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Regularization of the semilinear sideways heat equation

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Abstract

A classical physical example of the sideways heat equation is represented by re-entry vehicles in the atmosphere where the temperature at the nozzle of a rocket is so high that any thermocouple attached to it would be destroyed. Instead one could measure both the temperature and heat flux, i.e. Cauchy data, at an interior boundary inward the capsule. In addition, we assume that there exists a heat source which is significantly dependent on space, time and temperature, and hence it cannot be neglected. This gives rise to a non-characteristic Cauchy inverse boundary value problem in the sense that the interior accessible boundary is overspecified, whilst the exterior hostile boundary is underspecified as nothing is prescribed on it. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the Cauchy data. In order to obtain a stable numerical solution, we propose two regularization methods to solve the semilinear problem in which the heat source is a Lipschitz function of temperature. We show rigorously, with error estimates provided, that the corresponding regularized solutions converge to the true solution strongly in L^2 uniformly with respect to the space coordinate under some *a priori* assumptions on the solution. These assumptions place no serious restrictions on the applicability of the results since in practice we always have some control and knowledge about how large the absolute temperature and heat flux are likely to be. Finally, in order to increase the significance of the study, numerical results are presented and discussed illustrating the theoretical findings in terms of accuracy and stability.

Keywords and phrases: Nonlinear heat equation; Ill-posed problem; Cauchy problem; Contraction principle; Regularization method.
MSC codes: 65N15, 65N20, 65N21, 35K05, 35K58

1. Introduction

Inverse heat conduction problems (IHCP) arise in many physical situations where a certain hostile part of the boundary of a body that is heated/cooled is inaccessible to measurement [3, 6, 11]. In this case the missing information is compensated for by additional observations made through measurement on the complementary accessible part of the boundary of the body or, even inside the body itself. Apart from the re-entry vehicle example mentioned at the beginning of the abstract, one can envisage other applications related to: (i) the accurate determination of the temperature 'spike' inside a cannon at firing, [23]; (ii) the safety analysis of elements of nuclear reactors where the temperature measurement at the adiabatic inner wall of a hollow cylinder is used to find the unspecified/unavailable temperature and heat flux at the outer wall that is abruptly cooled, [11]; and (iii) the determination of ther temperature and heat flux at the surface of a particle board, on which a thin layer of lacquer coating is applied, [4]. All these practical applications can be mathematically modelled by the following IHCP:

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find the temperature $u(x,t)$ for $(x,t) \in [0,L] \times [0,2\pi]$ from known boundary temperature $u(L,t) = g(t)$ and heat flux $u_x(L,t) = h(t)$ measurements satifying the following problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t,u(x,t)), \quad 0 < x < L, \ 0 \le t \le 2\pi, \\ u(L,t) = g(t), \quad 0 \le t \le 2\pi, \\ u_x(L,t) = h(t), \quad 0 \le t \le 2\pi, \end{cases} \tag{1.1}$$

where g, h are given functions (usually in $L^2(0,2\pi)$). Moreover, to investigate stability and simulate the real situation of measured data, the Cauchy data g and h are perturbed so as to contain errors in the form of the input noisy Cauchy data g^ε and h^ε (also in $L^2(0,2\pi)$) satisfying

$$\left\| g^\varepsilon - g \right\| + \left\| h^\varepsilon - h \right\| \le \varepsilon, \tag{1.2}$$

where $\|\cdot\|$ denotes the $L^2(0,2\pi)$-norm and $\varepsilon > 0$ is a small positive number representing the level of noise. Although the semilinear problem (1.1) is formulated in the one-dimensional setting of a finite slab of length $L > 0$, it can be also extended to higher-dimensional cuboids with a preferential sideways coordinate.

The main difference with the classical linear IHCP is that in the governing semilinear heat equation in (1.1), the given source F may depend on not only the independent variables (x,t) but also on the dependent variable u. The time interval $t \in [0,T]$, where $T > 0$ is a given finite time of interest, does not necessarily require that $T = 2\pi$, which herein is taken only for the convenience of the Fourier series development in Section 2. Note also that we have no intial condition prescribed at $t = 0$, which may occur when the heat conducting device is already in service and the initial temperature is an extra unknown to be determined. We also mention here the case of internal measurements when it is sometimes necessary to determine the surface temperature and heat flux from a measured temperature history at a fixed location $x_0 \in (0,L)$ inside the body, [8, 21]. Note that in this case the initial temperature at $t = 0$ has to be supplied, at least on the space interval (x_0, L). However, internal measurements are intrusive and may damage the material. In this case, non-destructive testing, where measurements are taken at the boundary only, is preferred and our formulation in (1.1) models such a situation.

In the linear case, i.e. $F = F(x,t)$ does not depend on u, the problem (1.1) has at most one solution using classical analytical sideways continuation for the parabolic heat equation. It can also easily be remarked that in this linear case one can take $F = 0$ by superposition with the solution of a direct and well-posed problem with heat source $F(x,t)$, and homogeneous initial and boundary conditions. Then, if $F = 0$, the existence of a solution holds if and only if the function $t \mapsto h(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{g'(t)}{\sqrt{t-\tau}} d\tau$ is a function of class two, [12].

In the semilinear case, i.e. $F = F(x,t,u)$, for $F \in C^1([0,L] \times [0,2\pi] \times \mathbb{R})$, uniqueness of the solution of problem (1.1) follows from the well-known uniqueness theorem for general parabolic partial differential equations of second-order with lateral Cauchy data, see, e.g., Chapter 4 of [17].

However, even if uniqueness holds, the problem is still ill-posed in the sense that the solution, if it exists, it does not depend continuously on the data. Any small perturbation in the observation data at $x = L$ can cause large errors in the solution which are increasing with decreasing x from L to 0. Therefore, most classical numerical methods often fail to give an acceptable approximation of the solution and regularization techniques are required to restore stability, [13, 25].

In recent years, the linear homogeneous sideways heat equation, i.e. $F = 0$ in the first equation in (1.1), has been researched by many authors and various numerical methods have been proposed, e.g. the boundary element Tikhonov regularization method [18], the conjugate gradient method [14], and, following the footsteps of Professor L. Elden, a few others based on filtering methods of regularization, e.g. the difference regularization method [27], the 'optimal filtering' method [23], the sequential windowing of the data [4], the Fourier method [28], the quasi-reversibility method [8, 20], the wavelet, wavelet-Galerkin and spectral regularization methods [9, 22], to mention only a few. However, the more important but challenging semilinear sideways heat equation with the heat source depending nonlinearly on the temperature, which occurs in many applications related to reaction-diffusion, combustion and radiation processes, is yet to be investigated from the filtering perspective, although it is worth citing here [11, 19], who used the finite difference method and a Lie-group differential algebraic equations algorithm, respectively, but both without any regularization, and the more rigorous study [16] of Professor Klibanov and his colleagues, who proposed minimizing a strictly convex Tikhonov-type functional with Carleman weight functions embedded in it. Therefore, our study is a major extension of the linear case, which requires novel filtering proposals, see equations (2.14) and (3.39), and proofs of new convergences theorems, see Theorems 3.1 and 3.2, based on the contraction mapping principle.

In summary, we propose two new methods that are based on nonlinear integral equations to regularize problem (1.1) under two *a priori conditions* on the solution. As will be shown in next section, for the semilinear sideways heat problem (1.1), its solution (true solution) can be represented as an integral equation which contains some instability terms. In order to restore stability we replace these instability terms by some regularization ones and show that the solution of our regularized problem converges to the solution of the original semilinear problem (if such solution exists), as the regularization parameter tends to zero.

The paper is organized as follows. In Section 2, the formulation of the problem and the regularization methods are given. In Section 3, a stability estimate is proved under *a priori* conditions on the solution and the Lipschitz source term. Numerical results are presented and discussed in Section 4 and finally, conclusions are summarised in Section 5.

2. Mathematical analysis

Let $\langle\cdot\rangle$ denote the inner product in $L^2(0,2\pi)$. For $f \in L^2(0,2\pi)$, we have the complex Fourier series $f(t) \sim \sum_{n\in\mathbb{Z}} \left\langle f(t), e^{-int}\right\rangle e^{int}$, where $\left\langle f(t), e^{-int}\right\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t)e^{-int}dt$. For time being, the symbol $\sim$ denotes that the right-hand side is the Fourier series of the left-hand side. It is well-known that the Fourier series of $f \in L^2(0,2\pi)$ converges in $L^2(0,2\pi)$ to f. Pointwise convergence holds if f is continuous in $[0,2\pi]$ and if $f(0) = f(2\pi)$. In case the latter condition does not hold we consider the $2\pi-$periodic extension f^* of f defined by $f^*(t) = f(t)$ for $t \in [0,2\pi]$, $f^*(2\pi) = f^*(0)$ and $f^*(t+2\pi) = f^*(t)$ for all $t \in \mathbb{R}$. This will not affect the values of any integrals over the interval $[0,2\pi]$ (since $\int_0^{2\pi} f(t)dt = \int_0^{2\pi} f^*(t)dt = \int_a^{2\pi+a} f^*(t)dt$ for any $a \in \mathbb{R}$), though it may change the value of f at $t = 2\pi$.

The $L^2(0,2\pi)$-norm of f is

$$\|f\|^2 = 2\pi \sum_{n\in\mathbb{Z}} \left|\left\langle f(t), e^{-int}\right\rangle\right|^2. \tag{2.3}$$

The principal value of $\sqrt{in}$ is

$$\sqrt{in} = \begin{cases} (1+i)\sqrt{|n|/2}, & n \geq 0, \\ (1-i)\sqrt{|n|/2}, & n < 0. \end{cases} \tag{2.4}$$

Let the solution of problem (1.1) be represented by the complex Fourier series

$$u(x,t) \sim u^*(x,t) = \sum_{n\in\mathbb{Z}} u_n(x)e^{int}, \text{ with } u_n(x) = \left\langle u(x,t), e^{-int}\right\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} u(x,t)e^{-int}dt, \tag{2.5}$$

where, as above, $u^*(x,\cdot)$ is a $2\pi-$periodic extension in t of the function $u(x,\cdot)$. Since $u^*(x,\cdot)$ can be represented as a Fourier series we know that this series converges to $u(x,\cdot)$ in $L^2(0,2\pi)$. Based on this agrgument, from now on, for simplicity, we identify u^* with u.

From (1.1), we have the following systems of second-order ordinary differential equations:

$$\begin{cases} -\dfrac{d^2u_n}{dx^2}(x) + inu_n(x) = F_n(u)(x), \\ u_n(L) = g_n = \left\langle g(t), e^{-int}\right\rangle, \\ \dfrac{du_n}{dx}(L) = h_n = \left\langle h(t), e^{-int}\right\rangle, \end{cases} \tag{2.6}$$

where $F_n(u)(x) = \left\langle F(x,t,u(x,t)), e^{-int} \right\rangle = \frac{1}{2\pi} \int\limits_0^{2\pi} F(x,t,u(x,t)) e^{-int} dt$ for all $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}\backslash\{0\}$, multiplying the first equation in (2.6) by $\frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ and integrating both sides from x to L, we obtain

$$u_n(x) = \cosh\left((L-x)\sqrt{in}\right) u_n(L) - \frac{\sinh\left((L-x)\sqrt{in}\right)}{\sqrt{in}} u_n'(L) - \int\limits_x^L \frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(u)(z) dz, \quad n \in \mathbb{Z}\backslash\{0\}. \tag{2.7}$$

In the case $n = 0$, multiplying the first equation in (2.6) by $z - x$ and integrating both sides from x to L, we obtain

$$u_0(x) = u_0(L) - (L-x) u_0'(L) + \int\limits_x^L (z-x) F_0(u)(z) dz. \tag{2.8}$$

Denoting

$$\widetilde{F}(\mathrm{g}, \mathrm{h}, v)(x) := \mathrm{g} - (L-x)\mathrm{h} + \int\limits_x^L (z-x) F_0(v)(z) dz, \tag{2.9}$$

from (2.5), (2.7) - (2.9) the exact form of u is given by

$$u(x,t) = \sum_{n \in \mathbb{Z}\backslash\{0\}} \left[\cosh\left((L-x)\sqrt{in}\right) g_n - \frac{\sinh\left((L-x)\sqrt{in}\right)}{\sqrt{in}} h_n - \int\limits_x^L \frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(u)(z) dz \right] e^{int} + \widetilde{F}(g_0, h_0, u)(x). \tag{2.10}$$

We remark that the term $\cosh\left((L-x)\sqrt{in}\right)$ satisfies

$$\left|\cosh\left((L-x)\sqrt{in}\right)\right| = \frac{1}{2}\left| e^{\sqrt{in}(L-x)} + e^{-\sqrt{in}(L-x)} \right| \le \frac{e^{\sqrt{|n|/2}(L-x)} + e^{-\sqrt{|n|/2}(L-x)}}{2} \le e^{\sqrt{|n|/2}(L-x)} \tag{2.11}$$

and

$$\left|\cosh\left((L-x)\sqrt{in}\right)\right| = \frac{1}{2}\left| e^{\sqrt{in}(L-x)} + e^{-\sqrt{in}(L-x)} \right| \ge \frac{e^{\sqrt{|n|/2}(L-x)} - e^{-\sqrt{|n|/2}(L-x)}}{2} \ge \frac{e^{\sqrt{|n|/2}(L-x)} - 1}{2}. \tag{2.12}$$

Also,

$$\frac{e^{\sqrt{|n|/2}(z-x)} - 1}{2\sqrt{|n|}} \le \left| \frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \right| \le \frac{e^{\sqrt{|n|/2}(z-x)}}{\sqrt{|n|}}, \quad 0 \le x \le z \le L. \tag{2.13}$$

Thus, we obtain that the three functions

$$\cosh\left((L-x)\sqrt{in}\right), \quad \frac{\sinh\left((L-x)\sqrt{in}\right)}{\sqrt{in}}, \quad \frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$$

in (2.10) are unbounded, as functions of the variable n, for $x \in [0, L)$. Consequently, small errors in high frequency components can blow up and completely destroy the solution for $x \in [0, L)$. A natural idea to stabilize the problem is to eliminate all high frequencies or to replace them by a bounded approximation. With this in mind, we replace $\cosh\left((L-x)\sqrt{in}\right)$, $\frac{\sinh\left((L-x)\sqrt{in}\right)}{\sqrt{in}}$, $\frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ by $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$, $\frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}$, $\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$, respectively. Our idea of regularization method is of constructing two new kernels, $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ and

$\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ which have the following two properties:

(A) If $\gamma = \gamma(\varepsilon) > 0$ is fixed, the terms $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ and $\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ are bounded, as functions of $n \in \mathbb{Z}\backslash\{0\}$.
(B) If the parameter $\gamma > 0$ is small, then for small n, the kernel $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ is close to $\cosh\left((L-x)\sqrt{in}\right)$ and the kernel $\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ is close to $\frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$.

