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Affine Formation Maneuver Control of Multi-Agent Systems with Directed Interaction Graphs

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Abstract: The affine formation maneuver control problem of a leader-follower type multi-agent systems with the directed interaction graphs is studied in this paper. This paper firstly gives and proves a sufficient and necessary condition of achieving the affine localizability. Then, under the $(d+1)$-reachable condition of the given d-dimensional nominal formation with $d+1$ leaders, a formation of agents can be reshaped in arbitrary dimension by only controlling these leaders. In the sequel, a novel distributed control method for the followers with single-integrator dynamics is proposed to achieve the desired time-varying maneuvers, and the global stability is also proved. Corresponding simulations are carried out to verify the theoretical results, which show that these followers are tracking the time-varying references accurately and continuously.

Key Words: Affine formation control, Multi-agent systems, Directed graph, Signed Laplacian

1 Introduction

This paper deals with the affine formation maneuver control problem for leader-follower multi-agent systems with the directed interaction graphs to achieve the time-varying maneuvers, and it is a new topic and has not been explored sufficiently. This control task can be divided into two subtasks: The first one is to select the number of leaders and fully control the entire formation to achieve an desired affine transformation from any initial configuration, thus it acts as a shape control part; the second is to design a distributed control approach for the followers to steer the whole formation to fulfil time-varying maneuvers continuously, thus it is utilized to execute the desired maneuvers.

Since the affine transformation has a useful property to represent a translation, rotation, scale, shear, or the combination of them as a linear transformation in arbitrary dimensional space. Then the affine formation control approach proposed in [1] is based on the affine transformation and can fulfil the formation reshaped in arbitrary dimensional space by only controlling a small number of agents. It provides a powerful tool to drive the group of agents to achieve various time-varying formation maneuvers. Therefore, the affine formation maneuver control is important for a formation of agents to react to the dynamical environment, for example, to shrink the scale of the formation by the affine transformation to pass through a narrow passage. In order to show the advantages of the affine formation control approach, the existing approaches are compared. In the survey [2], the formation control approaches can be classified into types based on displacement, distance, and bearing under different target formations of constant constraints. The displacement-based method injects constraints into the inter-agent displacement of the target formation. It is suitable to track a time-varying translation but not for a rotation or scale. As for the distance-based method, each agent has its own local coordinate with inter-agent distance and orientation being its constraints. A time-varying translation or rotation can be achieved [3], but it cannot be applied to a time-varying scale. Although the bearing-based approach can track a time-varying translation or scale [4], a time-varying rotation cannot be achieved. Researchers in [1], [5], [6] have proposed many advanced methods to compensate these limitations recently. The appealing similar formation in [5], [7] can achieve the translational, rotational, and scaling formation maneuvers simultaneously, but it is only confined in the planar.

The problem of the affine formation maneuver control under an undirected graphical condition has been studied in [8], but the directed case has not been solved yet. In the real implementation, the interaction graph between agents is usually oriented, thus the directed condition is more suitable. For the directed problem, there are following tasks to be solved in this paper: 1) The undirected problem in [8] needs strong connectivity of generic universal rigidity to obtain the equilibrium set, but it is not suitable for real implementation due to the limits of onboard power of agents. Thus a loose precondition for the directed case needs to be found in this paper; 2) In [8] there exists a stress matrix is associated to the undirected graph, and this matrix has the properties of both positive and negative weights, positive semi-definite and symmetric. Under a directed graphical condition, a signed Laplacian is also defined with both positive and negative entries, but it does not satisfy the properties of positive semi-definite and symmetric. Thus the property of the signed Laplacian should be studied and adapted; 3) The directed interaction graph usually means fewer control effects of the following agents, thus the distributed control protocol for these followers would be different from the undirected case in [8] and needs to be redesigned.

The main contributions of this paper are twofold. First, a sufficient and necessary condition corresponding to the

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affine localizability with a directed interaction graph is given and proved. If the condition is satisfied, and only giving the references to these leaders can steer the whole group to achieve the desired affine formation shapes in the arbitrary dimensions. Second, based on various types of measurements, we propose the distributed control laws for these agents with single-integrator dynamics, and the global stability is also proved. Through the simulations, we verify that the proposed control laws can achieve time-varying affine formation maneuvers.

Notations: $\mathbb{R}$ denotes the set of real numbers. $\mathbf{1}_n$ stands for a n-dimensional vector of ones and I_d represents a $d \times d$ identity matrix. Denote $\otimes$ as the Kronecker product, and $\text{diag}(\cdot)$ the diagonal matrix.

