Research Article | Vol. 26, No. 24 | 26 Nov 2018 | OPTICS EXPRESS 31351
Optics EXPRESS

# Theoretical consideration on convergence of the fixed-point iteration method in CIE mesopic photometry system MES2

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**Abstract:** Currently the fixed-point iteration method with initial guess $m_0 = 0.5$ is officially recommended by the CIE MES2 system [CIE 191:2010] in order to compute the adaptation coefficient $m$ and the mesopic luminance $L_{mes}$. However, recently, Gao *et al.* [Opt. Express **25**, 18365 (2017)] and Shpak *et al.* [Lighting Res. Technol. **49**, 111 (2017)] have numerically found that the fixed-point iteration method could be not convergent for large values of $S/P$. Shpak *et al.* suspected that, to achieve convergence, the $S/P$ ratio cannot be greater than 17. In this paper, a theoretical consideration for the CIE MES2 system is given. Namely, it is shown that the ratio $S/P$ be smaller than a constant $C_2$ $(\approx 18.1834)$ is a sufficient condition for the convergence of the fixed-point iteration method. In addition, a new initial guess strategy, achieving faster convergence, is proposed.



## 1. Introduction

The CIE MES2 system [1] was proposed in 2010 as an intermediate between the USP-system developed by Rea *et al.* [2] in 2004, and the Move-system developed by Goodman *et al.* [3] in 2007. In the MES2 system, the spectral luminance efficiency function in the mesopic range from 0.005 $cd\,m^{-2}$ to 5.0 $cd\,m^{-2}$ is denoted by $V_{mes}(\lambda)$, and defined as

$$M(m)V_{mes} = mV(\lambda) + (1-m)V'(\lambda), \tag{1}$$

where $m$ is a coefficient of adaptation in the range $0 \le m \le 1$, $M(m)$ is a normalization constant such that $V_{mes}(\lambda)$ attains a maximum value of 1, and $V(\lambda)$ and $V'(\lambda)$ are the CIE spectral luminous efficiency functions for photopic and scotopic visions, respectively. Hence the mesopic luminance $L_{mes}$ (in $cd\,m^{-2}$), for a given light source with a spectral radiance $E(\lambda)$ (in $W\,m^{-2}\,sr^{-1}$), is given by

$$L_{mes} = \frac{683}{V_{mes}(\lambda_0)} \int_{380}^{780} V_{mes}(\lambda)E(\lambda)\,d\lambda, \tag{2}$$

where $\lambda_0 = 555\,nm$.

If we let

#340120 https://doi.org/10.1364/OE.26.031351
Journal © 2018 Received 23 Jul 2018; revised 11 Oct 2018; accepted 25 Oct 2018; published 14 Nov 2018

$$C = V'(\lambda_0), \qquad L_p = 683\int_{380}^{780} V(\lambda)E(\lambda)\,d\lambda, \qquad L_s = 1700\int_{380}^{780} V'(\lambda)E(\lambda)\,d\lambda, \qquad (3)$$

and since $V(\lambda_0) = 1$ and $V'(\lambda_0) = 683/1699 \approx 0.402$, we have

$$L_{mes}(m) = \frac{mL_p + (1-m)L_s C}{m + (1-m)C}. \qquad (4)$$

Moreover, the mesopic luminance $L_{mes}$, and the parameter $m$ are related by

$$m = a + b\,log_{10}(L_{mes}), \qquad (5)$$

where the values for the parameters $a$ and $b$ adopted by CIE [1] are

$$a = 0.7670, \qquad b = 0.334. \qquad (6)$$

Thus, if we let

$$F(m) = \frac{mL_p + (1-m)L_s C}{m + (1-m)C} - 10^{\frac{m-a}{b}}, \qquad (7)$$

then the coefficient of adaptation $m$, defined by (4) and (5), should be also the solution of the equation $F(m) = 0$.

Recently, Gao *et al.* [4] have shown that, with the values for $a$ and $b$ given by (6), the equation $F(m) = 0$ may have either no solution or more than one, and, in agreement with Shpak *et al.* [5], they have recommended that the values for the parameters $a$ and $b$ should be better replaced by

$$a = 1 - \frac{log_{10}5}{3}, \qquad b = \frac{1}{3}. \qquad (8)$$

Gao *et al.* [4] have shown that, with the new values for $a$ and $b$ given by (8), the equation $F(m) = 0$ has a unique solution between $0$ and $1$, when the following condition is satisfied:

$$L_s > 0.005\,cd\,m^{-2} \quad \text{and} \quad L_p < 5.0\,cd\,m^{-2}. \qquad (9)$$

Thus, in this paper we will use the values for $a$ and $b$ given by (8), together with the remaining equations of the CIE MES2 system.