Property (B) describes how close the kernels $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ and $\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ are to $\cosh\left((L-x)\sqrt{in}\right)$ and $\frac{\sinh\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$, respectively, in the low frequency components. Obviously, the smaller the parameter γ, the closer is the agreement. Property (A) describes the degree of continuous dependence, i.e., when the terms $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ and $\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}$ are bounded, the regularized solution will depend continuously on the data.

To approximate u, we introduce the first regularized solution u^ε satisfying

$$\begin{aligned} u^\varepsilon(x,t) = &\sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right) g_n^\varepsilon - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}} h_n^\varepsilon\right] e^{int} \\ &- \sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(u^\varepsilon)(z)dz\right] e^{int} + \widetilde{F}(g_0^\varepsilon, h_0^\varepsilon, u^\varepsilon)(x). \end{aligned} \tag{2.14}$$

Here, $\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)$ and $\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)$ are defined for all $n \in \mathbb{Z}\backslash\{0\}$ by

$$\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right) := \cosh\left((L-x)\sqrt{in}\right) + \sqrt{in}\,\widetilde{P}^{\gamma(\varepsilon)}(x,L,n), \tag{2.15}$$

$$\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right) := \sinh\left((z-x)\sqrt{in}\right) + \sqrt{in}\,\widetilde{P}^{\gamma(\varepsilon)}(x,z,n), \tag{2.16}$$

where $\widetilde{P}^{\gamma(\varepsilon)}(x,z,n)$ is given by

$$\widetilde{P}^{\gamma(\varepsilon)}(x,z,n) = \frac{\left[R^{\gamma(\varepsilon)}(L,n) - 1\right] e^{\sqrt{in}(z-x)}}{2\sqrt{in}}. \tag{2.17}$$

The well-posedness of the solution to the integral equation (2.14) depends on the filter function $R^{\gamma(\varepsilon)}(L,n)$ and the regularization parameter $\gamma(\varepsilon) > 0$. In this paper, we assume that the filter function $R^{\gamma(\varepsilon)}(L,n)$ satisfies

$$\left|R^{\gamma(\varepsilon)}(L,n)\right| e^{\sqrt{|n|/2}y} \le \gamma(\varepsilon)^{-y/L}, \quad y \in [0,L], \tag{2.18}$$

$$\left|R^{\gamma(\varepsilon)}(L,n) - 1\right| e^{-\sqrt{|n|/2}y} \le \gamma(\varepsilon)^{y/L}, \quad y \in [0,L]. \tag{2.19}$$

From (2.15) and (2.17), we have

$$\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right) = \frac{R^{\gamma(\varepsilon)}(L,n) e^{\sqrt{in}(L-x)} + e^{-\sqrt{in}(L-x)}}{2},$$

so that one can see that the filtering applies to the unbounded component.

For illustration, we give a couple of examples of $R^{\gamma(\varepsilon)}$ which satisfy the conditions (2.18) and (2.19).

Example 1. Let $R_1^{\gamma(\varepsilon)}$ be as

$$R_1^{\gamma(\varepsilon)}(L,n) = \frac{e^{-\sqrt{|n|/2}L}}{\gamma(\varepsilon) + e^{-\sqrt{|n|/2}L}}. \tag{2.20}$$

First, we deduce the following inequality:

$$\left|R_1^{\gamma(\varepsilon)}(L,n)\right| e^{\sqrt{|n|/2}y} = \frac{e^{-\sqrt{|n|/2}L}}{\gamma(\varepsilon)+e^{-\sqrt{|n|/2}L}} e^{\sqrt{|n|/2}y}$$

$$= \frac{e^{-\sqrt{|n|/2}(L-y)}}{\left(\gamma(\varepsilon)+e^{-\sqrt{|n|/2}L}\right)^{(L-y)/L}\left(\gamma(\varepsilon)+e^{-\sqrt{|n|/2}L}\right)^{y/L}} \le \left(\frac{1}{\gamma(\varepsilon)+e^{-\sqrt{|n|/2}L}}\right)^{y/L} \le \gamma(\varepsilon)^{-y/L}, \tag{2.21}$$

which shows that (2.18) holds.

To prove that (2.19) holds, consider the function $\chi : [0, L] \to \mathbb{R}_+$ defined by

$$\chi(y) := \gamma(\varepsilon)^{-y/L} e^{-\sqrt{|n|/2}y} = e^{y\Gamma}, \quad y \in [0, L],$$

where $\Gamma := -\sqrt{|n|/2} - \ln \gamma(\varepsilon)^{1/L}$. Depending on the sign of Γ, we have that χ is increasing and $\max_{y\in[0,L]} \chi(y) = \chi(L) = \frac{e^{-\sqrt{|n|/2}L}}{\gamma(\varepsilon)}$ if $\Gamma > 0$, and χ is decreasing and $\max_{y\in[0,L]} \chi(y) = \chi(0) = 1$ if $\Gamma \le 0$. In both cases, we obtain that

$$\chi(y) := \gamma(\varepsilon)^{-y/L} e^{-\sqrt{|n|/2}y} \le \max_{y\in[0,L]} \chi(y) \le 1 + \frac{e^{-\sqrt{|n|/2}L}}{\gamma(\varepsilon)}. \tag{2.22}$$

Using (2.20) and (2.22) we obtain that

$$\left|R_1^{\gamma(\varepsilon)}(L,n) - 1\right| e^{-\sqrt{|n|/2}y} = \gamma(\varepsilon)\frac{e^{-\sqrt{|n|/2}y}}{\gamma(\varepsilon)+e^{-\sqrt{|n|/2}L}} \le \gamma(\varepsilon)^{y/L}, \quad y \in [0, L], \tag{2.23}$$

which shows that (2.19) holds.

Example 2: Let us choose $R_2^{\gamma(\varepsilon)}$ as follows:

$$R_2^{\gamma(\varepsilon)}(L,n) = \begin{cases} 1, & \text{if } |n| \le N_\varepsilon, \\ 0, & \text{if } |n| > N_\varepsilon, \end{cases} \tag{2.24}$$

with N_ε satisfying $\lim_{\varepsilon\to0} N_\varepsilon = +\infty$. It then follows that

$$\left|R_2^{\gamma(\varepsilon)}(L,n)\right| e^{\sqrt{|n|/2}y} \le e^{\sqrt{N_\varepsilon/2}y} \quad \text{and} \quad \left|R_2^{\gamma(\varepsilon)}(L,n) - 1\right| e^{-\sqrt{|n|/2}y} \le e^{-\sqrt{N_\varepsilon/2}y}, \quad y \in [0, L]. \tag{2.25}$$

Therefore, $R_2^{\gamma(\varepsilon)}$ given in (2.24) satisfies (2.18) and (2.19) with $\gamma(\varepsilon) = e^{-L\sqrt{N_\varepsilon/2}}$.

Before we establish the properties of the regularized solution, let us introduce some notation first. For a Hilbert space B, we denote

$$L^\infty(0, L; \mathrm{B}) = \left\{ f : [0, L] \to \mathrm{B} \,\middle|\, \operatorname*{ess\,sup}_{0\le z\le L} \|f(z)\|_{\mathrm{B}} < \infty \right\}$$

with the norm

$$\|f\|_{L^\infty(0,L;\mathrm{B})} = \operatorname*{ess\,sup}_{0\le z\le L} \|f(z)\|_{\mathrm{B}}.$$

For $r \ge 0$ and δ, let us also introduce the spaces

$$G_\delta^r(0, 2\pi) = \left\{ \theta \in L^2(0, 2\pi); \sum_{n\in\mathbb{Z}} |n|^{2r} e^{\sqrt{2|n|}\delta} \left|\langle \theta(t), e^{-int} \rangle\right|^2 < \infty \right\}, \tag{2.26a}$$

$$V^r(0, 2\pi) = \left\{ \theta \in L^2(0, 2\pi); \sum_{n\in\mathbb{Z}} |n|^{2r} \left|\langle \theta(t), e^{-int} \rangle\right|^2 < \infty \right\}, \tag{2.26b}$$

and their norms given by

$$\|\theta\|_{G^r_\delta(0,2\pi)} = \sqrt{\sum_{n\in\mathbb{Z}} |n|^{2r} e^{\sqrt{2|n|}\delta} \left|\langle\theta(t), e^{-int}\rangle\right|^2}, \tag{2.27a}$$

$$\|\theta\|_{V^r(0,2\pi)} = \sqrt{\sum_{n\in\mathbb{Z}} |n|^{2r} \left|\langle\theta(t), e^{-int}\rangle\right|^2}. \tag{2.27b}$$

It is easy to see that $G^r_0(0, 2\pi) = V^r(0, 2\pi)$ and $V^0(0, 2\pi) = L^2(0, 2\pi)$.

3. Regularization and error estimates

Throughout this section, we assume that $F : [0, L] \times [0, 2\pi] \times L^2(0, 2\pi) \to L^2(0, 2\pi)$ is a Lipschitz function, i.e. there exists a constant $\mathbb{K}_F \geq 0$ such that

$$\|F(x, \cdot, u(x, \cdot)) - F(x, \cdot, v(x, \cdot))\| \leq \mathbb{K}_F \|u(x, \cdot) - v(x, \cdot)\|, \quad \forall x \in [0, L],\ \forall u, v \in C\left([0, L]; L^2(0, 2\pi)\right). \tag{3.28}$$

The next theorem and remark give error estimates between the true solution u of problem (1.1) and the regularized solution u^ε satisfying (2.14).

Theorem 3.1. *Let $\gamma(\varepsilon)$ be a regularization parameter such that $0 < \gamma(\varepsilon) < e^{-L}$ and*

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \gamma(\varepsilon) = 0, \\ \lim\limits_{\varepsilon\to 0} \dfrac{\varepsilon}{\gamma(\varepsilon)} \text{ is a non-negative real number.} \end{cases} \tag{3.29}$$

Then, the integral equation (2.14) has a unique solution $u^\varepsilon \in C\left([0, L]; L^2(0, 2\pi)\right)$.

Let $R^{\gamma(\varepsilon)}$ satisfy conditions (2.18) and (2.19).

(a) Suppose that the problem (1.1) has a solution u satisfying

$$\|u\|_{L^\infty\left(0,L;G^0_L(0,2\pi)\right)} + \|u_x\|_{L^\infty\left(0,L;G^0_L(0,2\pi)\right)} \leq \mathcal{I}_1, \tag{3.30}$$

for some known constant $\mathcal{I}_1 \geq 0$. Then,

$$\left\|u^\varepsilon(x, \cdot) - u(x, \cdot)\right\| \leq E_1 \gamma(\varepsilon)^{x/L}, \quad x \in [0, L], \tag{3.31}$$

where

$$E_1 \geq \sqrt{3}\frac{\varepsilon}{\gamma(\varepsilon)} \exp\left(\frac{3\mathbb{K}_F^2 L^2}{2}\right) + \sqrt{2\pi}\mathcal{I}_1 \exp\left(\mathbb{K}_F^2 L^2\right). \tag{3.32}$$

(b) Assume that there exists $r > 0$ such that

$$\left|R^{\gamma(\varepsilon)}(L, n) - 1\right| |n|^{-r} e^{-\sqrt{|n|/2}y} \leq \widetilde{M}(\varepsilon, r)\gamma(\varepsilon)^{y/L}, \quad \forall n \in \mathbb{Z}\backslash\{0\},\ \forall y \in [0, L], \tag{3.33}$$

where $\widetilde{M}(\varepsilon, r) \to 0$ as $\varepsilon \to 0$. Suppose that the problem (1.1) has a solution satisfying

$$\|u\|_{L^\infty\left(0,L;G^r_L(0,2\pi)\right)} + \|u_x\|_{L^\infty\left(0,L;G^r_L(0,2\pi)\right)} \leq \mathcal{I}_2, \tag{3.34}$$

for some known constant $\mathcal{I}_2 \geq 0$. Then,

$$\left\|u^\varepsilon(x, \cdot) - u(x, \cdot)\right\| \leq E_2 \gamma(\varepsilon)^{x/L}, \quad x \in [0, L], \tag{3.35}$$

where

$$E_2 \geq \sqrt{3}\frac{\varepsilon}{\gamma(\varepsilon)} \exp\left(\frac{3\mathbb{K}_F^2 L^2}{2}\right) + \sqrt{2\pi}\widetilde{M}(\varepsilon, r)\mathcal{I}_2 \exp\left(\mathbb{K}_F^2 L^2\right). \tag{3.36}$$

