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Simultaneous reconstruction of the perfusion coefficient and initial temperature from time-average integral temperature measurements

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Abstract

Inverse coefficient identification formulations give rise to some of the most important mathematical problems because they tell us how to determine the unknown physical properties of a given medium under inspection from appropriate extra measurements. Such an example occurs in bio-heat transfer where the knowledge of the blood perfusion is of critical importance for calculating the temperature of the blood flowing through the tissue. Furthermore, in many related applications the initial temperature of the diffusion process is also unknown. Therefore, in this framework the simultaneous reconstruction of the space-dependent perfusion coefficient and initial temperature from two linearly independent weighted time-integral observations of temperature is investigated. The quasi-solution of the inverse problem is obtained by minimizing the least-squares objective functional, and the Fréchet gradients with respect to both of the two unknown space-dependent quantities are derived. The stabilisation of the conjugate gradient method (CGM) is established by regularising the algorithm with the discrepancy principle. Three numerical tests for one- and two-dimensional examples are illustrated to reveal the accuracy and stability of the numerical results.

Keywords: Inverse problem; Parabolic equation; Conjugated gradient method; Initial temperature; Perfusion coefficient

NOMENCLATURE

d_q^n, d_ϕ^n	search directions	δ	Dirac delta function
E_1, E_2	accuracy errors	ϵ	noise level
f	heat source	λ	adjoint function
J	objective functional	μ	heat flux
J'_q, J'_ϕ	gradients of J	ν	outward unit normal to $\partial\Omega$
k	thermal conductivity tensor	σ	standard deviation
n	number of iterations	ϕ	initial temperature
q	perfusion coefficient	ϕ_1, ϕ_2	exact integral observations
T	final time	ϕ_1^ϵ, ϕ_2^ϵ	measured data
u	temperature	ω_1, ω_2	weight functions
α	surface heat transfer coefficient	Ω	bounded domain
β_q^n, β_ϕ^n	step sizes	$\partial\Omega$	boundary of Ω
γ_q^n, γ_ϕ^n	conjugate coefficients		

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1. Introduction

The inverse problem of identifying the space-dependent perfusion/radiative coefficient from integral observation was previously studied in [1, 2, 3]. This unknown coefficient was numerically determined in the one-dimensional bio-heat equation with heat flux or time-average temperature measurement by minimising the Tikhonov regularisation functional using the NAG routine E04FCF together with the finite-difference method (FDM), [4]. Recently, the space-dependent perfusion coefficient was recovered by the CGM from the final or time-average temperature measurement in [5]. Also, the inverse problem of determining the initial temperature from temperature measurements at a later time was extensively studied, e.g. [6, 7]. Besides, there are many numerical techniques that had been developed to reconstruct the unknown initial temperature, including the iterative CGM [8, 9], the boundary element method (BEM) with regularisation [10], the elliptic approximation together with the BEM [11], the Tikhonov regularisation approach [12], the Fourier regularisation method [13] and the self-adaptive Lie-group adaptive method [14].

In [15], the space-dependent radiative coefficient and the initial temperature were simultaneously reconstructed from temperature measurements at a fixed time $\theta > 0$ and in $\omega \times (0, T)$, where ω is a subregion of the space domain Ω; the stability of the inverse problem was established, the existence of the minimizer of Tikhonov's first-order regularisation functional was proved, and the numerical results were obtained by using a nonlinear gradient multigrid technique. Similarly, the determination of the radiative coefficient, the Robin coefficient in a convection boundary condition and the initial temperature from the final observation of temperature and the prior knowledge of the radiative coefficient in $\omega \subset \Omega$, was investigated in [16] where the uniqueness and stability of the inverse problem were established.

In this paper, we address the inverse heat transfer problem of simultaneously identifying the unknown space-dependent perfusion coefficient $q(x)$ and the initial temperature $\phi(x)$ from the integral observations $\phi_1(x)$ and $\phi_2(x)$ in (7) and (8) below, generated by two linearly independent weight functions $\omega_1(t)$ and $\omega_2(t)$. This formulation generalises some of the previously-posed inverse models, which can be obtained by particular choices of the weights ω_1 and ω_2, and it has been investigated before. For the numerical stable reconstruction, the least-squares objective functional is minimised to obtain the quasi-solution of the two unknown quantities. The existence of the minimizer for the objective functional is presented, and the Fréchet gradients are derived. In addition, we show that these Fréchet gradients are Lipschitz continuous. These gradients and the adjoint problem are utilized in the CGM to reconstruct the unknown quantities simultaneously. The global convergence of the CGM with the Fletcher-Reeves formula [17] is established according to the arguments in [18] obtained from the Lipschitz continuous property of the Fréchet gradients. Since the inverse problem discussed in our work is nonlinear and unstable, our CGM is regularised by the discrepancy principle [8].

The paper is organized as follows: Section 2 presents the mathematical formulation of the inverse heat transfer problem of reconstructing the unknown radiative coefficient and the initial temperature, together with the objective functional to be minimized, and several properties of this functional are presented. The CGM is introduced in Section 4 according to the Fréchet gradients obtained in Section 3, and the global convergence of the algorithm is obtained. Three numerical examples are discussed in Section 5. Finally, Section 6 highlights the conclusions of this paper.

2. Mathematical formulation

Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, be a bounded domain with a sufficiently smooth boundary $\partial\Omega$ representing the issue in a biomechanical engineering situation. In the cylinder $Q := \Omega \times (0, T)$, where $T > 0$ is a final time of interest, we consider the bio-heat transfer process governed by the parabolic equation (Pennes' equation [19])

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot (k(x)\nabla u(x,t)) - q(x)u(x,t) + f(x,t), \quad (x,t) \in Q, \tag{1}$$

where $u(x,t)$ is the tissue temperature, $k(x)$ is the thermal conductivity tensor which is symmetric and positive definite, $q(x) \geq 0$ is the space-dependent coefficient denoting the blood perfusion, and f is a metabolic heat source. For simplicity, the heat capacity was assumed to be constant and taken to be unity. The above fundamental governing bio-heat equation (1) represents a balance between the accumulation of energy (in the left-hand side of (1)) and the superposition of heat conduction (diffusion), heat transfer effect due to the blood flowing through the capillary network and heat generation due to the cell metabolism. The inverse linear problem of finding the metabolic heat source f has been considered elsewhere, [20, 21, 22], herein we address the more difficult nonlinear problem of finding the blood perfusion coefficient $q(x)$. The importance of the blood perfusion contribution to the heat generation in tissue has been stressed in carcinogenic skin and brest tumours because of the increased nutrition and oxygen demand [23]. Therefore, knowing $q(x)$ as it varies through the tissue $x \in \Omega$, would be beneficial to explain and understand the heat transfer through such biological tissues. In another application related to fin heat transfer in heat exchangers, q denotes the domain heat transfer coefficient, [24].

For the boundary condition we assume that this of Robin convection type

$$k(x)\frac{\partial u}{\partial \nu}(x,t) + \alpha(x)u(x,t) = \mu(x,t), \quad (x,t) \in S := \partial\Omega \times (0,T), \tag{2}$$

where ν is the outward unit normal to $\partial\Omega$, μ is a given heat flux and $\alpha(x) \geq 0$ is the surface heat transfer coefficient, which also includes the case of a Neumann heat flux boundary condition obtained when $\alpha(x) \equiv 0$.

Let

$$u(x,0) = \phi(x), \quad x \in \Omega, \tag{3}$$

denote the initial temperature at $t = 0$.

Several basic functional spaces [25], which shall be used in this paper, are presented. The space $L_p(\Omega)$, $p \in [1,\infty)$, consists all p-integrable functions $u(x)$ over Ω, endowed with the norm

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

The space $L_\infty(\Omega)$ comprises all essentially bounded functions $u(x)$ in Ω, equipped with the norm

$$\|u\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| := \inf\{M \geq 0 : |u(x)| \leq M, \text{ a.e. } x \in \Omega\}.$$

The spaces $L_p(Q)$ and $L_\infty(Q)$ can be defined similarly. We denote by $H^{1,0}(Q)$ the normed space of all functions $u(x,t) \in L_2(Q)$ having weak first-order derivatives with respect to x in $L_2(Q)$, endowed with the norm

$$\|u\|_{H^{1,0}(Q)} = \left(\|u\|^2_{L_2(Q)} + \|\nabla u\|^2_{L_2(Q)}\right)^{1/2}.$$

The space $H^{1,1}(Q)$, defined by $H^{1,1}(Q) = \left\{u \in L_2(Q) : \frac{\partial u}{\partial t}, \nabla u \in L_2(Q)\right\}$, is a normed space with

$$\|u\|_{H^{1,1}(Q)} = \left(\|u\|^2_{L_2(Q)} + \|\nabla u\|^2_{L_2(Q)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(Q)}\right)^{1/2}.$$

The space $C([0,T];L_2(\Omega))$ consists of all real-valued functions $u(x,t)$, square integrable with respect to $x\in\Omega$ for every $t\in[0,T]$, and continuous in t with respect to the norm of $L_2(\Omega)$, i.e., $\|u(\cdot,t+\Delta t)-u(\cdot,t)\|_{L_2(\Omega)}\to 0$ for $\Delta t\to 0$. The norm of such space is given by

$$\|u\|_{C([0,T],L_2(\Omega))}=\max_{t\in[0,T]}\|u(\cdot,t)\|_{L_2(\Omega)}.$$

We denote by $V_2^{1,0}(Q)$ the space $H^{1,0}(Q)\cap C([0,T];L_2(\Omega))$, equipped with the norm

$$\|u\|_{V_2^{1,0}(Q)}=\max_{t\in[0,T]}\|u(\cdot,t)\|_{L_2(\Omega)}+\|\nabla u\|_{L_2(Q)}.$$

Throughout this work, the operator $\mathcal{L}:=\frac{\partial}{\partial t}-\nabla\cdot(k\nabla)+q\mathcal{I}$, where $\mathcal{I}$ is the identity, is assumed to be uniformly parabolic, i.e.,

$$\upsilon_1|\xi|^2\le\sum_{i,j=1}^{N}k_{ij}(x)\xi_i\xi_j\le\upsilon_2|\xi|^2,\quad \text{a.e. } x\in\Omega,\ \forall\xi=(\xi_i)_{i=\overline{1,N}}\in\mathbb{R}^N, \tag{4}$$

for some given positive constants υ_1 and υ_2. We further assume that k is symmetric, i.e., $k_{ij}=k_{ji}$.