Note first that $a$ and $b$ given by (8), also satisfy

$$0.005 = 10^{a/b}, \qquad 5 = 10^{(1-a)/b}. \qquad (10)$$

Now from (4), we note that when $m = 0$, we have $L_{mes} = L_s$, and when $m = 1$, we have $L_{mes} = L_p$. Hence, for the continuity of the luminance scale, from scotopic via mesopic to photopic visions, we should have:

$$\text{when} \quad L_s \le 0.005\,cd\,m^{-2}, \qquad m = 0 \quad \text{and} \quad L_{mes} = L_s \qquad (11)$$

$$\text{when} \quad L_p \ge 5\,cd\,m^{-2}, \qquad m = 1 \quad \text{and} \quad L_{mes} = L_p. \qquad (12)$$

Henceforth, in this paper all luminance units ($cd\,m^{-2}$) will be missed for simplicity.

Let

$$g(m) = a + b\,log_{10}\left[L_{mes}(m)\right], \qquad (13)$$

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where $L_{mes}(m)$ is defined by (4). To compute the value $m$, satisfying $m = g(m)$, the CIE [1] has recommended the iteration method

$$m_{n+1} = g(m_n), \qquad \text{for} \qquad n = 0,1,\ldots \tag{14}$$

with $m_0 = 0.5$, until 'convergence' is achieved. The term 'convergence' in this algorithm (14) means that the iteration process is stopped when two consecutive values $m_n$ and $m_{n+1}$ are close enough, i.e., the difference among them in absolute value is smaller than a prefixed small tolerance $\varepsilon$. Therefore, when we have

$$\left|m_{n+1} - m_n\right| \le \varepsilon, \tag{15}$$

the value $m_{n+1}$ is accepted as an approximation of the solution of the equation $m = g(m)$.

Note that, if the sequence $\{m_n\}_{n\ge 0}$, generated by the iteration method (14), converges to $m^*$, then we have

$$m^* = \lim_{n\to\infty} m_{n+1} = \lim_{n\to\infty} g(m_n) = g\left(\lim_{n\to\infty} m_n\right) = g(m^*), \tag{16}$$

since $g$ is a continuous function.

Hence, $m^*$ is a fixed point of the function $g$, and this is the reason this iteration method is also named in the literature [6] as fixed-point iteration.

It is clear that the function $g$, or the fixed-point iteration method, is dependent on both, $L_p$ and the ratio $L_s / L_p$. In this paper, the ratio $L_s / L_p$. will be denoted in an abbreviated form as $S/P$, i.e.,

$$S/P = L_s / L_p. \tag{17}$$

Recently, Gao *et al.* [4] and Shpak *et al.* [5] have reported that the convergence of the fixed-point iteration method depends on the ratio $S/P$, and for large values of $S/P$ the method does not converge. Shpak *et al.* [5] suspected that, to achieve convergence, the ratio $S/P$ cannot be larger than 17. Since currently the fixed-point iteration method is officially recommended by the CIE MES2 system [1], and it may be also implemented in automatic devices, it is appropriate to provide a full theoretical consideration on the convergence of such method. This is the main goal of this paper.

## 2. Convergence analysis for fixed-point iteration method

We start quoting a result about a sufficient condition for the convergence of the fixed-point iteration method.

**Lemma 1**: (Fixed-Point Theorem [6, page 62, Chapter 2])

Let $f$ be a continuous function defined on $[c,d] \subset \mathbf{R}$, such that $f(x) \in [c,d]$ for all $x \in [c,d]$. Suppose, in addition, that $f'$, the derivative of $f$, exists on $(c,d)$, and there is a constant $k \in (0,1)$ such that $\left|f'(x)\right| \le k$, for all $x \in [c,d]$. Then, for any number $x_0$ in $[c,d]$, the sequence $\{x_n\}_{n\ge 0}$ defined by

$$x_n = f(x_{n-1}), \qquad n \ge 1, \tag{18}$$

converges to the unique fixed point $x^*$ of the function $f$ in $[c,d]$.

Now, we provide another sufficient condition for the convergence of the fixed-point iteration method.

**Theorem 1**: Let $f$ be a monotonically increasing and continuous function defined on $[c,d] \subset \mathbf{R}$, such that $f(x) \in [c,d]$ for all $x \in [c,d]$. Then, for any $x_0$ in $[c,d]$, the sequence $\{x_n\}_{n\ge 0}$ defined by (18) converges to a fixed point $x^*$, in $[c,d]$, of the function $f$.

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**Proof**: It is obvious that $\{x_n\}_{n\geq 0} \subset [c,d]$. If $x_0 = x_1 = f(x_0)$, then $x_n = x_0$ for all $n \geq 1$, and $x_0$ is a fixed point of the function $f$. Now, suppose $x_0 < x_1$. In this case, it can be shown that the sequence $\{x_n\}_{n\geq 0}$ is monotonically increasing, and therefore has a limit $x^* \in [c,d]$. Due to the continuity of $f$, we have

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\left(\lim_{n\to\infty} x_{n-1}\right) = f(x^*), \tag{19}$$

and $x^*$ is a fixed point of $f$. If $x_0 > x_1$, the sequence $\{x_n\}_{n\geq 0}$ is monotonically decreasing, and we get the same conclusion.