Remark 3.1. *Under assumption* (3.30)*, the estimate* (3.31)*, via* (3.32)*, does not yield the continuous dependence of the solution at* $x = 0$*. Therefore, we need a stronger assumption on* u*, as in* (3.34)*, to obtain the error estimate* (3.35)*, via* (3.36)*, at* $x = 0$*. To obtain the approximation of the solution at* $x = 0$ *with assumption* (3.30)*, we select a number* $x_\varepsilon \in (0, L)$ *such that* $\lim_{\varepsilon\to 0} x_\varepsilon = 0$ *and then we have*

$$\left\|u^\varepsilon(x_\varepsilon, \cdot) - u(0, \cdot)\right\| \le \left\|u^\varepsilon(x_\varepsilon, \cdot) - u(x_\varepsilon, \cdot)\right\| + \|u(x_\varepsilon, \cdot) - u(0, \cdot)\| \le E_1\gamma(\varepsilon)^{x_\varepsilon/L} + x_\varepsilon E_3, \tag{3.37}$$

where $E_3 = \sup_{0\le x\le L} \|u_x(x, \cdot)\|$*. It is easy to show that for every* $\gamma(\varepsilon) > 0$*, there exists a unique* $x_\varepsilon \in (0, L)$ *such that* $\lim_{\varepsilon\to 0} x_\varepsilon = 0$ *and* $x_\varepsilon = \gamma(\varepsilon)^{x_\varepsilon/L}$*. This implies that* $\frac{\ln x_\varepsilon}{x_\varepsilon} = \frac{\ln \gamma(\varepsilon)}{L}$*. Using the inequality* $\ln x > -\frac{1}{x}$ *for every* $x > 0$*, we obtain* $x_\varepsilon < \sqrt{\frac{L}{\ln\left(\frac{1}{\gamma(\varepsilon)}\right)}}$*, which yields*

$$\left\|u^\varepsilon(x_\varepsilon, \cdot) - u(0, \cdot)\right\| \le (E_1 + E_3)\sqrt{\frac{L}{\ln\left(\frac{1}{\gamma(\varepsilon)}\right)}}. \tag{3.38}$$

In order to obtain error estimates under easier to check and weaker assumptions than (3.30) and (3.34), next we develop a second regularized solution U^ε satisfying the integral equation

$$\begin{aligned} U^\varepsilon(x, t) = &\sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\cosh^{\gamma(\varepsilon)}\left((L - x)\sqrt{in}\right)g_n^\varepsilon - \frac{\sinh^{\gamma(\varepsilon)}\left((L - x)\sqrt{in}\right)}{\sqrt{in}}h_n^\varepsilon\right]e^{int} \\ -&\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z - x)\sqrt{in}\right)}{\sqrt{in}}F_n(U^\varepsilon)(z)dz + \int_0^x \widetilde{P}^{\gamma(\varepsilon)}(x, z, n)F_n(U^\varepsilon)(z)dz\right]e^{int} + \widetilde{F}(g_0^\varepsilon, h_0^\varepsilon, U^\varepsilon)(x). \end{aligned} \tag{3.39}$$

Instead of conditions (3.30) and (3.34), we will assume

$$\|u(0, \cdot)\| + \|u_x(0, \cdot)\| \le \mathcal{I}_3, \tag{3.40a}$$

$$\|u(0, \cdot)\|_{V^r(0,2\pi)} + \|u_x(0, \cdot)\|_{V^r(0,2\pi)} \le \mathcal{I}_4, \quad \text{with } r > 0, \tag{3.40b}$$

for some known non-negative constants $\mathcal{I}_3$ and $\mathcal{I}_4$, respectively. We then obtain error estimates between the true solution u and the regularized solution U^ε, as given by the next theorem and remarks.

Theorem 3.2. *Let* $\gamma(\varepsilon)$ *be as in Theorem 3.1 and assume that* $L\mathbb{K}_F < 1$*. Then the integral equation* (3.39) *has a unique solution* $U^\varepsilon \in C([0, L]; L^2(0, 2\pi))$*.*

Let $R^{\gamma(\varepsilon)}$ *satisfy conditions* (2.18) *and* (2.19)*.*

(a) Suppose that the problem (1.1) *has a solution* u *satisfying* (3.40a)*. Then,*

$$\left\|U^\varepsilon(x, \cdot) - u(x, \cdot)\right\| \le E_4(\widetilde{\alpha})\sqrt{\mathcal{I}_3^2 + 4\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2}\,\gamma(\varepsilon)^{x/L}, \quad x \in [0, L], \tag{3.41}$$

for some $\widetilde{\alpha} \in \left(0, \frac{1}{\mathbb{K}_F^2 L^2} - 1\right)$*, where*

$$E_4(\widetilde{\alpha}) := \sqrt{\frac{1 + \frac{1}{\widetilde{\alpha}}}{1 - (1 + \widetilde{\alpha})\mathbb{K}_F^2 L^2}}. \tag{3.42}$$

(b) Assume that there exists $r > 0$ *such that* (3.33) *holds. Suppose that the problem* (1.1) *has a solution* u *satisfying* (3.40b)*. Then,*

$$\left\|U^\varepsilon(x, \cdot) - u(x, \cdot)\right\| \le E_4(\widetilde{\alpha})\sqrt{2\pi\widetilde{M}^2(\varepsilon, r)\mathcal{I}_4^2 + 4\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2}\,\gamma(\varepsilon)^{x/L}, \quad x \in [0, L]. \tag{3.43}$$

Remark 3.2. *Under assumption* (3.40a), *the estimate* (3.41) *does not yield the continuous dependence of the solution at* $x = 0$. *Therefore, we need a stronger assumptions of u as in* (3.40b), *to obtain the error estimate* (3.43) *at* $x = 0$. *In the same way as in Remark 3.1, there exists* $x_\varepsilon \in (0, L)$ *such that* $\lim_{\varepsilon\to 0} x_\varepsilon = 0$ *and*

$$\left\|U^\varepsilon(x_\varepsilon, \cdot) - u(0, \cdot)\right\| \le (E_3 + E_4(\widetilde{\alpha}))\sqrt{\frac{L}{\ln\left(\frac{1}{\gamma(\varepsilon)}\right)}}. \tag{3.44}$$

Remark 3.3. *For the the cut-off regularizing filter* (2.24), *condition* (3.33) *is satisfied with* $\widetilde{M}(\varepsilon, r) = N_\varepsilon^{-r}$ *and* $\gamma(\varepsilon) = e^{-L\sqrt{N_\varepsilon/2}}$.

First, we have the following lemmas which will be useful in proving Theorems 3.1 and 3.2.

Lemma 3.1. *For* $0 < \gamma(\varepsilon) < e^{-L}$ *we have*

$$\gamma(\varepsilon)^{\frac{x-L}{L}} \ge 1, \quad 0 \le x \le L, \tag{3.45a}$$

$$x \le \gamma(\varepsilon)^{-\frac{x}{L}}, \quad 0 \le x \le L. \tag{3.45b}$$

The proof is omitted.

Lemma 3.2. *For* $n \in \mathbb{Z}\backslash\{0\}$ *and* $0 < \gamma(\varepsilon) < e^{-L}$, *we have the following inequalities:*

$$\left|\widetilde{P}^{\gamma(\varepsilon)}(x, z, n)\right| \le \gamma(\varepsilon)^{\frac{x-z}{L}}, \quad 0 \le z \le x \le L, \tag{3.46a}$$

$$\left|\cosh^{\gamma(\varepsilon)}\left((L - x)\sqrt{in}\right)\right| \le \gamma(\varepsilon)^{\frac{x-L}{L}}, \quad x \in [0, L], \tag{3.46b}$$

$$\left|\frac{\sinh^{\gamma(\varepsilon)}\left((z - x)\sqrt{in}\right)}{\sqrt{in}}\right| \le \gamma(\varepsilon)^{\frac{x-z}{L}}, \quad 0 \le x \le z \le L. \tag{3.46c}$$

Proof. From (2.17) and (2.19), we obtain (3.46a), as follows:

$$\left|\widetilde{P}^{\gamma(\varepsilon)}(x, z, n)\right| = \left|\frac{\left(1 - R^{\gamma(\varepsilon)}(L, n)\right)e^{\sqrt{in}(z-x)}}{2\sqrt{in}}\right| \le \frac{\gamma(\varepsilon)^{\frac{x-z}{L}}}{2\sqrt{|n|}} \le \gamma(\varepsilon)^{\frac{x-z}{L}}, \quad 0 \le z \le x \le L.$$

From (2.15), (2.17), (2.18) and (3.45a), we obtain (3.46b), as follows:

$$\left|\cosh^{\gamma(\varepsilon)}\left((L - x)\sqrt{in}\right)\right| \le \frac{1}{2}\left|R^{\gamma(\varepsilon)}(L, n)e^{\sqrt{in}(L-x)}\right| + \frac{1}{2}\left|e^{-\sqrt{in}(L-x)}\right| \le \frac{1}{2}\gamma(\varepsilon)^{\frac{x-L}{L}} + \frac{1}{2}e^{-\sqrt{\frac{|n|}{2}}(L-x)} \le \gamma(\varepsilon)^{\frac{x-L}{L}}.$$

From (2.16), (2.18) and (3.45b), we also obtain (3.46c), as follows:

$$\left|\frac{\sinh^{\gamma(\varepsilon)}\left((z - x)\sqrt{in}\right)}{\sqrt{in}}\right| \le \frac{\left|R^{\gamma(\varepsilon)}(L, n)e^{\sqrt{in}(z-x)}\right|}{2\sqrt{|n|}} + \frac{\left|e^{-\sqrt{in}(z-x)}\right|}{2\sqrt{|n|}}$$
$$\le \frac{1}{2\sqrt{|n|}}\gamma(\varepsilon)^{\frac{x-z}{L}} + \frac{1}{2\sqrt{|n|}}e^{-\sqrt{\frac{|n|}{2}}(z-x)} \le \gamma(\varepsilon)^{\frac{x-z}{L}}, \quad 0 \le x \le z \le L.$$

□

Lemma 3.3. *For* $0 < \gamma(\varepsilon) < e^{-L}$, *we have the following inequalities:*

$$\left|\widetilde{F}(g_1, h_1, w_1)(x) - \widetilde{F}(g_2, h_2, w_2)(x)\right|$$
$$\le \gamma(\varepsilon)^{\frac{x-L}{L}}\left(|g_1 - g_2| + |h_1 - h_2|\right) + \int_x^L \gamma(\varepsilon)^{\frac{x-z}{L}}\left|F_0(w_1)(z) - F_0(w_2)(z)\right| dz, \quad x \in [0, L] \tag{3.47}$$

and for $n \in \mathbb{Z}\backslash\{0\}$

$$2\pi \sum_{n\in\mathbb{Z}\backslash\{0\}} \left| \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \Big(F_n(w_1)(z) - F_n(w_2)(z)\Big) dz \right|^2$$

$$+2\pi \left|\widetilde{F}(g_0,h_0,w_1)(x) - \widetilde{F}(g_0,h_0,w_2)(x)\right|^2 \le \mathbb{K}_F^2 (L-x) \int_x^L \gamma(\varepsilon)^{\frac{2x-2z}{L}} \|w_1(z,\cdot) - w_2(z,\cdot)\|^2 \, dz, \quad x \in [0,L]. \quad (3.48)$$

Proof. We invoke (2.9) and Lemma 3.1 to deduce that

$$\left|\widetilde{F}(g_1,h_1,w_1)(x) - \widetilde{F}(g_2,h_2,w_2)(x)\right| = \left|(g_1-g_2) - (L-x)(h_1-h_2) + \int_x^L (z-x)\Big(F_0(w_1)(z) - F_0(w_2)(z)\Big) dz\right|$$

$$\le \gamma(\varepsilon)^{\frac{x-L}{L}} \left[|g_1-g_2| + |h_1-h_2|\right] + \int_x^L \gamma(\varepsilon)^{\frac{x-z}{L}} \left|F_0(w_1)(z) - F_0(w_2)(z)\right| dz,$$

as required.

Using (3.28), (3.46c), (3.47) and Hölder's inequality, we obtain

$$2\pi \sum_{n\in\mathbb{Z}\backslash\{0\}} \left| \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \Big(F_n(w_1)(z) - F_n(w_2)(z)\Big) dz \right|^2$$

$$+2\pi \left|\widetilde{F}(g_0,h_0,w_1)(x) - \widetilde{F}(g_0,h_0,w_2)(x)\right|^2 \le 2\pi(L-x) \sum_{n\in\mathbb{Z}\backslash\{0\}} \int_x^L \left| \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \right|^2 |F_n(w_1)(z) - F_n(w_2)(z)|^2 \, dz$$

$$+2\pi(L-x) \int_x^L \gamma(\varepsilon)^{\frac{2x-2z}{L}} \left|F_0(w_1)(z) - F_0(w_2)(z)\right|^2 dz \le (L-x) \int_x^L \gamma(\varepsilon)^{\frac{2x-2z}{L}} \|F(z,\cdot,w_1(z,\cdot)) - F(z,\cdot,w_2(z,\cdot))\|^2 \, dz$$

$$\le \mathbb{K}_F^2 (L-x) \int_x^L \gamma(\varepsilon)^{\frac{2x-2z}{L}} \|w_1(z,\cdot) - w_2(z,\cdot)\|^2 \, dz,$$

as required. ☐

Lemma 3.4. *For* $n \in \mathbb{Z}\backslash\{0\}$, *we have*

$$u_n(x) - \frac{u_n'(x)}{\sqrt{in}} = e^{\sqrt{in}(L-x)} \left(g_n - \frac{h_n}{\sqrt{in}}\right) - \int_x^L \frac{e^{\sqrt{in}(z-x)}}{\sqrt{in}} F_n(u)(z) dz, \quad x \in [0,L]. \quad (3.49)$$

Proof. Differentiating (2.7) with respect to x gives

$$-\frac{u_n'(x)}{\sqrt{in}} = \sinh\left((L-x)\sqrt{in}\right) g_n - \frac{\cosh\left((L-x)\sqrt{in}\right)}{\sqrt{in}} h_n - \int_x^L \frac{\cosh\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(u)(z) dz, \quad (3.50)$$

and adding (3.50) to (2.10) we complete the proof. ☐

We are now in a position to prove Theorems 3.1 and 3.2.

3.1. Proof of Theorem 3.1

The proof is divided into two steps.

Step 1. *The existence and uniqueness of the solution $u^\varepsilon \in C\Big([0,L];L^2(0,2\pi)\Big)$ to the integral equation* (2.14).