Definition 1. *A function $u(x,t)\in V_2^{1,0}(Q)$ is called as a weak solution to the direct initial-boundary value problem* (1)–(3) *if*

$$\begin{aligned}&\int_Q\left(-u\frac{\partial\eta}{\partial t}+(k\nabla u)\cdot\nabla\eta+qu\eta\right)dxdt+\int_S\alpha u\eta dsdt\\&=\int_Q f\eta dxdt+\int_S\mu\eta dsdt+\int_\Omega\phi\eta(\cdot,0)dx,\quad\forall\eta\in H^{1,1}(Q)\text{ with }\eta(\cdot,T)=0.\end{aligned} \tag{5}$$

The existence and uniqueness of the weak solution $u(x,t)\in V_2^{1,0}(Q)$ to the initial-boundary value direct problem (1)–(3) is presented as follows ([25] p.373):

Lemma 1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and suppose that $f\in L_2(Q)$, $0\le\alpha\in L_\infty(\partial\Omega)$, $\mu\in L_2(S)$ and $\phi\in L_2(\Omega)$. Let k satisfy* (4) *and $k_{ij}\in L_\infty(\Omega)$, $i,j=\overline{1,N}$, and $q\in L_\infty(\Omega)$, $0<q^-\le q(x)\le q^+$, a.e. $x\in\Omega$, where, q^-,q^+ are two positive constants. Then the initial-boundary value direct problem* (1)–(3) *has a unique weak solution $u\in H^{1,0}(Q)$ that belongs to $V_2^{1,0}(Q)$.*

Note that by the direct problem (1)–(3) for a.e., $t\in[0,T]$, we know

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L_2(\Omega)}+\int_\Omega(k\nabla u\cdot\nabla u+qu^2)dx+\int_{\partial\Omega}\alpha u^2ds=\int_\Omega fudx+\int_{\partial\Omega}\mu uds.$$

By (4), $q\ge q^->0$ and $\alpha\ge 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L_2(\Omega)}+\min\{q^-,\upsilon_1\}\|u(\cdot,t)\|^2_{H^1(\Omega)}\le c(\|u(\cdot,t)\|^2_{L_2(\Omega)}+\|f(\cdot,t)\|^2_{L_2(\Omega)}+\|\mu(\cdot,t)\|^2_{L_2(\partial\Omega)}),$$

where c is a positive constant depending on Ω. Using the Gronwall's inequality, we can obtain

$$\max_{t\in[0,T]}\|u(\cdot,t)\|_{L_2(\Omega)}+\|u\|_{H^{1,0}(Q)}\le C_0\left(\|f\|_{L_2(Q)}+\|\mu\|_{L_2(S)}+\|\phi\|_{L_2(\Omega)}\right) \tag{6}$$

where $C_0(q^-, \upsilon_1, \Omega, T)$ is a positive constant.

The inverse problem is to determine the triplet $(q(x), \phi(x), u(x,t))$ satisfying (1) and (2) together with the time-integral temperature measurements,

$$\int_0^T \omega_1(t) u(x,t) dt = \phi_1(x), \quad x \in \Omega, \tag{7}$$

$$\int_0^T \omega_2(t) u(x,t) dt = \phi_2(x), \quad x \in \Omega, \tag{8}$$

where $\omega_1(t)$ and $\omega_2(t) \in L_\infty(0,T)$ are two given linearly independent weight functions, and $\phi_1(x)$ and $\phi_2(x)$ are given data which may be subjected to noise due to measurement errors. We are actually recovering the solution to the inverse problem (1), (2), (7) and (8) from the noisy data $(\phi_1^\epsilon, \phi_2^\epsilon)$ satisfying

$$\|\phi_1^\epsilon - \phi_1\|_{L_2(\Omega)} \le \epsilon, \quad \|\phi_2^\epsilon - \phi_2\|_{L_2(\Omega)} \le \epsilon, \tag{9}$$

where ϵ represents the noise level.

Note that $\phi_1(x)$ may mimic the temperature measurement at a instant time $t_1 \in (0,T]$ if $\omega_1(t) = \delta(t - t_1)$, namely,

$$u(x, t_1) = \phi_1(x), \quad x \in \Omega, \tag{10}$$

and $\phi_2(x)$ the temperature at another instant time $t_2 \in (0, t_1)$ if $\omega_2(t) = \delta(t - t_2)$, namely,

$$u(x, t_2) = \phi_2(x), \quad x \in \Omega, \tag{11}$$

where δ is the Dirac delta function, and the inverse problem of finding the triplet $(q(x), \phi(x), u(x,t))$ satisfying (1), (2), (10) and (11) has recently been investigated by the authors in [26]. The Dirac delta function $\delta(t - t_1)$ can be approximated by the function $\delta_a(t) = \frac{1}{a\sqrt{\pi}} e^{-(t-t_1)^2/a^2}$ with small positive parameter a, e.g., $a = 10^{-3}$, and so does $\delta(t - t_2)$, such that the approximated weighted functions belong to the space $L_\infty(0,T)$.

Other cases of potential interest may be obtained by taking the weights as cut-off functions, e.g.,

$$\omega_1(t) = \tilde{\omega}_1(t) \mathcal{X}_{[t_1,T]}(t), \quad \omega_2(t) = \tilde{\omega}_2(t) \mathcal{X}_{[0,t_1]}(t), \quad t \in [0,T], \tag{12}$$

where $\mathcal{X}_D$ denotes the characteristic function of the domain D and $\tilde{\omega}_1(t)$ and $\tilde{\omega}_2(t) \in L_2(0,T)$, in which case (7) and (8) yield

$$\int_{t_1}^T \tilde{\omega}_1(t) u(x,t) dt = \phi_1(x), \quad x \in \Omega, \tag{13}$$

$$\int_0^{t_1} \tilde{\omega}_2(t) u(x,t) dt = \phi_2(x), \quad x \in \Omega. \tag{14}$$

The uniqueness of the general inverse problem given by (1), (2) supplemented with (7) and (8) is still to be established, but under some of the particular cases (10)–(14) the inverse problem can be split in two separate inverse problem, namely, first identifying $q(x)$ and after that $\phi(x)$. For example, when solving the inverse problem given by (1), (2), (10) and (11), one can first identify $q(x)$ by solving this in the layer $\Omega \times (t_2, t_1)$ followed by retrieving the initial data $\phi(x)$ in (3) by solving the backward heat conduction problem (BHCP) (1), (2) and (11) in the layer $\Omega \times (0, t_2)$. Similarly, when solving the inverse problem given by (1), (2), (10) and (13), for $t_1 < T$, one can first identity $q(x)$ by solving this in the layer $\Omega \times (t_1, T)$ followed by retrieving the initial data

$\phi(x)$ in (3) by solving the BHCP (1), (2) and (10) in the layer $\Omega \times (0, t_1)$. We finally mention that uniqueness results for the retrieval of the perfusion coefficient $q(x)$ from final time or time-average temperature measurements can be found in [1, 2, 3, 27, 28, 29] with numerical reconstructions performed in [4, 30, 31, 32], to mention only a few.

From the above discussion it can be realised that the choice of the weight functions in the extra conditions (7) and (8) is important in order to extract useful information on the inverse problem solution. An obvious necessary condition is that $\omega_1(t)$ and $\omega_2(t)$ are linearly independent such that (7) and (8) are non-redundant, but is this enough? We try to gain some insight by taking $\omega_1(t) = 1$ and $\omega_2(t) = t$ (which will also be numerically investigated in Section 5) such that (7) and (8) read as

$$\int_0^T u(x,t)dt = \phi_1(x), \quad \int_0^T tu(x,t)dt = \phi_2(x), \quad x \in \Omega. \tag{15}$$

For this choice of the weight functions (and also for $\omega_2(t) = e^t$), it is possible to eliminate the perfusion coefficient from the inverse problem. To see this, assuming that the functions involved are as differentiable as required by the process of their manipulation, we proceed formally to yield

$$\begin{aligned}\phi_1(x) =& tu(x,t)|_{t=0}^{t=T} - \int_0^T tu_t(x,t)dt = Tu(x,T) - \int_0^T t\left(\nabla \cdot (k(x)\nabla u) - q(x)u + f(x,t)\right)dt \\ =& Tu(x,T) - \nabla \cdot (k(x)\nabla \phi_2(x)) + q(x)\phi_2(x) - \int_0^T tf(x,t)dt,\end{aligned} \tag{16}$$

$$u(x,T) - \phi(x) = \nabla \cdot (k(x)\nabla \phi_1(x)) - q(x)\phi_1(x) + \int_0^T f(x,t)dt. \tag{17}$$

Assuming further that $\Phi(x) := T\phi_1(x) - \phi_2(x) \neq 0$, $\forall x \in \Omega$, solving (16) and (17) yields

$$q(x) = \frac{T\phi(x) - \phi_1(x) + \nabla \cdot (k\nabla \Phi(x)) + \int_0^T (T-t)f(x,t)dt}{\Phi(x)}, \tag{18}$$

$$\begin{aligned}u(x,T) =& \frac{\phi_1(x)\left(\phi_1(x) + \nabla \cdot (k(x)\nabla \phi_2(x)) + \int_0^T tf(x,t)dt\right)}{\Phi(x)} \\ &- \frac{\phi_2(x)\left(\phi(x) + \nabla \cdot (k(x)\nabla \phi_1(x)) + \int_0^T f(x,t)dt\right)}{\Phi(x)}.\end{aligned} \tag{19}$$

So, $q(x)$ (and also $u(x,T)$) is expressible in terms of $\phi(x)$.

Note that if an extra integral condition with a weight function $\omega_3(t) = t^2$ would be available in the form

$$\int_0^T t^2 u(x,t) = \phi_3(x), \quad x \in \Omega, \tag{20}$$

then, (15) and (20) would yield

$$2\phi_2(x) = T^2 u(x,T) - \nabla \cdot (k(x)\nabla \phi_3(x)) + q(x)\phi_3(x) - \int_0^T t^2 f(x,t)dt, \tag{21}$$

and the system of 3 equations (16), (17) and (21) would uniquely yield a solution $(\phi(x), u(x,T), q(x))$ provided that the determinant

$$0 \neq \begin{vmatrix} -1 & 1 & \phi_1(x) \\ 0 & T & \phi_2(x) \\ 0 & T^2 & \phi_3(x) \end{vmatrix} = \phi_2(x)T^2 - T\phi_3(x)$$

or, $\phi_3(x) - T\phi_2(x) \neq 0$, $\forall x \in \Omega$. However, as only (15) is available, introducing (18) into (1) we obtain

$$\frac{\partial u}{\partial t} = \nabla \cdot (k(x)\nabla u) + f(x,t) - \frac{T\phi(x) + \mathcal{A}(x)}{\Phi(x)} u, \tag{22}$$

where $\mathcal{A}(x) := -\phi_1(x) + \nabla \cdot (k(x)\nabla\Phi(x)) + \int_0^T (T-t) f(x,t)dt$. Not that the new problem given by equations (2), (3), (19) and (22) is simpler than the original one, as it is still nonlinear, but it only involves finding the pair solution $(\phi(x), u(x,t))$.

For the numerical reconstruction we employ a variation formulation, as described next.

3. Variational formulation

Let $u(q,\phi) := u(x,t;q,\phi)$ denote the weak solution to the initial-boundary value problem (1)–(3) subject to a particular pair $(q(x),\phi(x)) \in L_\infty(\Omega) \times L_2(\Omega)$. Then, given ϕ_1^ϵ and ϕ_2^ϵ in $L_2(\Omega)$ temperature measurements satisfying (9), the quasi-solution of the inverse problem (1), (2), (7) and (8) can be obtained by minimizing the following least-squares objective functional:

$$J(q,\phi) := \frac{1}{2}\left\| \int_0^T \omega_1(t) u(q,\phi) dt - \phi_1^\epsilon \right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\| \int_0^T \omega_2(t) u(q,\phi) dt - \phi_2^\epsilon \right\|^2_{L^2(\Omega)}, \tag{23}$$

subject to $u \in V_2^{1,0}(Q)$ satisfying the variational equality (5), over the admissible set $\mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 = \{q \in L_\infty(\Omega) : 0 < q^- \le q(x) \le q^+, \text{ a.e. } x \in \Omega\}$, $\mathcal{A}_2 = \{\phi \in L_2(\Omega) : |\phi(x)| \le \kappa, \text{ a.e. } x \in \Omega\}$, for a positive constant κ.

The existence of a minimizer to the optimization problem (23) over the admissible set $\mathcal{A}_1 \times \mathcal{A}_2$ is established in the following theorem, according to the approaches utilized in [15, 33].

Theorem 1. *There exists at least one minimizer to the optimization problem* (23).

