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Discussion of “Birnbaum-Saunders distribution: A review of models, analysis and applications”

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1 Introduction

The fifty years since the birth of the Birnbaum-Saunders distribution can be broken-down into three distinct periods. The first 30 years (1969–1999) showed slow development and few applications; see, for example, Birnbaum and Saunders (1969), Rieck and Nedelman (1991), Dupuis and Mills (1998) and Owen and Padgett (1999). The next ten years (2000–2010), in contrast, represented rapid methodological development in terms of estimation, diagnostics and computational aspects, as well as generalisations and novel modeling based on arguments of cumulative effects; see, for example, Volodin and Dzhungurova (2000), Galea et al. (2004) and Vilca et al. (2010). Work during the most recent period (2011 to the present) is characterized by the breaking of the previously ever-present link with lifetime modelling and the resultant move to new areas of application such as: agriculture, business, econometrics, environment, industry, management, medicine, neurology and seismology; see the book by Leiva (2016) for details of these developments and applications of the univariate Birnbaum-Saunders distribution. Leiva (2016), however, does not emphasise multivariate versions of the Birnbaum-Saunders distribution.

In their paper, Balakrishnan and Kundu (2018) conducted a comprehensive review of the Birnbaum-Saunders distribution, which includes mathematical and statistical properties, physical interpretations, analysis of the distribution shape, relations with other distributions, regression modeling and generalisations for the univariate case, as well as extensions to the multivariate and matrix-variate cases. Our discussion updates and complements the references on the topic and we also provide more details on multivariate Birnbaum-Saunders models and discuss a novel application in economics.

Other recent articles include a review of multivariate Birnbaum-Saunders distributions and applications by Aykroyd et al. (2018). Work on multivariate Birnbaum-Saunders models and their applications was presented in Marchant et al. (2016a,b, 2018) and applications of multivariate Birnbaum-Saunders distributions in spatial modelling appear in Garcia-Papani et al. (2017, 2018a,b). Also, multivariate cumulative damage models, and their relationship with time of occurrence in the context of multicomponent systems, were recently derived by Fierro et al. (2018).

In our discussion, we provide further aspects of the multivariate Birnbaum-Saunders distribution reviewed by Balakrishnan and Kundu (2018), considering its properties and features, modeling, diagnostics, as well as new opportunities and future applications.

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2 Multivariate model definitions

2.1 The multivariate Birnbaum-Saunders distribution

The random vector $\underline{T} = (T_1, \ldots, T_m)^\top \in \mathbb{R}_+^m$ follows a multivariate Birnbaum-Saunders distribution with parameters $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)^\top \in \mathbb{R}_+^m$, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)^\top \in \mathbb{R}_+^m$ and covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{m\times m}$ if $T_j = \lambda_j(\alpha_j V_j/2 + (((\alpha_j V_j/2)^2 + 1)^{1/2})^2$, for $j = 1, \ldots, m$, and $\underline{V} = (V_1, \ldots, V_m)^\top \in \mathbb{R}^m \sim \mathrm{N}_m(\underline{0}_{m\times 1}, \mathbf{\Gamma})$, with $\underline{0}_{m\times 1}$ being an $m \times 1$ vector of zeros and $\mathbf{\Gamma} \in \mathbb{R}^{m\times m}$ being a correlation matrix. Furthermore, since for the Birnbaum-Saunders case $\sigma_{kk} = 1$, for all $k = 1, \ldots, m$, then

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{12} & 1 & \cdots & \rho_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1m} & \rho_{2m} & \cdots & 1 \end{pmatrix} = \mathbf{\Gamma}. \tag{2.1}$$