For $w \in C\Big([0,L];L^2(0,2\pi)\Big)$, we consider the following function

$$\widetilde{\mathcal{J}}(x,t,w(x,t)) := \sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\cosh^{\gamma(\varepsilon)}\Big((L-x)\sqrt{in}\Big)g_n^\varepsilon - \frac{\sinh^{\gamma(\varepsilon)}\Big((L-x)\sqrt{in}\Big)}{\sqrt{in}}h_n^\varepsilon\right]e^{int}$$
$$-\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\Big((z-x)\sqrt{in}\Big)}{\sqrt{in}}F_n(w)(z)dz\right]e^{int} + \widetilde{F}(g_0^\varepsilon, h_0^\varepsilon, w)(x). \tag{3.51}$$

and we aim to apply the Banach fixed point theorem. For this, we have to show that there exists an integer number m_0 such that the m_0-compound power $\widetilde{\mathcal{J}}^{m_0}$ is a contraction mapping. In fact, we will prove that for every $w_1, w_2 \in C\Big([0,L];L^2(0,2\pi)\Big)$ and $m \geq 1$, we have

$$\left\|\widetilde{\mathcal{J}}^m(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{J}}^m(x,\cdot,w_2(x,\cdot))\right\|^2 \leq \left(\frac{L\mathbb{K}_F^2}{\gamma^2(\varepsilon)}\right)^m \frac{(L-x)^m}{m!}\, |||\, w_1 - w_2\, |||^2, \tag{3.52}$$

where $|||\cdot|||$ is supremum norm in $C\Big([0,L];L^2(0,2\pi)\Big)$. We shall prove this inequality by induction. Indeed, for $m=1$, using (3.48), we have

$$\left\|\widetilde{\mathcal{J}}(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{J}}(x,\cdot,w_2(x,\cdot))\right\|^2 \leq 2\pi\sum_{n\in\mathbb{Z}\setminus\{0\}}\left|\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\Big((z-x)\sqrt{in}\Big)}{\sqrt{in}}\Big(F_n(w_1)(z) - F_n(w_2)(z)\Big)dz\right|^2$$
$$+2\pi\left|\widetilde{F}(g_0^\varepsilon,h_0^\varepsilon,w_1)(x) - \widetilde{F}(g_0^\varepsilon,h_0^\varepsilon,w_2)(x)\right|^2 \leq \mathbb{K}_F^2(L-x)\int_x^L \gamma(\varepsilon)^{\frac{2x-2z}{L}}\,\|w_1(z,\cdot)-w_2(z,\cdot)\|^2\,dz$$
$$\leq \frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}(L-x)\,|||w_1-w_2|||^2.$$

Thus, (3.52) holds for $m=1$. Next, supposing that (3.52) holds for $m=p$, we prove that it also holds for $m=p+1$. We have

$$\left\|\widetilde{\mathcal{J}}^{p+1}(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{J}}^{p+1}(x,\cdot,w_2(x,\cdot))\right\|^2 \leq \frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}\int_x^L \left\|\widetilde{\mathcal{J}}^p(z,\cdot,w_1(z,\cdot)) - \widetilde{\mathcal{J}}^p(z,\cdot,w_2(z,\cdot))\right\|^2 dz$$
$$\leq \frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}\int_x^L \left(\frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}\right)^p \frac{(L-z)^p}{p!}\,|||\,w_1-w_2\,|||^2\,dz \leq \left(\frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}\right)^{p+1}\frac{(L-x)^{p+1}}{(p+1)!}\,|||w_1-w_2|||^2.$$

Therefore, by the induction principle, we obtain (3.52).

We consider $\widetilde{\mathcal{J}} : C\Big([0,L];L^2(0,2\pi)\Big) \to C\Big([0,L];L^2(0,2\pi)\Big)$ defined by (3.51) and satisfying (3.52). Since

$$\lim_{m\to\infty}\sqrt{\left(\frac{\mathbb{K}_F^2 L}{\gamma^2(\varepsilon)}\right)^m \frac{(L-x)^m}{m!}} = 0, \quad \forall x \in [0,L],$$

there exists a positive integer number m_0 such that $\sqrt{\left(\frac{\mathbb{K}_F^2 L}{\gamma(\varepsilon)^2}\right)^{m_0}\frac{(L-x)^{m_0}}{m_0!}} < 1$. It means that $\widetilde{\mathcal{J}}^{m_0}$ is a contraction. It follows that the equation $\widetilde{\mathcal{J}}^{m_0}(w) = w$ has a unique solution $u^\varepsilon \in C\Big([0,L];L^2(0,2\pi)\Big)$. We claim that $\widetilde{\mathcal{J}}(u^\varepsilon) = u^\varepsilon$.

In fact, we have $\widetilde{\mathcal{J}}^{m_0}\left(\widetilde{\mathcal{J}}(u^\varepsilon)\right) = \widetilde{\mathcal{J}}(u^\varepsilon)$ because of $\widetilde{\mathcal{J}}\left(\widetilde{\mathcal{J}}^{m_0}(u^\varepsilon)\right) = \widetilde{\mathcal{J}}(u^\varepsilon)$. Then, the uniqueness of the fixed point of $\widetilde{\mathcal{J}}^{m_0}$ leads to $\widetilde{\mathcal{J}}(v^\varepsilon) = v^\varepsilon$; i.e., the equation $\widetilde{\mathcal{J}}(w) = w$ has a unique solution $u^\varepsilon \in C\left([0,L]; L^2(0,2\pi)\right)$. Finally, from (2.14) and (3.52) it then follows the desired conclusion that the integral equation (2.14) has a unique solution $u^\varepsilon \in C\left([0,L]; L^2(0,2\pi)\right)$.

Step 2. *Estimate the errors* (3.31) *and* (3.35) *between the first regularization u^ε and the true solution u.*

Proof of part (a). Using the triangle inequality, we have

$$\left\|u^\varepsilon(x,\cdot) - u(x,\cdot)\right\| \le \left\|u^\varepsilon(x,\cdot) - v^\varepsilon(x,\cdot)\right\| + \left\|v^\varepsilon(x,\cdot) - u(x,\cdot)\right\| = \left|\widetilde{\mathcal{A}}_1(x)\right| + \left|\widetilde{\mathcal{A}}_2(x)\right|, \tag{3.53}$$

where v^ε is defined by

$$\begin{aligned} v^\varepsilon(x,t) = &\sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right) g_n - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}} h_n\right] e^{int} \\ &- \sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(v^\varepsilon)(z)dz\right] e^{int} + \widetilde{F}(g_0, h_0, v^\varepsilon)(x). \end{aligned} \tag{3.54}$$

From the proof of Step 1, we know that the nonlinear integral equation (3.54) has unique solution $v^\varepsilon \in C\left([0,L]; L^2(0,2\pi)\right)$. We first estimate the term $\widetilde{\mathcal{A}}_1$. For $n \in \mathbb{Z}\backslash\{0\}$, combining to (2.14) and (3.54), we get

$$\begin{aligned} \left|\widetilde{\mathcal{A}}_1(x)\right| &= 2\pi \sum_{n\in\mathbb{Z}\backslash\{0\}} \left|u_n^\varepsilon(x) - v_n^\varepsilon(x)\right|^2 + 2\pi\left|\widetilde{F}(g_0^\varepsilon, h_0^\varepsilon, u^\varepsilon)(x) - \widetilde{F}(g_0, h_0, v^\varepsilon)(x)\right|^2 \\ &\le 6\pi \sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\left|\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)\right|^2 \left|g_n^\varepsilon - g_n\right|^2 + \left|\frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}\right|^2 \left|h_n^\varepsilon - h_n\right|^2\right] \\ &\quad + 6\pi \sum_{n\in\mathbb{Z}\backslash\{0\}} \left|\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \left(F_n(u^\varepsilon)(z) - F_n(v^\varepsilon)(z)\right) dz\right|^2 \\ &\quad + 2\pi\left|\widetilde{F}(g_0^\varepsilon, h_0^\varepsilon, u^\varepsilon)(x) - \widetilde{F}(g_0, h_0, v^\varepsilon)(x)\right|^2, \end{aligned}$$

where we have used the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$.

We now apply Lemmas 3.2 and 3.3 to obtain

$$\begin{aligned} \left|\widetilde{\mathcal{A}}_1(x)\right|^2 &\le 6\pi\gamma(\varepsilon)^{\frac{2x-2L}{L}} \left[\left(\sum_{n\in\mathbb{Z}\backslash\{0\}} \left|g_n^\varepsilon - g_n\right|^2 + \left|g_0^\varepsilon - g_0\right|^2\right) + \left(\sum_{n\in\mathbb{Z}\backslash\{0\}} \left|h_n^\varepsilon - h_n\right|^2 + \left|h_0^\varepsilon - h_0\right|^2\right)\right] \\ &\quad + 6\pi L\gamma(\varepsilon)^{\frac{2x}{L}} \int_x^L |\gamma(\varepsilon)|^{\frac{-2z}{L}} \left(\sum_{n\in\mathbb{Z}\backslash\{0\}} \left|F_n(u^\varepsilon)(z) - F_n(v^\varepsilon)(z)\right|^2 + \left|F_0(u^\varepsilon)(z) - F_0(v^\varepsilon)(z)\right|^2\right) dz \\ &\le 3\gamma(\varepsilon)^{\frac{2x-2L}{L}} \left(\left\|g^\varepsilon - g\right\|^2 + \left\|h^\varepsilon - h\right\|^2\right) + 3\mathbb{K}_F^2 L\gamma(\varepsilon)^{\frac{2x}{L}} \int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}} \left\|u^\varepsilon(z,\cdot) - v^\varepsilon(z,\cdot)\right\|^2 dz. \end{aligned} \tag{3.55}$$

Multiplying by $\gamma(\varepsilon)^{-\frac{2x}{L}}$ both sides of (3.55) and using (1.2), we get

$$\gamma(\varepsilon)^{\frac{-2x}{L}} \left|\widetilde{\mathcal{A}}_1(x)\right|^2 \le 3\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + 3\mathbb{K}_F^2 L \int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}} \left|\widetilde{\mathcal{A}}_1(z)\right|^2 dz.$$

Applying Gronwall's inequality to this yields

$$\gamma(\varepsilon)^{\frac{-2x}{L}}\left|\widetilde{\mathcal{A}}_1(x)\right|^2 \le 3\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 \exp\left(3\mathbb{K}_F^2 L(L-x)\right).$$

Therefore, we obtain

$$\left\|u^\varepsilon(x,\cdot)-v^\varepsilon(x,\cdot)\right\| \le \sqrt{3}\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)\exp\left(\frac{3\mathbb{K}_F^2 L(L-x)}{2}\right)\gamma(\varepsilon)^{\frac{x}{L}}. \tag{3.56}$$

Next, we estimate $|\widetilde{\mathcal{A}}_2(x)|$. From (2.10), (2.15)-(2.17) and (3.49), we have

$$\begin{aligned}
u_n(x) &= \cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)g_n - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}h_n - \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}F_n(u)(z)dz \\
&\quad + \frac{1}{2}\left(1-R^{\gamma(\varepsilon)}(L,n)\right)\left[e^{\sqrt{in}(L-x)}\left(g_n-\frac{h_n}{\sqrt{in}}\right) - \int_x^L \frac{e^{\sqrt{in}(z-x)}}{\sqrt{in}}F_n(u)(z)dz\right] \\
&= \cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)g_n - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}h_n - \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}F_n(u)(z)dz \\
&\quad + \frac{1}{2}\left(1-R^{\gamma(\varepsilon)}(L,n)\right)\left[u_n(x)-\frac{u_n'(x)}{\sqrt{in}}\right], \quad \forall n \in \mathbb{Z}\backslash\{0\}.
\end{aligned} \tag{3.57}$$

Combining (3.54) and (3.57) yields

$$v_n^\varepsilon(x)-u_n(x) = \Psi_n(x) - \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}\left(F_n(v^\varepsilon)(z)-F_n(u)(z)\right)dz, \quad \forall n \in \mathbb{Z}\backslash\{0\}, \tag{3.58}$$

where

$$\Psi_n(x) = \frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n)-1\right)e^{-\sqrt{in}x}\left[e^{\sqrt{in}x}u_n(x) - e^{\sqrt{in}x}\frac{u_n'(x)}{\sqrt{in}}\right], \quad \forall n \in \mathbb{Z}\backslash\{0\}. \tag{3.59}$$

The term $\widetilde{\mathcal{A}}_2$ can be estimated as follows:

$$|\widetilde{\mathcal{A}}_2(x)|^2 = 2\pi\sum_{n\in\mathbb{Z}\backslash\{0\}}\left|v_n^\varepsilon(x)-u_n(x)\right|^2 + 2\pi\left|\widetilde{F}(g_0,h_0,v^\varepsilon)-\widetilde{F}(g_0,h_0,u)\right|^2 \le \widetilde{J}_1(x)+\widetilde{J}_2(x), \tag{3.60}$$

where

$$\widetilde{J}_1(x) = 4\pi\sum_{n\in\mathbb{Z}\backslash\{0\}}\left|\frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n)-1\right)e^{-\sqrt{in}x}\right|^2\left|e^{\sqrt{in}x}u_n(x)-e^{\sqrt{in}x}\frac{u_n'(x)}{\sqrt{in}}\right|^2, \tag{3.61}$$

$$\widetilde{J}_2(x) = 4\pi\sum_{n\in\mathbb{Z}\backslash\{0\}}\left|\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}\left(F_n(v^\varepsilon)(z)-F_n(u)(z)\right)dz\right|^2 + 2\pi\left|\widetilde{F}(g_0,h_0,v^\varepsilon)-\widetilde{F}(g_0,h_0,u)\right|^2. \tag{3.62}$$

Using (2.19) we have

$$\begin{aligned}
\widetilde{J}_1(x) &\le 2\pi\gamma(\varepsilon)^{\frac{2x}{L}}\left(\sum_{n\in\mathbb{Z}\backslash\{0\}} e^{\sqrt{2|n|}L}|u_n(x)|^2 + \sum_{n\in\mathbb{Z}\backslash\{0\}} e^{\sqrt{2|n|}L}\frac{|u_n'(x)|^2}{|n|}\right) \\
&\le 2\pi\gamma(\varepsilon)^{\frac{2x}{L}}\left[\|u\|^2_{L^\infty(0,L;G_L^0(0,2\pi))} + \|u_x\|^2_{L^\infty(0,L;G_L^0(0,2\pi))}\right] \le 2\pi\gamma(\varepsilon)^{\frac{2x}{L}}\mathcal{I}_1^2.
\end{aligned} \tag{3.63}$$

It readily follows from (3.48) that

$$\widetilde{J_2}(x) \le 2\mathbb{K}_F^2(L-x)\gamma(\varepsilon)^{\frac{2x}{L}} \int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}} \left\|v^\varepsilon(z,\cdot) - u(z,\cdot)\right\|^2 dz. \tag{3.64}$$

Using (3.63) and (3.64) into (3.60) yields

$$\left|\widetilde{\mathcal{A}_2}(x)\right|^2 \le 2\pi\gamma(\varepsilon)^{\frac{2x}{L}} \mathcal{I}_1^2 + 2\mathbb{K}_F^2 L\gamma(\varepsilon)^{\frac{2x}{L}} \int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}} \left|\widetilde{\mathcal{A}_2}(z)\right|^2 dz. \tag{3.65}$$

Multiplying by $\gamma(\varepsilon)^{\frac{-2x}{L}}$ both sides and using Gronwall's inequality we obtain

$$\left\|v^\varepsilon(x,\cdot) - u(x,\cdot)\right\| \le \sqrt{2\pi}\mathcal{I}_1 \exp\left(\mathbb{K}_F^2 L(L-x)\right)\gamma(\varepsilon)^{\frac{x}{L}}. \tag{3.66}$$

Combining (3.53), (3.56) and (3.66), we deduce that

$$\left\|u^\varepsilon(x,\cdot) - u(x,\cdot)\right\| \le \left[\sqrt{3}\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)\exp\left(\frac{3\mathbb{K}_F^2 L(L-x)}{2}\right) + \sqrt{2\pi}\mathcal{I}_1 \exp\left(\mathbb{K}_F^2 L(L-x)\right)\right]\gamma(\varepsilon)^{\frac{x}{L}}, \tag{3.67}$$

which implies (3.31) and completes the proof of part (a) of Theorem 3.1.

