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Stochastic differential equations in a scale of Hilbert spaces

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Abstract

A stochastic differential equation with coefficients defined in a scale of Hilbert spaces is considered. The existence and uniqueness of finite time solutions is proved by an extension of the Ovsyannikov method. This result is applied to a system of equations describing non-equilibrium stochastic dynamics of (real-valued) spins of an infinite particle system on a typical realization of a Poisson or Gibbs point process in $\mathbb{R}^n$.

1 Introduction

Evolution differential and stochastic differential equations in Banach spaces play hugely important role in many parts of mathematics and its applications. This class of equations unifies infinite systems of ordinary differential equations and partial differential equations (realized in l_p-type spaces of sequences and Sobolev-type spaces, respectively), and their stochastic counterparts, see e.g. [15], [13] and references therein and modern developments in e.g. [8].

So let us consider a stochastic differential equation (SDE) of the form

$$d\xi(t) = f(\xi(t))dt + B(\xi(t))dW(t) \tag{1.1}$$

in a Banach space X, where f and B are given vector and operator fields on X respectively and W a suitable Wiener process in X. The standard approach to such equations usually requires that $f = A + \phi$, where (C1) A is a generator of a C_0-semigroup in X, and (C2) ϕ and B satisfy certain Lipschitz or dissipativity conditions in X. Then the existence, uniqueness and regularity of solutions of the corresponding Cauchy problem can be proved.

This classical theory does not cover some important examples motivated by e.g. problems of statistical mechanics and hydrodynamics. In particular, there are situations where A fails to satisfy condition (C1) but is instead bounded in a scale of Banach spaces X_α, $\alpha \in \mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}^1$ is an interval and $X_\alpha \subset X_\beta$ if $\alpha \leq \beta$. That is, A is a bounded operator acting from X_α to X_β for any $\alpha < \beta$, and

$$\|Ax\|_{X_\beta} \leq c\,(\beta - \alpha)^{-1}\,\|x\|_{X_\alpha} \tag{1.2}$$

for all $x \in X_\alpha$ and some constant $c > 0$ (independent of α and β but possibly dependent on the interval $\mathcal{A}$).

In this framework, equation (1.1) with no diffusion term ($B \equiv 0$) has been studied by Ovsyannikov's method, see e.g. [15] and modern developments and references in [16], [4]. Moreover, instead of (C2), the non-linear drift term ϕ is allowed to satisfy a generalized Lipschitz condition in the scale $(X_\alpha)_{\alpha\in\mathcal{A}}$ with singularity of the type as in (1.2) (see [25, 29, 4]). The price to pay here is that the existence of a solution with initial value in X_α can only be proved in the bigger space X_β, $\beta > \alpha$. The lifetime of this solution depends on α and β (and the interval $\mathcal{A}$ itself).

The aim of the present work is to extend Ovsyannikov's method to the case of stochastic differential equations. We require the drift f to be a map from X_α to X_β for any $\alpha < \beta$ and satisfy a generalized Lipschitz condition with singularity $(\beta - \alpha)^{-1/2}$ (and make similar assumption about the diffusion coefficient B), see Condition 2.1 given in the next section, and prove the existence and uniqueness of finite time solutions of the corresponding Cauchy problem. Observe that the singularity allowed here is weaker than in the deterministic case (cf. (1.2)), which is related to the specifics of the Ito integral estimates. As in the deterministic case, the solution will live in the scale X_α, $\alpha \in \mathcal{A}$. For simplicity, we assume that all X_α are Hilbert spaces, although all our results hold in a more general situation of suitable Banach spaces. The proof is based on the contractivity of the corresponding integral transformation of a weighted space of trajectories in $\cup_{\alpha\in\mathcal{A}} X_\alpha$ (constructed similar to the ones used in [25, 29, 4]).

Our main example is motivated by the study of countable systems of particles randomly distributed in a metirc space $\mathfrak{X}$ (which in this paper is supposed to be a Euclidean space, $\mathfrak{X} = \mathbb{R}^n$). Each particle is characterized by its position x and an internal parameter (spin) $\sigma_x \in S = \mathbb{R}^1$. For a given fixed ("quenched") configuration γ of particle positions, which is a locally finite subset of $\mathbb{R}^n$, we consider a system of stochastic differential equations describing (non-equilibrium) dynamics of spins σ_x, $x \in \gamma$. Two spins σ_x and σ_y are allowed to interact via a pair potential if the distance between x and y is no more than a fixed interaction radius r, that is, they are neighbors in the geometric graph defined by γ and r. Vertex degrees of this graph are typically unbounded, which implies that the coefficients of the corresponding equations cannot be controlled in a single Hilbert or Banach space (in contrast to spin systems on a regular lattice, which have been well-studied, see e.g. [14] and modern developments in [19], and references therein). However, under mild conditions on the density of γ (holding for e.g. Poisson and Gibbs point processes in $\mathbb{R}^n$), it is possible to apply the approach discussed above and construct a solution in the scale of Hilbert spaces S_α^γ of weighted sequences $(q_x)_{x\in\gamma} \in S^\gamma$ such that $\sum_{x\in\gamma} |q_x|^2 e^{-\alpha|x|} < \infty,\ \alpha > 0$.

Construction of non-equilibrium stochastic dynamics of infinite particle systems of the aforementioned type has been a long-standing problem, even in the case of linear drift and a single-particle diffusion coefficient. It has become important in the framework of analysis on spaces $\Gamma(\mathfrak{X}, S)$ of configurations $\{(x, \sigma_x)\}_{x\in\gamma}$ with marks (see e.g. [12]), and is motivated by a variety of applications, in particular in modeling of non-crystalline (amorphous) substances, e.g. ferrofluids and amorphous magnets, see e.g. [27], [26, Section 11], [6] and [10, 11]. $\Gamma(\mathfrak{X}, S)$ possesses a fibration-like structure over the space $\Gamma(\mathfrak{X})$ of position configurations γ, with the fibres identified with S^γ, see [10]. Thus the construction of spin dynamics of a quenched system (in S^γ) is complementary to that of the dynamics in $\Gamma(\mathfrak{X})$.