Now, from Theorem 1, we have:

**Theorem 2**: If $S/P \leq 1$, then the fixed-point iteration method given by (14) is convergent.

**Proof**: First, we note that when $S/P = 1$, the fixed-point iteration method (14) is convergent. In fact, when $S/P = 1$, we have $L_{mes} = L_p$, for any $m$, and therefore

$$g(m) = a + b\,log_{10}(L_p). \tag{20}$$

Thus, for any $m_0 \in (0,1)$, we have $m_n = g(m_{n-1}) = a + b\,log_{10}(L_p)$, for all $n \geq 1$. Therefore, the sequence $\{m_n\}_{n\geq 0}$ converges to $a + b\,log_{10}(L_p)$, which is the unique fixed point of $g$ in $[0,1]$.

Now, suppose that $S/P < 1$, and let

$$U(m) = m + (1-m)(S/P)C, \quad \text{and} \quad W(m) = m + (1-m)C. \tag{21}$$

It is easy to check that $g'(m)$ is given by

$$g'(m) = \frac{bC(1-S/P)}{log10\,U(m)W(m)}. \tag{22}$$

Since $S/P < 1$, and $U(m)$ and $W(m)$ are positive for $m \in (0,1)$, we have $g'(m) > 0$ for all $m \in (0,1)$, and therefore, $g$ is a monotonically increasing and continuous function on $[0,1]$. Moreover, since $S/P < 1$, then $0.005 \leq L_s < L_p < 5$, and therefore $0 \leq g(0) < g(1) < 1$, i.e., $g(m) \in [0,1]$ for all $m \in [0,1]$. And by Theorem 1, the fixed-point iteration method given by (14) is convergent.

We note that Lemma 1 cannot be applied to prove Theorem 2, since $g'(m)$ is greater than 1, when $S/P$ and $m$ are sufficiently small.

Now, in order to investigate the convergence of the fixed-point iteration method when $S/P > 1$, we need the second derivative of the function $g$, given by

$$g''(m) = \frac{bC(1-S/P)}{log10\,U(m)^2\,W(m)^2}(Am + B), \tag{23}$$

where

$$A = 2(1-C)((S/P)C - 1) \quad \text{and} \quad B = C((S/P)C - 1) + (S/P)C(C-1). \tag{24}$$

Thus, it is clear that both $A$ and $B$ are linear functions of $S/P$, and therefore $m_w = -B/A$ is a function of the ratio $S/P$, i.e., $m_w = m_w(S/P)$. Fig. 1 shows the variation of the function $m_w = m_w(S/P)$ (vertical axis) with ratio $S/P$ (horizontal axis). The dotted vertical line corresponds to $S/P = 1/C$. It can be seen that when $S/P < 1/C$, we have that $m_w(S/P)$ is negative, and approaches to $-\infty$ when $S/P$ approaches from bellow to $1/C$. However, when $S/P > 1/C$, $m_w(S/P)$ is positive and monotonically decreasing with respect to $S/P$; and $m_w(S/P)$ approaches to $\infty$ when $S/P$ approaches from above to $1/C$. Furthermore, it can be easily seen that

$$m_w\left(\frac{2-C}{C}\right)=1 \quad \text{and} \quad \lim_{S/P\to\infty} m_w\left(S/P\right)=\frac{1-2C}{2\left(1-C\right)}<1. \tag{25}$$

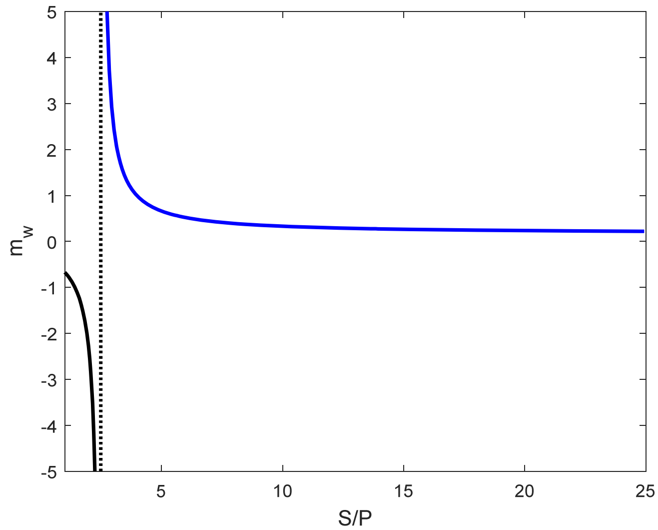


Fig. 1. The function $m_w\left(S/P\right)$. Vertical dotted line is $S/P=1/C$.