2 Preliminaries and Problem Statement

2.1 Directed Graph Theory

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains a node set $\mathcal{V} = \{1, 2, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If there exists an ordered pair $(j, i) \in \mathcal{E}$, then the node v_j is said to be the tail (where the arrow starts) of the edge, while node v_i is its head, and the node v_j denotes the *in-neighbour* of v_i and the converse is called *out-neighbour*. Denote $\mathcal{N}_i$ as the in-neighbour set of node v_i, where $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. Throughout this paper, it assumes that a directed graph does not have any self-loop and is a fixed topology. In a directed graph $\mathcal{G}$, a *path* is an alternating sequence of node v_i and edge e_i, and these nodes in the sequence are different.

Then some useful concepts are defined as follows. In a directed graph $\mathcal{G}$, a node v is set as κ*-reachable* ($\kappa \geq 2$) from a nonsingleton set $\mathcal{U}$ of nodes when there is a path from a node in $\mathcal{U}$ to v after deleting any $\kappa - 1$ nodes except v. A directed graph $\mathcal{G}$ is κ*-rooted* if there exists a subset of κ nodes (*roots*), from which every other node is κ-reachable. For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a spanning κ*-tree* rooted at $\mathcal{R} = \{r_1, r_2, \dots, r_\kappa\} \subset \mathcal{V}$ is a spanning subgraph $\mathcal{T} = (\mathcal{V}, \bar{\mathcal{E}})$ that satisfies:

(i) every node $r \in \mathcal{R}$ has no in-neighbour;
(ii) every node $r \notin \mathcal{R}$ has κ in-neighbours;
(iii) every node $r \notin \mathcal{R}$ is κ-reachable form $\mathcal{R}$.

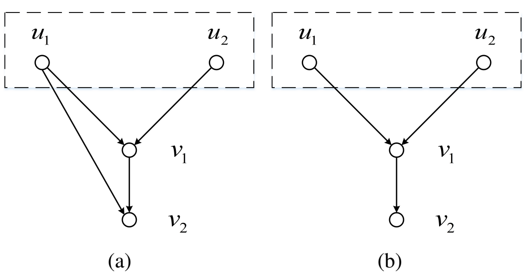

Fig. 1: Examples of 2-reachable and non 2-reachable

Fig. 1 shows the examples of κ-reachable, in Fig. 1(a) both of the nodes v_1 and v_2 are 2-reachable form the root set $\mathcal{U} = \{u_1, u_2\}$, while in Fig. 1(b) the node v_2 is not 2-reachable form set $\mathcal{U}$ after removing v_1. The directed graph in Fig. 1(a) is 2-rooted, and it has a spanning 2-tree.

Finally, the notion of signed Laplacian for a directed graph is introduced. A *signed Laplacian* L^s refers to a Laplacian matrix associated to a graph with both positive and negative real off-diagonal entries. The matrix L^s of a directed graph is defined as follows

$$L^s(i, j) = \begin{cases} -\omega_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i, \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i, \\ \sum_{k \in \mathcal{N}_i} \omega_{ik} & \text{if } i = j. \end{cases} \tag{1}$$

where $\omega \in \mathbb{R}$ may be a positive or negative real weight attributed on the edge (j, i), and L^s is normally a nonsymmetric matrix.

2.2 Problem Statement

Consider a group of n mobile agents in $\mathbb{R}^d$ and assume $d \geq 2$ and $n \geq d + 2$. Denote $p_i \in \mathbb{R}^d$ as the point of agent i, and the whole group of points $p = [p_1^T, p_2^T, \dots, p_n^T]^T \in \mathbb{R}^{dn}$ of n agents constitute a configuration. These agents exchange information by a directed graph $\mathcal{G}$. The directed graph $\mathcal{G}$ containing n nodes represents the *interaction graph* of the group of agents, an edge (j, i) shows that agent i can measure the relative position of agent j.

Denote the first n_ℓ of agents as the leaders and the rest $n_f = n - n_\ell$ as the followers. Thus the leaders' subset is $\mathcal{V}_\ell = \{1, 2, \dots, n_\ell\}$ and the followers' subset terms $\mathcal{V}_f = \mathcal{V} \backslash \mathcal{V}_\ell$. The points of leaders are $p_\ell = [p_1^T, p_2^T, \dots, p_{n_\ell}^T]^T$ and those of followers are $p_f = [p_{n_\ell+1}^T, p_{n_\ell+2}^T, \dots, p_n^T]^T$ respectively, thus $p = [p_\ell^T, p_f^T]^T$. The leaders do not interact with the following agents, and do not need to access the information from these followers.