In order to numerically obtain the minimizer of the objective functional $J(q,\phi)$ (23), the CGM can be applied together with the Fréchet gradient. Thus the adjoint problem to (1), (2), (7) and (8) is introduced and given by

$$\begin{cases} \frac{\partial \lambda}{\partial t} = -\nabla \cdot (k\nabla\lambda) + q\lambda - \omega_1(t)\left(\int_0^T \omega_1(\tau) u(x,\tau) d\tau - \phi_1^\epsilon(x)\right) \\ \qquad -\omega_2(t)\left(\int_0^T \omega_2(\tau) u(x,\tau) d\tau - \phi_2^\epsilon(x)\right), & (x,t) \in Q, \\ k(x)\frac{\partial\lambda}{\partial\nu} + \alpha\lambda = 0, & (x,t) \in S, \\ \lambda(x,T) = 0, & x \in \overline{\Omega}. \end{cases} \tag{24}$$

Its weak solution $\lambda \in V_2^{1,0}(Q)$ to the adjoint problem (24) is defined as satisfying

$$\begin{aligned} &\int_Q \left(\lambda \frac{\partial \eta}{\partial t} + (k\nabla\lambda)\cdot\nabla\eta + q\lambda\eta\right) dxdt + \int_S \alpha\lambda\eta dsdt \\ &= \int_\Omega \int_0^T \omega_1(t)\eta(x,t)dt \left(\int_0^T \omega_1(\tau) u(x,\tau) d\tau - \phi_1^\epsilon(x)\right) dx \\ &+ \int_\Omega \int_0^T \omega_2(t)\eta(x,t)dt \left(\int_0^T \omega_2(\tau) u(x,\tau) d\tau - \phi_2^\epsilon(x)\right) dx, \quad \forall \eta \in H^{1,1}(Q) \text{ with } \eta(\cdot,0) = 0. \end{aligned} \tag{25}$$

Theorem 2. *The objective functional $J(q,\phi)$ is Fréchet differentiable, and $J'_q(q,\phi)$ and $J'_\phi(q,\phi)$ are given by*

$$J'_q(q,\phi) = -\int_0^T u(x,t)\lambda(x,t)dt, \tag{26}$$

$$J'_\phi(q,\phi) = \lambda(x,0). \tag{27}$$

Proof. Take $\Delta q \in L_\infty(\Omega)$ such that $q+\Delta q \in \mathcal{A}_1$, and denote by $\Delta u_q := u(q+\Delta q,\phi) - u(q,\phi)$ the increment of u with respect to q. According to the initial-boundary value problem (1)–(3), this increment satisfies the sensitivity problem:

$$\begin{cases} \frac{\partial(\Delta u_q)}{\partial t} = \nabla\cdot(k\nabla(\Delta u_q)) - q\Delta u_q - u(q+\Delta q,\phi)\Delta q, & (x,t)\in Q, \\ k\frac{\partial(\Delta u_q)}{\partial \nu} + \alpha\Delta u_q = 0, & (x,t)\in S, \\ \Delta u_q(x,0) = 0, & x\in\Omega, \end{cases} \tag{28}$$

and using the estimate (6) for the above parabolic problem, we have

$$\|\Delta u_q\|_{L_2(Q)} \le C_0\|u\Delta q\|_{L_2(Q)} \le C_0\|\Delta q\|_{L_\infty(\Omega)}\|u\|_{L_2(Q)}.$$

Denote $\Delta J_q := J(q+\Delta q,\phi) - J(q,\phi)$, then we have

$$\begin{aligned} \Delta J_q =& \frac{1}{2}\left\|\int_0^T \omega_1(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\int_0^T \omega_2(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} \\ &+ \int_Q \omega_1(t)\Delta u_q(x,t)\left(\int_0^T \omega_1(\tau)u(x,\tau)d\tau - \phi_1^\epsilon(x)\right)dxdt \\ &+ \int_Q \omega_2(t)\Delta u_q(x,t)\left(\int_0^T \omega_2(\tau)u(x,\tau)d\tau - \phi_2^\epsilon(x)\right)dxdt. \end{aligned}$$

By the adjoint problem (24) and the sensitivity problem (28), we have

$$\begin{aligned} \Delta J_q =& \frac{1}{2}\left\|\int_0^T \omega_1(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\int_0^T \omega_2(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} \\ &+ \int_Q \Delta u_q\left\{-\frac{\partial\lambda}{\partial t} - \nabla\cdot(k\nabla\lambda) + q\lambda\right\}dxdt, \end{aligned}$$

and

$$\begin{aligned} \int_Q \Delta u_q\left\{-\frac{\partial\lambda}{\partial t} - \nabla\cdot(k\nabla\lambda) + q\lambda\right\}dxdt =& -\int_\Omega \Delta u_q\lambda|_0^T dx \\ &+ \int_Q \lambda\left\{\frac{\partial(\Delta u_q)}{\partial t} - \nabla\cdot(k\nabla(\Delta u_q)) + q\Delta u_q\right\}dxdt + \int_S\left\{k\frac{\partial(\Delta u_q)}{\partial\nu}\lambda - k\frac{\partial\lambda}{\partial\nu}\Delta u_q\right\}dsdt \\ =& -\int_Q \Delta q u(q+\Delta q,\phi)\lambda dxdt = -\int_Q \Delta q\Delta u_q\lambda dxdt - \int_Q \Delta q u\lambda dxdt, \end{aligned}$$

thus

$$\begin{aligned} \Delta J_q =& \frac{1}{2}\left\|\int_0^T \omega_1(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\int_0^T \omega_2(t)\Delta u_q(x,t)dt\right\|^2_{L_2(\Omega)} \\ &- \int_Q \Delta q\Delta u_q\lambda dxdt - \int_Q \Delta q u\lambda dxdt. \end{aligned}$$

We have

$$\left\|\int_0^T \omega_1(t)\Delta u_q(x,t)dt\right\|_{L_2(\Omega)}^2 \le c\|\omega_1\|_{L_\infty(0,T)}^2\|\Delta u_q\|_{L_2(Q)}^2 \le cC_0^2\|\omega_1\|_{L_\infty(0,T)}^2\|u\|_{L_2(Q)}^2\|\Delta q\|_{L_\infty(\Omega)}^2,$$

where $c>0$ depends on Ω, and similarly

$$\left\|\int_0^T \omega_2(t)\Delta u_q(x,t)dt\right\|_{L_2(\Omega)}^2 \le cC_0^2\|\omega_2\|_{L_\infty(0,T)}^2\|u\|_{L_2(Q)}^2\|\Delta q\|_{L_\infty(\Omega)}^2,$$

$$\left|\int_Q \Delta q\Delta u_q\lambda dxdt\right| \le \|\Delta q\|_{L_\infty(\Omega)}\|\Delta u_q\|_{L_2(Q)}\|\lambda\|_{L_2(Q)} \le C_0\|u\|_{L_2(Q)}\|\lambda\|_{L_2(Q)}\|\Delta q\|_{L_\infty(\Omega)}^2.$$

Finally,

$$\Delta J_q = -\int_Q \Delta q u\lambda dxdt + o\left(\|\Delta q\|_{L_\infty(\Omega)}\right), \tag{29}$$

which means that the Fréchet derivative $J_q'(q,\phi)$ is given by (26).

Similarly, take $\Delta\phi \in L_2(\Omega)$ such that $\phi+\Delta\phi \in \mathcal{A}_2$, and denote by $\Delta u_\phi := u(q,\phi+\Delta\phi)-u(q,\phi)$ the increment of u with respect to ϕ, then this increment satisfies the sensitivity problem

$$\begin{cases} \frac{\partial(\Delta u_\phi)}{\partial t} = \nabla\cdot(k\nabla(\Delta u_\phi)) - q\Delta u_\phi, & (x,t)\in Q, \\ k\frac{\partial(\Delta u_\phi)}{\partial \nu} + \alpha\Delta u_\phi = 0, & (x,t)\in S, \\ \Delta u_\phi(x,0) = \Delta\phi, & x\in\Omega. \end{cases} \tag{30}$$

Then, we can obtain that the Fréchet derivative $J_\phi'(q,\phi)$ is given by (27) by the same approach. The theorem is proved. □

4. Conjugate gradient method

The following iteration process based on the CGM scheme is applied for the reconstruction of the two unknown functions $q(x)$ and $\phi(x)$ by minimizing the objective functional $J(q,\phi)$ in (23):

$$q^{n+1}(x) = q^n(x) + \beta_q^n d_q^n(x), \quad \phi^{n+1}(x) = \phi^n(x) + \beta_\phi^n d_\phi^n(x), \quad n = 0,1,2,\cdots \tag{31}$$

with the search directions d_q^n and d_ϕ^n given by

$$d_q^n = \begin{cases} -J_q'^0, \\ -J_q'^n + \gamma_q^n d_q^{n-1}, \end{cases} \qquad d_\phi^n = \begin{cases} -J_\phi'^0, \\ -J_\phi'^n + \gamma_\phi^n d_\phi^{n-1}, \end{cases} \qquad n = 1,2,\cdots \tag{32}$$

where n is the subscript which denotes the number of iterations, $J_q'^n = J_q'(q^n,\phi^n)$, $J_\phi'^n = J_\phi'(q^n,\phi^n)$, q^0 and ϕ^0 are the initial guesses, β_q^n and β_ϕ^n are the step sizes for q and ϕ in passing from iteration n to the next iteration $n+1$. In this work, the Fletcher-Reeves formula in [17] is utilized for the conjugate coefficients γ_q^n and γ_ϕ^n, and they are given by

$$\gamma_q^n = \frac{\|J_q'^n\|_{L_2(\Omega)}^2}{\|J_q'^{n-1}\|_{L_2(\Omega)}^2}, \quad \gamma_\phi^n = \frac{\|J_\phi'^n\|_{L_2(\Omega)}^2}{\|J_\phi'^{n-1}\|_{L_2(\Omega)}^2}, \quad n = 1,2,\cdots \tag{33}$$

To determine the step sizes β_q^n and β_ϕ^n, the exact line search is utilized, i.e.,

$$J^{n+1} = J(q^n+\beta_q^n d_q^n, \phi^n+\beta_\phi^n d_\phi^n) = \min_{\beta_q,\beta_\phi\ge 0} J(q^n+\beta_q d_q^n, \phi^n+\beta_\phi d_\phi^n), \quad n = 0,1,2\cdots. \tag{34}$$

By (29), (31) and the gradient $J_q'^{n+1}$ (26), we have

$$\begin{aligned}\frac{\partial J}{\partial \beta_q^n} &= \frac{\partial J}{\partial q^{n+1}} \cdot \frac{\partial q^{n+1}}{\partial \beta_q^n} = \lim_{\beta_q^n \to 0} \frac{J(q^{n+1}, \phi^{n+1}) - J(q^n, \phi^{n+1})}{\beta_q^n d_q^n} d_q^n = \lim_{\beta_q^n \to 0} \frac{J(q^{n+1}, \phi^{n+1}) - J(q^n, \phi^{n+1})}{\beta_q^n} \\ &= \lim_{\beta_q^n \to 0} \frac{1}{\beta_q^n} \left(- \int_Q u(q^n, \phi^{n+1}) \lambda(q^n, \phi^{n+1}) \beta_q^n d_q^n dx dt + o(\|\beta_q^n d_q^n\|_{L_\infty(\Omega)}) \right) \\ &= - \int_Q u(q^{n+1}, \phi^{n+1}) \lambda(q^{n+1}, \phi^{n+1}) d_q^n dx dt = \int_\Omega J_q'^{n+1} d_q^n dx,\end{aligned}$$

and similarly, we have

$$\frac{\partial J}{\partial \beta_\phi^n} = \int_\Omega J_\phi'^{n+1} d_\phi^n dx.$$

Thus, condition (34) implies that the step sizes β_q^n and β_ϕ^n satisfy the following conditions:

$$\langle J_q'^{n+1}, d_q^n \rangle = 0, \quad \langle J_\phi'^{n+1}, d_\phi^n \rangle = 0, \tag{35}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the space $L_2(\Omega)$.