With the expression of $g''(m)$ given by (23), we have:

**Theorem 3**: If $1<S/P\le(2-C)/C$, then $g''(m)\ge 0$ for $0\le m\le 1$. If $S/P>(2-C)/C$, then

$$g''(m)=\begin{cases} >0 & \text{for} \quad 0\le m<m_w\left(S/P\right) \\ 0 & \text{for} \quad m=m_w\left(S/P\right) \\ <0 & \text{for} \quad m_w\left(S/P\right)<m\le 1 \end{cases} \tag{26}$$

**Proof**: Suppose that $1<S/P<1/C$. Then, from (24), we have $A<0$ and $B<0$. And from (23), it is obvious that $g''(m)\ge 0$ if, and only if, $Am+B\le 0$, i.e., $m\ge -B/A$. Since $-B/A\le 0$, we have $g''(m)\ge 0$ for $0\le m\le 1$.

For $S/P=1/C$, it is clear from (24) that $A=0$ and $B<0$. Therefore, from (23), $g''(m)>0$.

Now, suppose that $1/C<S/P\le(2-C)/C$. Then, from (24), we have $A>0$ and $B<0$. And from (23), it is obvious that $g''(m)\ge 0$ if and only if $Am+B\le 0$, i.e., $m\le -B/A$. Since $-B/A=m_w\left(S/P\right)$ is a decreasing function of $S/P$, as it can be easily shown, we have

$$-B/A=m_w\left(S/P\right)\ge m_w\left(\frac{2-C}{C}\right)=1, \tag{27}$$

and, again $g''(m)\ge 0$ for $0\le m\le 1$.

Finally, suppose $S/P>(2-C)/C$. Then, from (24), we have $A>0$ and $B<0$, and now, since $-B/A=m_w\left(S/P\right)$ is a decreasing function of $S/P$, we get

$$0\le -B/A=m_w\left(S/P\right)\le m_w\left(\frac{2-C}{C}\right)=1. \tag{28}$$

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Therefore, since $A > 0$, we have $Am + B < 0$ for $0 \le m < m_w(S/P)$, $Am + B = 0$ for $m = m_w(S/P)$, and $Am + B > 0$ for $m_w(S/P) < m \le 1$; and consequently $g''(m)$ is positive in $[0, m_w(S/P))$, negative in $(m_w(S/P), 1]$, and equal zero when $m = m_w(S/P)$.

By using Theorem 3 above, we can prove the following result about the derivative of the function $g$ given by (13).

**Theorem 4**: Let

$$C_1 = \frac{Cb}{log10} \approx 0.0582, \quad \text{and} \quad C_2 = \frac{1+C_1}{C_1} \approx 18.1834. \tag{29}$$

If $1 < S/P < C_2$, then $g'(m)$, given by (22), is negative for $0 \le m \le 1$. Moreover, there exists a constant $k \in (0,1)$ such that $|g'(m)| \le k$, for $0 \le m \le 1$.

**Proof**: From (22), we have

$$g'(m) = \frac{bC(1 - S/P)}{log10\, U(m) W(m)}, \tag{30}$$

where $U(m) = m + (1-m)(S/P)C$, and $W(m) = m + (1-m)C$. Therefore, if $1 < S/P < C_2$, it is obvious that $g'(m) < 0$ for $0 \le m \le 1$. Moreover, from Theorem 3, if $1 < S/P \le (2-C)/C$, we have $g''(m) \ge 0$ for $0 \le m \le 1$, and therefore $g'(m)$ is an increasing function of $m$. Thus, for $0 \le m \le 1$, we have

$$|g'(m)| = -g'(m) \le -g'(0) = \frac{C_1(S/P - 1)}{C^2(S/P)} < \frac{C_1}{C^2} = k_1 \approx 0.3601 < 1. \tag{31}$$

Finally, from Theorem 3, if $S/P > (2-C)/C$, then $g'(m)$ is increasing in $[0, m_w(S/P))$, and decreasing in $(m_w(S/P), 1]$. Hence, we have

$$|g'(m)| \le \max\{-g'(0), -g'(1)\} = \max\left\{\frac{C_1(S/P-1)}{C^2(S/P)},\ C_1(S/P-1)\right\} = h(S/P) =$$
$$= \begin{cases} \dfrac{C_1(S/P-1)}{C^2(S/P)} & \text{for} \quad (2-C)/C < S/P \le 1/C^2 \\ C_1(S/P-1) & \text{for} \quad 1/C^2 < S/P \end{cases} \tag{32}$$

Figure 2 shows $h(S/P)$ versus the ratio $S/P$ for $S/P > (2-C)/C \approx 3.9751$. It is obvious that $h(S/P)$ is an increasing function of the ratio $S/P$. It can also be checked that $h(C_2) = 1$. Since $S/P < C_2$, there exists $\varepsilon$ $(0 < \varepsilon < 1)$, such that $S/P < C_2 - \varepsilon$. Thus, from (31), we have, for $0 \le m \le 1$,

$$|g'(m)| \le h(S/P) \le h(C_2 - \varepsilon) = C_1(C_2 - \varepsilon - 1) = 1 - \varepsilon C_1 = k_2 < 1, \tag{33}$$

and the proof is concluded.