The *affine transformation* is a general linear transformation. The types may be a translation, rotation, scale, shear or combinations of them, and there may exist some special conditions of collinear or coplanar. A formation $(\mathcal{G}, p)$ is a directed graph $\mathcal{G}$ with its vertex i of mapped to p_i. Then the *nominal formation* associated to $\mathcal{G}$ is defined as $(\mathcal{G}, r)$, and $r = [r_1^T, r_2^T, \dots, r_n^T]^T = [r_\ell^T, r_f^T]^T \in \mathbb{R}^{dn}$ is constant and named *nominal configuration*. The *affine image* of the nominal configuration can be defined as

$$\begin{aligned} \mathcal{A}(r) = \{p \in \mathbb{R}^{dn} : p = (I_n \otimes A)r + \mathbf{1}_n \otimes b, \\ \forall A \in \mathbb{R}^{d \times d}, \forall b \in \mathbb{R}^d\}, \end{aligned} \tag{2}$$

where (A, b) is affine transformation.

Definition 1. (Affine Localizability) *The nominal formation $(\mathcal{G}, r)$ is said to be affinely localizable if both of the following conditions are satisfied:*

1) For any $p = [p_\ell^T, p_f^T]^T \in \mathcal{A}(r)$ in $\mathbb{R}^d$, where p_f can be determined by p_ℓ uniquely;

2) For $\mathcal{G}$ and p, there exists a signed Laplacian $L^s \in \mathbb{R}^{n \times n}$ associated with $\mathcal{G}$ such that the equilibrium set

$$(L^s \otimes I_d)p = 0. \tag{3}$$

The first condition is for the leader selection, and the second one is about the directed graphical condition. Then, with the definition of the signed Laplacian, it can obtain

$$L^s \mathbf{1}_n = 0. \tag{4}$$

The objective of affine formation maneuver control is to drive the group of n agents to track the time-varying target formation under a distributed control law u_i of agent i.

Definition 2. **(Target Formation)** *The time-varying target formation has the expression of*

$$p^*(t) = [I_n \otimes A(t)]\, r + \mathbf{1}_n \otimes b(t), \tag{5}$$

where $A(t) \in \mathbb{R}^{d\times d}$ and $b(t) \in \mathbb{R}^d$ are continuous of t and time-varying, if the time-varying target can be tracked successfully, $p^*(t)$ is in $\mathcal{A}(r)$ for all t.

Since the leaders' points have a one-to-one corresponding to the affine transformation, the entire formation can be achieved by controlling only the leaders. The leaders' number is usually small, in this paper we assume the leaders can be controlled by human pilots or intelligent planners and their points are equal to desired target formation, i.e., $p_\ell(t) = p_\ell^*(t)$ for all t. Then, the control objective is changed to drive these followers to achieve $p_f(t) = p_f^*(t)$ as $t \to \infty$.

In this paper we will study the following two problems:

P1: What is the necessary and sufficient condition of the nominal formation $(\mathcal{G}, r)$ such that a group of n agents can achieve the affine localizability in arbitrary dimensional space?

P2: If given a nominal formation $(\mathcal{G}, r)$ satisfying conditions of the affine localizability, how to design the affine formation maneuver control law of these followers with single-integrator dynamics so as to drive them to maneuver continuously in the time-varying target formation in arbitrary dimensional space?

3 Affine Localizability of Directed Graph

Both the leader selection and the directed graphical condition of the affine localizability are given and proved in this section.

To get the affine localizability, it needs to introduce a term *affine span* $\mathcal{S}$ of the given point set $\{p_i\}_{i=1}^n$ of n agents in $\mathbb{R}^d$, which denotes

$$\mathcal{S} = \left\{ \sum_{i=1}^{n} a_i p_i : a_i \in \mathbb{R} \text{ for all } i \text{ and } \sum_{i=1}^{n} a_i = 1 \right\}, \tag{6}$$

where this span can always be translated to get a linear space, and the dimension of the obtained linear space is just the dimension of the affine span. If these scalars $\{a_i\}_{i=1}^n$ of not all zero ones can not satisfy that $\sum_{i=i}^n a_i p_i = 0$ and $\sum_{i=i}^n a_i = 0$, this condition is called as *affinely independent*.