4.1. Global convergence

For the exact data (7) and (8), the global convergence of the CGM over the admissible set $\mathcal{A}_1 \times \mathcal{A}_2$ is established in the following sense:

$$\liminf_{n\to\infty} \|J_q'^n\|_{L_2(\Omega)} = 0, \quad \liminf_{n\to\infty} \|J_\phi'^n\|_{L_2(\Omega)} = 0. \tag{36}$$

First, we will prove that the Fréchet gradients J_q' and J_ϕ' are Lipschitz continuous over $\mathcal{A}_1 \times \mathcal{A}_2$ under the following stronger assumption on the input data than in Lemma 1.

Assumption 1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain of class $C^{2,\beta}$ for some $\beta > 0$, i.e. the boundary $\partial\Omega$ is a $(N-1)$-dimensional manifold of class $C^{2,\beta}$ such that Ω lies locally on one side of $\partial\Omega$, (a function is of class $C^{2,\beta}$ if it is of class C^2 and its partial derivative of second-order are Hölder continuous of order β). Let $p > 1 + N/2$ and $r > N + 1$ and assume that $f \in L_p(Q)$ and $\mu \in L_r(S)$.*

Then we have the following lemma, see Proposition 3.3 of [34].

Lemma 2. *Let the Assumption 1 on Ω, f and μ hold. Let also the other assumptions of Lemma 1 on data α, k and q hold, and also let $\phi \in \mathcal{A}_2 \subset L_\infty(\Omega)$. Then, the weak solution $u(x,t) \in V_2^{1,0}(Q)$ of the initial-boundary value direct problem (1)–(3) also belongs to $L_\infty(Q)$ and there exists a positive constant $C = C(N, p, r, q^-, \Omega, T)$ such that*

$$\|u\|_{L_\infty(Q)} \leq C(\|f\|_{L_p(Q)} + \|\mu\|_{L_r(S)} + \|\phi\|_{L_\infty(\Omega)}). \tag{37}$$

For the adjoint problem we also have the following lemma.

Lemma 3. *Let the assumptions of Lemma 2 hold and let ω_1 and ω_2 be given weights in $L_\infty(0,T)$. Then, there exists a unique weak solution $\lambda(x,t) \in V_2^{1,0}(Q) \cap L_\infty(Q)$ to the adjoint problem (24) with $\epsilon = 0$, which satisfies*

$$\|\lambda\|_{L_\infty(Q)} \leq C_1 \|u\|_{L_\infty(Q)} \tag{38}$$

for some positive constant C_1 depending on $N, p, r, q^-, \Omega, T, \omega_1$ and ω_2.

Proof. First, through the change of time variable $t \mapsto T-t$, the adjoint problem (24) can be seen of the same form as the problem (1)–(3) with $\mu = \phi = 0$ and the source

$$f(x,t) = \tilde{f}(x,t) := \omega_1(t)\left(\int_0^T \omega_1(\tau)u(x,\tau)d\tau - \phi_1(x)\right) + \omega_2(t)\left(\int_0^T \omega_2(\tau)u(x,\tau)d\tau - \phi_2(x)\right).$$

From Lemma 2 it follows that $u \in L_\infty(Q)$ and since ω_1 and $\omega_2 \in L_\infty(0,T)$, and using also (7) and (8), we obtain that $\tilde{f} \in L_\infty(Q)$. Moreover, from (7) and (8), and using the inequality (37) of Lemma 2 for the function λ satisfying the adjoint problem (24) with $\epsilon = 0$, we obtain

$$\|\lambda\|_{L_\infty(Q)} \le C\|\tilde{f}\|_{L_p(Q)} \le C\|\tilde{f}\|_{L_\infty(Q)} \le 2C(\|\omega_1\|^2_{L_\infty(0,T)} + \|\omega_2\|^2_{L_\infty(0,T)})\|u\|_{L_\infty(Q)},$$

which implies that (38) holds. □

Theorem 3. *Under the assumptions of Lemma 3, the gradients J'_q in (26) and J'_ϕ in (27) are Lipschitz continuous, namely, there exist two positive constants M_q and M_ϕ such that*

$$\|J'_q(q^1,\phi^1) - J'_q(q^2,\phi^2)\|_{L_2(\Omega)} \le M_q(\|q^1-q^2\|_{L_2(\Omega)} + \|\phi^1-\phi^2\|_{L_2(\Omega)}), \tag{39}$$
$$\|J'_\phi(q^1,\phi^1) - J'_\phi(q^2,\phi^2)\|_{L_2(\Omega)} \le M_\phi(\|q^1-q^2\|_{L_2(\Omega)} + \|\phi^1-\phi^2\|_{L_2(\Omega)}), \tag{40}$$

for any $q^1, q^2 \in \mathcal{A}_1$, $\phi^1, \phi^2 \in \mathcal{A}_2$.

Proof. By Lemma 2 and using the estimate (37), it is easy to see that

$$\|u(q,\phi)\|_{L_\infty(Q)} \le C(\|f\|_{L_p(Q)} + \|\mu\|_{L_r(S)} + \kappa) =: K_1 \tag{41}$$

for any $q \in \mathcal{A}_1$ and $\phi \in \mathcal{A}_2$, and K_1 is a positive constant depending on $N, p, r, q^-, \kappa, \Omega, T, f$ and μ (independent of q and ϕ). Similarly, using Lemma 3, (38) and (41), we have

$$\|\lambda(q,\phi)\|_{L_\infty(Q)} \le C_1K_1 =: K_2, \tag{42}$$

where K_2 is a positive constant depending on $N, p, r, q^-, \kappa, \Omega, T, \omega_1, \omega_2, f$ and μ (independent of q and ϕ).

Denote $u_q := u(q^1,\phi^1) - u(q^2,\phi^1)$ and by the direct problem (1)–(3), we have

$$\begin{cases} \frac{\partial u_q}{\partial t} = \nabla\cdot(k\nabla u_q) - q^1u_q - (q^1-q^2)u(q^2,\phi^1), & (x,t)\in Q,\\ k\frac{\partial u_q}{\partial \nu} + \alpha u_q = 0, & (x,t)\in S,\\ u_q(x,0) = 0, & x\in\Omega. \end{cases}$$

Since $q^1, q^2 \in \mathcal{A}_1 \subset L_\infty(\Omega)$, then $q^1 - q^2 \in L_\infty(\Omega) \subset L_2(\Omega)$, and by using the estimate (6), we have

$$\|u_q\|_{L_2(Q)} \le C_0\|(q^1-q^2)u(q^2,\phi^1)\|_{L_2(Q)} \le C_0K_1\|q^1-q^2\|_{L_2(\Omega)}.$$

Similarly, denoting $u_\phi := u(q^2,\phi^1) - u(q^2,\phi^2)$, we have

$$\begin{cases} \frac{\partial u_\phi}{\partial t} = \nabla\cdot(k\nabla u_\phi) - q^2u_\phi, & (x,t)\in Q,\\ k\frac{\partial u_\phi}{\partial \nu} + \alpha u_q = 0, & (x,t)\in S,\\ u_\phi(x,0) = \phi^1-\phi^2, & x\in\Omega, \end{cases}$$

and $\|u_\phi\|_{L_2(Q)} \le C_0\|\phi^1-\phi^2\|_{L_2(\Omega)}$.

Define $\lambda_q := \lambda(q^1,\phi^1) - \lambda(q^2,\phi^1)$ and by the adjoint problem (24), we have

$$\begin{cases} \frac{\partial \lambda_q}{\partial t} = -\nabla\cdot(k\nabla\lambda_q) + q^1\lambda_q + (q^1-q^2)\lambda(q^2,\phi^1) \\ \qquad\qquad -\omega_1(t)\int_0^T \omega_1(\tau)u_q(x,\tau)d\tau - \omega_2(t)\int_0^T \omega_2(\tau)u_q(x,\tau)d\tau, & (x,t)\in Q, \\ k\frac{\partial\lambda_q}{\partial\nu} + \alpha\lambda_q = 0, & (x,t)\in S, \\ \lambda_q(x,T) = 0, & x\in\Omega, \end{cases}$$

and by Lemma 1, we have

$$\begin{aligned} \|\lambda_q\|_{L_2(Q)} \le & C_0 \left\| (q^1-q^2)\lambda(q^2,\phi^1) + \omega_1\int_0^T \omega_1(\tau)u_q(\cdot,\tau)d\tau + \omega_2\int_0^T \omega_2(\tau)u_q(\cdot,\tau)d\tau \right\|_{L_2(Q)} \\ \le & C_0K_2\|q^1-q^2\|_{L_2(\Omega)} + C_0\left(\|\omega_1\|^2_{L_\infty(0,T)} + \|\omega_2\|^2_{L_\infty(0,T)}\right)\|u_q\|_{L_2(Q)} \le K_3\|q^1-q^2\|_{L_2(\Omega)}, \end{aligned}$$

where $K_3 := C_0K_2 + C_0^2K_1\left(\|\omega_1\|^2_{L_\infty(0,T)} + \|\omega_2\|^2_{L_\infty(0,T)}\right)$. Similarly, denoting $\lambda_\phi := \lambda(q^2,\phi^1) - \lambda(q^2,\phi^2)$, we have

$$\begin{cases} \frac{\partial \lambda_\phi}{\partial t} = -\nabla\cdot(k\nabla\lambda_\phi) + q^2\lambda_\phi \\ \qquad\qquad -\omega_1(t)\int_0^T \omega_1(\tau)u_\phi(x,\tau)d\tau - \omega_2(t)\int_0^T \omega_2(\tau)u_\phi(x,\tau)d\tau, & (x,t)\in Q, \\ k\frac{\partial\lambda_\phi}{\partial\nu} + \alpha\lambda_\phi = 0, & (x,t)\in S, \\ \lambda_\phi(x,T) = 0, & x\in\Omega, \end{cases}$$

and

$$\begin{aligned} \|\lambda_\phi\|_{L_2(Q)} \le & C_0 \left\| \omega_1\int_0^T \omega_1(\tau)u_\phi(\cdot,\tau)d\tau + \omega_2\int_0^T \omega_2(\tau)u_\phi(\cdot,\tau)d\tau \right\|_{L_2(Q)} \\ \le & C_0\left(\|\omega_1\|^2_{L_\infty(0,T)} + \|\omega_2\|^2_{L_\infty(0,T)}\right)\|u_\phi\|_{L_2(Q)} \le K_4\|\phi^1-\phi^2\|_{L_2(\Omega)}, \end{aligned}$$

where $K_4 := C_0^2\left(\|\omega_1\|^2_{L_\infty(0,T)} + \|\omega_2\|^2_{L_\infty(0,T)}\right)$.

Denote $\Delta J'_q := J'_q(q^1,\phi^1) - J'_q(q^2,\phi^2)$, then we have

$$\|\Delta J'_q\|_{L_2(\Omega)} = \left\| \int_0^T [u(q^1,\phi^1)\lambda(q^1,\phi^1) - u(q^2,\phi^2)\lambda(q^2,\phi^2)]dt \right\|_{L_2(\Omega)},$$

and

$$\begin{aligned} u(q^1,\phi^1)\lambda(q^1,\phi^1) - u(q^2,\phi^2)\lambda(q^2,\phi^2) = [u(q^1,\phi^1) - u(q^2,\phi^1) + u(q^2,\phi^1) - u(q^2,\phi^2)]\lambda(q^1,\phi^1) \\ +[\lambda(q^1,\phi^1) - \lambda(q^2,\phi^1) + \lambda(q^2,\phi^1) - \lambda(q^2,\phi^2)]u(q^2,\phi^2) = (u_q+u_\phi)\lambda(q^1,\phi^1) + (\lambda_q+\lambda_\phi)u(q^2,\phi^2), \end{aligned}$$

thus

$$\begin{aligned} \|\Delta J'_q\|_{L_2(\Omega)} \le & \left\| \int_0^T (u_q+u_\phi)\lambda(q^1,\phi^1)dt \right\|_{L_2(\Omega)} + \left\| \int_0^T (\lambda_q+\lambda_\phi)u(q^2,\phi^2)dt \right\|_{L_2(\Omega)} \\ \le & c(\|u_q\|_{L_2(Q)} + \|u_\phi\|_{L_2(Q)})\|\lambda(q^1,\phi^1)\|_{L_\infty(Q)} + c(\|\lambda_q\|_{L_2(Q)} + \|\lambda_\phi\|_{L_2(\Omega)})\|u(q^2,\phi^2)\|_{L_\infty(Q)} \\ \le & c(C_0K_1K_2 + K_1K_3)\|q^1-q^2\|_{L_2(\Omega)} + c(C_0K_2 + K_1K_4)\|\phi^1-\phi^2\|_{L_2(\Omega)} \\ \le & M_q(\|q^1-q^2\|_{L_2(\Omega)} + \|\phi^1-\phi^2\|_{L_2(\Omega)}), \end{aligned}$$

where c is a positive constant depending on Ω and T, and $M_q := c \times \max\{C_0K_1K_2 + K_1K_3, C_0K_2 + K_1K_4\} > 0$, which is independent of q^1, q^2, ϕ^1 and ϕ^2.