We denote the m-variate Birnbaum-Saunders distribution by $\underline{T} \sim \mathrm{BS}_m(\underline{\alpha}, \underline{\lambda}, \mathbf{\Gamma})$. Thus, the cumulative distribution and probability density functions (CDF and PDF) of $\underline{T}$ are given by $F_{\underline{T}}(\underline{t}; \underline{\alpha}, \underline{\lambda}, \mathbf{\Gamma}) = \Phi_m(\underline{A}; \mathbf{\Gamma})$ and $f_{\underline{T}}(\underline{t}; \underline{\alpha}, \underline{\lambda}, \mathbf{\Gamma}) = \phi_m(\underline{A}; \mathbf{\Gamma})\underline{a}$, respectively, for $\underline{t} = (t_1, \ldots, t_m)^\top \in \mathbb{R}_+^m$, where ϕ_m and Φ_m are the m-variate standard normal PDF and CDF, $\underline{A} = (A_1, \ldots, A_m)^\top$ and $\underline{a} = \prod_{j=1}^m a_j$, with $A_j = (1/\alpha_j)((t_j/\lambda_j)^{1/2} - (\lambda/t)^{1/2})$ and $a_j = (1/(2\alpha_j\lambda_j))\,((\lambda_j/t_j)^{1/2} + (\lambda_j/t_j)^{3/2})$, for $j = 1, \ldots, m$. Let $\underline{T} = (T_1, \ldots, T_m)^\top \sim \mathrm{BS}_m(\underline{\alpha}, \underline{\lambda}, \mathbf{\Gamma})$. Then, $\underline{Y} = (Y_1, \ldots, Y_m)^\top = (\log(T_1), \ldots, \log(T_m))^\top$ follows a multivariate log-Birnbaum-Saunders distribution with shape parameters $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)^\top$, mean vector $\underline{\mu} = \mathrm{E}[\underline{Y}] = (\mathrm{E}[Y_1], \ldots, \mathrm{E}[Y_m])^\top = (\log(\lambda_1), \ldots, \log(\lambda_m))^\top \in \mathbb{R}^m$, and correlation matrix $\mathbf{\Gamma} \in \mathbb{R}^{m\times m}$ given in (2.1). This is denoted by $\underline{Y} \sim \log\text{-}\mathrm{BS}_m(\underline{\alpha}, \underline{\mu}, \mathbf{\Gamma})$. The CDF and PDF of $\underline{Y}$ are defined as $F_{\underline{Y}}(\underline{y}; \underline{\alpha}, \underline{\mu}, \mathbf{\Gamma}) = \Phi_m(\underline{B}; \mathbf{\Gamma})$ and $f_{\underline{Y}}(\underline{y}; \underline{\alpha}, \underline{\mu}, \mathbf{\Gamma}) = \phi_m(\underline{B}; \mathbf{\Gamma})\underline{b}$, for $\underline{y} \in \mathbb{R}^m$, where $\underline{b} = \prod_{j=1}^m b_j$ and $\underline{B} = (B_1, \ldots, B_m)^\top$, with $b_j = (1/\alpha_j)\cosh((y_j - \mu_j)/2)$ and $B_j = (2/\alpha_j)\sinh((y_j - \mu_j)/2)$, for $j = 1, \ldots, m$.

2.2 The multivariate Birnbaum-Saunders regression model

The Birnbaum-Saunders log-linear regression model is formulated as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{E}, \tag{2.2}$$

where $\boldsymbol{Y} = (Y_{ij}) \in \mathbb{R}^{n\times m}$ is the log-response matrix and $\boldsymbol{X} = (x_{is}) \in \mathbb{R}^{n\times p}$ is the model design matrix of rank p, which contains values of p covariates. Here, $\boldsymbol{X}, \boldsymbol{Y}$ are linked by a coefficient matrix $\boldsymbol{\beta} = (\beta_{sj}) = (\underline{\beta}_1, \ldots, \underline{\beta}_m) \in \mathbb{R}^{p\times m}$, and $\boldsymbol{E} = (\varepsilon_{ij}) \in \mathbb{R}^{n\times m}$ is the error matrix. Furthermore, let $\underline{Y}_i^\top$, $\underline{x}_i^\top$ and $\underline{\varepsilon}_i^\top$ be the i-th rows of $\boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{E}$, respectively. Thus, the model defined in (2.2) can be rewritten as

$$\underline{Y}_i = \underline{\mu}_i + \underline{\varepsilon}_i = \boldsymbol{\beta}^\top \underline{x}_i + \underline{\varepsilon}_i, \quad i = 1, \ldots, n, \tag{2.3}$$