Then, the *configuration matrix* can be defined as $P \in \mathbb{R}^{n\times d}$ and denote corresponding augmented matrix $\bar{P} \in \mathbb{R}^{n\times(d+1)}$ as

$$P(p) = \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix}, \bar{P}(p) = \begin{bmatrix} p_1^T & 1 \\ \vdots & \vdots \\ p_n^T & 1 \end{bmatrix} = [P(p), \mathbf{1}_n], \tag{7}$$

where these points of $\{p_i\}_{i=1}^n$ are affinely independent if and only if the rows of $\bar{P}(p)$ are linearly independent, thus there exist at most $d+1$ points and they are affinely independent in $\mathbb{R}^d$.

With the above definitions, two lemmas are given as below.

Lemma 1. *The point set $\{p_i\}_{i=1}^n$ of n agents has d-dimensional affine span if and only if $n \geq d+1$ and* $\operatorname{rank}(\bar{P}(p)) = d+1$.

Lemma 2. *[8] The affine image $\mathcal{A}(r)$ is a linear space of dimension $d^2 + d$ if and only if $\{r_i\}_{i=1}^n$ is d-dimensional affine span.*

Now an assumption of the nominal formation should be given.

Assumption 1. *Assume that the given nominal formation $(\mathcal{G}, r)$ of n agents satisfies $\{r_i\}_{i=1}^n$ has d-dimensional affine span.*

After obtaining these lemmas and assumption before, a theorem of the necessary and sufficient condition to fulfil the affine localizability can be deduced.

Theorem 1. *Under Assumption 1, the affine localizability of the given nominal formation $(\mathcal{G}, r)$ of n agents can be achieved if and only if the leaders' subset $\mathcal{V}_\ell$ has $d+1$ leaders and every follower in $\mathcal{V}_f$ is $(d+1)$-reachable from $\mathcal{V}_\ell$.*

Proof. The proof is omitted here due to page limit. □

From Theorem 1, the associated signed Laplacian L^s can be rewritten as the following blocks

$$L^s = \left[\begin{array}{c|c} 0_{\ell\ell}^{(d+1)\times(d+1)} & 0_{\ell f}^{(d+1)\times(n-d-1)} \\ \hline L_{f\ell}^{s\,(n-d-1)\times(d+1)} & L_{ff}^{s\,(n-d-1)\times(n-d-1)} \end{array} \right]. \tag{8}$$

Because any $p = [p_\ell^T, p_f^T]^T \in \mathcal{A}(r)$ satisfies $(L^s \otimes I_d)p = 0$, it can obtain that

$$\bar{L}_{f\ell}^s p_\ell + \bar{L}_{ff}^s p_f = 0, \tag{9}$$

where $\bar{L}_{f\ell}^s = L_{f\ell}^s \otimes I_d$ and $\bar{L}_{ff}^s = L_{ff}^s \otimes I_d$.

Now, another assumption is given about the affine localizability of the nominal formation.

Assumption 2. *Assume that the given nominal formation $(\mathcal{G}, r)$ can achieve the affine localizability.*

Till now, it can recall that the control objective is to drive these followers to achieve $p_f(t) \to p_f^*(t)$ as $t \to \infty$. In the light of above Theorem 1, it can find $p_f^*(t) = -\bar{L}_{ff}^{s^{-1}} \bar{L}_{f\ell}^s p_\ell^*(t)$, although L_{ff}^s is nonsingular, its eigenvalues may not all locate in the right-half complex plane, and there are fixed $d+1$ zero eigenvalues corresponding to roots of the directed interaction graph $\mathcal{G}$, thus it implies that L^s may contain negative eigenvalues. A transformation method should be found to make L^s to get all nonnegative eigenvalues. After being transformed, two blocks of $\bar{L}_{ff}^s$ and $\bar{L}_{fl}^s$ corresponding matrices become $\tilde{L}_{ff}^s$ and $\tilde{L}_{fl}^s$. Define the tracking error of following agents as

$$\delta_{p_f}(t) = p_f(t) - p_f^*(t) = p_f(t) + \tilde{L}_{ff}^{s^{-1}} \tilde{L}_{f\ell}^s p_\ell^*(t). \tag{10}$$

Then the control law $u_i(t), i \in \mathcal{V}_f$ needs to be designed for the followers to make $\delta_{p_f}(t) \to 0$ as $t \to \infty$. The next section will give the design process of the distributed control approach to achieve affine formation maneuvers of single-integrator multi-agent systems.