Denote $\Delta J'_\phi = J'_\phi(q^1, \phi^1) - J'_\phi(q^2, \phi^2)$, then by (6) we have

$$\begin{aligned}\|\Delta J'_\phi\|_{L_2(\Omega)} =&\|\Delta\lambda(x,0)(q^1,\phi^1) - \lambda(x,0)(q^2,\phi^2)\|_{L_2(\Omega)} \leq \|\lambda_q(x,0)\|_{L_2(\Omega)} + \|\lambda_\phi(x,0)\|_{L_2(\Omega)}\\ \leq& C_0\left\|(q^1-q^2)\lambda(q^2,\phi^1) + \omega_1\int_0^T \omega_1(\tau)u_q(\cdot,\tau)d\tau + \omega_2\int_0^T \omega_2(\tau)u_q(\cdot,\tau)d\tau\right\|_{L_2(Q)}\\ &+ C_0\left\|\omega_1\int_0^T \omega_1(\tau)u_\phi(\cdot,\tau)d\tau + \omega_2\int_0^T \omega_2(\tau)u_\phi(\cdot,\tau)d\tau\right\|_{L_2(Q)}\\ \leq& K_3\|q^1-q^2\|_{L_2(\Omega)} + K_4\|\phi_1-\phi_2\|_{L_2(\Omega)} \leq M_\phi(\|q^1-q^2\|_{L_2(\Omega)} + \|\phi^1-\phi^2\|_{L_2(\Omega)}),\end{aligned}$$

where $M_\phi := \max\{K_3, K_4\} > 0$ independent of q^1, q^2, ϕ^1 and ϕ^2. The theorem is proved. □

From the Lipschitz continuity of the gradients J'_q and J'_ϕ, following the arguments of [18, 35, 36, 37], we can obtain that

$$\sum_{n\geq 0}\frac{\|J'^n_q\|^4_{L_2(\Omega)}}{\|d^n_q\|^2_{L_2(\Omega)}} < \infty, \quad \sum_{n\geq 0}\frac{\|J'^n_\phi\|^4_{L_2(\Omega)}}{\|d^n_\phi\|^2_{L_2(\Omega)}} < \infty. \tag{43}$$

Theorem 4. *Under the assumptions of Theorem 3, the CGM either terminates at a stationary point or converges in the following senses:*

$$\liminf_{n\to\infty}\|J'^n_q\|_{L_2(\Omega)} = 0, \quad \liminf_{n\to\infty}\|J'^n_\phi\|_{L_2(\Omega)} = 0. \tag{44}$$

Proof. Assume by absurd that $\liminf_{n\to\infty}\|J'^n_q\|_{L_2(\Omega)} \neq 0$. Then, there exists a constant $c > 0$ and a natural number $n_0 > 0$ such that $\|J'^n_q\|_{L_2(\Omega)} \geq c$ for $n \geq n_0$. Then, (33) and (35) imply that $\|d^n_q\|^2_{L_2(\Omega)} = \|J'^n_q\|^2_{L_2(\Omega)} + \frac{\|J'^n_q\|^4_{L_2(\Omega)}}{\|J'^{n-1}_q\|^4_{L_2(\Omega)}}\|d^{n-1}_q\|^2_{L_2(\Omega)}$ for $n > n_0$. Dividing both sides by $\|J'^n_q\|^4_{L_2(\Omega)}$ we obtain

$$\frac{\|d^n_q\|^2_{L_2(\Omega)}}{\|J'^n_q\|^4_{L_2(\Omega)}} = \frac{1}{\|J'^n_q\|^2_{L_2(\Omega)}} + \frac{\|d^{n-1}_q\|^2_{L_2(\Omega)}}{\|J'^{n-1}_q\|^4_{L_2(\Omega)}} = \sum_{i=n_0}^{n}\frac{1}{\|J'^i_q\|^2_{L_2(\Omega)}} \leq \frac{n-n_0+1}{c}, \quad n > n_0.$$

Then,

$$\sum_{n\geq 0}\frac{\|J'^n_q\|^4_{L_2(\Omega)}}{\|d^n_q\|^2_{L_2(\Omega)}} \geq \sum_{n>n_0}\frac{\|J'^n_q\|^4_{L_2(\Omega)}}{\|d^n_q\|^2_{L_2(\Omega)}} \geq c\sum_{n\geq 1}\frac{1}{n+1} = \infty,$$

which is in contradiction with the first inequality in (43). Thus, the first result in (44) holds, and the second result in (44) can be obtained by the same method. The proof is complete. □

4.2. CGM

Based on the above discussions, all the coefficients of the iteration process (31) and (32) are expressed in explicit form except for the search step sizes β^n_q and β^n_ϕ which satisfy the exact line search conditions (35). These can be found by minimizing

$$J(q^{n+1},\phi^{n+1}) =\frac{1}{2}\int_{\Omega}\left(\int_0^T \omega_1 u(q^n+\beta_q^n d_q^n,\phi^n+\beta_\phi^n d_\phi^n)dt-\phi_1^\epsilon\right)^2 dx$$
$$+\frac{1}{2}\int_{\Omega}\left(\int_0^T \omega_2 u(q^n+\beta_q^n d_q^n,\phi^n+\beta_\phi^n d_\phi^n)dt-\phi_2^\epsilon\right)^2 dx.$$

Since the above expression shows that the step sizes β_q^n and β_ϕ^n are in implicit form, the Taylor series expression can be applied to approximate $J(q^{n+1},\phi^{n+1})$ such that the step sizes β_q^n and β_ϕ^n become explicit in the new formulation. Therefore, setting $\Delta q^n = d_q^n$ and $\Delta\phi^n = d_\phi^n$, the temperature $u(x,t;q^n+\beta_q^n d_q^n,\phi^n+\beta_\phi^n d_\phi^n)$ is linearised by a Taylor series expression in the form

$$u(x,t;q^n+\beta_q^n d_q^n,\phi^n+\beta_\phi^n d_\phi^n)\approx u(x,t;q^n,\phi^n)+\beta_q^n d_q^n\frac{\partial u(x,t;q^n,\phi^n)}{\partial q^n}+\beta_\phi^n d_\phi^n\frac{\partial u(x,t;q^n,\phi^n)}{\partial \phi^n}$$
$$\approx u(x,t;q^n,\phi^n)+\beta_q^n\Delta u_q(x,t;q^n,\phi^n)+\beta_\phi^n\Delta u_\phi(x,t;q^n,\phi^n).$$

Here the functions $\Delta u_q(x,t;q^n,\phi^n)$ and $\Delta u_\phi(x,t;q^n,\phi^n)$ can be obtained by solving the sensitivity problems (28), and (30). Then, we rewrite

$$u_1^n=\int_0^T \omega_1 u(q^n,\phi^n)dt,\quad u_2^n=\int_0^T \omega_2 u(q^n,\phi^n)dt,$$
$$\Delta u_{q,1}^n=\int_0^T \omega_1 \Delta u_q(q^n,\phi^n)dt,\quad \Delta u_{q,2}^n=\int_0^T \omega_2 \Delta u_q(q^n,\phi^n)dt,$$
$$\Delta u_{\phi,1}^n=\int_0^T \omega_1 \Delta u_\phi(q^n,\phi^n)dt,\quad \Delta u_{\phi,2}^n=\int_0^T \omega_2 \Delta u_\phi(q^n,\phi^n)dt,$$

and then

$$J(q^{n+1},\phi^{n+1}) =\frac{1}{2}\int_{\Omega}\left\{\left(u_1^n+\beta_q^n\Delta u_{q,1}^n+\beta_\phi^n\Delta u_{\phi,1}^n-\phi_1^\epsilon\right)^2+\left(u_2^n+\beta_q^n\Delta u_{q,2}^n+\beta_\phi^n\Delta u_{\phi,2}^n-\phi_2^\epsilon\right)^2\right\}dx.$$

The partial derivatives of the objective functional $J(q^{n+1},\phi^{n+1})$ with respect to β_q^n and β_ϕ^n are given by

$$\frac{\partial J(q^{n+1},\phi^{n+1})}{\partial\beta_q^n}=C_1\beta_q^n+C_2\beta_\phi^n+C_3,\quad \frac{\partial J(q^{n+1},\phi^{n+1})}{\partial\beta_\phi^n}=C_2\beta_q^n+C_4\beta_\phi^n+C_5,$$

where

$$C_1=\int_\Omega\left[(\Delta u_{q,1}^n)^2+(\Delta u_{q,2}^n)^2\right]dx,\quad C_2=\int_\Omega\left(\Delta u_{q,1}^n\Delta u_{\phi,1}^n+\Delta u_{q,2}^n\Delta u_{\phi,2}^n\right)dx,$$
$$C_3=\int_\Omega\left[(u_1^n-\phi_1^\epsilon)\Delta u_{q,1}^n+(u_2^n-\phi_2^\epsilon)\Delta u_{q,2}^n\right]dx,\quad C_4=\int_\Omega\left[(\Delta u_{\phi,1}^n)^2+(\Delta u_{\phi,2}^n)^2\right]dx,$$
$$C_5=\int_\Omega\left[(u_1^n-\phi_1^\epsilon)\Delta u_{\phi,1}^n+u_2^n-\phi_2^\epsilon)\Delta u_{\phi,2}^n\right]dx.$$

According to the conditions (35), we set

$$\frac{\partial J(q^{n+1},\phi^{n+1})}{\partial\beta_q^n}=\frac{\partial J(q^{n+1},\phi^{n+1})}{\partial\beta_\phi^n}=0,$$

and then obtain the search step sizes β_q^n and β_ϕ^n as follows:

$$\beta_q^n = \frac{C_3C_4 - C_2C_5}{C_2^2 - C_1C_4}, \quad \beta_\phi^n = \frac{C_1C_5 - C_2C_3}{C_2^2 - C_1C_4}. \tag{45}$$

The iteration process given by (31) does not provide the CGM with the stabilisation necessary for the minimizing of the objective functional (23) to be classified as well-posed because of the errors inherent in the time-average temperature measurements (7) and (8). However, the method may become well-posed if the discrepancy principle is applied to stop the iteration procedure. According to the discrepancy principle, the iterative procedure is stopped when the following criterion is satisfied:

$$J(q^n, \phi^n) \approx \frac{1}{2}\left(\|\phi_1^\epsilon - \phi_1\|_{L_2(\Omega)}^2 + \|\phi_2^\epsilon - \phi_2\|_{L_2(\Omega)}^2\right) \le \epsilon^2, \tag{46}$$

where ϕ_1^ϵ and ϕ_2^ϵ are noisy perturbations of the data ϕ_1 and ϕ_2, respectively, satisfying (9). Then, the CGM for the numerical reconstruction of the perfusion coefficient $q(x)$ and initial temperature $\phi(x)$ is shown as follows:

- S1 Set $n = 0$ and choose initial guesses q^0 and ϕ^0 for the unknowns q and ϕ, respectively.
- S2 Solve the initial-boundary value direct problem (1)–(3) numerically by applying the FDM to compute the temperature $u(x, t; q^n, \phi^n)$, and the objective functional $J(q^n, \phi^n)$ by (23).
- S3 Solve the adjoint problem (24) to get the function $\lambda(x, t; q^n, \phi^n)$, and the gradients $J'_q(q^n, \phi^n)$ in (26) and $J'_\phi(q^n, \phi^n)$ in (27). Compute the conjugate coefficients γ_q^n and γ_ϕ^n in (33), and the search directions (32).
- S4 Solve the sensitivity problems given by (28) for $\Delta u_q(x, t; q^n, \phi^n)$, and (30) for $\Delta u_\phi(x, t; q^n, \phi^n)$ by taking $\Delta q^n = d_q^n$ and $\Delta\phi^n = d_\phi^n$, and compute the step sizes β_q^n and β_ϕ^n by (45).
- S5 Compute q^{n+1} and ϕ^{n+1} by (31).
- S6 If the condition (46) is satisfied, then go to S7. Else set $n = n + 1$, and go to S2.
- S7 End.