Various aspects of the study of deterministic (Hamiltonian) and stochastic evolution of configurations $\gamma \in \Gamma(\mathfrak{X})$ have been discussed by many authors, see e.g. [23, 18, 5, 3, 17] and references given there. It is anticipated that (some of) these results can be combined with the approach proposed in the present paper allowing to build stochastic dynamics on the marked configuration space $\Gamma(\mathfrak{X}, S)$.

Another potential field of applications of the present results is the study of stochastic perturbations of certain (non-local) partial differential equations, cf. [4] and [7].

Observe that the family $X_\alpha = S_\alpha^\gamma$, $\alpha > 0$, forms the dual to nuclear space $\Phi' = \cup_\alpha X_\alpha$. SDEs on such spaces were considered in [20], [21]. The existence of solutions to the corresponding martingale problem was proved under assumption of continuity of coefficients on Φ' and their linear growth (which, for the diffusion coefficient, is supposed to hold in each α-norm). Moreover, the existence of strong solutions requires a dissipativity-type estimate in each α-norm, too, which does not hold in our framework.

In the last subsection, we prove the uniqueness of the infinite-particle dynamics using more classical methods, which generalise those applied to deterministic systems in [24], [9].

2 Setting

Let us consider a family of separable Hilbert spaces X_α indexed by $\alpha \in [\alpha_*, \alpha^*]$ with fixed $0 \le \alpha_*, \alpha^* < \infty$, and denote by $\|\cdot\|_\alpha$ the corresponding norms. We assume that

$$X_\alpha \subset X_\beta \text{ and } \|u\|_\beta \le \|u\|_\alpha \text{ if } \alpha < \beta,\ u \in X_\beta, \tag{2.1}$$

where the embedding means that X_α is a vector subspace of X_β. When speaking of these spaces and related objects, we will always assume that the range of indices is $[\alpha_*, \alpha^*]$, unless stated otherwise.

Let $W(t)$ be a cylinder Wiener process in a separable Hilbert space $\mathcal{H}$ defined on a suitable filtered probability space. Introduce notation

$$H_\beta \equiv HS(\mathcal{H}, X_\beta) := \{\text{Hilbert-Schmidt operators } \mathcal{H} \to X_\beta\}.$$

We will denote by $\|\cdot\|_{H_\beta}$ its standard norm. Our aim is to construct a strong solution of equation (1.1), that is, a solution of the stochastic integral equation

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t B(u(s))dW(s), \tag{2.2}$$

with coefficients acting in the scale of spaces (2.1). More precisely, we assume that $f : X_\alpha \to X_\beta$ and $B : X_\alpha \to H_\beta$ for any $\alpha < \beta$, and the following Lipschitz-type condition is satisfied.

Condition 2.1 *There exists a constant L such that*

$$\|f(u) - f(v)\|_\beta \le \frac{L}{|\beta - \alpha|^{1/2}} \|u - v\|_\alpha \tag{2.3}$$

and

$$\|B(u) - B(v)\|_{H_\beta} \le \frac{L}{|\beta - \alpha|^{1/2}} \|u - v\|_\alpha \tag{2.4}$$

for any $\alpha < \beta$ and all $u, v \in X_\alpha$.

We denote by $\mathcal{GL}^{(1)}$ and $\mathcal{GL}^{(2)}$ the sets of mappings f and B under conditions (2.3) and (2.4), respectively.

Remark 2.2 *The Lipschitz constant L may depend on α^* and α_*, as usually happens in applications.*

Remark 2.3 *In contrast to the classical Ovsyannikov method for deterministic equations, where the right-hand side of (2.3) is proportional to* $(\beta-\alpha)^{-1}$*, we have to require stronger bounds with the singularity* $(\beta-\alpha)^{-1/2}$*. This is due to the presence of the Ito stochastic integral in (2.2).*

Remark 2.4 *Setting* $v=0$ *in (2.3) and (2.4), we obtain linear growth conditions*

$$\|f(u)\|_\beta \le \frac{K}{|\beta-\alpha|^{1/2}}\left(1+\|u\|_\alpha\right)$$

and

$$\|B(u)\|_{H_\beta} \le \frac{K}{|\beta-\alpha|^{1/2}}\left(1+\|u\|_\alpha\right)$$

for some constant K*, any* $\alpha<\beta$ *and all* $u\in X_\alpha$*.*

Remark 2.5 *Assume that* ϕ *is Lipschitz continuous in each* X_α *with a uniform Lipschitz constant* M*. Then* $\phi\in\mathcal{GL}^{(1)}$ *with* $L=\sqrt{\alpha^*-\alpha_*}M$*.*

Remark 2.6 *Some authors have used the scale* X_α *such that* $X_\alpha\subset X_\beta$ *if* $\alpha>\beta$*. This framework can be transformed to our setting by an appropriate change of the parametrization, e.g.* $\alpha\mapsto\alpha^*-\alpha$*.*

3 Main results

Let us fix $b>0$ and define the function

$$p_b(\alpha,t) := 1-\left((\alpha-\alpha_*)\,b\right)^{-1}t,\ \alpha\in(\alpha_*,\alpha^*],\ t\in[0,(\alpha-\alpha_*)\,b).$$

Obviously, $p_b(\alpha,t)$ is decreasing in t and increasing in α, and satisfies inequality $0<p_b(\alpha,t)\le 1$.

We introduce the space M_b of square-integrable progressively measurable random processes $u:[0,(\alpha^*-\alpha_*)\,b)\to X_{\alpha^*}$ such that $u(t)\in X_\alpha$ for $t<(\alpha-\alpha_*)\,b$, and

$$|||u|||_b := \sup\left\{\left(\mathbb{E}\,\|u(t)\|_\alpha^2\,p_b(\alpha,t)\right)^{1/2} :\ \alpha\in(\alpha_*,\alpha^*],t\in[0,(\alpha-\alpha_*)\,b)\right\}<\infty.$$

Thus for any $u\in M_b$ there exists $C>0$ such that

$$\mathbb{E}\,\|u(t)\|_\alpha^2 \le \frac{C}{1-\left((\alpha-\alpha_*)\,b\right)^{-1}t},\ t<(\alpha-\alpha_*)\,b.$$