4 Affine Formation Maneuver Control Laws of Single-Integrator Dynamics

In this section, the distributed affine formation maneuver control laws based on different kinds of information flows are proposed. The condition of single-integrator dynamics agent model is considered: $\dot{p}_i = u_i$, where u_i is the control input that needs to be designed.

1) Leaders with Zero Velocities If all of these leaders are stationary, then the target formation is also stationary, the following control law of these agents $i \in \mathcal{V}_f$ is proposed

$$\dot{p}_i = -d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j), \tag{11}$$

where ω_{ij} may be positive or negative, d_i is the nonzero control parameter to be designed.

Then a theorem is given as below, which gives the stability of the proposed distributed control law. To prove this theorem, a lemma is needed.

Lemma 3. *[9] Let A be an $n \times n$ real matrix, all of its leading principal minors are nonzero. Then there exist an $n\times n$ diagonal real matrix D, all the roots of DA are positive and simple.*

Theorem 2. *Under Assumptions 1-2, if the leaders have the zero velocities $\dot{p}^*_\ell(t)$, then the tracking error $\delta_{p_f}(t)$ of these followers under the control law (11) converges globally and exponentially fast to zero.*

Proof. On the basis of Lemma 3, since the real matrix $D \in \mathbb{R}^{n\times n}$ and $D = \text{diag}(d_i), i \in \mathcal{V}$ is diagonal and invertible, it can imply that the null space of DL^s is the same as the one of L^s. Then these eigenvalues of DL^s are not localized in the left-half complex plane, which indicates that DL^s can obtain all positive real-part eigenvalues. In the sequel, the solution to finding D is tackled.

Under Assumptions 1-2 such that L^s_{ff} satisfies nonsingular, there exists a permutation matrix P such that all of the leading principal minors of $PL^s_{ff}P^T$ are nonzero. According to Lemma 3, it can obtain that a diagonal matrix D' makes all of the eigenvalues of $D'PL^s_{ff}P^T$ localize in the right-half complex plane. The permutation matrix P has the proposition of $P^{-1} = P^T$, thus $D'PL^s_{ff}P^T = P(P^TD'PL^s_{ff})P^T$. From both sides of this equation, it implies that $D'PL^s_{ff}P^T$ and $P^TD'PL^s_{ff}$ have the same eigenvalues. And $P^TD'P$ is also diagonal, then let $D'' = P^TD'P$, and these diagonal entries of D corresponding to zero eigenvalues as I_{d+1}, the form of D has the expression of

$$D = \begin{bmatrix} I_{d+1} & 0 \\ 0 & D'' \end{bmatrix}, \tag{12}$$

where the zero vector has corresponding rows, and the following parts of this paper is the same for the reason of simplification.

After premultiplying the diagonal D, these eigenvalues of DL^s have $d+1$ zeros and the rest with positive real parts. Denote this diagonal matrix D as the stabilizing matrix.

Since $\dot{p}^*_\ell = 0$, a matrix form of (11) can be expressed as

$$\begin{bmatrix} 0 \\ \dot{\delta}_{p_f} \end{bmatrix} = -\left[(DL^s) \otimes I_d\right] \begin{bmatrix} 0 \\ \delta_{p_f} \end{bmatrix}. \tag{13}$$

After partitioning DL^s to blocks, it has a expression of

$$\left[\begin{array}{c|c} I_{d+1} & 0 \\ \hline 0 & D'' \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline L^s_{f\ell} & L^s_{ff} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline D''L^s_{f\ell} & D''L^s_{ff} \end{array}\right]. \tag{14}$$

In the sequel, denote $\tilde{L}^s_{f\ell} = (D''L^s_{f\ell}) \otimes I_d$ and $\tilde{L}^s_{ff} = (D''L^s_{ff}) \otimes I_d$, then $\dot{\delta}_{p_f} = -\tilde{L}^s_{ff}\delta_{p_f}$, it implies that there exists a gradient-decent control law for the Lyapunov function $V = 1/2\delta^T_{p_f}(G\tilde{L}^s_{ff} + \tilde{L}^{sT}_{ff}G)\delta_{p_f}$ with a diagonal positive definite matrix G, and if and only if $D''L^s_{ff}$ contain all positive eigenvalues, then the tracking error δ_{p_f} converges to zero globally and exponentially fast. □

2) Leaders with Constant Velocities If these leaders are moving with nonzero constant velocities, then additional integral term is needed to compensate (11). A proportional-integral control law of these agents $i \in \mathcal{V}_f$ is proposed

$$\begin{aligned} \dot{p}_i = & \underbrace{-\,\alpha d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j)}_{\text{proportional term}} \\ & \underbrace{-\,\beta \int_0^t d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i(\tau) - p_j(\tau))d\tau}_{\text{integral term}}, \end{aligned} \tag{15}$$

where α and β are positive constant control gains to be designed.