where $\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_n$ are independently and identically $\log\text{-}\mathrm{BS}_m(\alpha\underline{1}_{m\times 1}, \underline{0}_{m\times 1}, \mathbf{\Gamma})$ distributed, with $\underline{1}_{m\times 1}$ being an $m \times 1$ vector of ones. Consider a sample from a multivariate log-Birnbaum-Saunders distribution, $\boldsymbol{Y} = (\underline{Y}_1, \ldots, \underline{Y}_n)^\top$, with $\mathrm{E}[\underline{Y}_i] = \boldsymbol{\beta}^\top \underline{x}_i$ and observations $\boldsymbol{y} = (\underline{y}_1, \ldots, \underline{y}_n)^\top$. Then, the log-likelihood function for $\underline{\theta} = (\alpha, \mathrm{vec}(\boldsymbol{\beta})^\top, \mathrm{svec}(\mathbf{\Gamma})^\top)^\top$, with 'vec' representing the vectorization of a general matrix and 'svec' denotes vectorization of a symmetric matrix, is given by

$$\ell(\underline{\theta}) = -m\log(2) - \frac{m}{2}\log(2\pi) - \frac{1}{2}\log(|\mathbf{\Gamma}|) - \frac{1}{2}\underline{\phi}_i^\top \mathbf{\Gamma}^{-1}\underline{\phi}_i + \sum_{j=1}^{m}\log(\xi_{ij}), \tag{2.4}$$

where $\underline{\phi}_i = (\phi_{i1}, \ldots, \phi_{im})^\top$, with $\phi_{ij} = (2/\alpha)\sinh((y_{ij} - \mu_{ij})/2)$, and $\xi_{ij} = (2/\alpha)\cosh((y_{ij} - \mu_{ij})/2)$ with $\mu_{ij} = \underline{\beta}_j^\top \underline{x_i}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The ML estimate $\widehat{\underline{\theta}}$ of $\underline{\theta}$ is defined as the values maximizing the log-likelihood function defined in (2.4). When this corresponds to a stationary point, it can be obtained from the solution of a homogeneous system of equations given by $\partial\ell(\underline{\theta})/\partial\alpha = 0$, $\partial\ell(\underline{\theta})/\partial\underline{\beta}^* = \underline{0}_{(pm)\times 1}$ and $\partial\ell(\underline{\theta})/\partial\underline{\Gamma}^* = \underline{0}_{(m(m-1)/2)\times 1}$, where $\underline{\beta}^* = \mathrm{vec}(\boldsymbol{\beta})^\top$ and $\underline{\Gamma}^* = \mathrm{svec}(\mathbf{\Gamma})^\top$; see the details in Marchant et al. (2016a). As this system cannot be solved analytically, the ML estimate $\widehat{\underline{\theta}}$ of $\underline{\theta}$ may be found using the Broyden-Fletcher-Goldfarb-Shanno procedure; see Lange (2001).

As an alternative form of modeling, we can use a parameterization of the Birnbaum-Saunders distribution based on the mean to introduce new multivariate regression models, as proposed in Leiva et al. (2014) and Santos-Neto et al. (2016), of the type $T \sim \mathrm{BS}(\alpha, \mu)$, where $\mathrm{E}[T] = \mu$. This allows the development of generalized linear models for the multivariate Birnbaum-Saunders distribution and the corresponding derivation of influence methods, as well as the incorporation of temporal and spatial components for the modeling within a standard framework.

Another alternative form of modeling can be established by considering quantile regression; see Furno and Vistocco (2018). In the model presented in (2.3), $\mathrm{Var}[\varepsilon_i]$ is constant and $\mathrm{Cov}[\varepsilon_l, \varepsilon_k] = 0$, for $l \neq k$ (that is, the errors have equal variance and are uncorrelated). This implies that a regression model describes the conditional mean $\mathrm{E}[T|\underline{X} = \underline{x}] = \underline{x}_i^\top\underline{\beta}$. In this perspective, one can even postulate a general expression as $\varrho_i = h(\underline{x}_i^\top\underline{\beta})$, where h is an invertible function, such as in generalized linear modelling. Then, considering the conditional Birnbaum-Saunders distribution, we may establish the formulation

$$T_i|\underline{X}_i = \underline{x}_i \sim f(t; \alpha, h(\underline{x}_i^\top\underline{\beta})), \quad i = 1, \ldots, n, \tag{2.5}$$

where f is now the conditional PDF. Thus, we can model the median of the conditional distribution presented in (2.5) instead of the mean. Hence, it is possible to use quantile regression models, which offer a mechanism to estimate and predict the median response, as well as other quantiles. This class of regression models is based on the quantile function given by $Q(\tau; \underline{\theta}) = \inf\{t\colon F(t; \underline{\theta}) \geq \tau\}$, where F is the Birnbaum-Saunders CDF, $\underline{\theta}^\top = (\alpha, \underline{\beta})$ and $0 < \tau < 1$; see Sánchez et al. (2018) for Birnbaum-Saunders quantile regression models.