The pair M_b, $|||\cdot|||_b$ forms a separable Banach space. For any $a>b$ there is a natural map $O_{ab}:M_a\to M_b$ given by the restriction

$$O_{ab}u = u\restriction_{[0,(\alpha^*-\alpha_*)b)}.$$

Remark 3.1 *Similar spaces of deterministic functions* $u : [0, (\alpha^* - \alpha_*)\, b) \to X_{\alpha^*}$ *where used in [25, 29, 4].*

Remark 3.2 *For any fixed* $b > 0$, $T < (\alpha^* - \alpha_*)\, b$ *and* $\beta \in (Tb^{-1} + \alpha_*, \alpha^*]$ *consider the spaces* $E_{\beta,T}$ *and* $H_{\beta,T}$ *of square-integrable progressively measurable random processes* $u : [0,T) \to X_\beta$ *and* $h : [0,T) \to H_\beta$ *with finite norms*

$$\|u\|_{E_{\beta,T}} := \sup_{t\in[0,T)} \left(\mathbb{E}\, \|u(t)\|_\beta^2\right)^{1/2} \text{ and } \|h\|_{H_{\beta,T}} := \sup_{t\in[0,T)} \left(\mathbb{E}\, \|u(t)\|_{H_\beta}^2\right)^{1/2},$$

respectively. Let $u^{(T)} := u \restriction_{[0,T)}$ *be the restriction of a process* $u \in M_b$ *to time interval* $[0,T)$. *Observe that* $p_b(\beta, t) \geq c$ *for some constant* $c > 0$ *and all* $t \leq T$. *Thus* $\left\|u^{(T)}\right\|_{E_{\beta,T}} \leq c^{-1} |||u|||_b^2$ *and so* $u^{(T)} \in E_{\beta,T}$. *Moreover, it is clear that for any* $f \in \mathcal{GL}^{(1)}$ *and* $B \in \mathcal{GL}^{(2)}$ *we have* $f(u^{(T)}) \in E_{\beta,T}$ *and* $B(u^{(T)}) \in H_{\beta,T}$. *Indeed, we can fix* $\alpha \in (Tb^{-1} + \alpha_*, \beta)$ *(so that* $u^{(T)} \in E_{\alpha,T}$*) and apply estimates from Remark 2.4, which will show that* $\|f(u))\|_{E_{\beta,T}}, \|B(u(t))\|_{H_{\beta,T}} < \infty$.

From now on, we fix $f \in \mathcal{GL}^{(1)}$ and $B \in \mathcal{GL}^{(2)}$. For any $u \in M_b$ define

$$F(u)(t) = \int_0^t f(u(s))ds + \int_0^t B(u(s))dW(s) \in X_{\alpha^*},\ t < b\alpha^*. \tag{3.1}$$

According to Remark 3.2, $f(u^{(t)}(\cdot)) \in E_{\beta,t}$ and $B(u^{(t)}(\cdot)) \in H_{\beta,t}$ for any $\beta > b^{-1}t + \alpha_*$. Thus the right-hand side is well-defined in X_β with $\beta > b^{-1}t + \alpha_*$.

Consider equation

$$u = u_0 + F(u) \tag{3.2}$$

with $u_0 \in X_{\alpha^*}$, cf. (2.2), and set $b^* := \frac{\sqrt{1+(\alpha^*-\alpha_*)L^{-1}}-1}{2(\alpha^*-\alpha_*)}$. The following theorem states the main existence result of this paper.

Theorem 3.3 (Existence) *Equation (3.2) has a solution* $u \in M_b$ *for any* $b < b^*$. *It is unique in the following sense: if* $u_1 \in M_{b_1}$ *and* $u_2 \in M_{b_2}$ *are two solutions and* $b_1 \leq b_2 < b^*$ *than* $O_{b_2 b_1} u_2 = u_1$.

Proof. It is sufficient to show that the map

$$u \mapsto u_0 + F(u)$$

is contractive in M_b with $b < b^*$, which in turn will imply the existence of its (unique) fixed point. It is straightforward that if u is the fixed point in M_{b_1} then $O_{b_2 b_1} u$ is the fixed point in M_{b_2}. Thus the statement of the theorem follows from Theorem 4.1 and Corollary 4.2, which will be proved in the next section. □

Of course the choice of the weight function p_b is somehow ambiguous. The following statement is a corollary of Theorem 3.3 formulated in a slightly more invariant form (although with some loss of information).

Corollary 3.4 *Equation (3.2) has a solution* $u : [0, (\alpha^* - \alpha_*)\, b^*) \to X_{\bar{\alpha}}$. *Moreover,* $u \restriction_{[0,T)} \in E_{\beta,T}$ *for any* $T < (\alpha^* - \alpha_*)\, b^*$ *and* $\beta \in (T/b^* + \alpha_*, \alpha^*]$.

Theorem 3.3 establishes the uniqueness of the solution in M_b. A natural question that arises here is whether there might be a solution that does not belong to any M_b. An answer is given by the following (somewhat stronger) uniqueness result.

Theorem 3.5 (Uniqueness) *Fix* $\beta \in \left[\alpha_*, \frac{\alpha_*+\alpha^*}{2}\right]$ *and* $b < b^*$ *and assume that* $u \in E_{\beta,T}$*, where* $T = (\alpha^* - \alpha_*)\, b$*, is a solution of equation (3.2). Then* $u \in M_b$ *and coincides in this space with the solution from Theorem 3.3.*

Proof. First observe that $E_{\alpha_*,T} \subset M_b$, which implies the statement for $\beta = \alpha_*$.

Let now $\beta \in (\alpha_*, \alpha^*)$ and us consider the Banach space $M_{b,\beta}$ defined by replacing α_* with β in the definition of M_b (so that $M_b = M_{b,\alpha_*}$). Then we clearly have $OE_{\beta,T} \subset M_{b,\beta}$, with the operator O given by the restriction to time interval $[0, (\alpha^* - \beta)\, b)$. Moreover, $OM_b \subset M_{b,\beta}$. Indeed, for any $v \in M_b$ and $t \in [0, (\alpha^* - \beta)\, b)$ we have $v(t) \in \bigcap_{\alpha > tb^{-1}+\alpha_*} X_a \subset \bigcap_{\alpha > tb^{-1}+\beta} X_a$ because $\beta > \alpha_*$. A direct check shows that $|||u|||_b \geq |||u|||_{b,\beta}$.