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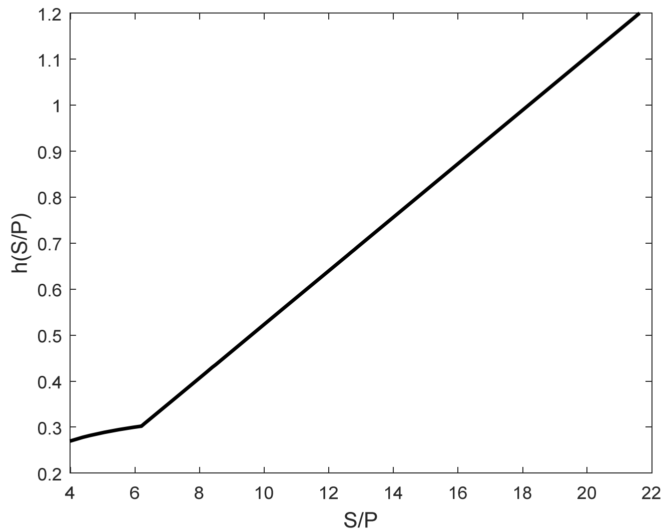


Fig. 2. The function $h(S/P)$ for $S/P > (2-C)/C \approx 3.9751$ .

Let $g$ be given by (13). Since $L_p < 5,$ we have

$$g_0 = g(0) = a + b\,log_{10}(L_s) \qquad \text{and} \qquad g_1 = g(1) = a + b\,log_{10}(L_p) < 1. \qquad (34)$$

Note that with the values for $a$ and $b$ given by (8), we have $a + b\,log_{10}5 = 1,$ and therefore $g_0 \le 1$ when $L_s \le 5,$ and $g_0 > 1$ when $L_s > 5.$ From Theorem 4, if $1 < S/P < C_2,$ then $g(m)$ is a decreasing function of $m$. Therefore, there exists $m_L \in (0,1),$ depending on the ratio $S/P,$ such that $g(m_L) = 1.$ For $m_L = m_L(S/P),$ it can be shown that

$$m_L(S/P) = \frac{C(S/P - 5/L_p)}{C(S/P) + (1-C)(5/L_p) - 1}, \qquad \text{for} \qquad L_s > 5. \qquad (35)$$

Figure 3 shows $m_L(S/P)$ (solid curve), for some given $L_p$ ( $L_p = 3.0$ in black, and $L_p = 0.5$ in blue), versus $S/P,$ between $5/L_p$ and $C_2$. It is also shown $g_1$ (dotted line), and it can be seen that $m_L(S/P)$ is less than $g_1,$ for a given $L_p$.

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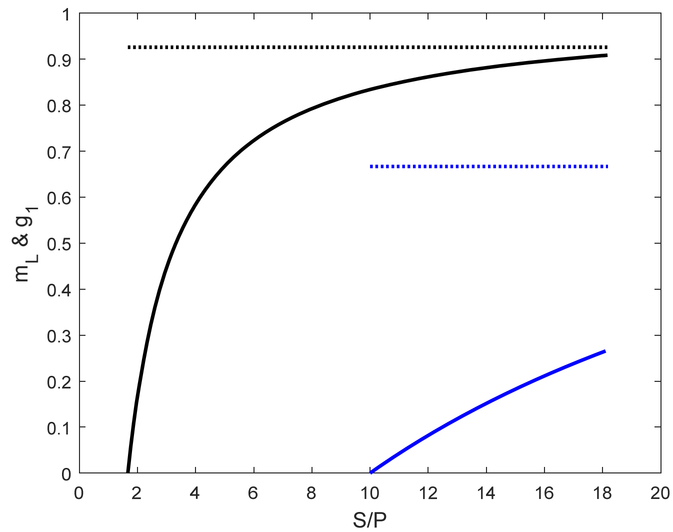


Fig. 3. The functions $m_L(S/P)$ (solid curves) and $g_1$ (dotted lines) for $S/P$ between $5/L_p$ and $C_2$. Black and blue curves correspond to $L_p = 3.0$, and $L_p = 0.5$ respectively.

Now, we can state the following result regarding the function $g$.

**Theorem 5**: Suppose a fixed $L_p < 5$, and $1 < S/P < C_2$. Then, we have:

(i) If $L_s \le 5$, then $g(m) \in [g_1, g_0] \subseteq [0,1]$, for $m \in [0,1]$.

(ii) If $L_s > 5$, then $g(m) \in [g_1, 1] \subseteq [m_L(S/P), 1]$, for $m \in [m_L(S/P), 1]$.

**Proof**: Suppose that $1 < S/P < C_2$. Then, from Theorem 4, $g$ is a decreasing function on $[0,1]$, and therefore $g_0 > g_1 > 0$.