Then a theorem is given as below, which gives the stability of the proposed distributed control law.

Theorem 3. *Under Assumptions 1-2, if the leaders have the constant velocities $\dot{p}^*_\ell(t)$, then the tracking error $\delta_{p_f}(t)$ of these followers under the control law (15) converges globally and exponentially fast to zero.*

Proof. To prove this theorem, a new intermediate state ζ is introduced and (15) can be rewritten as

$$\begin{aligned} \dot{p}_i &= -\alpha d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j) - \beta\zeta_i, \\ \dot{\zeta}_i &= d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j). \end{aligned} \tag{16}$$

Then a matrix form of (16) can be expressed as

$$\begin{aligned} \dot{p}_f &= -\alpha\tilde{L}^s_{f\ell}p^*_\ell - \alpha\tilde{L}^s_{ff}p_f - \beta\zeta, \\ \dot{\zeta} &= \tilde{L}^s_{f\ell}p^*_\ell + \tilde{L}^s_{ff}p_f. \end{aligned} \tag{17}$$

Differentiate (10) and substitute (17) can obtain

$$\begin{aligned} \dot{\delta}_{p_f} &= \dot{p}_f + \tilde{L}^{s^{-1}}_{ff}\tilde{L}^s_{f\ell}\dot{p}^*_\ell \\ &= -\alpha\tilde{L}_{ff}\delta_{p_f} - \beta\zeta + \tilde{L}^{s^{-1}}_{ff}\tilde{L}^s_{f\ell}\dot{p}^*_\ell. \end{aligned} \tag{18}$$

The matrix combining the tracking error δ_{p_f} and the intermediate state ζ can be formulated as

$$\begin{bmatrix} \dot{\delta}_{p_f} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} -\alpha\tilde{L}^s_{ff} & -\beta I_{dn_f} \\ \tilde{L}^s_{ff} & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_f} \\ \zeta \end{bmatrix} + \begin{bmatrix} \tilde{L}^{s^{-1}}_{ff}\tilde{L}^s_{f\ell} \\ 0 \end{bmatrix} \dot{p}^*_\ell. \tag{19}$$

Denote $\varphi = [\delta^T_{p_f}, \zeta^T]^T$, M as the state matrix containing α and β, the second term C of (19) is constant since $\dot{p}^*_\ell$ is constant, then (19) can be rewritten as

$$\dot{\varphi} = M\varphi + C. \tag{20}$$

Then φ can converge if and only if M is Hurwitz, it implies that α, β are positive constant and $\tilde{L}^s_{ff}$ contain all positive eigenvalues. The infinite state of (19) can be written as

$$\begin{bmatrix} -\alpha\tilde{L}^s_{ff} & -\beta I_{dn_f} \\ \tilde{L}^s_{ff} & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_f}(\infty) \\ \zeta(\infty) \end{bmatrix} + \begin{bmatrix} \tilde{L}^{s^{-1}}_{ff}\tilde{L}^s_{f\ell} \\ 0 \end{bmatrix} \dot{p}^*_\ell = 0. \tag{21}$$

where $\delta_{p_f}(\infty) = 0$ and $\zeta(\infty)$ cancels the steady-state error induced by $\dot{p}^*_\ell$. □

3) Leaders with Time-Varying Velocities If the absolute velocity of every agent can be measured, when leaders are moving with time-varying velocities, a term collecting all neighbouring information is needed, the following control law of these agent $i \in \mathcal{V}_f$ is proposed

$$\dot{p}_i = -\frac{1}{\gamma_i} d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}[(p_i - p_j) - \dot{p}_j], \tag{22}$$

where $\gamma_i = d_i \sum_{j\in\mathcal{N}_i} \omega_{ij}$ and needs to satisfy $\gamma_i \neq 0$.

Observe (22) that the term $d_i \neq 0$ can be reduced, and it can be simplified as

$$\dot{p}_i = -\frac{1}{\gamma_i{}'} \sum_{j\in\mathcal{N}_i} \omega_{ij}[(p_i - p_j) - \dot{p}_j], \tag{23}$$

where $\gamma_i{}' = \sum_{j\in\mathcal{N}_i} \omega_{ij}$ and needs to satisfy $\gamma_i{}' \neq 0$.