Observe that the proof of Theorem 4.1 (and thus of Theorem 3.3) can be accomplished in the space $M_{b,\beta}$ instead of M_b, which implies that Ou is the unique solution of (3.2) in $M_{b,\beta}$. Let now $v \in M_b$ be the solution constructed in Theorem 3.3. By the uniqueness part of that theorem, we have $Ou = Ov$, which means that $u(t) = v(t)$, $t \in [0, (\alpha^* - \beta)\, b)$. Observe that the assumption $\beta \leq \frac{\alpha_*+\alpha^*}{2}$ implies that $(\alpha^* - \beta)\, b \geq (\beta - \alpha_*)\, b$. By Lemma 3.6 below we have $u \in M_b$, and the statement of the theorem follows from the uniqueness in M_b. □

Lemma 3.6 *Let* $\beta \in (\alpha_*, \alpha^*)$, $u \in E_{\beta,T}$ *and there exist* $v \in M_b$ *such that*

$$u \restriction_{[0,(\beta-\alpha_*)b)} = v \restriction_{[0,(\beta-\alpha_*)b)} . \tag{3.3}$$

Then $u \in M_b$.

Proof. $u \in M_b$ iff $\exists C > 0$ such that $\forall \alpha \in (\alpha_*, \alpha^*)$ we have $u(t) \in X_\alpha$ for $t < (\alpha - \alpha_*)\, b$ and $\sup_{t<(\alpha-\alpha_*)b} \mathbb{E} \|u(t)\|_\alpha^2\, p_b(\alpha, t) < C$. In our case, this holds for $\alpha < \beta$ because of (3.3) and for $\alpha \geq \beta$ because of the inclusion $u \in E_{\beta,T}$ and the bound $p_b(\alpha, t) < 1$. □

Our main example is given by an infinite system of SDEs describing stochastic dynamics of certain infinite particle spin system and will be discussed in Section 5. Here, we provide an example of a very different type, which can also be dealt with by much simpler methods and thus clarifies up to some extend the statement of Theorem 3.3.

Remark 3.7 *For simplicity, we required X_α to be Hilbert spaces. This is in fact not essential and the case of a scale of suitable Banach spaces can be treated in a similar way.*

Example 3.8 *Consider the following SPDE on the 1-dimensional torus $\mathbb{T}$:*

$$du(t) = cu_x(t)dW(t), \tag{3.4}$$

where $u(t) \in C^1(\mathbb{T})$, $u_x(t,x) := \frac{\partial}{\partial x}u(t,x)$, $x \in \mathbb{T}$, $c \in \mathbb{R}$ and W is a real-valued Wiener process. Denote by $\widehat{v}(k)$, $k \in \mathbb{Z}$, the Fourier coefficients of $v \in L^2(\mathbb{T})$ and define the scale of Hilbert spaces

$$X_\alpha := \left\{ v \in L^2(\mathbb{T}) : \|v\|_\alpha := \left(\sum_{k\in\mathbb{Z}} |\widehat{v}(k)|^2 e^{\alpha|k|^2} \right)^{1/2} < \infty \right\},\ \alpha > 0.$$

It is clear that $X_\alpha \subset X_\beta,\ \alpha > \beta$ (cf. Remark 2.6). Let $\mathcal{H} := \mathbb{R}$ and define $B : X_\alpha \to HS(\mathcal{H}, X_\beta)$ by the formula $B(v)h = cv_x h$, $v \in X_\alpha$, $h \in \mathcal{H}$. Equation (3.4) can now be written in the form (2.2). Moreover, it can be shown by a direct computation that B satisfies condition (2.4). Thus, by Theorem 3.3 adopted to this setting, for any $\beta < \alpha$ and an initial condition $u(0) \in X_\alpha$ there exists a solution $u(t) \in X_\beta$, $t < \tau(\alpha - \beta)$, where τ is a constant (independent of α and β but possibly dependent on their allowed range).

Observe that equation (3.4) can be solved explicitly. Indeed, the Fourier coefficients of $u(t)$ satisfy the equation

$$d\widehat{u}(t,k) = ick\widehat{u}(t,k)dW(t),\ k \in \mathbb{Z},$$

so that

$$\widehat{u}(t,k) = e^{tc^2k^2/2}e^{ickW(t)}\widehat{u}(0,k),\ k \in \mathbb{Z},$$

which in turn implies the equality

$$|\widehat{u}(t,k)|^2 = e^{tc^2k^2} |\widehat{u}(0,k)|^2,\ k \in \mathbb{Z}. \tag{3.5}$$

Fix any $\beta < \alpha$ and an initial condition $u(0) \in X_\alpha$. It follows directly from (3.5) that the solution $u(t)$ belongs to X_β for $t < c^{-2}(\alpha - \beta)$. It is also clear that the solution does not live in the scale of standard Sobolev spaces. Neither of course does B satisfy condition (2.4) in such a scale.

4 Proof of the contractivity.

In this section, we will show that F is a contraction in M_b with b sufficiently small.

Theorem 4.1 *For any $b > 0$, formula (3.1) defines the map $F : M_b \to M_b$. Moreover, F is Lipschitz continuous with Lipschitz constant $2bL\sqrt{(\alpha^* - \alpha_*) + b^{-1}}$.*

Proof. Let $u, v \in M_b$ and fix $\beta \leq \alpha^*$ and $t \in (0, b\beta)$. Then $F(u)(t), F(v)(t) \in X_\beta$, and we have the estimate

$$
\begin{aligned}
\mathbb{E}\, \|F(u)(t) - F(v)(t)\|_\beta^2 &\leq t\mathbb{E} \int_0^t \|f(u(s)) - f(v(s))\|_\beta^2 \, ds \\
&\quad + \mathbb{E} \int_0^t \|B(u(s)) - B(v(s))\|_{H_\beta}^2 \, ds \\
&\leq cL^2 \mathbb{E} \int_0^t \|u(s) - v(s)\|_{\alpha(s)}^2 \, (\beta - \alpha(s))^{-1} \, ds
\end{aligned}
$$

with $c = (b\,(\alpha^* - \alpha_*) + 1)$, for any $\alpha(s)$ satisfying $b^{-1}s + \alpha_* < \alpha(s) < \beta$. Then

$$
\begin{aligned}
\mathbb{E}\, \|F(u)(t) - F(v)(t)\|_\beta^2 &\leq cL^2 \mathbb{E} \int_0^t \|u(s) - v(s)\|_{\alpha(s)}^2 \, p_b(\alpha(s), s) \\
&\quad \times p_b(\alpha(s), s)^{-1} \, (\beta - \alpha(s))^{-1} \, ds \\
&\leq cL^2 \, |||u - v|||_b^2 \int_0^t p_b(\alpha(s), s)^{-1} \, (\beta - \alpha(s))^{-1} \, ds. \quad (4.1)
\end{aligned}
$$