If $L_s \le 5$, then $g_0 \le 1$, and $g(m) \in [g_1, g_0] \subseteq [0,1]$, for $m \in [0,1]$.

If $L_s > 5$, then $g_0 > 1$, and, since $g$ is a decreasing function on $[0,1]$, there exists $m_L = m_L(S/P) \in (0,1)$ such that $g(m_L) = 1$. By solving the equation $g(m) = 1$, it can be found that $m_L = m_L(S/P)$ is given by (35). Moreover, we have

$$m_L'(S/P) = \frac{C(5/L_p - 1)}{\left(C(S/P) + (1-C)(5/L_p) - 1\right)^2} > 0. \quad (36)$$

Therefore, for any fixed $L_p < 5$, $m_L$ is an increasing function of $S/P$ as shown by Fig. 3, where the solid black curve corresponds to $L_p = 3$, and the solid blue curve corresponds to $L_p = 0.5$. It can be seen that for $S/P < C_2$, we have $m_L(S/P) < g_1$. For proving this, it is enough to show that $m_L(C_2) \le g_1$, since $m_L$ is an increasing function of $S/P$. If we define the function

$$q(L_p) = C(C_2 - 5/L_p) - \left[CC_2 + (1-C)(5/L_p) - 1\right]\left(a + b\,log_{10}(L_p)\right), \quad (37)$$

then, $m_L(C_2) \le g_1$ is equivalent to $q(L_p) \le 0$. It is obvious that $q(5) = 0$. Moreover, since $L_p < 5$, $L_s > 5$, and $1 < S/P < C_2$, we have $5/C_2 < L_p < 5$. Now, we want to show that $q$ is an increasing function of $L_p$ in the interval $(5/C_2, 5)$. Since

$$q'(L_p) = \frac{5\left[C + (1-C)\left(a + b\,log_{10}(L_p)\right)\right]}{L_p^2} - \frac{b\left[CC_2 + (1-C)(5/L_p) - 1\right]}{L_p log10}, \quad (38)$$

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it is clear that $q'(L_p) \geq 0$ if, and only if,

$$C + (1-C)\left(a + b\,log_{10}(L_p)\right) \geq \frac{bL_p\left[CC_2 + (1-C)(5/L_p) - 1\right]}{5log10}, \tag{39}$$

which is also equivalent to

$$\frac{C}{1-C} + a - \frac{b}{log10} \geq -b\,log_{10}(L_p) + \frac{bL_p(CC_2 - 1)}{5(1-C)log10}. \tag{40}$$

The left-hand side of the above inequality is a constant, while the right-hand side depends on $L_p$. If we let

$$p(L_p) = -b\,log_{10}(L_p) + \frac{bL_p(CC_2 - 1)}{5(1-C)log10}, \tag{41}$$

then,

$$p'(L_p) = \begin{cases} < 0 & \text{when} \quad L_p < C_3 \\ 0 & \text{when} \quad L_p = C_3 \\ > 0 & \text{when} \quad L_p > C_3 \end{cases} \tag{42}$$

where $C_3 = 5(1-C)/(CC_2 - 1) \approx 0.4739$.

Hence, $p$ is a decreasing function of $L_p$ in the interval $(5/C_2, C_3)$, and increasing in the interval $(C_3, 5)$. Furthermore, it can be verified that

$$\frac{C}{1-C} + a - \frac{b}{log10} = p(5) > p(5/C_2). \tag{43}$$

Therefore, the inequality (40) is true for $L_p \in (5/C_2, 5)$, and consequently $q$, given by (37), is an increasing function of $L_p$ in the interval $(5/C_2, 5)$. Then, $q(L_p) \leq q(5) = 0$, which is equivalent to $m_L(C_2) \leq g_1$, and then, $m_L(S/P) < g_1$ for $S/P < C_2$, which concludes the proof.

We are now ready to state another convergence theorem for the fixed-point iteration method when $S/P > 1$.

**Theorem 6**: Suppose a fixed $L_p < 5$, and $1 < S/P < C_2$. Then, the fixed-point iteration method given by (14), is convergent for any $m_0 \in [0,1]$ when $L_s \leq 5$, and for any $m_0 \in \left[m_L(S/P), 1\right]$ when $L_s > 5$.

**Proof**: From Theorem 4, there exists a constant $k \in (0,1)$ such that $|g'(m)| \leq k$, for $0 \leq m \leq 1$. Moreover, if $L_s \leq 5$, from Theorem 5 we have $g(m) \in [0,1]$, for $m \in [0,1]$. Then, by using Lemma 1, the fixed-point iteration method given by (14) is convergent for any $m_0 \in [0,1]$. In the case $L_s > 5$, again from Theorem 5 we have $g(m) \in \left[m_L(S/P), 1\right]$, for $m \in \left[m_L(S/P), 1\right]$, and according to Lemma 1, the fixed-point iteration method given by (14) is convergent for any $m_0 \in \left[m_L(S/P), 1\right]$.