Proof of part (b). This part can be proved similarly as part (a). Rewrite (3.59) as

$$\Psi_n(x) = \frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n) - 1\right) n^{-r} e^{-\sqrt{in}x}\left[n^r e^{\sqrt{in}x} u_n(x) - n^r e^{\sqrt{in}x}\frac{u_n'(x)}{\sqrt{in}}\right], \quad r>0,$$

and observe in passing that (3.33) implies that

$$\left|\frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n) - 1\right)\right| |n|^{-r}\left|e^{-\sqrt{in}x}\right| \le \frac{1}{2}\widetilde{M}(\varepsilon,r)\gamma(\varepsilon)^{\frac{x}{L}}, \quad n \in \mathbb{Z}\backslash\{0\}. \tag{3.68}$$

As in (3.63), we obtain

$$\begin{aligned}\widetilde{J_1}(x) &\le 2\pi\widetilde{M}^2(\varepsilon,r)\gamma(\varepsilon)^{\frac{2x}{L}}\left(\sum_{n\in\mathbb{Z}\backslash\{0\}} |n|^{2r} e^{\sqrt{2|n|}L}|u_n(x)|^2 + \sum_{n\in\mathbb{Z}\backslash\{0\}} |n|^{2r} e^{\sqrt{2|n|}L}\frac{|u_n'(x)|^2}{|n|}\right)\\ &\le 2\pi\widetilde{M}^2(\varepsilon,r)\gamma(\varepsilon)^{\frac{2x}{L}}\left[\|u\|^2_{L^\infty(0,L;G_L^r(0,2\pi))} + \|u_x\|^2_{L^\infty(0,L;G_L^r(0,2\pi))}\right] \le 2\pi\widetilde{M}^2(\varepsilon,r)\gamma(\varepsilon)^{\frac{2x}{L}}\mathcal{I}_2^2.\end{aligned} \tag{3.69}$$

Combining (3.64) and (3.69) yields

$$\gamma(\varepsilon)^{\frac{-2x}{L}}\left|\widetilde{\mathcal{A}_2}(x)\right|^2 \le 2\pi\widetilde{M}^2(\varepsilon,r)\mathcal{I}_2^2 + 2\mathbb{K}_F^2 L\int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}}\left|\widetilde{\mathcal{A}_2}(z)\right|^2 dz.$$

Applying Gronwall's inequality, we deduce that

$$\left\|v^\varepsilon(x,\cdot) - u(x,\cdot)\right\| \le \sqrt{2\pi}\widetilde{M}(\varepsilon,r)\mathcal{I}_2 \exp\left(\mathbb{K}_F^2 L(L-x)\right)\gamma(\varepsilon)^{\frac{x}{L}}. \tag{3.70}$$

Combining (3.53), (3.56) and (3.70), we deduce that

$$\left\|u^\varepsilon(x,\cdot) - u(x,\cdot)\right\| \le \left[\sqrt{3}\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)\exp\left(\frac{3\mathbb{K}_F^2 L(L-x)}{2}\right) + \sqrt{2\pi}\widetilde{M}(\varepsilon,r)\mathcal{I}_2 \exp\left(\mathbb{K}_F^2 L(L-x)\right)\right]\gamma(\varepsilon)^{\frac{x}{L}},$$

which implies (3.35) and completes the proof of part (b) of Theorem 3.1.

3.2. Proof of Theorem 3.2

The proof of Theorem 3.2 consists of two steps.

Step 1. *The existence and uniqueness of solution* $U^{\varepsilon} \in C\left([0,L]; L^2(0,2\pi)\right)$ *to* (3.39).

For any $w \in C\left([0,L]; L^2(0,2\pi)\right)$, we define

$$\widetilde{\mathcal{G}}(x,t,w(x,t)) := \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)g_n^{\varepsilon} - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}} h_n^{\varepsilon}\right] e^{int}$$

$$- \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} F_n(U^{\varepsilon})(z)dz + \int_0^x \widetilde{P}^{\gamma(\varepsilon)}(x,z,n) F_n(U^{\varepsilon})(z)dz\right] e^{int} + \widetilde{F}(g_0^{\varepsilon}, h_0^{\varepsilon}, w)(x). \tag{3.71}$$

The proof of this step is nontrivial and it is different from the proof of Step 1 in Theorem 3.1. We have to prove that the mapping $\widetilde{\mathcal{G}}$ is a contraction mapping by using a new norm. Let us define the following norm on $C\left([0,L]; L^2(0,2\pi)\right)$:

$$\|f\|_1 = \sup_{0\le x\le L} \left\{\gamma(\varepsilon)^{-\frac{x}{L}} \|f(x,\cdot)\|\right\}, \quad \forall f \in C\left([0,L]; L^2(0,2\pi)\right). \tag{3.72}$$

It is easy to show that $\|\cdot\|_1$ is a norm on $C\left([0,L]; L^2(0,2\pi)\right)$. We claim that for every $w_1, w_2 \in C\left([0,L]; L^2(0,2\pi)\right)$, we have

$$\left\|\widetilde{\mathcal{G}}(w_1) - \widetilde{\mathcal{G}}(w_2)\right\|_1 \le \mathbb{K}_F L \|w_1 - w_2\|_1 . \tag{3.73}$$

First, from (3.47) we have

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}} \left(F_n(w_1)(z) - F_n(w_2)(z)\right) dz\right|^2 + \left|\widetilde{F}(g_0^{\varepsilon}, h_0^{\varepsilon}, w_1)(x) - \widetilde{F}(g_0^{\varepsilon}, h_0^{\varepsilon}, w_2)(x)\right|^2$$
$$\le \frac{1}{2\pi}\mathbb{K}_F^2 (L-x)^2 \gamma(\varepsilon)^{\frac{2x}{L}} \|w_1 - w_2\|_1^2, \quad x \in [0,L]. \tag{3.74}$$

Second, using (3.46a), as in the proof of Lemma 3.3, we have

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\int_0^x \widetilde{P}^{\gamma(\varepsilon)}(x,z,n) \left(F_n(w_1)(z) - F_n(w_2)(z)\right) dz\right|^2 \le x\gamma(\varepsilon)^{\frac{2x}{L}} \int_0^x \gamma(\varepsilon)^{\frac{-2z}{L}} \sum_{n\in\mathbb{Z}} |F_n(w_1)(z) - F_n(w_2)(z)|^2 dz$$
$$\le \frac{1}{2\pi} x \mathbb{K}_F^2 \gamma(\varepsilon)^{\frac{2x}{L}} \int_0^x \gamma(\varepsilon)^{\frac{-2z}{L}} \|w_1(z,\cdot) - w_2(z,\cdot)\|^2 dz \le \frac{1}{2\pi} x^2 \mathbb{K}_F^2 \gamma(\varepsilon)^{\frac{2x}{L}} \|w_1 - w_2\|_1^2 . \tag{3.75}$$

Then, for $0 < x < L$, using the inequality $(a_1 + a_2)^2 \le (1 + \widetilde{\alpha})\, a_1^2 + \left(1 + \frac{1}{\widetilde{\alpha}}\right) a_2^2$, for any real numbers $a_1, a_2, \widetilde{\alpha} > 0$ together with (2.3), we conclude that

$$\left\|\widetilde{\mathcal{G}}(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{G}}(x,\cdot,w_2(x,\cdot))\right\|^2 \le \gamma(\varepsilon)^{\frac{2x}{L}} \mathbb{K}_F^2 (1+\widetilde{\alpha}) x^2 \|w_1 - w_2\|_1^2 + \gamma(\varepsilon)^{\frac{2x}{L}} \mathbb{K}_F^2 \left(1 + \frac{1}{\widetilde{\alpha}}\right)(L-x)^2 \|w_1 - w_2\|_1^2 .$$

By choosing $\widetilde{\alpha} = \frac{L-x}{x}$, we obtain

$$\gamma(\varepsilon)^{-\frac{2x}{L}} \left\|\widetilde{\mathcal{G}}(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{G}}(x,\cdot,w_2(x,\cdot))\right\|^2 \le \mathbb{K}_F^2 L^2 \|w_1 - w_2\|_1^2, \quad \forall x \in (0,L). \tag{3.76}$$

On the hand, letting $x = L$ in (3.75), we have

$$\gamma(\varepsilon)^{-2}\left\|\widetilde{\mathcal{G}}(L,\cdot,w_1(L,\cdot)) - \widetilde{\mathcal{G}}(L,\cdot,w_2(L,\cdot))\right\|^2 \le \mathbb{K}_F^2 L^2 \|w_1 - w_2\|_1^2, \tag{3.77}$$

and letting $x = 0$ in (3.74), we have

$$\left\|\widetilde{\mathcal{G}}(0,\cdot,w_1(0,\cdot)) - \widetilde{\mathcal{G}}(0,\cdot,w_2(0,\cdot))\right\|^2 \le \mathbb{K}_F^2 L^2 \|w_1 - w_2\|_1^2. \tag{3.78}$$

Combining (3.76)-(3.78), we obtain

$$\gamma(\varepsilon)^{-\frac{x}{L}}\left\|\widetilde{\mathcal{G}}(x,\cdot,w_1(x,\cdot)) - \widetilde{\mathcal{G}}(x,\cdot,w_2(x,\cdot))\right\| \le \mathbb{K}_F L \|w_1 - w_2\|_1, \quad \forall x \in [0,L],$$

which leads to (3.73). Since $\mathbb{K}_F L < 1$ it means that $\widetilde{\mathcal{G}}$ is a contraction. It follows that the equation $\widetilde{\mathcal{G}}(w) = w$ has a unique solution $w \in C\left([0,L]; L^2(0,2\pi)\right)$ and this completes the proof of Step 1.

Step 2. *For establishing* (3.41) *we estimate the error* $\|U^\varepsilon - u\|_1$ *in the norm* (3.72) *of* $C\left([0,L]; L^2(0,2\pi)\right)$.

Proof of part (a). Consider the function

$$\widetilde{\mathcal{W}}(x,t) = \gamma(\varepsilon)^{-\frac{x}{L}} \sum_{n\in\mathbb{Z}} \left[U_n^\varepsilon(x) - u_n(x)\right] e^{int}. \tag{3.79}$$

Let us find an upper bound for $\left\|\widetilde{\mathcal{W}}\right\|_1 = \sup_{x\in[0,L]}\left\|\widetilde{\mathcal{W}}(x,\cdot)\right\|$. The norm exists because the two functions U^ε and u belong to $C\left([0,L]; L^2(0,2\pi)\right)$.