5. Numerical results and discussions

In this section, the perfusion coefficient $q(x)$ and the initial temperature $\phi(x)$ are reconstructed numerically and simultaneously by the nonlinear CGM proposed in Section 4. The FDM, based on the Crank-Nicolson scheme for the one-dimensional ($N = 1$) case and the alternating direction implicit (ADI) scheme for the two-dimensional ($N = 2$) case, are applied to solve the direct, sensitivity and adjoint problems involved. The Simpson's rule is utilized to deal with all the integrals involved. The accuracy errors, as functions of the iteration number n, are defined as

$$E_1(q^n) = \|q^n - q\|_{L_2(\Omega)}, \tag{47}$$
$$E_2(\phi^n) = \|\phi^n - \phi\|_{L_2(\Omega)}, \tag{48}$$

where q^n and ϕ^n are the numerical results obtained by the CGM at the iteration number n, and q and ϕ are the analytical perfusion coefficient and initial temperature, if available.

The integral temperature observations ϕ_1^ϵ and ϕ_2^ϵ are corrupted by Gaussian additive noise as

$$\phi_1^\epsilon = \phi_1 + \sigma \times \text{random}(1), \quad \phi_2^\epsilon = \phi_2 + \sigma \times \text{random}(1), \tag{49}$$

where $\sigma = \frac{p}{100} \max_{x\in\overline{\Omega}} \{|\phi_1|, |\phi_2|\}$ is the standard deviation, $p\%$ represents the percentage of noise, and the term random(1) generates random values from the normal distribution with zero mean and standard deviation equal to unity.

In the following sections, numerical examples are considered in one- and two-dimensions.

5.1. Example 1

In the the one-dimensional ($N = 1$) case, we take $\Omega = (0, 1)$

$$k \equiv 1, \quad \alpha \equiv 1, \quad f(x,t) = \left(x^2(1 + \pi + \sin(\pi x)) + \pi^2 \sin(\pi x)\right) e^{-t}, \quad \mu(0,t) = \mu(1,t) = e^{-t}, \tag{50}$$

and the analytical solution given by

$$q(x) = 1 + x^2, \quad \phi(x) = 1 + \pi + \sin(\pi x), \quad u(x,t) = (1 + \pi + \sin(\pi x))e^{-t}. \tag{51}$$

The initial guesses are chosen arbitrary, say $q^0(x) = 1.5$ and $\phi^0(x) = 1$.

We first fix $T = 1$, $\omega_1(t) = 1$ and $\omega_2(t) = 2t$ such that (7) and (8) become

$$\phi_1(x) = (1 - e^{-T})(1 + \pi + \sin(\pi x)), \quad \phi_2(x) = 2(1 - (1 + T)e^{-T})(1 + \pi + \sin(\pi x)), \tag{52}$$

and investigate, for exact data, the influence of the mesh size of the Crank-Nicolson FDM that is used to solve the problems (direct, sensitivity and adjoint problems) in the CGM, which is run for 50 iterations. Then, the obtained errors (47) and (48) were $E_1(q^{50}) \in \{0.0214, 0.0397, 0.0691\}$ and $E_2(\phi^{50}) \in \{0.0344, 0.0478, 0.0529\}$ for the three mesh sizes $\Delta x = \Delta t \in \{0.01, 0.05, 0.1\}$, respectively. These results indicate a monotonic decreasing convergence of the numerical solutions for $q(x)$ and $\phi(x)$, as the FDM mesh size decreases.

Next, we investigate the influence of the final time T, as for the classical backward heat conduction problem, with final data at $t = T$, the reconstruction of the initial temperature (3) becomes more (exponentially) ill-posed with increasing T. For various $T \in \{1, 2, 4\}$, the obtained errors (47) and (48), with $\Delta x = \Delta t = 0.01$, were $E_1(q^{50}) \in \{0.0214, 0.0168, 0.0253\}$ and $E_2(\phi^{50}) \in \{0.0344, 0.0366, 0.0506\}$. These results indicate only some slight decrease in accuracy of the initial temperature (3), as T increases, because the imposed extra data (52) represent an average temperature measurement rather than the temperature measurement at a latter time. In support to this conclusion, supposing that $q(x)$ has been determined or is known, it is interesting to comment on solving a new backward average heat conduction problem consisting of reconstructing the initial temperature (3) from a time integral measurement, e.g. consider solving

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,1) \times (0,T), \\ u|_{\partial\Omega\times(0,T)} = 0, \\ \int_0^T u(x,t)dt = \phi_T(x), & x \in (0,1). \end{cases} \tag{53}$$

Then, by the semi-group theory, or simply by the method of separating variables, one obtains the exact solution of the problem (53) in the Fourier sine series form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where

$$A_n = \frac{2n^2\pi^2}{1 - e^{-n^2\pi^2 T}} \int_0^1 \phi_T(x) \sin(n\pi x) dx, \quad n \in \mathbb{N}^*,$$

which shows that the problem (53) is only mildly ill-posed, as opposed to the classical backward heat conduction problem, which is exponentially ill-posed.

Finally, fixing $T = 1$ and $\Delta x = \Delta t = 0.01$, we investigate the influence of the choices of the weight functions in (7) and (8). The obtained errors (47) and (48) were $E_1(q^{50}) \in \{0.0214, 0.0357, 0.0170\}$ and $E_2(\phi^{50}) \in \{0.0344, 0.1093, 0.2216\}$ for the choices $(\omega_1(t), \omega_2(t)) \in \{(1, 2t), (1, t^2), (2t, t^2)\}$, respectively. These results indicate that lower-order moments (in t) contain more information than the higher-order moments for the recovery of the initial temperature (3).

In the remainer of this section we fix $T = 1$, $\Delta x = \Delta t = 0.01$ and $\omega_1(t) = 1$, $\omega_2(t) = 2t$, such that the measurement information (7) and (8) is given by (52).

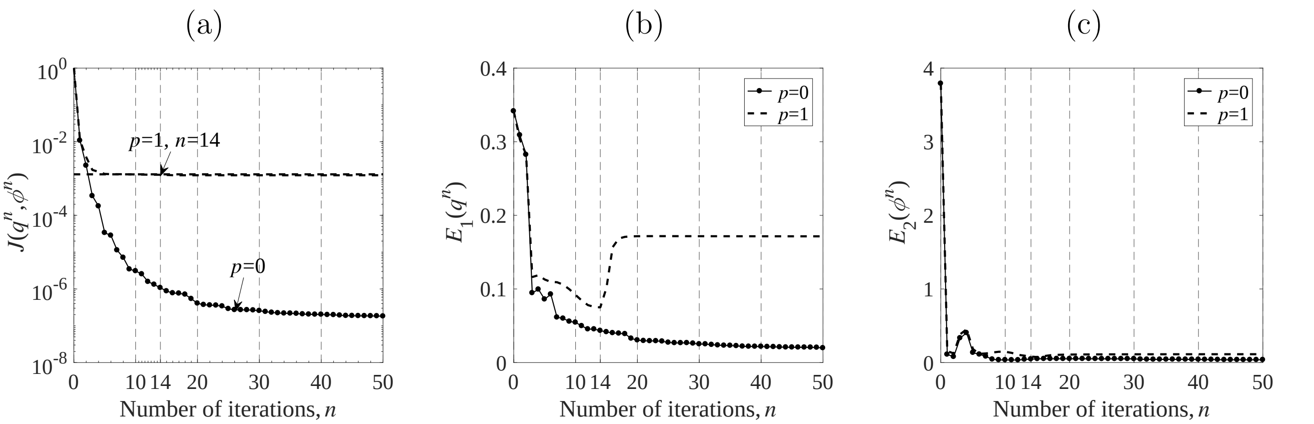

Figure 1: (a) The objective functional (23), the accuracy errors (b) (47) and (c) (48), with $p \in \{0, 1\}$ noise, for Example 1.

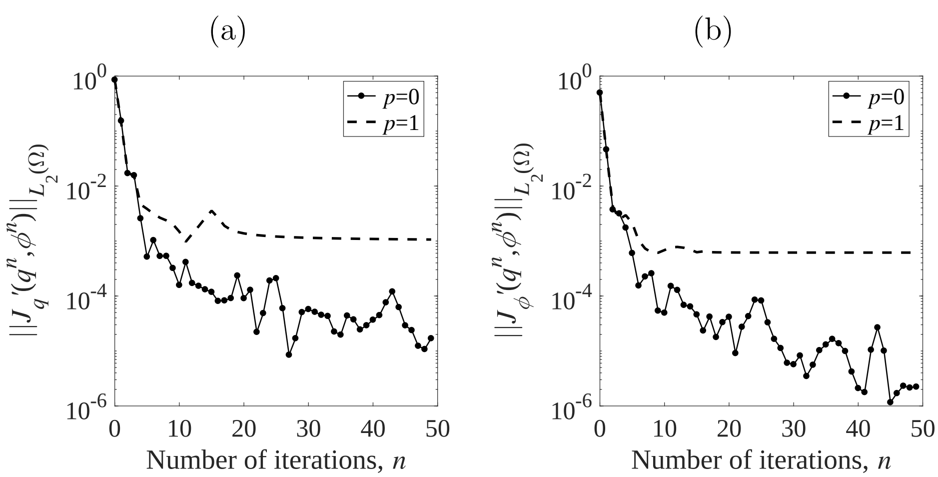

Figure 2: The norm of gradients (a) $\|J_q'(q^n, \phi^n)\|_{L_2(\Omega)}$ and (b) $\|J_\phi'(q^n, \phi^n)\|_{L_2(\Omega)}$, with $p \in \{0, 1\}$ noise, for Example 1.

Figures 1(a)–1(c) show the objective functional $J(q^n, \phi^n)$ given by (23) and the accuracy errors $E_1(q^n)$ given by (47) and $E_2(\phi^n)$ given by (48), for the reconstruction of the two unknown functions, simultaneously, in case of no noise, i.e., $p = 0$, and with $p = 1$ noise. Figure 1(a) illustrates the rapid monotonic decreasing convergence of the objective functional, as a function of iteration number n. The stopping number of the iterative process is 50 for exact data, i.e., for $p = 0$, whilst the iteration process is stopped at iteration number 14 according to the discrepancy principle (46) for $p = 1$ noise. On comparing Figures 1(a)–1(c) it can be seen that there is some consistency and agreement between the stopping iteration numbers given by the discrepancy principle (46) and the optimal iteration numbers given by the minimum of the errors (47) and (48).