2.3 Diagnostics

Outlying and influential observations are often present in data. Global influence in multivariate Birnbaum-Saunders regression models can be assessed by the Mahalanobis distance; see Marchant et al. (2016b). Local influence methods also play a central role for diagnostics in statistical modelling. These methods allow us to evaluate the effect of perturbations on the estimates of parameters and then to detect potentially influential cases in multivariate Birnbaum-Saunders regression models; see Marchant et al. (2016b).

3 An application in economics

Next, we analyze a real-world data set from economics presented in Leiva et al. (2016), which corresponds to the demand for textile products. We consider an inverse regression with the following responses: (i) deflated price index of clothing (T_1) and (ii) real per capita income (T_2); and as covariate we include the per capita textile consumption per year (X). Figure 1(a) shows scatter-plots for log-responses Y_1, Y_2 and the covariate X. From this figure, we detect adequate levels of correlation between the responses and the covariate, justifying the use of a bivariate linear regression model, as well as a weak correlation between (X, Y_1) and a strong correlation between (X, Y_2).

Consider the bivariate Birnbaum-Saunders regression model, to describe (Y_1, Y_2) as a function of X, given by

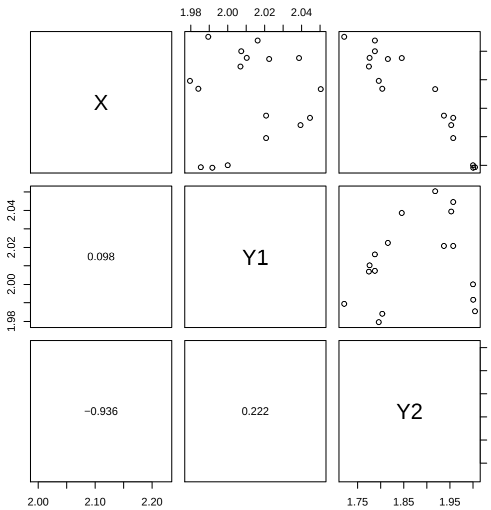
$$\underline{Y}_i = \boldsymbol{\beta}^\top \underline{x}_i + \underline{\varepsilon}_i, \quad i = 1, \ldots, n,$$

where $\underline{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2})^\top \sim$ log-BS$_2(\alpha \underline{1}_{2\times1}, \underline{0}_{2\times1}, \boldsymbol{\Gamma})$. ML estimates for the model parameters (based on a sample with $n = 17$) are reported in Table 1, with corresponding estimated standard errors and p-values. Note that: (i) ρ is statistically significant at the 5% level, confirming the conjecture from the exploratory analysis; and (ii) the regression coefficients β_{11}, β_{12} and β_{22} should be included in the model, whereas the coefficient β_{21} can be discarded in the prediction of Y_1, again in agreement with the exploratory analysis.

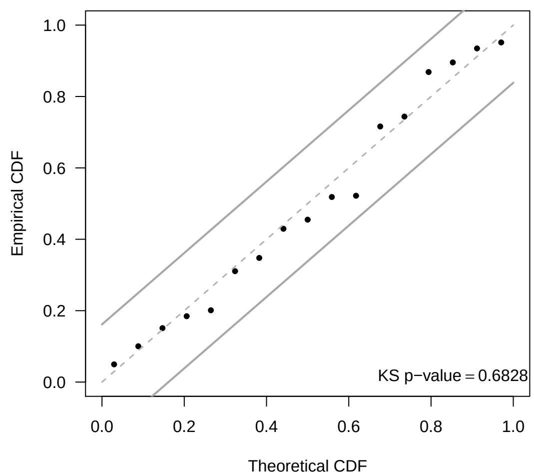
Table 1: ML estimates of the indicated parameters with corresponding estimated standard errors and p-values using textile products data.