We set

$$
\alpha(s) = \frac{1}{2} \left(\beta + b^{-1}s + \alpha_* \right).
$$

Then

$$
\beta - \alpha(s) = \frac{1}{2} \left(\hat{\beta} - b^{-1}s \right), \ \hat{\beta} := \beta - \alpha_*,
$$

and

$$
p_b(\alpha(s), s) = \left(\hat{\beta} - b^{-1}s \right) \left(\hat{\beta} + b^{-1}s \right)^{-1},
$$

and the integral term of (4.1) obtains the form

$$
\begin{aligned}
I &:= 2 \int_0^t \left(\hat{\beta} - b^{-1}s \right)^{-2} \left(\hat{\beta} + b^{-1}s \right) ds \\
&\leq 2b \left[\left(\hat{\beta} - b^{-1}t \right)^{-1} - \hat{\beta}^{-1} \right] \left(\hat{\beta} + b^{-1}t \right) \\
&\leq 2b \left(\hat{\beta} - b^{-1}t \right)^{-1} \hat{\beta} \left(1 + \hat{\beta}^{-1} b^{-1} t \right) \\
&= 2b p_b(\beta, t)^{-1} \left(1 + \hat{\beta}^{-1} b^{-1} t \right).
\end{aligned}
$$

The bound $\hat{\beta}^{-1}b^{-1}t < 1$ implies that

$$I \leq 4bp_b(\beta, t)^{-1}.$$

Thus it follows from (4.1) that

$$|||F(u) - F(v)|||_b \leq 2\sqrt{c}L\, |||u - v|||_b\,. \tag{4.2}$$

Let us now show that F preserves the space M_b. For this, we set $\mathbf{u}_0(t) = 0 \in X_{\alpha_*}$. Then $\mathbf{u}_0 \in M_b$ so that $F(u) - F(\mathbf{u}_0) \in M_b$ provided $u \in M_b$. Moreover,

$$F(\mathbf{u}_0)(t) = tf(0) + B(0)W(t),$$

and so

$$\mathbb{E}\, \|F(\mathbf{u}_0)(t)\|_\beta^2 \leq 2t^2\, \|f(0)\|_\beta^2 + 2t\, \|B(0)\|_{H_\beta}^2 \leq 2(t^2 + t)K^2\beta^{-1}.$$

In the second inequality we used Remark 2.4 with $u = 0$ and $\alpha = 0$. Then

$$|||F(\mathbf{u}_0)|||_b^2 \leq \sup_{\beta,t:\ t<b(\beta-\alpha_*)} p_b(\beta,t)2(t^2+t)K^2\beta^{-1} \leq 2cK^2 < \infty,$$

because $p_b(\beta, t) \leq 1$ and $t < b\beta \leq b\alpha^*$. Thus $F(\mathbf{u}_0) \in M_b$ and

$$F(u) = (F(u) - F(u_0)) + F(u_0) \in M_b.$$

This together with (4.2) implies the result. □

Corollary 4.2 *The map F is contractive in every M_b with $b < \frac{\sqrt{1+(\alpha^*-\alpha_*)L^{-1}}-1}{2(\alpha^*-\alpha_*)}$.*

5 Stochastic spin dynamics of a quenched particle system

Our main example is motivated by the study of stochastic dynamics of interacting particle systems. Let $\gamma \subset \mathfrak{X} = \mathbb{R}^n$ be a locally finite set (configuration) representing a collection of point particles. Each particle with position $x \in X$ is characterized by an internal parameter (spin) $\sigma_x \in S = \mathbb{R}^1$.

We fix a configuration γ and look at the time evolution of spins $\sigma_x(t)$, $x \in \gamma$, which is described by a system of stochastic differential equations in S of the form

$$d\sigma_x(t) = f_x(\bar{\sigma})dt + B_x(\bar{\sigma})dW_x(t),\ x \in \gamma, \tag{5.1}$$

where $\bar{\sigma} = (\sigma_x)_{x\in\gamma}$ and $W = (W_x)_{x\in\gamma}$ is a collection of independent Wiener processes in S. We assume that both drift and diffusion coefficients f_x and B_x

depend only on spins σ_y with $|y-x|<r$ for some fixed interaction radius $r>0$ and have the form

$$f_x(\bar{\sigma}) = \sum_{y\in\gamma} \varphi_{xy}(\sigma_x, \sigma_y), \;\; B_x(\bar{\sigma}) = \sum_{y\in\gamma} \Psi_{xy}(\sigma_x, \sigma_y), \tag{5.2}$$

where the mappings $\varphi_{xy} : S\times S \to S$ and $\Psi_{xy} : S\times S \to S$ satisfy finite range and uniform Lipschitz conditions, see Definition 5.3 and Condition 5.5 below.

Our aim is to realise system (5.1) as an equation in a suitable scale of Hilbert spaces and apply the results of previous sections in order to find its strong solutions.

We introduce the following notations:

- $S^\gamma := \prod_{x\in\gamma} S_x \ni \bar{\sigma} = (\sigma_x)_{x\in\gamma}, \; \sigma_x \in S_x = S$;
- $\gamma_{x,r} := \{y\in\gamma : |x-y|<r\}, \; x\in\gamma$;
- $n_x \equiv n_{x,r}(\gamma) :=$ number of points in $\gamma_{x,r}$ ($=$ number of particles interacting with particle in position x).

Observe that, although the number n_x is finite, it is in general unbounded function of x. We assume that it satisfies the following regularity condition.

Condition 5.1 *There exists a constant $a(\gamma, r)$ such that*

$$n_{x,r}(\gamma) \le a(\gamma,r)\,(1+|x|)^{1/2} \tag{5.3}$$

for all $x\in X$.