Theorems 2 and 6 provide a sufficient condition for the convergence of the fixed-point iteration method (14), namely $S/P < C_2$. However, the fixed-point iteration method may also be convergent when this condition fails, i.e., $S/P > C_2$.

Moreover, in both Theorems 2 and 6, the convergence is guaranteed when choosing properly the initial value $m_0$. CIE recommended the initial guess $m_0 = 0.5$ for the fixed-point iteration method, i.e., the middle point of the interval $[0,1]$. Gao *et al.* [4] proved that the function $g$ has a unique fixed point in $[0,1]$, which is equivalent to assert that the equation

Optics EXPRESS

$F(m)=0$, where $F$ is given by (7), has a unique solution in $[0,1]$. However, when $L_s > 5$, Theorem 6 indicates that the initial guess $m_0$ should be in the interval $\left[m_L(S/P),1\right]$ to ensure the convergence of the fixed-point iteration method. It is well known that the performance of the iteration method depends on the initial guess $m_0$. Therefore, if $m_0$ is chosen "close" to the fixed point $m^*$, then the number of iterations to get a good estimation of $m^*$ will be smaller. From the results presented in this paper, it follows that

$$m^* \in [g_0, g_1] \quad \text{for} \quad S/P \le 1 \tag{44}$$

$$m^* \in [g_1, g_0] \quad \text{for} \quad S/P > 1 \quad \text{and} \quad L_s \le 5 \tag{45}$$

$$m^* \in [g_1, 1] \quad \text{for} \quad S/P > 1 \quad \text{and} \quad L_s > 5 \tag{46}$$

The above information can help to choose a "better" initial guess $m_0$, and will be discussed in the next section.

## 3. Performance of fixed-point iteration method with new initial strategy

We have shown that the fixed-point iteration method (14) is convergent for $S/P < C_2$, whenever a proper initial value $m_0$ is chosen. In order to test numerically this result, we have taken some values for $L_p$ from 0.1 to 4.9, namely 0.1, 0.3, 0.5, 0.7,…,4.9, that is a total number of 25 values for $L_p$. Similarly, we have taken some values for the ratio $S/P$ from 0.1 to 1, namely 0.1, 0.15, 0.2, 0.25,…,1, and from 1.1 to 18.1 we have taken the values 1.1, 2.1, 3.1, 4.1,…,18.1, which make a total number of 37 values for the ratio $S/P$. Therefore, we have considered $925 = 25\times 37$ cases for testing the performance of the fixed-point iteration method with the original initial guess $m_0 = 0.5$ (the current CIE MES2 method [1]), and also with a new initial strategy. Regarding the selected range of values for the ratio $S/P$ from 0.1 to 18.1, it must be remarked that, for most current conventional light sources, these values are low, in a range around 0.0-3.0 [7]. However, higher values, up to a maximum of around 73.0, which are associated to blue monochromatic lights, are also possible [5]. For example, Nizamoglu *et al*. [8] have reported $S/P$ values of 5.15 for nanocrystal hybridized LEDs, and previous researchers [4] have considered $S/P$ values up to 50, for theoretical light sources based on Hung *et al*. method [9]. The initial value $m_0$ in the new strategy, that we propose, is given by

$$m_0 = \begin{cases} 0.5(g_0 + g_1) & \text{if} \quad L_s \le 5 \\ 0.5(1+g_1) & \text{if} \quad L_s > 5 \end{cases} \tag{47}$$

where $g_0$ and $g_1$ are given by (34). We have fixed tolerance $\varepsilon = 10^{-5}$ for the convergence, and have limited the number of iterations to 200, in order to avoid the program running during a very long time.

Figure 4 shows the contours with the number of iterations needed for the convergence of the fixed-point iteration method, using the initial value $m_0 = 0.5$, as recommended by CIE [1]. In 921 cases, from the total of 925, the convergence is obtained when computing less than 100 iterations. In two cases, namely $S/P = 18.1$ and $L_p = 4.3$, and $S/P = 18.1$ and $L_p = 4.5$, have been necessary 120 and 157 iterations, respectively, for the convergence. And for $S/P = 18.1$ and $L_p = 4.7$, and $S/P = 18.1$ and $L_p = 4.9$, the convergence is not achieved after 200 iterations.

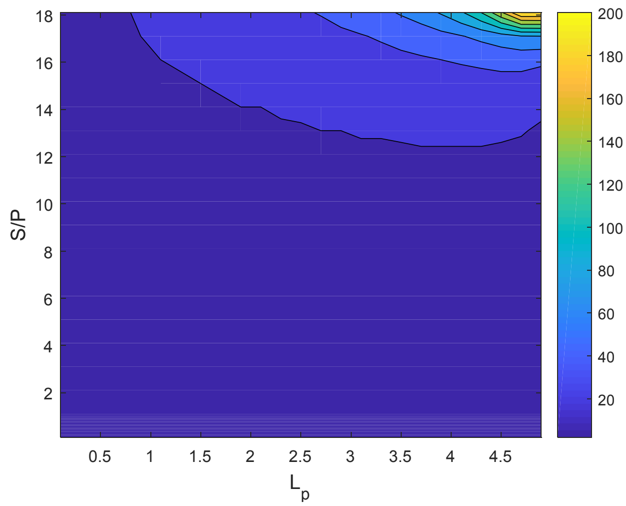


Fig. 4. Contours plots with the number of iterations for the convergence of the fixed-point iteration method with initial value $m_0 = 0.5$, as a function of $S/P$ and $L_p$. Different colors represent different numbers of iterations needed, as shown on the vertical bar on the right.