We first observe that

$$\left\|\widetilde{\mathcal{W}}(x,\cdot)\right\|^2 = 2\pi \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\widetilde{\mathcal{W}}_n(x)\right|^2 + 2\pi\gamma(\varepsilon)^{-\frac{2x}{L}}\left|\widetilde{F}\left(g_0^\varepsilon, h_0^\varepsilon, U^\varepsilon\right)(x) - \widetilde{F}\left(g_0, h_0, u\right)(x)\right|^2. \tag{3.80}$$

Applying (3.49) at $x = 0$, we have

$$u_n(0) - \frac{u_n'(0)}{\sqrt{in}} = e^{\sqrt{in}L}\left(g_n - \frac{h_n}{\sqrt{in}}\right) - \int_0^L \frac{e^{\sqrt{in}z}}{\sqrt{in}} F_n(u)(z)dz, \quad n \in \mathbb{Z}\setminus\|0\},$$

which implies that

$$e^{-\sqrt{in}L}\left(u_n(0) - \frac{u_n'(0)}{\sqrt{in}}\right) + \int_0^x \frac{e^{\sqrt{in}(z-L)}}{\sqrt{in}} F_n(u)(z)dz = g_n - \frac{h_n}{\sqrt{in}} - \int_x^L \frac{e^{\sqrt{in}(z-L)}}{\sqrt{in}} F_n(u)(z)dz, \quad n \in \mathbb{Z}\setminus\{0\}. \tag{3.81}$$

From (2.17), (3.57) and (3.81), we deduce that

$$\begin{aligned}
u_n(x) &= \cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)g_n - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}h_n - \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}F_n(u)(z)dz \\
&\quad + \frac{1}{2}\left(1 - R^{\gamma(\varepsilon)}(L,n)\right)e^{\sqrt{in}(L-x)}\left[e^{-\sqrt{in}L}\left(u_n(0) - \frac{u_n'(0)}{\sqrt{in}}\right) + \int_0^x \frac{e^{\sqrt{in}(z-L)}}{\sqrt{in}}F_n(u)(z)dz\right] \\
&= \cosh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)g_n - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}h_n - \int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}F_n(u)(z)dz \\
&\quad + \frac{1}{2}\left(1 - R^{\gamma(\varepsilon)}(L,n)\right)e^{-\sqrt{in}x}\left(u_n(0) - \frac{u_n'(0)}{\sqrt{in}}\right) - \int_0^x \widetilde{P}^{\gamma(\varepsilon)}(x,z,n)F_n(u)(z)dz.
\end{aligned} \tag{3.82}$$

From (3.39), (3.79) and (3.82), we have

$$\begin{aligned}\widetilde{\mathcal{W}}_n(x) &= \gamma(\varepsilon)^{-\frac{x}{L}}\left[U_n^\varepsilon(x) - u_n(x)\right]\\ &= \gamma(\varepsilon)^{-\frac{x}{L}}\left[\Phi_n(x) + \cosh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)(g_n^\varepsilon - g_n) - \frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}(h_n^\varepsilon - h_n)\right]\\ &\quad - \gamma(\varepsilon)^{-\frac{x}{L}}\left[\int_x^L \frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}\left(F_n(U^\varepsilon)(z) - F_n(u)(z)\right)dz\right]\\ &\quad - \gamma(\varepsilon)^{-\frac{x}{L}}\left[\int_0^x \widetilde{P}^{\gamma(\varepsilon)}(x,z,n)\left(F_n(U^\varepsilon)(z) - F_n(u)(z)\right)dz\right],\end{aligned} \tag{3.83}$$

where

$$\Phi_n(x) := \frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n) - 1\right)e^{-\sqrt{in}x}\left(u_n(0) - \frac{u_n'(0)}{\sqrt{in}}\right), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Then

$$\begin{aligned}|\widetilde{\mathcal{W}}_n(x)| &= \gamma(\varepsilon)^{-\frac{x}{L}}\left|U_n^\varepsilon(x) - u_n(x)\right|\\ &\le \gamma(\varepsilon)^{-\frac{x}{L}}\Bigg[|\Phi_n(x)| + \left|\cosh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)\right|\left|g_n^\varepsilon - g_n\right| + \left|\frac{\sinh^{\gamma(\varepsilon)}\left((L-x)\sqrt{in}\right)}{\sqrt{in}}\right|\left|h_n^\varepsilon - h_n\right|\\ &\quad + \int_x^L \left|\frac{\sinh^{\gamma(\varepsilon)}\left((z-x)\sqrt{in}\right)}{\sqrt{in}}\right|\left|F_n(U^\varepsilon)(z) - F_n(u)(z)\right|dz + \int_0^x \left|\widetilde{P}^{\gamma(\varepsilon)}(x,z,n)\left(F_n(U^\varepsilon)(z) - F_n(u)(z)\right)\right|dz\Bigg],\\ &\qquad n \in \mathbb{Z}\backslash\{0\}.\end{aligned} \tag{3.84}$$

From Lemma 3.2, we get

$$\begin{aligned}\left|\widetilde{\mathcal{W}}_n(x)\right| &\le \frac{1}{2}\left[|u_n(0)| + \left|u_n'(0)\right|\right] + \gamma(\varepsilon)^{-1}\left[\left|g_n^\varepsilon - g_n\right| + \left|h_n^\varepsilon - h_n\right|\right]\\ &\quad + \int_0^L \gamma(\varepsilon)^{-\frac{z}{L}}\left|F_n(U^\varepsilon)(z) - F_n(u)(z)\right|dz, \quad n \in \mathbb{Z}\backslash\{0\}.\end{aligned} \tag{3.85}$$

From the inequality

$$(a_1 + a_2 + a_3)^2 \le 2\left(1 + \frac{1}{\widetilde{\alpha}}\right)a_1^2 + 2\left(1 + \frac{1}{\widetilde{\alpha}}\right)a_2^2 + (1 + \widetilde{\alpha})a_3^2, \tag{3.86}$$

for any real numbers a_j, $(j = 1,2,3)$ and $\widetilde{\alpha} > 0$ and thanks to Hölder's inequality, we deduce that

$$\begin{aligned}2\pi \sum_{n\in\mathbb{Z}\backslash\{0\}}\left|\widetilde{\mathcal{W}}_n(x)\right|^2 &\le \sum_{n\in\mathbb{Z}\backslash\{0\}}\left[2\pi\left(1 + \frac{1}{\widetilde{\alpha}}\right)\left[|u_n(0)|^2 + \left|u_n'(0)\right|^2\right] + 8\pi\left(1 + \frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\left(\left|g_n^\varepsilon - g_n\right|^2 + \left|h_n^\varepsilon - h_n\right|^2\right)\right]\\ &\quad + \sum_{n\in\mathbb{Z}\backslash\{0\}}\left[2\pi(1 + \widetilde{\alpha})L\int_0^L \gamma(\varepsilon)^{\frac{-2z}{L}}\left|F_n(U^\varepsilon)(z) - F_n(u)(z)\right|^2 dz\right].\end{aligned} \tag{3.87}$$

Using again (3.86) in (3.47), and Hölder inequality we obtain

$$\begin{aligned}&2\pi\gamma(\varepsilon)^{-\frac{2x}{L}}\left|\widetilde{F}\left(g_0^\varepsilon, h_0^\varepsilon, U^\varepsilon\right)(x) - \widetilde{F}(g_0, h_0, u)(x)\right|^2\\ &\le 4\pi\left(1 + \frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\left(\left|g_0^\varepsilon - g_0\right|^2 + \left|h_0^\varepsilon - h_0\right|^2\right) + 2\pi(1 + \widetilde{\alpha})L\int_x^L \gamma(\varepsilon)^{\frac{-2z}{L}}\left|F_0(U^\varepsilon)(z) - F_0(u)(z)\right|^2 dz.\end{aligned} \tag{3.88}$$

We can now combine the results of (1.2), (3.28), (3.79), (3.80), (3.87) and (3.88), to obtain

$$\left\|\widetilde{\mathcal{W}}(x,\cdot)\right\|^2 \le 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + \left(1+\frac{1}{\widetilde{\alpha}}\right)\left[\|u(0,\cdot)\|^2 + \|u_x(0,\cdot)\|^2\right] + (1+\widetilde{\alpha})\,\mathbb{K}_F^2 L \int_0^L \left\|\widetilde{\mathcal{W}}(z,\cdot)\right\|^2 dz$$

$$\le 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + \left(1+\frac{1}{\widetilde{\alpha}}\right)\mathcal{I}_3^2 + (1+\widetilde{\alpha})\,\mathbb{K}_F^2 L^2 \left\|\widetilde{\mathcal{W}}\right\|_1^2, \quad x\in[0,L]. \tag{3.89}$$

The latter inequality holds for all $x \in [0, L]$ and the right-hand side of (3.89) is independent of x so, we get

$$\left\|\widetilde{\mathcal{W}}\right\|_1^2 \le 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\varepsilon^2 + \left(1+\frac{1}{\widetilde{\alpha}}\right)\mathcal{I}_3^2 + (1+\widetilde{\alpha})\mathbb{K}_F^2 L^2 \left\|\widetilde{\mathcal{W}}\right\|_1^2.$$

Then,

$$\left(1-(1+\widetilde{\alpha})\mathbb{K}_F^2L^2\right)\left\|\widetilde{\mathcal{W}}\right\|_1^2 \le 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\varepsilon^2 + \left(1+\frac{1}{\widetilde{\alpha}}\right)\mathcal{I}_3^2 = \left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\mathcal{I}_3^2 + 4\gamma(\varepsilon)^{-2}\varepsilon^2\right).$$

Since $\widetilde{\alpha} \in \left(0, \frac{1}{\mathbb{K}_F^2 L^2} - 1\right)$ it follows that the left-hand side bracket is positive. This implies that

$$\gamma(\varepsilon)^{-\frac{2x}{L}}\left\|U_n^\varepsilon(x) - u_n(x)\right\|^2 \le \left\|\widetilde{\mathcal{W}}\right\|_1^2 \le \frac{\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\mathcal{I}_3^2 + 4\gamma(\varepsilon)^{-2}\varepsilon^2\right)}{\left(1-(1+\widetilde{\alpha})\mathbb{K}_F^2L^2\right)}.$$

Thus (3.41) holds.

Proof of part (b). First, we re-write Φ_n as

$$\Phi_n(x) = \frac{1}{2}\left(R^{\gamma(\varepsilon)}(L,n) - 1\right) n^{-r} e^{-\sqrt{in}x}\left(n^r u_n(0) - n^r\frac{u_n'(0)}{\sqrt{in}}\right), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Using (3.68), as in (3.85), we obtain

$$\left|\widetilde{\mathcal{W}}_n(x)\right| \le \frac{1}{2}\widetilde{M}(\varepsilon,r)|n|^r\left[|u_n(0)| + \left|u_n'(0)\right|\right] + \gamma(\varepsilon)^{-1}\left[\left|g_n^\varepsilon - g_n\right| + \left|h_n^\varepsilon - h_n\right|\right]$$

$$+ \int_0^L \gamma(\varepsilon)^{-\frac{z}{L}}\left|F_n(U^\varepsilon)(z) - F_n(u)(z)\right|dz, \quad n \in \mathbb{Z}\backslash\{0\}.$$

Using inequality (3.86), as in (3.87), we obtain

$$2\pi\sum_{n\in\mathbb{Z}\backslash\{0\}}\left|\widetilde{\mathcal{W}}_n(x)\right|^2 \le \sum_{n\in\mathbb{Z}\backslash\{0\}}\left[2\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\widetilde{M}^2(\varepsilon,r)\left[|n|^{2r}|u_n(0)|^2 + |n|^{2r}\left|u_n'(0)\right|^2\right]\right]$$

$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}}\left[8\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\left(\left|g_n^\varepsilon - g_n\right|^2 + \left|h_n^\varepsilon - h_n\right|^2\right)\right]$$

$$+ \sum_{n\in\mathbb{Z}\backslash\{0\}}\left[2\pi(1+\widetilde{\alpha})L\int_0^L\gamma(\varepsilon)^{\frac{-2z}{L}}\left|F_n(U^\varepsilon)(z) - F_n(u)(z)\right|^2dz\right].$$

Finally, as in (3.88), we obtain

$$\left\|\widetilde{\mathcal{W}}(x,\cdot)\right\|^2 \le 2\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\widetilde{M}^2(\varepsilon,r)\left[\|u(0,\cdot)\|^2_{V^r(0,2\pi)} + \|u_x(0,\cdot)\|^2_{V^r(0,2\pi)}\right]$$

$$+ 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + (1+\widetilde{\alpha})\,\mathbb{K}_F^2L\int_0^L\left\|\widetilde{\mathcal{W}}(z,\cdot)\right\|^2dz$$

$$\le 2\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\widetilde{M}^2(\varepsilon,r)\mathcal{I}_4^2 + 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + (1+\widetilde{\alpha})\,\mathbb{K}_F^2L^2\left\|\widetilde{\mathcal{W}}\right\|_1^2.$$

We obtain

$$\left\|\widetilde{\mathcal{W}}\right\|_1^2 \le 2\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\widetilde{M}^2(\varepsilon,r)\mathcal{I}_4^2 + 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\left(\frac{\varepsilon}{\gamma(\varepsilon)}\right)^2 + (1+\widetilde{\alpha})\,\mathbb{K}_F^2L^2\left\|\widetilde{\mathcal{W}}\right\|_1^2.$$

Finally, we get

$$\gamma(\varepsilon)^{-\frac{2x}{L}}\left\|U^\varepsilon(x,\cdot)-u(x,\cdot)\right\|^2 \le \left\|\widetilde{\mathcal{W}}\right\|_1^2 \le \frac{2\pi\left(1+\frac{1}{\widetilde{\alpha}}\right)\widetilde{M}^2(\varepsilon,r)\mathcal{I}_4^2 + 4\left(1+\frac{1}{\widetilde{\alpha}}\right)\gamma(\varepsilon)^{-2}\varepsilon^2}{1-(1+\widetilde{\alpha})\,\mathbb{K}_F^2L^2}.$$

Hence, inequality (3.43) holds. This concludes the proof of Theorem 3.2.

4. Numerical results and discussion

As showed in Sections 2 and 3, the two regularized solutions u^ε and U^ε satisfying the integral equations (2.14) and (3.39), respectively, can be resolved uniquely by fixed-point iteration.

Let us introduce the uniform mesh

$$x_k = k\Delta x, \quad \Delta x = \frac{L}{K}, \quad k=\overline{0,K}, \quad K\in\mathbb{N}\backslash\{0\},$$
$$t_m = m\Delta t, \quad \Delta t = \frac{2\pi}{M}, \quad m=\overline{0,(M-1)}, \quad M\in\mathbb{N}\backslash\{0\},$$

where K and M are given positive integers and, for convenience, M is even.

The outline of the remainder of this section is as follows. First, we explain modelling a data function from its discrete values. Second, we give some numerical details of the approximations of the right-hand sides of equations (2.14) and (3.39). Finally, a couple of examples are presented and discussed.