Then a theorem is given as below, which gives the stability of the proposed distributed control law.

Theorem 4. *Under Assumptions 1-2, if the leaders have the time-varying and continuous velocities $\dot{p}^*_\ell(t)$, then the tracking error $\delta_{p_f}(t)$ of these followers under the control law (23) converges globally and exponentially fast to zero.*

Proof. Multiply both sides of (23) can get

$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(\dot{p}_i - \dot{p}_j) = -\sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j). \tag{24}$$

Then (24) can be rewritten to a matrix form as

$$\bar{L}^s_{ff}\dot{\delta}_{p_f} = -\bar{L}^s_{ff}\delta_{p_f}. \tag{25}$$

In the sequel, denote $\eta_f = \bar{L}^s_{ff}\delta_{p_f}$, it implies that $\dot{\eta}_f = -\eta_f$ that if and only if $\bar{L}^s_{ff}$ is nonsingular, then η_f converges to zero globally and exponentially fast. If $\eta_f = 0$ then $\bar{L}^s_{ff}p_f + \bar{L}^s_{f\ell}p^*_\ell = 0$, it means that $\bar{L}^s_{ff}\delta_{p_f} = 0$. Consequently, δ_{p_f} can converge to zero if and only if $\bar{L}^s_{ff}$ is nonsingular. It can obtain that $\gamma_i{}' = \sum_{j\in\mathcal{N}_i} \omega_{ij}$ for all $i \in \mathcal{V}_f$. Because $\bar{L}^s_{ff}$ is nonsingular, then $\gamma_i{}'$ for all $i \in \mathcal{V}_f$ is a diagonal entry of L^s_{ff}, which infers that $\gamma_i{}' \neq 0$. □

Remark 1. *To be specific, when the leaders have time-varying velocities, the stabilizing matrix D is not necessary to be designed, it implies that the control laws designed are the same as the undirected graphical condition as [8].*

5 Calculation of Signed Laplacian and Stabilizing Matrix

In this section, it needs to find the algorithms to calculate the signed Laplacian L^s and the stabilizing matrix D.

When a given nominal formation $(\mathcal{G}, r)$ satisfies Assumptions 1-2, from Theorem 1, it shows that a signed Laplacian L^s with every weight ω_{ij} exists and satisfies $\det(L^s_{ff}) \neq 0$ and $\text{rank}(L^s) = n - d - 1$ if the directed graph $\mathcal{G}$ satisfies every following agent of $\mathcal{V}_f$ is $(d+1)$-reachable from the leaders' set $\mathcal{V}_\ell$ with $d+1$ leaders. The given nominal configuration is $r = [r_1^T, \cdots, r_n^T]^T$, and $(L^s \otimes I_d)r = 0$. Then the off-diagonal weight ω_{ij} can be computed by

$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(r_j - r_i) = 0, i \in \mathcal{V}_f. \tag{26}$$

Due to the directed graph $\mathcal{G}$ has $(d+1)$-spanning tree, and according to the definition of κ-spanning tree, there are exactly $d+1$ in-neighbours, which infers that (26) must have a solution and can be solved by linear programming efficiently. Then using $L^s\mathbf{1}_n = 0$ such that all of these diagonal entries of L^s can be calculated. Till now the whole entries of L^s can be found and the solution of L^s is not unique.

Algorithm 1 Stabilizing matrix D calculation

Input: L^s_{ff} satisfying all nonzero leading principal minors.
Output: Stabilizing matrix D.
for $i = 1, \cdots, n-d-1$ **do**
Find d_{i+d+1} to assign these eigenvalues of $\text{diag}(d_{d+2}, \cdots, d_{i+d+1})L^s_{ff}{}^{(1\sim i)}$ in the right-half complex plane.
end for
return $D = \text{diag}(1, \cdots, 1, d_{d+2}, \cdots, d_n)$.

The existence of the stabilizing matrix D is assured by Lemma 3 and above solution of L^s. Give an algorithm above to calculate D in an iterative way, where $L^s_{ff}{}^{(1\sim i)}$ denotes the block of first i rows and columns of L^s_{ff}. This procedure requires the centralized computation.

6 Simulations

In this section, the proposed distributed control laws are verified through the simulations.