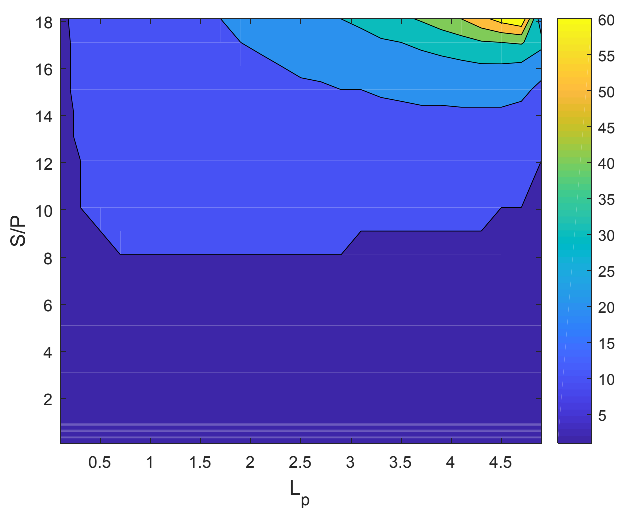


Fig. 5. Contours plots with the number of iterations for the convergence of the fixed-point iteration method with the new initial strategy (see (47)) as a function of $S/P$ and $L_p$. Different colors represent different numbers of iterations needed, as shown on the vertical bar on the right.

Figure 5 shows the contours with the number of iterations needed for the convergence of the fixed-point iteration method, using as initial value $m_0$, in each case, the value provided by the new strategy proposed in (47). Now, in all 925 cases under study, the convergence is obtained when computing less than 70 iterations. This fact proves the validity of our convergence analysis. In addition, let $N_1$ and $N_2$ be the number of iterations needed for the convergence, when using $m_0 = 0.5$ and the initial value provided by (47), respectively. It has been found that $N_2 = N_1 + 1$ only for 3 cases, $N_2 = N_1$ for 175 cases, and $N_2 < N_1$ for 747

Optics EXPRESS

cases. Thus, with the proposed new initial strategy (47), the fixed-point iteration method converges faster than with the CIE recommended initial value [1] in the $80\%$ of the cases.

## 4. Conclusions

The MES2 system was recommended by CIE [1] to compute the mesopic luminance, using a fixed-point iteration method (see (14)). In this process of computation of the mesopic luminance, a numerical solution of a nonlinear equation $F(m)=0$ is searched (see (7)). Shpak *et al.* [5] have proposed new values for the parameters $a$ and $b$ involved in that equation (see (8)). With these new values for $a$ and $b$, Gao *et al.* [4] have shown that the nonlinear equation $F(m)=0$ has a unique solution in $(0,1)$, whenever a condition for $L_s$ and $L_p$, given by (3), is satisfied (see (9)). However, Gao *et al.* [4] and Shpak *et al.* [5] have found that the fixed-point iteration method may be not convergent for large values of $S/P=L_s/L_p$. Shpak *et al.* [5] pointed out that this ratio should not be larger than 17 in order to have convergence. In this paper a theoretical consideration on the convergence has been given, and it has been found that the fixed-point iteration method converges when using appropriate initial values, and the ratio $S/P$ is smaller than $C_2 \approx 18.1834$. For values of $S/P$ larger than $C_2$, there is no guarantee for the convergence of the fixed-point iteration method. Values of the ratio $S/P$ for current light sources are usually lower than 3.0 [7], but Nizamoglu *et al*. [8] have reported higher $S/P$ values of 5.15 for nanocrystal hybridized LEDs. The theoretical upper limit for the ratio $S/P$ is around 73.0 [5]. Therefore, our current analyses considering sources with high $S/P$ values make sense, because we are proposing a valid CIE method for both current and future light sources. Moreover, for values of $S/P$ smaller than $C_2$, a new strategy (47) for the choice of the initial value $m_0$ has been proposed. In many cases, this new strategy produces a remarkable reduction of the number of iterations needed to achieve convergence, compared when using $m_0=0.5$ as initial value, as currently recommended in CIE MES2 method [1].

## Funding

National Natural Science Foundation of China (Grant numbers: 61575090, 61775169); Ministry of Economy and Competitiveness of Spain (Research projects: FIS2016-80983-P and MTM2014-60594-P), with contribution of the European Regional Development Fund (ERDF).

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