	Parameter					
	α	β_{11}	β_{12}	β_{21}	β_{22}	ρ
ML estimate	0.022851	1.953439	4.285798	0.027697	-1.137174	0.895837
Standard error	0.001973	0.153330	0.072205	0.153323	0.072201	0.016190
p-value	< 0.0001	< 0.0001	< 0.0001	0.8566	< 0.0001	< 0.0001

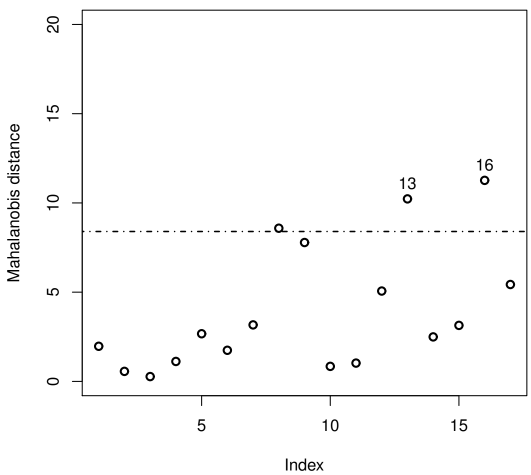
Validation of the model is based on the Mahalanobis distance, after being transformed to normality using the Wilson–Hilferty approximation; see Marchant et al. (2016a). Probability-probability plots and acceptance bands associated with the Kolmogorov–Smirnov (KS) statistic for testing normality are constructed; for details, see Marchant et al. (2016a). From Figure 1(b), note that the Birnbaum-Saunders model provides a good fit, which agrees with the p-value of 0.6828 from the associated KS test. Figure 1(c) displays the Mahalanobis distance index plot for the bivariate Birnbaum-Saunders regression model, with cases #13 and #16 being identified as potential multivariate outliers. Figure 1(d) shows the index plots of total local influence (C_i) (see Marchant et al., 2016b) under the case-weight perturbation for $\widehat{\boldsymbol{\beta}}$. From this figure, note that case #16 has a large influence on the bivariate Birnbaum-Saunders model. This case coincides with that detected by the Mahalanobis distance. In addition, case #17 seems to have a large influence on the model under case-weight for $\widehat{\boldsymbol{\beta}}$. However, when these potentially influential cases are removed, no inferential changes are produced. Therefore, cases #13, #16 and #17 do not need to be removed, and the prediction model can be estimated considering them.



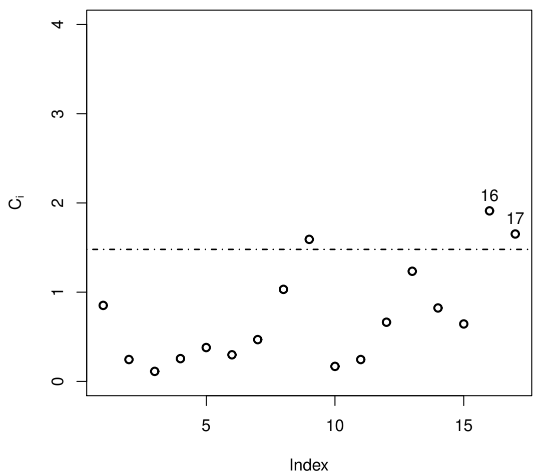
(a) Data scatter-plots



(b) Probability-probability plot



(c) Mahalanobis plot for Birnbaum-Saunders model



(d) Index plot of total local influence for $\widehat{\boldsymbol{\beta}}$

Figure 1: Indicated plot for the textile products data.

4 Conclusions

New work on the multivariate Birnbaum-Saunders distribution was developed in 2016-2018, hence this is a new relatively topic and so here we have presented more details of the multivariate Birnbaum-Saunders distribution, compared to Balakrishnan and Kundu (2018), mainly focussing on modelling, diagnostics and applications. We have given a new application of the multivariate Birnbaum-Saunders regression model in economics, an area in which data following asymmetric distributions are often present. With respect to future research on the multivariate Birnbaum-Saunders distribution, some new ideas on modeling have been presented throughout our discussion which can be exploited in further investigations of the multivariate Birnbaum-Saunders distribution.

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References

Aykroyd, R. G., Leiva, V., and Marchant, C. (2018). Multivariate birnbaum-saunders distributions: Modelling and applications. *Risks*, 6-1(article 21):1–25.