Remark 5.2 *Condition (5.3) holds if γ is a typical realization of a Poisson or Gibbs (Ruelle) point process in X. For such configurations, stronger (logarithmic) bound holds:*

$$n_{x,r}(\gamma) \le c(\gamma)\,[1+\log(1+|x|)]\,r^d,$$

see e.g. [28] and [22, p. 1047].

5.1 Existence of the dynamics

Our dynamics will live in the scale of Hilbert spaces

$$X_\alpha = S^\gamma_\alpha := \left\{ \bar{q}\in S^\gamma : \|\bar{q}\|_\alpha := \sqrt{\sum_{x\in\gamma} |q_x|^2\, e^{-\alpha|x|}} < \infty \right\}, \; \alpha>0.$$

Let us define the corresponding spaces $\mathcal{GL}^{(1)}$ and $\mathcal{GL}^{(2)}$ (cf. Condition 2.1) and set

$$\mathcal{H} = S^\gamma_0 := \left\{ \bar{q}\in S^\gamma : \|\bar{q}\|_0 := \sqrt{\sum_{x\in\gamma} |q_x|^2} < \infty \right\}.$$

Observe that $W(t) := (W_x(t))_{x\in\gamma}$ is a cylinder Wiener process in $\mathcal{H}$.

Let $\mathcal{V}$ be a family of mappings $V_{xy} : S^2 \to S$, $x, y \in \gamma$.

Definition 5.3 *We call the family $\mathcal{V}$ admissible if it satisfies the following two assumptions:*

- finite range: there exists constant $r > 0$ such that $V_{xy} \equiv 0$ if $|x - y| \geq r$;
- uniform Lipschitz continuity: there exists constant $C > 0$ such that

$$|V_{xy}(q_1', q_2') - V_{xy}(q_1'', q_2'')| \leq C\left(|q_1' - q_1''| + |q_2' - q_2''|\right) \qquad (5.4)$$

  for all $x, y \in \gamma$ and $q_1', q_2', q_1'', q_2'' \in S$.

Define a map $\overline{V} : S^\gamma \to S^\gamma$ and a linear operator $\widehat{V}(\bar{q}) : S^\gamma \to S^\gamma$, $\bar{q} \in S^\gamma$, by the formula

$$\overline{V}_x(\bar{q}) = \sum_{y\in\gamma} V_{xy}(q_x, q_y),$$

and

$$\left(\widehat{V}(\bar{q})\bar{\sigma}\right)_x := \overline{V}_x(\bar{q})\sigma_x,\ x \in \gamma,\ \bar{\sigma} \in S^\gamma,$$

respectively.

Lemma 5.4 *Assume that $\mathcal{V}$ is admissible. Then $\overline{V} \in \mathcal{GL}^{(1)}$ and $\widehat{V} \in \mathcal{GL}^{(2)}$.*

The proof of this Lemma is quite tedious and will be given in Section 6.

Now we can return to the discussion of system (5.1). Assume that the following condition holds.

Condition 5.5 *The families of mappings $\{\varphi_{xy}\}_{x,y\in\gamma}$ and $\{\Psi_{xy}\}_{x,y\in\gamma}$ from (5.2) are admissible.*

By Lemma 5.4 we have $\overline{\varphi} \in \mathcal{GL}^{(1)}$ and $\widehat{\Psi} \in \mathcal{GL}^{(2)}$. Thus we can write (5.1) in the form

$$d\bar{\sigma}(t) = \overline{\varphi}(\bar{\sigma})dt + \widehat{\Psi}(\bar{\sigma})dW(t),$$

where $W(t) = (W_x(t))_{x\in\gamma}$, and apply the results of Section 3 to its integral counterpart. We summarize the existence results in the following theorem, which follows directly from Theorem 3.3.

Theorem 5.6 *System (5.1) has a strong solution $u : [0, (\alpha^* - \alpha_*) b^*) \to X_{\alpha^*}$. Moreover, $u(T) \in \bigcap_{\alpha > T/b^* + \alpha_*} X_\alpha$ for any $T < (\alpha^* - \alpha_*) b^*$, and the restriction of u to the time interval $[0, T)$ belongs to M_b with $b = (\alpha^* - \alpha_*)^{-1} T$.*

Remark 5.7 *Theorem 5.6 can also be proved in the scale of Banach spaces*

$$S^\gamma_{\alpha,p} := \left\{ \bar{q} \in S^\gamma : \|\bar{q}\|_\alpha := \left(\sum_{x \in \gamma} |q_x|^p e^{-\alpha|x|} \right)^{1/p} < \infty \right\}, \ \alpha > 0, \ p > 2,$$

cf. Remark 3.7.

5.2 The uniqueness

In this section we establish a stronger uniqueness result, extending to our situation the method applied to deterministic systems in [24], [9]. As before, the main ingredients here are the bound on the density of configuration γ (Condition 5.1) and uniform Lipschitz continuity of the maps φ_{xy} and Ψ_{xy} (Condition 5.5). However, in contrast to the previous section, we will consider solutions of a more general type.

Let $E(S, T)$ be the space of square-integrable progressively measurable random processes $q : [0, T) \to S$ such that $\sup_{t \in [0,T)} \mathbb{E} \|u(t)\|^2_\beta < \infty$.

Definition 5.8 *We call a random process $\bar{q} : [0, T) \to S^\gamma$ a pointwise (strong) solution of system (5.1) if $q_x(\cdot) \in E(S, T)$ and satisfies integral equation*

$$q_x(t) = q_x(0) + \int_0^t f_x(\bar{q}(s)) ds + \int_0^t B_x(\bar{q}(s)) dW_x(s)$$

for each $x \in \gamma$.

It is clear that the solution constructed in Theorem 5.6 is a pointwise strong solution.