4.1. Modelling data

Lemma 4.1. *Let $v \in H^2(0,2\pi)$ and set $v_m := v(t_m)$ for $m=\overline{0,(M-1)}$. A function $\tilde{v}$ is called interpolating function of data sets $(v_m)_{m=\overline{0,(M-1)}}$ for v, if it satisfies, [15],*

$$\tilde{v}(t) = \sum_{n=-M/2+1}^{M/2}\hat{v}_n e^{int} \quad \text{for } t\in[0,2\pi), \quad \text{where } \hat{v}_n = \frac{1}{M}\sum_{m=0}^{M-1} v_m e^{-mn\frac{2\pi i}{M}}. \tag{4.90}$$

Then, we have

$$\tilde{v}(t_m) = \sum_{n=-M/2+1}^{M/2}\hat{v}_n e^{mn\frac{2\pi i}{M}} = v_m \quad \text{for } m=\overline{0,(M-1)}. \tag{4.91}$$

Moreover, the error between v and $\tilde{v}$ is bounded by

$$\|v-\tilde{v}\| \le C(\Delta t)^2\|v''\|, \tag{4.92}$$

where C is a positive constant independent of v and Δt.

Proof. One can find the proof of this lemma in textbooks, see e.g. [15], Chapter 2. The relationship between $\hat{v}_n$ and v_m shown in equations (4.90) and (4.91) is well-known as the discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT), respectively, [5]. ☐

Definition 5.1 *Now assume the Cauchy data g and $h \in H^2(0, 2\pi)$. Let $(g^\varepsilon_m, h^\varepsilon_m)$ be the discrete data on the time grid $(t_m)_{m=\overline{0,(M-1)}}$ measured with pointwise errors*

$$g^\varepsilon_m = g_m + \frac{\varepsilon}{2}\, rand(m), \quad h^\varepsilon_m = h_m + \frac{\varepsilon}{2}\, rand(m), \tag{4.93}$$

where $(g_m, h_m) := (g(t_m), h(t_m))$, the function $rand(\cdot)$ generates a vector of M random numbers from a uniform distribution in $[-1, 1]$ and $\varepsilon \geq 0$ indicates the level of noise disturbing the data. Using the discrete data $(g^\varepsilon_m, h^\varepsilon_m)$ we construct the noisy data functions $(\tilde{g^\varepsilon}, \tilde{h^\varepsilon})$ as follows:

$$\tilde{g^\varepsilon}(t) = \sum_{n=-M/2+1}^{M/2} \hat{g^\varepsilon_n} e^{int} \quad \text{for } t \in [0, 2\pi), \quad \text{where } \hat{g^\varepsilon_n} = \frac{1}{M}\sum_{m=0}^{M-1} g^\varepsilon_m e^{-mn\frac{2\pi i}{M}} \tag{4.94}$$

and

$$\tilde{h^\varepsilon}(t) = \sum_{n=-M/2+1}^{M/2} \hat{h^\varepsilon_n} e^{int} \quad \text{for } t \in [0, 2\pi), \quad \text{where } \hat{h^\varepsilon_n} = \frac{1}{M}\sum_{m=0}^{M-1} h^\varepsilon_m e^{-mn\frac{2\pi i}{M}}. \tag{4.95}$$

Now, we have the following lemma which shows that $(\tilde{g^\varepsilon}, \tilde{h^\varepsilon})$ is a noisy data function of (g, h).

Lemma 4.2. *We have the following estimate:*

$$\|g - \tilde{g^\varepsilon}\| + \|h - \tilde{h^\varepsilon}\| \leq \varepsilon_0, \tag{4.96}$$

where $\varepsilon_0 := \varepsilon + 2C(\Delta t)^2 \max\{\|g''\|, \|h''\|\}$.

Proof. Using the modelling formula (4.90) and the discrete form of Parseval's identity, see [5], Chapter 3, i.e.

$$\sum_{n=-M/2+1}^{M/2} |\hat{v}_n|^2 = \frac{1}{M}\sum_{m=0}^{M-1} |v_m|^2,$$

we have

$$\|\tilde{g} - \tilde{g^\varepsilon}\| = \sum_{n=-M/2+1}^{M/2} |\hat{g^\varepsilon_n} - \hat{g}_n|^2 = \frac{1}{M}\sum_{m=0}^{M-1} |g^\varepsilon_m - g_m|^2 \leq (\varepsilon/2)^2.$$

Using (4.92), we have

$$\|g - \tilde{g}\| \leq C(\Delta t)^2 \|g''\|.$$

These imply that

$$\|g - \tilde{g^\varepsilon}\| \leq \|\tilde{g} - \tilde{g^\varepsilon}\| + \|g - \tilde{g}\| \leq \frac{\varepsilon}{2} + C(\Delta t)^2 \|g''\|$$

and a similar inequality holds for $\|h - \tilde{h^\varepsilon}\|$. Thus (4.96) holds. ☐

4.2. Numerical details

As mentioned before, the numerical solutions to equations (2.14) and (3.39) can be found by a fixed-point convergent iteration. To compute u^ε and U^ε, we need to evaluate the integrals in the right-hand sides of these equations. For each $k = \overline{1, K}$, we need to approximate the integral

$$\int_{X_l}^{X_u} \alpha(n, x_k, z) \langle F(z, t, u^\varepsilon(z, t)), e^{int} \rangle dz = \int_{X_l}^{X_u} \phi(z) dz,$$

for two cases: $[X_l, X_u] = [0, x_k]$ or $[X_l, X_u] = [x_k, L]$.

Given u^ε, a previous numerical solution of the fixed-point iteration, the integrand $\phi = \phi(u^\varepsilon, x_k, z, n)$ has only discrete values with repect to $z \in [X_l, X_u]$. Using the Newton-Cotes formulas, we have

$$\int_{X_l}^{X_u} \phi(z)dz \approx \Delta x \sum_{j=1}^{p} A_{p,j}\phi_j, \quad p = \overline{2, K},$$

where the coefficients $A_{p,j}$ are given in [1], p.886, or in [26].

In our numerical practice, computation is implemented in Fortran programming language with double precision floating point numbers. Since the aforementioned transforms, such as DFT: $(v_m) \mapsto (\hat{v}_n)$ (equation (4.90)) and IDFT: $(\hat{v}_n) \mapsto (v_m)$ (equation (4.91)) can be performed efficiently using the fast Fourier transform (FFT) technique, we adopt the subroutines $cfft1f$ and $cfft1b$ of FFTPACKS, [10], to accomplish the DFT and IDFT, respectively.

4.3. Numerical tests

In this subsection we test a couple of numerical examples in order to assess and verify the accuracy and stability of the regularized solutions u^ε and U^ε.

Example 1. We take $L = 0.5$ and consider the analytical solution given by

$$u(x,t) = \exp\left(-3x\left(t - \frac{5\pi}{6}\right)^2\right) =: \mathcal{U}(x,t), \quad (x,t) \in [0, 0.5] \times [0, 2\pi], \tag{4.97}$$

with the nonlinear Lipschitz source (with Lipschitz constant $\mathbb{K}_F = 1$)

$$F(x,t,u) = \frac{|u|}{u^2+1} + R(x,t), \tag{4.98}$$

where $R = \mathcal{U}_t - \mathcal{U}_{xx} - \frac{|\mathcal{U}|}{\mathcal{U}^2+1}$. The nonlinearity in (4.98) is characteristic to a reaction-diffusion equation.

From (4.97), the Cauchy data is given by

$$g(t) = \mathcal{U}(0.5,t) = \exp\left(-\frac{3}{2}\left(t - \frac{5\pi}{6}\right)^2\right), \quad h(t) = \mathcal{U}_x(0.5,t) = -3\left(t - \frac{5\pi}{6}\right)^2 \exp\left(-\frac{3}{2}\left(t - \frac{5\pi}{6}\right)^2\right), \quad t \in [0, 2\pi]. \tag{4.99}$$

The graph of the exact solution (4.97) is shown in Figure 1(a). The exact Cauchy data (4.99) are plotted in Figure 2(a) together with the desired solution at the boundary $x = 0$ given by $u(0,t) \equiv 1$.

The numerical solutions u^ε and U^ε solving (2.14) and (3.39), respectively, are obtained with the mesh $M \times K = 100 \times 50$. The number of iterations (starting from the zero trivial initial guess) was 7 when the relative error between two subsequent iterations was less than 10^{-9} tolerance, and the iterative process was terminated. Furthermore, in both cases $\varepsilon = 0$ and 2×10^{-2}, the root mean square errors between the two regularized solutions u^ε, U^ε and the exact solution (4.97) were obtained equally accurate when the regularization parameter $\gamma(\varepsilon)$ becomes smaller than 10^{-2}; in other words, the two regularization methods have the same order of accuracy. This can also be visualised from Figures 1(b)-1(e) which show the graphs of u^ε and U^ε for noisy ($\varepsilon = 2 \times 10^{-2}$) Cauchy data (4.93). From top to bottom of figures one can see that the regularized solutions are in better agreement with the exact solution (shown in Figure 1(a)), as $\gamma(\varepsilon)$ decreases. However, from Figures 1(b)-1(e), one can observe the sensitivity of the numerical solutions to noise ($\varepsilon = 2 \times 10^{-2}$) via the fluctuating countour lines, especially near the boundary $x = 0$. This is to be expected since the stability decreases with x marching sideway leftwards from the overprescribed boundary $x = L$. The numerical results of $u(0,t)$ also degrade close to the end of the time interval $t = 2\pi$ which is consistent with the remarks made in [7]. Despite the fluctuations occuring, the shaping of the numerically retrieved solutions confirm the estimates predicted in Theorems 3.1 and 3.2; a proper choice of the regularization parameter $\gamma(\varepsilon)$ concerns the term $\varepsilon/\gamma(\varepsilon)$ in expressions (3.32) and (3.36) of Theorem 3.1, and (3.41) and (3.43) of Theorem 3.2. That is, once this term is kept under control, the convergence and stability of the numerical solutions can be guaranteed. The best choice of $\gamma(\varepsilon)$ in our numerical experiments for Example 1 is $\gamma(\epsilon) = \varepsilon/20$.

Example 2. In contrast to Example 1, in this example an analytical solution for $u(x,t)$ is not explicitly available. Consider a non-smooth function $b(t)$ with compact support given by

$$b(t) = \begin{cases} 3\left(\frac{1}{2} - \left|t - \frac{2\pi}{5}\right|\right), & \text{if } \left|t - \frac{2\pi}{5}\right| \le \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.100}$$

Let $u(0,t) = b(t)$. Let $L = 5$ and consider the following well-posed linear problems:

$$\begin{cases} v_t - v_{xx} &= 0, & (x,t) \in (0,5) \times [0,2\pi], \\ v(x,0) &= 0, & x \in (0,5), \\ v(0,t) &= b(t), & t \in [0,2\pi], \\ v_x(5,t) &= 0, & t \in [0,2\pi]. \end{cases} \quad \text{and} \quad \begin{cases} w_t - w_{xx} &= \sin(v(x,t)), & (x,t) \in (0,5) \times [0,2\pi], \\ w(x,0) &= 0, & x \in (0,5), \\ w(0,t) &= 0, & t \in [0,2\pi], \\ w_x(5,t) &= 0, & t \in [0,2\pi], \end{cases} \tag{4.101}$$

Since we cannot obtain the exact solutions for v and w, we approximate them numerically using the second-order Crank-Nicolson finite difference method (FDM), [24]. Then, it is obvious that $\tilde{u}(x,t) = w(x,t) + v(x,t)$ satisfies the following nonlinear problem:

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} &= \sin(\tilde{u} - w(x,t)), & (x,t) \in (0,5) \times [0,2\pi], \\ \tilde{u}(x,0) &= 0, & x \in (0,5), \\ \tilde{u}(0,t) &= b(t), & t \in [0,2\pi], \\ \tilde{u}_x(5,t) &= 0, & t \in [0,2\pi]. \end{cases} \tag{4.102}$$

Let us define the Cauchy data such as $g(t) = \tilde{u}(1,t)$ and $h(t) = \tilde{u}_x(1,t)$ and consider solving (1.1) for a function $u(x,t)$ satisfying

$$\begin{cases} u_t - u_{xx} &= \sin(u - w(x,t)), & (x,t) \in (0,1) \times [0,2\pi], \\ u(1,t) &= g(t), & t \in [0,2\pi], \\ u_x(1,t) &= h(t), & t \in [0,2\pi]. \end{cases} \tag{4.103}$$

The nonlinearity in the first equation in (4.103) is characteristic to a sine-Gordon equation.

The graphs of the function (4.100) and the Cauchy data (g,h) obtained by solving the problem (4.102) are shown in Figure 2(b).

In Example 2, in order to obtain the "exact" solution and its Cauchy data, we have solved the two well-posed mixed direct problems in (4.101) with the mesh 1001×1001 using the FDM. Afterwards, the obtained results were interpolated to the computation domain $(x,t) \in [0,1] \times [0,2\pi]$ by adopting the subroutine `RGSF3P`, [2], where we employed two mesh resolutions: $K \times M = 100 \times 80$ and 100×160. Similarly to Example 1, the fixed-point iterative process was terminated after 7 iterations (starting from the zero trivial initial guess) with the error tolerance 10^{-9}. Again, the two regularization methods produced numerical solutions with the same order of accuracy and therefore, for brevity and clarity, only the numerical results for $u^{\varepsilon}(0,t)$ are illustrated in the next figure.

Figure 3 illustrates the convergence of the regularized solution $u^{\varepsilon}(0,t)$ to the exact solution (4.100), as $\gamma(\varepsilon)$ tends to zero (from the left to the right of the figure), for a fixed mesh size $K \times M = 100 \times 80$. Here the Cauchy data $(g_m^{\varepsilon}, h_m^{\varepsilon})$ in equation (4.93) are disturbed by $\varepsilon = 2 \times 10^{-2}$, 2×10^{-3} and 0 (from the top to the bottom of the figure). From Figure 3 it can be seen, as expected, that better data quality yields a more accurate and stable solution. The proper choice of regularization parameter $\gamma(\varepsilon)$ for this example is $\gamma(\varepsilon) \approx 5\varepsilon$.