Balakrishnan, N. and Kundu, D. (2018). Birnbaum-saunders distribution: A review of models, analysis and applications. *Applied Stochastic Models in Business and Industry*, page in press.

Birnbaum, Z. W. and Saunders, S. C. (1969). A new family of life distributions. *Journal of Applied Probability*, 6:319–327.

Dupuis, D. J. and Mills, J. E. (1998). Robust estimation of the Birnbaum-Saunders distribution. *IEEE Transactions on Reliability*, 1:88–95.

Fierro, R., Leiva, V., and Maidala, J. P. (2018). Cumulative damage and times of occurrence for a multicomponent system: A discrete time approach. *Journal of Multivariate Analysis*, page in press.

Furno, M. and Vistocco, D. (2018). *Quantile Regression: Estimation and Simulation*. Wiley, New York, US.

Galea, M., Leiva, V., and Paula, G. A. (2004). Influence diagnostics in log-Birnbaum-Saunders regression models. *Journal of Applied Statistics*, 31:1049–1064.

Garcia-Papani, F., Leiva, V., Ruggeri, F., and Uribe-Opazo, M. A. (2018a). Kriging with external drift in a Birnbaum-Saunders geostatistical model. *Stochastic Environmental Research and Risk Assessment*, 32:1517–1530.

Garcia-Papani, F., Leiva, V., Uribe-Opazo, M. A., and Aykroyd, R. G. (2018b). Birnbaum-Saunders spatial regression models: Diagnostics and application to chemical data. *Chemometrics and Intelligent Laboratory Systems*, 177:114–128.

Garcia-Papani, F., Uribe-Opazo, M. A., Leiva, V., and Aykroyd, R. G. (2017). Birnbaum-Saunders spatial modelling and diagnostics applied to agricultural engineering data. *Stochastic Environmental Research and Risk Assessment*, 31:105–124.

Lange, K. (2001). *Numerical Analysis for Statisticians*. Springer, New York, US.

Leiva, V. (2016). *The Birnbaum-Saunders Distribution*. Academic Press, New York, US.

Leiva, V., Liu, S., Shi, L., and Cysneiros, F. J. A. (2016). Diagnostics in elliptical regression models with stochastic restrictions applied to econometrics. *Journal of Applied Statistics*, 43:627–642.

Leiva, V., Santos-Neto, M., Cysneiros, F. J. A., and Barros, M. (2014). Birnbaum-Saunders statistical modelling: A new approach. *Statistical Modelling*, 14:21–48.

Marchant, C., Leiva, V., and Cysneiros, F. J. A. (2016a). A multivariate log-linear model for Birnbaum-Saunders distributions. *IEEE Transactions on Reliability*, 65:816–827.

Marchant, C., Leiva, V., Cysneiros, F. J. A., and Liu, S. (2018). Robust multivariate control charts based on Birnbaum-Saunders distributions. *Journal of Statistical Computation and Simulation*, 88:182–202.

Marchant, C., Leiva, V., Cysneiros, F. J. A., and Vivanco, J. F. (2016b). Diagnostics in multivariate generalized Birnbaum-Saunders regression models. *Journal of Applied Statistics*, 43:2829–2849.

Owen, W. and Padgett, W. (1999). Accelerated test models for system strength based on Birnbaum-Saunders distribution. *Lifetime Data Analysis*, 5:133–147.

Rieck, J. R. and Nedelman, J. R. (1991). A log-linear model for the Birnbaum-Saunders distribution. *Technometrics*, 3:51–60.

Sánchez, L., Leiva, V., and Galea, M. (2018). On quantile regression for the Birnbaum-Saunders distribution. *Under review*.

Santos-Neto, M., Cysneiros, F. J. A., Leiva, V., and Barros, M. (2016). Reparameterized Birnbaum-Saunders regression models with varying precision. *Electronic Journal of Statistics*, 10:2825–2855.

Vilca, F., Sanhueza, A., Leiva, V., and Christakos, G. (2010). An extended Birnbaum-Saunders model and its application in the study of environmental quality in Santiago, Chile. *Stochastic Environmental Research and Risk Assessment*, 24:771–782.

Volodin, I. and Dzhungurova, O. (2000). On limit distribution emerging in the generalized Birnbaum-Saunders model. *Journal of Mathematical Science*, 99:1348–1366.