sum_{i,j=1}^{N} a_{ij} y_i \psi(x_j, x_i) = -\frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(y_j - y_i)\psi(x_j, x_i).$$

LEMMA A.3 (see [15]). *Let $z \in \mathbb{R}^n$ and $0 < r < s$. Then the following norm equivalence property holds:*

$$\left(\sum_{i=1}^{n} |z_i|^s\right)^{\frac{1}{s}} \le \left(\sum_{i=1}^{n} |z_i|^r\right)^{\frac{1}{r}}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} |z_i|^s\right)^{\frac{1}{s}} \ge \left(\frac{1}{n}\sum_{i=1}^{n} |z_i|^r\right)^{\frac{1}{r}}.$$

DEFINITION A.4. *Let $M = (M_{ij}) \in \mathbb{R}^{n\times n}$ and $R_i'(M) = \sum_{j=1, j\neq i}^{n} |M_{ij}|$. We say that matrix M is irreducibly diagonally dominant if*

(a) *M is irreducible;*
(b) *M is diagonally dominant, that is, $|M_{ii}| \ge R_i'(M)$ $\forall i = 1, 2, \dots, n$; and*
(c) *for at least one value of i we have $|M_{ii}| > R_i'(M)$.*

LEMMA A.5 (Taussky's theorem [18, p. 363]). *Let $M = (M_{ij}) \in \mathbb{R}^{n\times n}$ be irreducibly diagonally dominant. If M is Hermitian (or more generally, if A has only real eigenvalues), and if all main diagonal entries are strictly positive, then all eigenvalues of M are strictly positive.*

LEMMA A.6 (see [5]). *Assume that the positive Lyapunov function $V(x)$ is defined on a neighborhood U of the origin, and*

$$\dot{V}(x) \le -\alpha V^p(x), \quad 0 < p < 1,$$

where $\alpha > 0$, then

$$V(x) \equiv 0 \qquad \forall t \ge T(x_0)$$

and the settling time $T(x_0)$ can be bounded from above as

$$T(x_0) \le \frac{V^{1-p}(x_0)}{\alpha(1-p)}.$$

Lemma A.7 (see [31]). *Consider the following equation:*

$$\dot{x} = f(t,x), x(0) = x_0, \tag{A.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function. Assume the origin is an equilibrium point of (A.1). *If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that*

(i) $V(z) = 0 \Longleftrightarrow z = 0$,

(ii) *for some positive numbers $\kappa, \varrho > 0, 0 < p < 1 < q$, any solution $z(t)$ satisfies the inequality*

$$\dot{V}(z(t)) \le -\kappa V^p(z(t)) - \varrho V^q(z(t)),$$

then the origin is globally fixed-time stable and $V(t) \equiv 0$ if

$$t \ge \frac{1}{\kappa(1-p)} + \frac{1}{\varrho(q-1)}.$$

To investigate the finite-time and fixed-time stochastic consensus, we need the following definitions and lemmas.

Consider an n-dimensional stochastic differential equation:

$$\mathrm{d}x = f(x)\mathrm{d}t + g(x)\mathrm{d}W(t), \tag{A.2}$$

where $x \in \mathbb{R}^n$ is the state vector, $W(t)$ is an m-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ generated by $W(t)$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are Borel measurable, continuous in x, and satisfy $f(0) = 0, g(0) = 0$ $\forall t \ge 0$. Clearly, (A.2) admits a trivial zero solution. For a twice continuously differentiable function V, the second-order differential operator of V with respect to (A.2) is defined as follows:

$$\mathcal{L}V = \frac{\partial V}{\partial x} \cdot f + \frac{1}{2}\mathrm{trace}\left[g^T \frac{\partial^2 V}{\partial x^2} g\right]. \tag{A.3}$$

Definition A.8 (see [43]). *The trivial solution of system* (2.1) *is said to be finite-time stable in probability if the solution exists for any initial state $x_0 \in \mathbb{R}^n$, denoted by $x(t, x_0)$, and the following statements hold:*

(i) *Finite-time attractiveness in probability: For every initial value' $x_0 \in \mathbb{R}^n \backslash \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t | x(t, x_0) = 0\}$, which is called the stochastic settling time, is finite almost surely, i.e., $P\{\tau_{x_0} < \infty\} = 1$.*

(ii) *Stability in probability: For every pair of $\varepsilon \in (0, 1)$ and $\vartheta > 0$, there exists a $\delta = \delta(\varepsilon, \vartheta) > 0$ such that $P\{|x(t, x_0)| < \vartheta\ \forall t \ge 0\} \ge 1-\varepsilon$, whenever $|x_0| < \delta$.*

Definition A.9 (see [43]). *A function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\nu(0) = 0$. A class $\mathcal{K}$ function is said to belong to class $\mathcal{K}_\infty$ if $\nu(r) \to \infty$ as $r \to \infty$.*

Lemma A.10 (see [43]). *Assume that* (A.2) *admits a unique solution. If there exists a twice continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}^+$, $\mathcal{K}_\infty$ class functions ν_1 and ν_2, and positive constants $\eta > 0$ and $0 < \rho < 1$, such that $\forall x \in \mathbb{R}^n$ and $t \ge 0$,*

$$\nu_1(x) < V(x) < \nu_2(x),$$
$$\mathcal{L}V(x) \le -\eta V^\rho(x),$$

then the origin of system (A.2) *is finite-time stable in probability, and the stochastic settling time satisfies*

$$\mathbb{E}[T_1(x_0)] \le \frac{V^{1-\rho}(x_0)}{\eta(1-\rho)}.$$

For stochastic multiagent systems, the settling time function T_1 not only depends on the initial state x_0 but also is a random variable. Hence, the finite-time property of T_1 ensues from $0 < \mathbb{E}(T_1) < +\infty$. The stochastic Lyapunov theorem in Lemma A.10 can be regarded as the stochastic counterpart of the finite-time stability theorem for deterministic nonlinear systems in Lemma A.6. The assumption of the uniqueness of solution for the stochastic system (A.2) has been slightly relaxed in [42] and only the existence of a solution is required. The following Lyapunov theorem is proposed in [45] and will be used to investigate the fixed-time stochastic consensus problem.

LEMMA A.11 (see [45]). *Consider system* (A.2). *If there exists a regular, positive definite, and radially unbounded function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *and positive constants* $\kappa, \varrho > 0, 0 < p < 1 < q$ *such that*

$$\mathcal{L}V(x) \leq -\kappa V^p(t) - \varrho V^q(t) \ \forall x \in \mathbb{R}^n,$$

then the origin of system (A.2) *is globally stochastically fixed-time stability in probability, and the stochastic settling time satisfies*

$$\mathbb{E}[T_1] \leq \frac{1}{\kappa(1-p)} + \frac{1}{\varrho(q-1)}.$$

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