Theorem 5.9 *Assume that Conditions 5.1 and 5.5 hold and let $\bar{q}^{(1)}(t), \bar{q}^{(2)}(t) \in S^\gamma_\beta$ be two pointwise strong solutions of (5.1) on $[0, T)$, and let $\bar{q}^{(1)}(0) = \bar{q}^{(2)}(0)$ a.s. Then $\bar{q}^{(1)}(t) = \bar{q}^{(2)}(t)$ a.s. for any $t \in [0, T)$.*

To proceed with the proof, we need the following Lemma, which will in turn be proved in Section 6. For any $n \in \mathbb{N}$ and $t \in [0, T)$ define

$$\delta_n(t) := \sup_{|x| \le nr} \mathbb{E} \left| q_x^{(1)}(t) - q_x^{(2)}(t) \right|^2 .$$

Lemma 5.10 *Assume that conditions of Theorem 5.9 hold. Then there exists $\mu >$ 0 such that*

$$\delta_n(t) \leq 2n(t+1)\mu \int_0^t \delta_{n+1}(s)ds \tag{5.5}$$

for any $t \in [0, T)$.

Proof of Theorem 5.9. The N-th iteration of bound (5.5) gives the estimate

$$\delta_n(t) \leq \frac{(2(t+1)t\mu)^N}{N!} n(n+1)....(n+N-1) \sup_{s \leq t} \delta_{n+N}(s) \tag{5.6}$$

for any $N = 2, 3, ...$. Set

$$R := \sup_{s \leq T} \left\{ \mathbb{E} \left\| \bar{q}^{(1)}(s) \right\|_\beta^2, \mathbb{E} \left\| \bar{q}^{(2)}(s) \right\|_\beta^2 \right\}.$$

Taking into account that $\bar{q}^{(1)}(t), \bar{q}^{(2)}(t) \in S_\beta^\gamma$ we obtain the bounds

$$\mathbb{E} \left| q_x^{(i)}(t) \right|^2 \leq e^{\beta|x|} \mathbb{E} \left\| \bar{q}^{(i)}(t) \right\|_\beta^2 \leq e^{\beta|x|} R, \; i = 1, 2,$$

which imply that

$$\delta_{n+N}(s) \leq 4e^{\beta(n+N)r} R$$

for any $s \in [0, T]$. It follows now from (5.6) that

$$\begin{aligned} \delta_n(t) &\leq 4e^{\beta(n+N)r} R \frac{(2(t+1)t\mu)^N}{N!} n(n+1)....(n+N-1) \\ &= 4e^{\beta(n+N)r} R \left(2(t+1)t\mu\right)^N \begin{pmatrix} n+N-1 \\ N \end{pmatrix} \\ &= 4e^{\beta nr} R \left[\left(2e^{\beta r+1}\mu(t+1)t\right) \frac{n+N-1}{N} \right]^N . \end{aligned}$$

Here we used the well-known inequality $\binom{M}{N} \leq \left(\frac{M\,e}{N}\right)^N$, $1 \leq N \leq M$. For $N > n-1$ we have $\frac{n+N-1}{N} < 2$ and so

$$\delta_n(t) < 4e^{\beta nr} R \left[4e^{\beta r+1}\mu(t+1)t\right]^N \to 0, \; N \to \infty,$$

provided $4e^{\beta r+1}\mu(t+1)t < 1$ (e.g. $t < t_0 := \frac{1}{4}\left(e^{\beta r+1}\mu(\alpha^*+1)b\right)^{-1}$). Thus

$$\sup_{|x| \leq nr} \mathbb{E} \left| q_x^{(1)}(t) - q_x^{(2)}(t) \right|^2 = 0, \; t < t_0,$$

for all $n \geq 1$, so that $\bar{q}^{(1)}(t) = \bar{q}^{(2)}(t)$ a.s. for any $t \in [0, t_0)$.

These arguments can be repeated on each of the time intervals $[t_k, t_{k+1})$ with $t_k := kt_0, \; k = 1, 2, ...,$ which shows that $\bar{q}^{(1)}(t) = \bar{q}^{(2)}(t)$ a.s. for any $t \in [0, T)$, and the proof is complete. □

6 Proofs of auxiliary results

In this section, we present proofs of two technical lemmas used in the previous section.

6.1 Proof of Lemma 5.4

Step 1. We first show that $\overline{V}$ is a mapping $S_\alpha^\gamma \to S_\beta^\gamma$ for any $\alpha < \beta$. For any $\bar{q} \in S_\alpha^\gamma$ we have

$$\begin{aligned} \left\|\overline{V}(\bar{q})\right\|_\beta^2 &= \sum_{x\in\gamma}\left|\sum_{y\in\gamma} V_{xy}(q_x, q_y)\right|^2 e^{-\beta|x|} \\ &\le 3C^2 \sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} n_x\left(1 + |q_x|^2 + |q_y|^2\right) e^{-\beta|x|}. \end{aligned}$$

The polynomial bound on the growth of n_x implies that

$$\sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} n_x e^{-\beta|x|} = \sum_{x\in\gamma} n_x^2 e^{-\beta|x|} \le \sum_{x\in\gamma} n_x^2 e^{-\alpha_*|x|} =: c(\gamma, \alpha_*) < \infty.$$

Next, we estimate

$$\begin{aligned} \sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} n_x |q_x|^2 e^{-\beta|x|} &= \sum_{x\in\gamma} n_x^2 |q_x|^2 e^{-(\beta-\alpha)|x|} e^{-\alpha|x|} \\ &\le \sup_{x\in\gamma}\left(n_x^2 e^{-(\beta-\alpha)|x|}\right)\|\bar{q}\|_\alpha^2 . \end{aligned}$$

Observe that $\sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} = \sum_{\substack{x,y\in\gamma \\ |x-y|<r}} = \sum_{y\in\gamma}\sum_{x\in\gamma_{y,r}}$, and so

$$\begin{aligned} \sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} n_x |q_y|^2 e^{-\beta|x|} &\le e^{\beta r}\sum_{y\in\gamma} N_y |q_y|^2 e^{-(\beta-\alpha)|y|} e^{-\alpha|y|} \\ &\le e^{\beta r}\sup_{y\in\gamma}\left(N_y e^{-(\beta-\alpha)|y|}\right)\|\bar{q}\|_\alpha^2 , \end{aligned}$$

where $N_y := \sum_{x\in\gamma_{y,r}} n_x$. Here we used inequality $|y| \le |y - x| + |x| \le r + |x|$ for $y \in \gamma_{x,r}$, so that $e^{-\beta|x|} \le e^{\beta r} e^{-\beta|y|}$. Condition 5.1 implies that