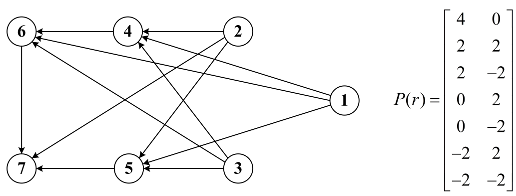

Fig. 2: Nominal formation of 7 agents

Given a nominal formation $(\mathcal{G}, r)$ of 7 agents with the configuration matrix $P(r)$ in $\mathbb{R}^2$ as shown in Fig. 2. The first three nodes denote the leaders and the rest four ones are the following agents, the leader number meets $3 = d+1$ and the directed interaction graph $\mathcal{G}$ is 3-reachable. The rank of the calculated signed Laplacian L^s satisfies 4, and the designed diagonal stabilizing matrix is $D = \text{diag}(1, 1, 1, -1, -1, -1, -1)$.

$$L^s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1.5 & 0.5 & -1 & 0 & 0 & 0 \\ -1 & 0.5 & 1.5 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0.5 & 2 & 0 & -1.5 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & -2 \end{bmatrix}.$$

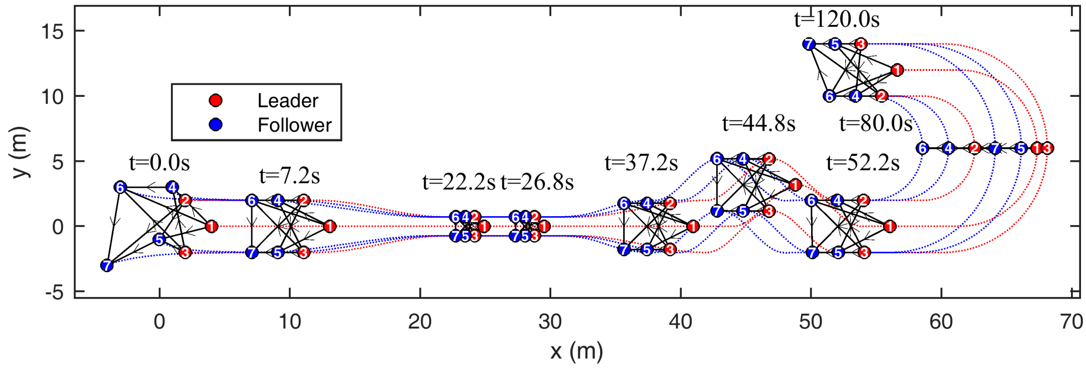

Fig. 3: Trajectories with time-varying transitional, scaling, and shearing affine formation maneuvers

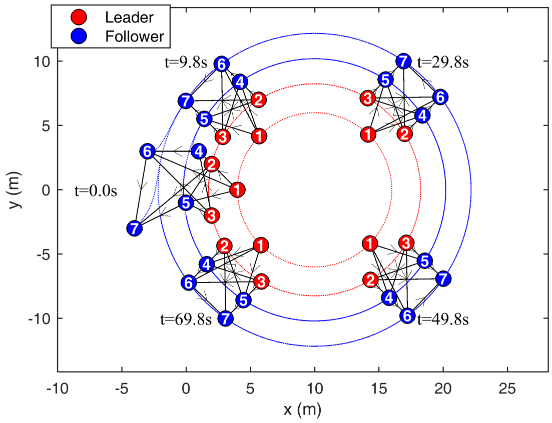

Fig. 4: Trajectories with time-varying rotational affine formation maneuvers

These agents with single-integrator dynamics are introduced, and the distributed control law of these following agents is given in (23). The resulting trajectories of time-varying affine formation maneuvers of these 7 agents are shown in Fig. 3 and Fig. 4. It should be remaindered that these leaders may occur the collinear condition during the maneuvering process, but it does not affect the localizability and stability of the affine formation. From these simulation results, the affine formation maneuvers can fulfil the translation, scale, shear, and rotation continuously.

7 Conclusions

This paper solves a multi-agent affine formation maneuver control problem of realizing the time-varying maneuvers via directed interaction graphs in arbitrary dimensional space. Each agent in the connected graph updates its own state via a signed Laplacian with both positive and negative weights of edges. The sufficient and necessary condition of achieving affine localizability of the directed interaction graph is proved to be $(d+1)$-rooted. A distributed affine formation maneuver control approach based on the single-integrator dynamics model is proposed and proved globally stable. According to designed control protocols, the time-varying maneuvers such as a translation, scale, shear, and rotation can be fulfilled when these references are only accessed by the leaders. Tracking performance of these following agents is accurate and continuous. In the future, the higher-order dynamics model of agents and the conditions for rigid formation need to be studied and tackled.

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