Clearly in Figure 3, there are some wiggles occuring near $t = 0$ and $t = 2\pi$, which degrade the accuracy of the numerical solutions near these endpoints. It is reasonable to believe that the issue is relevant to the periodicity of the input data, which in this example is violated; a comprehensive survey of this matter can be found in [5], Chapter 6. However, the error estimates of the regularized solutions in an interior of the interval $[0,2\pi]$ are actually much better than the courterparts evaluated fully in $[0,2\pi]$ including the endpoints, and moreover, they improve with either increasing M from $M_1 = 80$ to $M_2 = 160$ (for $\varepsilon = 2 \times 10^{-3}$ and $\varepsilon = 0$) or, decreasing ε from 2×10^{-2}

to zero. Indeed, in Table 1, we provide numerical evidence to support the latter assertion. Therein, two kinds of error estimates $(\dot{\Lambda}_{1,p}, \ddot{\Lambda}_{1,p})$ and $(\dot{\Lambda}_{2,p}, \ddot{\Lambda}_{2,p})$ are defined in form of the root mean square errors, as follows:

$$\dot{\Lambda}_{1,p} = \sqrt{\frac{1}{M_p}\sum_{m=0}^{M_p-1} |u^\varepsilon(0,t_m) - b(t_m)|^2}, \qquad \dot{\Lambda}_{2,p} = \sqrt{\frac{1}{M_p-9}\sum_{m=5}^{M_p-5} |u^\varepsilon(0,t_m) - b(t_m)|^2},$$
$$\ddot{\Lambda}_{1,p} = \sqrt{\frac{1}{M_p}\sum_{m=0}^{M_p-1} |U^\varepsilon(0,t_m) - b(t_m)|^2}, \qquad \ddot{\Lambda}_{2,p} = \sqrt{\frac{1}{M_p-9}\sum_{m=5}^{M_p-5} |U^\varepsilon(0,t_m) - b(t_m)|^2}, \tag{4.104}$$

for $p = 1, 2$, where $(\dot{\Lambda}_{1,p}, \ddot{\Lambda}_{1,p})$ plays the role of the error estimate on the full interval $[0, 2\pi]$, while $(\dot{\Lambda}_{2,p}, \ddot{\Lambda}_{2,p})$ stands for the error estimate well-inside the interior of the interval $[0, 2\pi]$.

$\gamma(\varepsilon) = 10^{-2}$	$M_1 = 80$				$M_2 = 160$			
ε	$\dot{\Lambda}_{1,1}$	$\ddot{\Lambda}_{1,1}$	$\dot{\Lambda}_{2,1}$	$\ddot{\Lambda}_{2,1}$	$\dot{\Lambda}_{1,2}$	$\ddot{\Lambda}_{1,2}$	$\dot{\Lambda}_{2,2}$	$\ddot{\Lambda}_{2,2}$
2×10^{-2}	1.061E-1	1.060E-1	8.226E-2	8.219E-2	2.088E-1	2.087E-1	1.491E-1	1.490E-1
2×10^{-3}	8.937E-2	8.925E-2	3.533E-2	3.537E-2	1.384E-1	1.383E-1	1.837E-2	1.845E-2
0	8.831E-2	8.819E-2	3.453E-2	3.457E-2	1.393E-1	1.392E-1	1.025E-2	1.041E-2

Table 1: Example 2, mean square errors $(\dot{\Lambda}_{1,p}, \ddot{\Lambda}_{1,p})$ and $(\dot{\Lambda}_{2,p}, \ddot{\Lambda}_{2,p})$ are defined in equation (4.104) corresponding to the regularized solution $(u^\varepsilon, U^\varepsilon)$, and $p = 1, 2$ means M_p. Here the regularization parameter $\gamma(\varepsilon) = 10^{-2}$ in all cases.

5. Conclusions

This study has achieved a major extension over the much more investigated linear sideways heat equation. The semilinear sideways heat equation, governing reaction-diffusion applications, has been solved using two new regularization methods based on (2.14) and (3.39) for the resulting nonlinear integral equation (2.10). Convergence and stability estimates, as the noise level tends to zero, have been formulated and proved. Numerical examples support the theoretical findings of the paper. Further work will consider extending the current study from Lipschitz heat sources to locally Lipschitz ones in order to allow for an even wider range of physical applications related, for example, to combustion and radiative heat transfer. Of additional interest would be to extend the inverse analysis to the sideways heat equation in which the nonlinear heat source F in (1.1) also depends on the gradient u_x.

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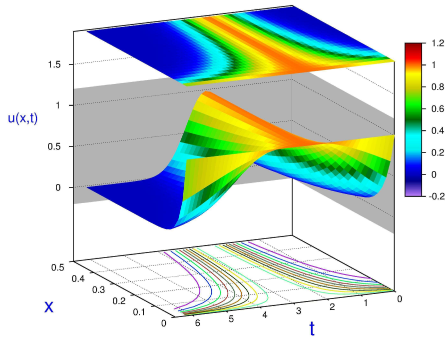
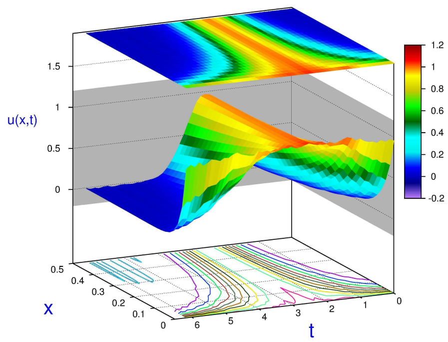
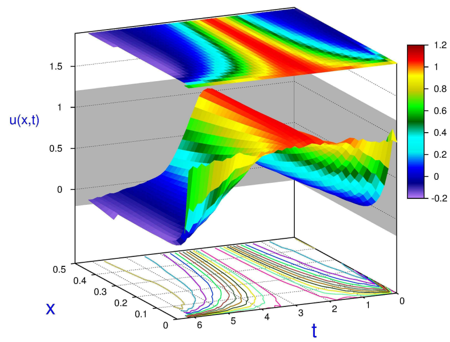


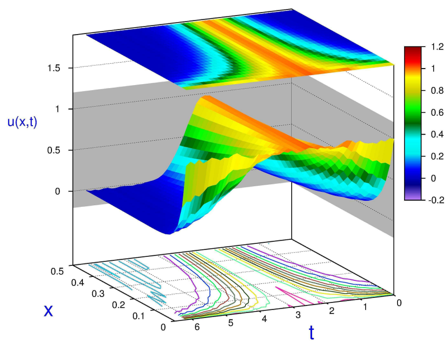
(a) Exact solution



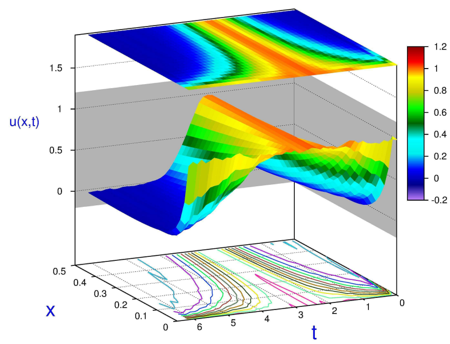
(b) $u^\varepsilon(x,t)$, $\gamma(\varepsilon) = 10^{-2}$



(c) $U^\varepsilon(x,t)$, $\gamma(\varepsilon) = 10^{-2}$

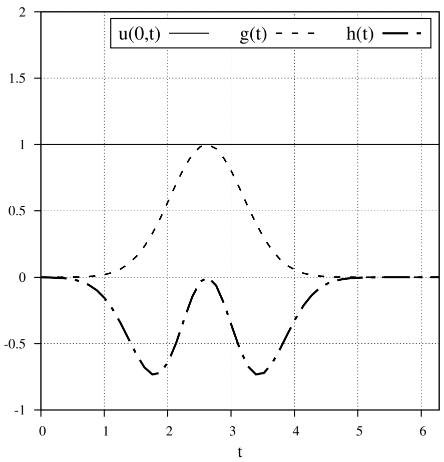


(d) $u^\varepsilon(x,t)$, $\gamma(\varepsilon) = 10^{-3}$

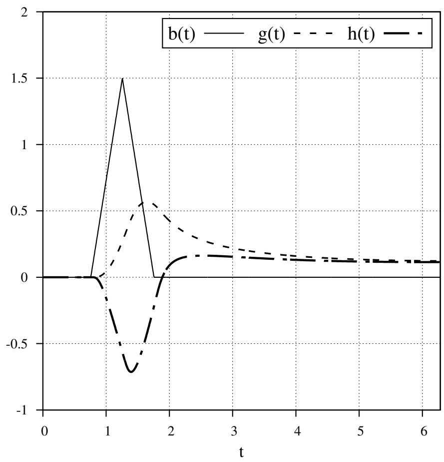


(e) $U^\varepsilon(x,t)$, $\gamma(\varepsilon) = 10^{-3}$

Figure 1: Example 1, graphs of the exact solution (4.97) and the numerical solutions $u^\varepsilon(x,t)$ and $U^\varepsilon(x,t)$ solving (2.14) and (3.39), respectively, obtained from noisy ($\varepsilon = 2 \times 10^{-2}$) Cauchy data (4.93).

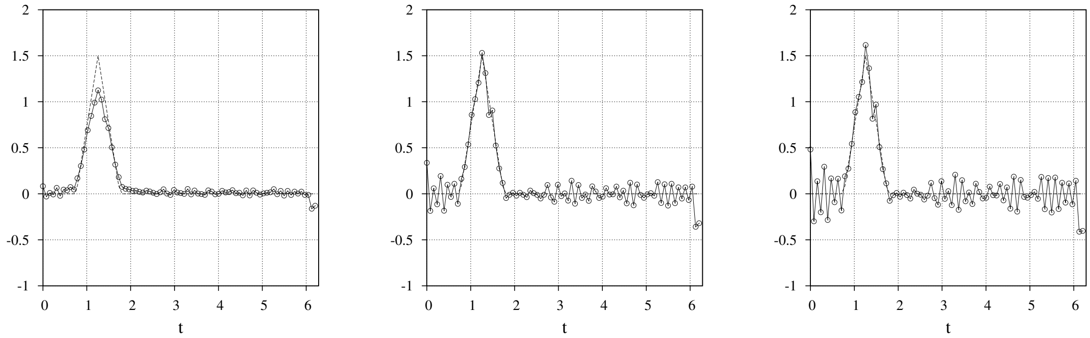


(a) Example 1

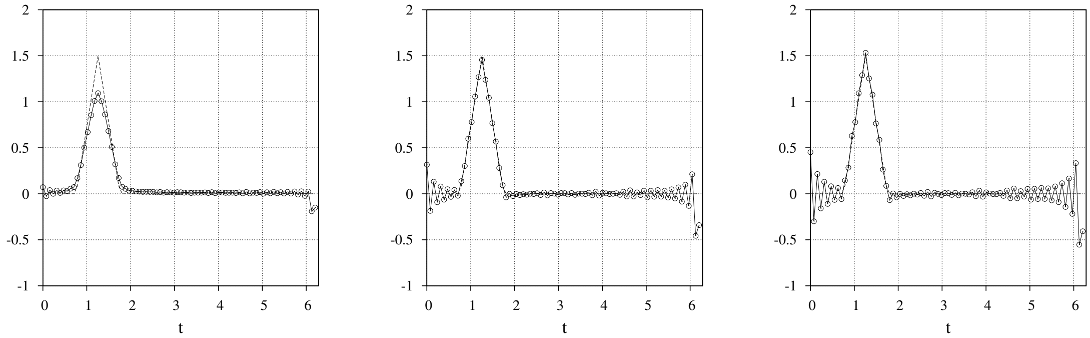


(b) Example 2

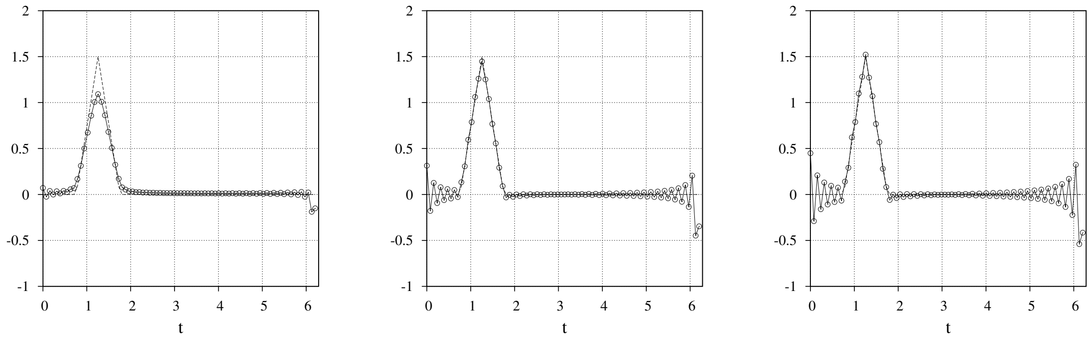
Figure 2: Graphs of exact solution $u(0,\cdot)$ and exact Cauchy data $(g, h) = (u(L,\cdot), u_x(L,\cdot))$ for Examples 1 and 2.



(a) $\varepsilon = 2 \times 10^{-2}$, $\gamma(\varepsilon) = 10^{-1}$. (b) $\varepsilon = 2 \times 10^{-2}$, $\gamma(\varepsilon) = 10^{-2}$. (c) $\varepsilon = 2 \times 10^{-2}$, $\gamma(\varepsilon) = 10^{-3}$.



(d) $\varepsilon = 2 \times 10^{-3}$, $\gamma(\varepsilon) = 10^{-1}$. (e) $\varepsilon = 2 \times 10^{-3}$, $\gamma(\varepsilon) = 10^{-2}$. (f) $\varepsilon = 2 \times 10^{-3}$, $\gamma(\varepsilon) = 10^{-3}$.



(g) $\varepsilon = 0$, $\gamma(\varepsilon) = 10^{-1}$. (h) $\varepsilon = 0$, $\gamma(\varepsilon) = 10^{-2}$. (i) $\varepsilon = 0$, $\gamma(\varepsilon) = 10^{-3}$.

Figure 3: Example 2, convergence tendency of the regularized solution $u^{\varepsilon}(0,t)$ $(-\circ-)$ to the exact solution (4.100) (——), as $\gamma(\varepsilon)$ decreases (from left to right). Here the Cauchy data $(g^{\varepsilon}, h^{\varepsilon})$ in (4.93) are disturbed by $\varepsilon = 2 \times 10^{-2}$, 2×10^{-3} and 0 (from top to bottom).