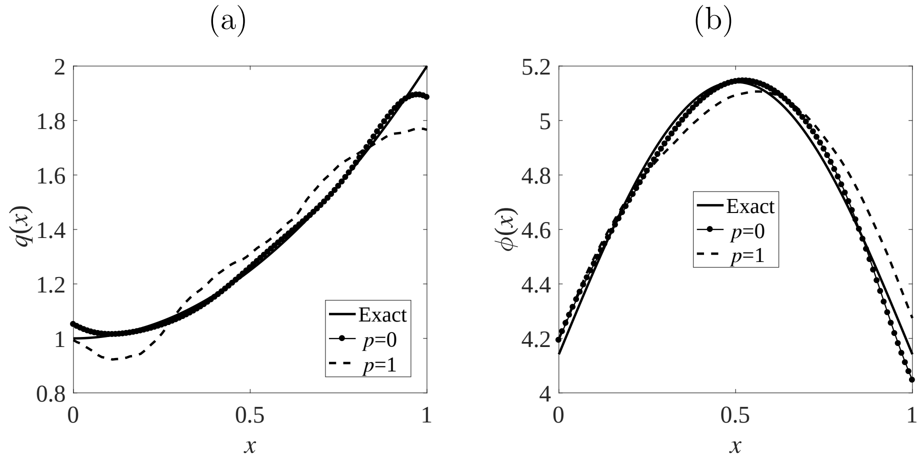

Figure 3: The exact and numerical results for (a) the perfusion coefficient $q(x)$ and (b) the initial temperature $\phi(x)$, with $p \in \{0, 1\}$ noise, for Example 1.

Figure 2 shows the convergence of the norms of gradients $\|J'_q(q^n, \phi^n)\|_{L_2(\Omega)}$, $\|J'_\phi(q^n, \phi^n)\|_{L_2(\Omega)}$ to small positive values with the increasing of the iteration number for $p = 0$. For $p\% = 1\%$ noise, the two norms are decreasing after the stopping iteration number 14, whilst the errors in Figures 1(b) and 1(c) are increasing after this discrepancy principle threshold. Such phenomenon means that while the CGM is convergent, the numerical solution is unstable, since the inverse problem is ill-posed. This is why the discrepancy principle (46) is applied to regularise the CGM to attain the stable solutions.

The numerical solutions of the perfusion coefficient $q(x)$ and the initial temperature $\phi(x)$ are presented in Figures 3(a) and 3(b) for $p \in \{0, 1\}$ noise. As previously inferred from Figure 1(a), the plotted results are after 30 iterations in the case of no noise, while for noisy data the results are plotted after 14 iterations. From Figure 3 it can be seen that the accurate and stable results are obtained for both perfusion coefficient $q(x)$ and the initial temperature $\phi(x)$.

5.2. Example 2

We take $\Omega = (0, 1)$, $T = 1$, $\omega_1(t) = 1$, $\omega_2(t) = 4t$ and

$$k \equiv 1, \quad \alpha \equiv 1, \quad \mu(0, t) = \mu(1, t) = e^{-t},$$

$$f(x, t) = 2e^{-t} + (2 + x - x^2)e^{-t} \times \begin{cases} 1 - x, & x \in [0, 0.3], \\ -x + 4x^2, & x \in (0.3, 0.7), \\ 2, & x \in [0.7, 1], \end{cases} \tag{54}$$

$$\phi_1(x) = (1 - e^{-1})(2 + x - x^2), \quad \phi_2(x) = (4 - 6e^{-1})(2 + x - x^2), \tag{55}$$

with this data the analytical solution of the inverse problem (1), (2), (7) and (8) is given by

$$q(x) = \begin{cases} 2 - x, & x \in [0, 0.3], \\ 1 - x + 4x^2, & x \in (0.3, 0.7), \\ 3, & x \in [0.7, 1], \end{cases} \qquad \phi(x) = 2 + x - x^2, \quad u(x,t) = (2 + x - x^2)e^{-t}. \quad (56)$$

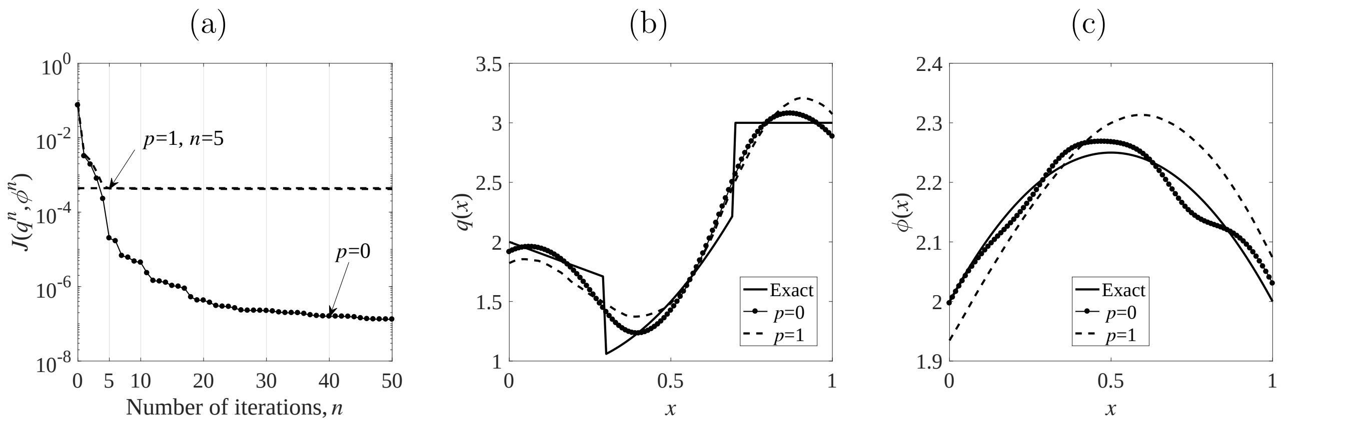

Figure 4: (a) The objective functional (23), and the exact and numerical results for (b) the perfusion coefficient $q(x)$ and (c) the initial temperature $\phi(x)$, with $p \in \{0, 1\}$ noise, for Example 2.

In comparison with the previous Example 1, this example is more severe since the perfusion coefficient to be retrieved is a discontinuous function. We take the initial guesses $q^0(x) = 1$ and $\phi^0(x) = 1$ and employ the Crank-Nicolson FDM with the mesh sizes $\Delta x = \Delta t = 0.01$. Figure 4 (a) illustrates the convergence of the objective functional (23) with the iterative procedure stopped at iteration numbers $\{50, 5\}$ for $p \in \{0, 1\}$, respectively.

The corresponding numerical solutions for the perfusion coefficient $q(x)$ and initial temperature $\phi(x)$ at these stopping iteration numbers are shown in Figures 4(b) and 4(c), respectively. From these figures it can be seen that the numerical solutions are stable and reasonably accurate bearing in mind the severe discontinuous perfusion coefficient that had to be retrieved simultaneously with the initial temperature.

5.3. Example 3

We now consider a two-dimensional example and take $\Omega = (0,1) \times (0,1)$, $T = 1$, $\omega_1(t) = 1$, $\omega_2(t) = 3t$ and

$$\begin{aligned} & k = \mathbf{I}_2, \quad \alpha \equiv 1, \quad \mu(0, x_2, t) = \mu(1, x_2, t) = \mu(x_1, 0, t) = \mu(x_2, 1, t) = e^{-t}, \\ & f(x_1, x_2, t) = (2\pi^2 + x_1^2 + x_2^2)(\sin^2(\pi x_1)\sin^2(\pi x_2) + 1)e^{-t} \\ & \qquad\qquad - 2\pi^2(\cos(2\pi x_1)\sin^2(\pi x_2) + \sin^2(\pi x_1)\cos(2\pi x_2))e^{-t}, \end{aligned} \quad (57)$$

$$\begin{aligned} & \phi_1(x_1, x_2) = (1 - e^{-1})(\sin^2(\pi x_1)\sin^2(\pi x_2) + 1), \\ & \phi_2(x_1, x_2) = (3 - 5e^{-1})(\sin^2(\pi x_1)\sin^2(\pi x_2) + 1). \end{aligned} \quad (58)$$

With this data, the analytical solution of the inverse problem (1), (2), (7) and (8) is given by

$$\begin{aligned} & q(x_1, x_2) = 1 + 2\pi^2 + x_1^2 + x_2^2, \quad \phi(x_1, x_2) = \sin^2(\pi x_1)\sin^2(\pi x_2) + 1, \\ & u(x_1, x_2, t) = (\sin^2(\pi x_1)\sin^2(\pi x_2) + 1)e^{-t}. \end{aligned} \quad (59)$$

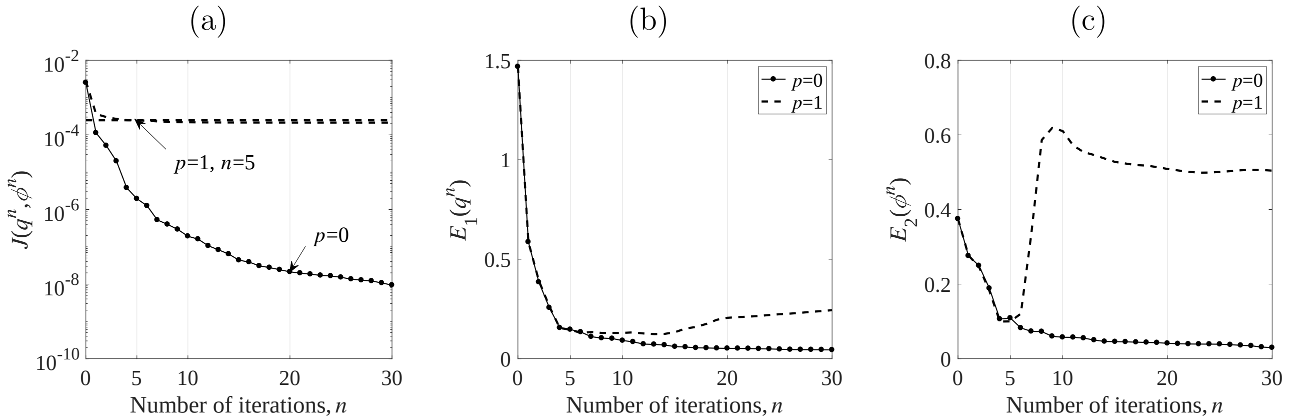

Figure 5: (a) The objective functional (23), the errors (b) (47) and (c) (48), with $p \in \{0, 1\}$ noise, for Example 3.

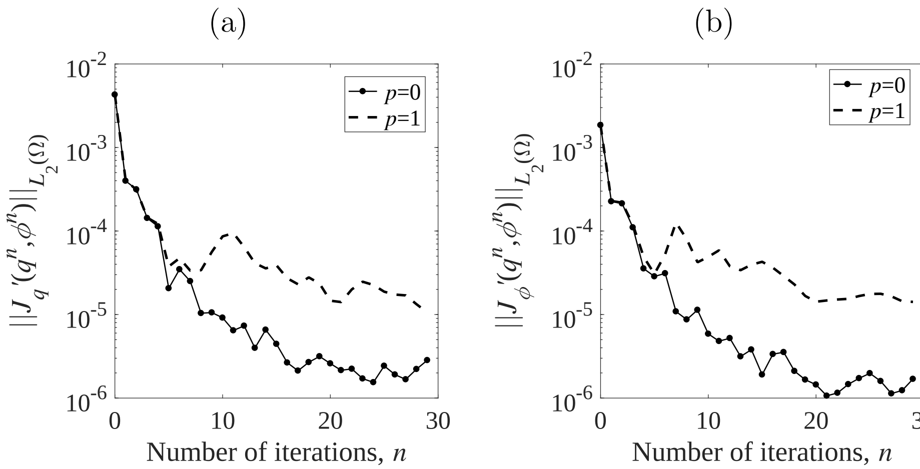

Figure 6: The norm of gradients (a) $\|J'_q(q^n, \phi^n)\|_{L_2(\Omega)}$ and (b) $\|J'_\phi(q^n, \phi^n)\|_{L_2(\Omega)}$, with $p \in \{0, 1\}$ noise, for Example 3.