$$N_x \le a(\gamma, r)^2 (1 + |x|)^{1/2} (1 + r + |x|)^{1/2} < a(\gamma, r)^2 (1 + r)^{1/2} (1 + |x|),$$

and

$$n_x^2 \le a(\gamma, r)^2 (1 + |x|)$$

for any $x \in \gamma$. Setting $c_2(\gamma, r) := a(\gamma, r)^2 \left[1 + e^{\alpha^* r}(1+r)^{1/2}\right]$ and $L^2 = 3C^2\,(c_1 + c_2)\, e^{\alpha^* - \alpha_* - 1}$ we obtain the bound

$$\left\|\overline{V}(\bar{q})\right\|_\beta^2 \le 3C^2\,(c_1 + c_2) \left[\sup_{s>0}(1+s)e^{-(\beta-\alpha)s}\right] \|\bar{q}\|_\alpha^2 \le L^2\,(\beta-\alpha)^{-1}\,\|\bar{q}\|_\alpha^2 < \infty.$$

Step 2. Lipschitz condition (5.4) implies the estimate

$$\begin{aligned}\left\|\overline{V}(\bar{q}') - \overline{V}(\bar{q}'')\right\|_\beta^2 &= \sum_{x\in\gamma}\left|\sum_{y\in\gamma} V_{xy}(q'_x, q'_y) - \sum_{y\in\gamma} V_{xy}(q''_x, q''_y)\right|^2 e^{-\beta|x|} \\ &\le 2C^2 \sum_{x\in\gamma}\sum_{y\in\gamma_{x,r}} n_x \left(\left|q'_x - q''_x\right|^2 + \left|q'_y - q''_y\right|^2\right) e^{-\beta|x|}\end{aligned}$$

for any $\bar{q}', \bar{q}'' \in S_\alpha^\gamma$. Similar to Step 1, we obtain the bound

$$\begin{aligned}\left\|\overline{V}(\bar{q}') - \overline{V}(\bar{q}'')\right\|_\beta^2 \le 2C^2 c_2 \left[\sup_{s>0}(1+s)e^{-(\beta-\alpha)s}\right] \|\bar{q}' - \bar{q}''\|_\alpha^2 \\ \le L^2\,(\beta-\alpha)^{-1}\,\|\bar{q}' - \bar{q}''\|_\alpha^2 < \infty.\end{aligned}$$

Step 3. The inclusion $\overline{V}(\bar{q}) \in S_\beta^\gamma$ implies that $\widehat{V}(\bar{q})\bar{\sigma} \in S_\beta^\gamma$ for any $\bar{\sigma} \in \mathcal{H} = S_0^\gamma$. A direct calculation shows that $\widehat{V}(\bar{q}) : \mathcal{H} \to S_\beta^\gamma$ is a Hilbert-Schmidt operator with the norm equal to $\left\|\bar{V}(\bar{q})\right\|_\beta$. Thus the inclusion $\overline{V} \in \mathcal{GL}^{(1)}$ implies that $\widehat{V} \in \mathcal{GL}^{(2)}$. □

6.2 Proof of Lemma 5.10

We start with the estimate of the distance between $q_x^{(1)}(t)$ and $q_x^{(2)}(t)$ for a fixed $x \in \gamma$ and $t \in [0, T)$. From (5.1) we obtain

$$\begin{aligned}\left|q_x^{(1)}(t) - q_x^{(2)}(t)\right|^2 \le 2t \int_0^t \left|f_x(\bar{q}^{(1)}(s)) - f_x(\bar{q}^{(2)}(s))\right|^2 ds \\ + 2\int_0^t \left|B_x(\bar{q}^{(1)}(s)) - B_x(\bar{q}^{(2)}(s))\right|^2 ds =: 2tI_{1,x}(t) + 2I_{2,x}(t), \quad (6.1)\end{aligned}$$

where $I_{1,x}(t)$ and $I_{2,x}(t)$ denote the first and second integral terms, respectively. Taking into account the explicit form (5.2) of f_x and B_x and using Condition 5.5

we obtain

$$
\begin{aligned}
I_{1,x}(t) &\le \int_0^t \left| \sum_{y\in\gamma_x} \left(\varphi_{xy}(q_x^{(1)}(s), q_y^{(1)}(s)) - \varphi_{xy}(q_x^{(2)}(s), q_y^{(2)}(s)) \right) \right|^2 ds \\
&\le n_x \int_0^t \sum_{y\in\gamma_x} \left| \varphi_{xy}(q_x^{(1)}(s), q_y^{(1)}(s)) - \varphi_{xy}(q_x^{(2)}(s), q_y^{(2)}(s)) \right|^2 ds \\
&\le 2n_x C^2 \int_0^t \sum_{y\in\gamma_x} \left[\left| q_x^{(1)}(s) - q_x^{(2)}(s) \right|^2 + \left| q_y^{(1)}(s) - q_y^{(2)}(s) \right|^2 \right] ds.
\end{aligned}
$$

Recall that

$$
n_x \le a(\gamma, r)\,(1+|x|)^{1/2}.
$$

Then for $|x| \le nr$

$$
\begin{aligned}
\mathbb{E}\left(I_{1,x}(t)\right) &\le 4n_x C^2 \int_0^t \sum_{y\in\gamma_x} \delta_{n+1}(s)ds = 4n_x^2 C^2 \int_0^t \delta_{n+1}(s)ds \\
&\le 4C^2 a(\gamma,r)^2\,(1+|x|) \int_0^t \delta_{n+1}(s)ds \le \mu n \int_0^t \delta_{n+1}(s)ds
\end{aligned}
$$

with $\mu := 4C^2 a(\gamma,r)^2\,(1+r)$. Similarly,

$$
\mathbb{E}\left(I_{2,x}(t)\right) \le \mu n \int_0^t \delta_{n+1}(s)ds,
$$

so that (6.1) implies the inequality

$$
\mathbb{E}\left| q_x^{(1)}(t) - q_x^{(2)}(t) \right|^2 \le 2(t+1)\mu n \int_0^t \delta_{n+1}(s)ds
$$

and, consequently,

$$
\delta_n(t) \le 2(t+1)\mu n \int_0^t \delta_{n+1}(s)ds.
$$

The proof is complete. □

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