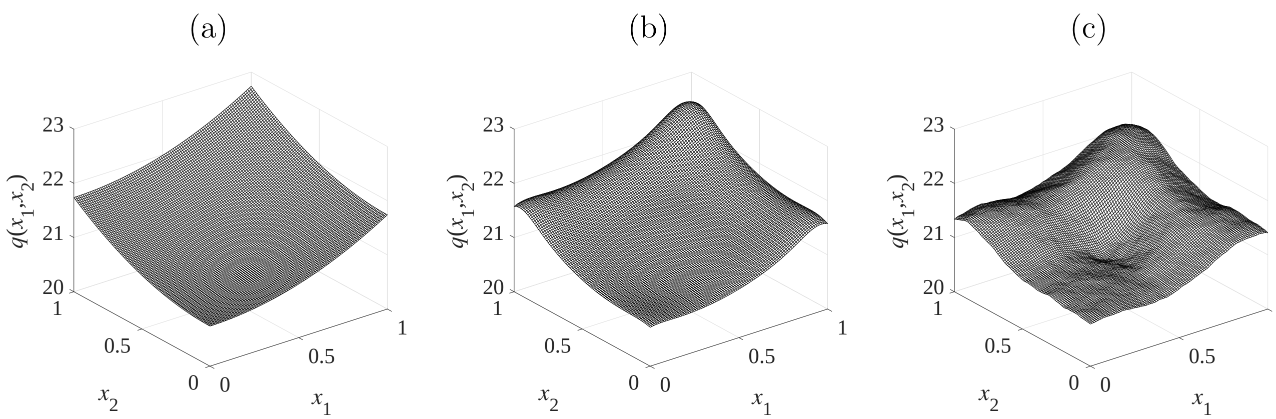

Figure 7: (a) The exact perfusion coefficient, and numerical results with (b) $p = 0$ and (c) $p = 1$, for Example 3.

The ADI scheme with mesh sizes $\Delta x_1 = \Delta x_2 = \Delta t = 0.01$ is used to obtain the numerical solutions for the direct, sensitivity and adjoint problems in the algorithm for the two-dimensional ($N = 2$) case. The initial guesses are chosen as $q^0(x_1, x_2) = 20$ and $\phi^0(x_1, x_2) = 1$. Figures 5–8 for Example 3 represent analogous quantities to Figures 1–3 of Example 1 and similar conclusions can be observed.

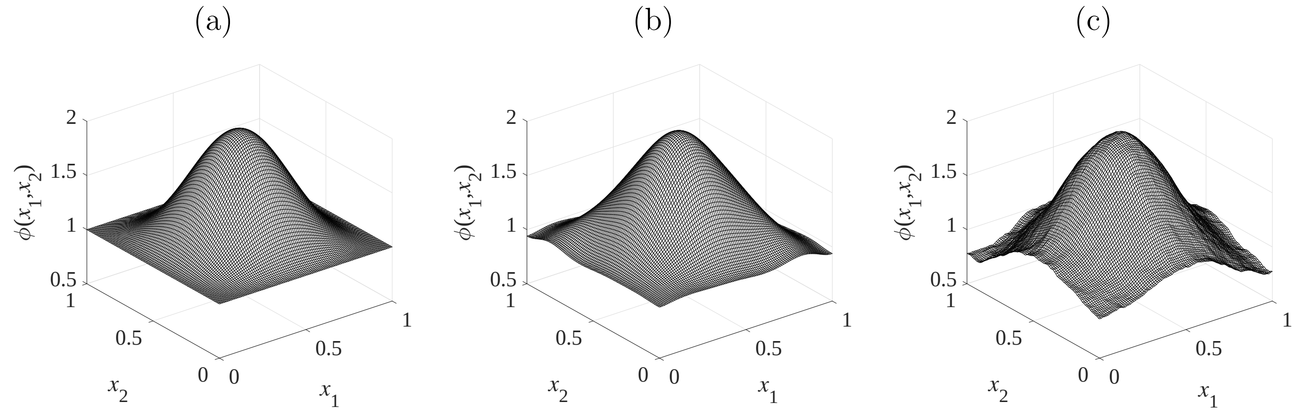

Figure 8: (a) The exact initial temperature, and numerical results with (b) $p = 0$ and (c) $p = 1$, for Example 3.

6. Conclusions

In this paper, the simultaneous retrieval of the space-dependent perfusion coefficient and initial temperature from time-integral weighted temperature observations has been investigated. The two unknown functions have been identified simultaneously by minimizing the least-squares objective functional using the CGM based on the newly derived adjoint problem (24), the sensitivity problems (28) and (30), and the gradient equations (26) and (27). Stability has been achieved by stopping the iterations according to the discrepancy criterion (46). Three numerical examples in both one- and two-dimensions have been presented, and discuss showing the accuracy and stability of the numerical reconstruction. Future work will consider the simultaneous retrieval of the space-dependent perfusion coefficient, metabolic heat source and initial temperature.

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References

[1] A. I. Prilepko, A. B. Kostin, On certain inverse problems for parabolic equations with final and integral observation, Russian Academy of Science Siberian Mathematics 75 (1993) 473–490.

[2] A. I. Kozhanov, A nonlinear loaded parabolic equation and a related inverse problem, Mathematical Notes 76 (5) (2004) 784–795.

[3] V. L. Kamynin, A. B. Kostin, Two inverse problems of finding a coefficient in a parabolic equation, Differential Equations 46 (3) (2010) 375–386.

[4] D. Trucu, D. B. Ingham, D. Lesnic, Space-dependent perfusion coefficient identification in the transient bio-heat equation, Journal of Engineering Mathematics 67 (4) (2010) 307–315.

[5] K. Cao, D. Lesnic, Reconstruction of the space-dependent perfusion coefficient from final time or time-average temperature measurements, Journal of Computational and Applied Mathematics 337 (2018) 150–165.

[6] W. L. Miranker, A well posed problem for the backward heat equation, Proceedings of the American Mathematical Society 12 (2) (1961) 243–247.

[7] J. R. Cannon, J. J. Douglas, The Cauchy problem for the heat equation, SIAM Journal on Numerical Analysis 4 (3) (1967) 317–336.

[8] O. M. Alifanov, Inverse Heat Transfer Problems, Springer Science & Business Media, Berlin, 2012.

[9] M. N. Ozisik, H. R. B. Orlande, Inverse Heat Transfer: Fundamentals and Applications, CRC Press, Taylor & Francis, New York, 2000.

[10] H. Han, D. Ingham, Y. Yuan, The boundary element method for the solution of the backward heat conduction equation, Journal of Computational Physics 116 (2) (1995) 292–299.

[11] D. Lesnic, L. Elliott, D. Ingham, An iterative boundary element method for solving the backward heat conduction problem using an elliptic approximation, Inverse Problems in Engineering 6 (4) (1998) 255–279.

[12] W. B. Muniz, H. F. de Campos Velho, F. M. Ramos, A comparison of some inverse methods for estimating the initial condition of the heat equation, Journal of Computational and Applied Mathematics 103 (1) (1999) 145–163.

[13] C.-L. Fu, X.-T. Xiong, Z. Qian, Fourier regularization for a backward heat equation, Journal of Mathematical Analysis and Applications 331 (1) (2007) 472–480.

[14] C.-S. Liu, A self-adaptive LGSM to recover initial condition or heat source of one-dimensional heat conduction equation by using only minimal boundary thermal data, International Journal of Heat and Mass Transfer 54 (7) (2011) 1305–1312.

[15] M. Yamamoto, J. Zou, Simultaneous reconstruction of the initial temperature and heat radiative coefficient, Inverse Problems 17 (4) (2001) 1181–1202.

[16] M. Choulli, M. Yamamoto, Uniqueness and stability in determining the heat radiative coefficient, the initial temperature and a boundary coefficient in a parabolic equation, Nonlinear Analysis: Theory, Methods & Applications 69 (11) (2008) 3983–3998.

[17] R. Fletcher, C. M. Reeves, Function minimization by conjugate gradients, The Computer Journal 7 (2) (1964) 149–154.

[18] G. Zoutendijk, Nonlinear programming, computational methods, Integer and Nonlinear Programming 143 (1) (1970) 37–86.

[19] H. H. Pennes, Analysis of tissue and arterial blood temperatures in the resting human forearm, Journal of Applied Physiology 1 (2) (1948) 93–122.

[20] L. Yang, J.-N. Yu, G.-W. Luo, Z.-C. Deng, Numerical identification of source terms for a two dimensional heat conduction problem in polar coordinate system, Applied Mathematical Modelling 37 (3) (2013) 939–957.

[21] L. Yang, J.-N. Yu, G.-W. Luo, Z.-C. Deng, Reconstruction of a space and time dependent heat source from finite measurement data, International Journal of Heat and Mass Transfer 55 (23-24) (2012) 6573–6581.

[22] L. Yang, Z.-C. Deng, Y.-C. Hon, Simultaneous identification of unknown initial temperature and heat source, Dynamic Systems and Applications 25 (4) (2016) 583–602.

[23] D. Trucu, Inverse Problems for Blood Perfusion Identification, Centre for Computational Fluid Dynamics, School of Mathematics, and School of Process, Environmental and Materials Engineering, The University of Leeds, 2009.

[24] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic, X. Wen, Analysis of polygonal fins using the boundary element method, Applied Thermal Engineering 24 (8) (2004) 1321–1339.

[25] F. Tröltzsch, Optimal Control of Partial Differential Equations: Theory, Methods, and Applications, Vol. 112, American Mathematical Society, 2010.

[26] K. Cao, D. Lesnic, J. J. Liu, Simultaneous reconstruction of space-dependent heat transfer coefficients and initial temperature, submitted to Journal of Computational and Applied Mathematics.

[27] W. Rundell, The determination of a parabolic equation from initial and final data, Proceedings of the American Mathematical Society 99 (4) (1987) 637–642.

[28] V. Isakov, Inverse parabolic problems with the final overdetermination, Communications on Pure and Applied Mathematics 44 (2) (1991) 185–209.

[29] A. I. Prilepko, V. V. Solov'ev, Solvability of the inverse boundary-value problem of finding a coefficient of a lower-order derivative in a parabolic equation, Differential Equations 23 (1) (1987) 101–107.

[30] L. Yang, J.-N. Yu, Z.-C. Deng, An inverse problem of identifying the coefficient of parabolic equation, Applied Mathematical Modelling 32 (10) (2008) 1984–1995.

[31] Z.-C. Deng, L. Yang, J.-N. Yu, Identifying the radiative coefficient of heat conduction equations from discrete measurement data, Applied Mathematics Letters 22 (4) (2009) 495–500.

[32] Q. Chen, J. Liu, Solving an inverse parabolic problem by optimization from final measurement data, Journal of Computational and Applied Mathematics 193 (1) (2006) 183–203.

[33] Y. L. Keung, J. Zou, Numerical identifications of parameters in parabolic systems, Inverse Problems 14 (1) (1998) 83–100.

[34] J. P. Raymond, H. Zidani, Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations, Applied Mathematics and Optimization 39 (2) (1999) 143–177.

[35] P. Wolfe, Convergence conditions for ascent methods, SIAM Review 11 (2) (1969) 226–235.

[36] P. Wolfe, Convergence conditions for ascent methods. II: Some corrections, SIAM Review 13 (2) (1971) 185–188.

[37] Y. H. Dai, Y. Yuan, Convergence properties of the Fletcher-Reeves method, IMA Journal of Numerical Analysis 16 (2) (1996) 155–164.