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Quaternion-based $\mathscr{H}_\infty$ attitude tracking control of rigid bodies with time-varying delay in attitude measurements

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***Abstract*— The problem of attitude and angular velocity tracking in the presence of exogenous disturbances and where feedback measurements are subjected to unknown time-varying delays is addressed. Sufficient conditions which guarantee stability and disturbance attenuation performance in the $\mathscr{H}_\infty$ sense are provided. Results are presented in the form of LMIs, which allow the conditions to be simply and efficiently computed. Using a simple quaternion-based linear state feedback controller and a feedforward term to compensate the nonlinearities of the system dynamics, simulation results illustrate that the control law is able to effectively track desired trajectories and reject disturbances even in the presence of large time-varying delays.**

INTRODUCTION

Rigid body attitude control is a critical issue to a breadth of engineering applications such as satellite attitude synchronization, aircraft systems and rigid robotic manipulators [2], [4], [10], [19]. The subject has been actively studied for decades, and extensive literature is available [9], [10], [15], [17], [18]. Nevertheless, only recently results concerning time-delays in the attitude feedback have been derived.

Time-delays often occur in several applications wherein attitude control is key. Indeed, multiple sources might be responsible for introducing time-delays into the system, such as communication and processing delays which cause feedback delays, as well as actuation delays due to the actuator dynamics. For instance, in the context of satellites, gas jet propulsion systems where electrical and mechanical delays occur in the valve circuits or by deactivation of magnetometers in the presence of magnetotorques. The influence of time-delays ranges from performance deterioration—such as oscillatory motion [15]—to actual system instability [1]. However, despite the detrimental effects caused by time-delays, in the context of attitude control, results are still scarce. Indeed, inherent non-linearities pose serious challenges which make existing time-delay analysis techniques [5], [8], [12]–[14], [20] not directly applicable to the attitude problem.

In [1], a velocity-free controller is proposed to the attitude regulation problem considering the effect of a known constant time-delay in the feedback loop. The authors of [3] also address attitude regulation, but the proposed controller uses both attitude and angular velocity measurements, and time-delay is unknown but constant and upper-bounded. Recently, a solution to the same problem from [3]—attitude regulation with unknown constant upper-bounded delays—was derived in [11] using a conventional proportional-derivative controller where delays affect both attitude and angular velocity measurements. Notwithstanding, exogenous disturbances and performance specifications were not taken into account. Such issues were addressed by the authors of [16] in the context of attitude regulation subjected to unknown time-varying delays wherein the presented Linear Matrix Inequality (LMI) conditions guarantee robust stability and disturbance rejection in the $\mathscr{H}_\infty$ sense. However, rigid body dynamics are neglected, that is, the conditions are only valid for kinematic control.

It should be noted that none of the previous works address the more general problem of attitude tracking. Indeed, to the best of author's knowledge, the work from [2] is the only to address the tracking problem in the presence of delay. The authors therein assume constant and known delays solely on the attitude feedback loop, and an unknown but constant inertia matrix. The inertia matrix uncertainty is dealt with by estimating the inertia matrix's independent elements in the regressor form. Discrepancies between the actual system and the error state are dealt as a disturbance artificially induced by the controller. However, the design is specific to this artificial disturbance and cannot be easily extended for limiting the impact of general exogenous disturbances in tracking performance.

In this sense, it is still an open problem an analysis suitable for real-world applications of attitude tracking where a rigid body must have a satisfactory performance in following a desired trajectory in spite of presence of time-varying delays and exogenous disturbances.

In this work, the problem of robust tracking attitude and angular velocity where attitude feedback measurements are liable to time-varying unknown delays is considered. To describe the attitude kinematics, the unit quaternion representation is adopted, which is nonminimal and singularity-free. The main result stems from a careful choice of Lyapunov-Krasovskii functional and exploiting particular characteristics from the unit quaternion manifold. The proposed criterion consists of sufficient conditions which, in addition to stability, provide an upper bound for exogenous disturbance attenuation. Results are presented in the form of LMIs, which can be readily tested using efficient computational tools. Simulations illustrate the effectiveness of the proposed controller under severe conditions of delay and aggressive profiles of desired trajectories, even when initial conditions provoke large initial errors.

PROBLEM STATEMENT

The attitude kinematics of a rigid body can be represented using unit quaternions. The quaternion algebra $\mathbb{H}$ is a four dimensional associative algebra over $\mathbb{R}$. An element $q \in \mathbb{H}$ can be written as $q = \left[\eta \, \epsilon^T\right]^T$, where $\eta \in \mathbb{R}$ and $\epsilon = \left[\epsilon_1 \, \epsilon_2 \, \epsilon_3\right]^T \in$

$\mathbb{R}^3$ denote the scalar and vector parts of q, respectively. A quaternion is said to be pure if its scalar part is zero, i.e., $\eta=0$. In particular, unit quaternions belong to the 3-dimensional unit sphere embedded in $\mathbb{R}^4$

$$\mathcal{S}^3=\left\{q\in\mathbb{R}^4|\eta^2+\|\epsilon\|^2=1\right\} \quad (1)$$

and form, under quaternion multiplication, a Lie group—Spin(3). The identity element of the group is $\begin{bmatrix}1 & \mathbf{0}\end{bmatrix}$ and the norm equals usual Euclidean norm, $\|q\|=\sqrt{q^Tq}$.

The quaternion-based attitude kinematics expressed in the body frame to a coordinate in the inertial frame can be described as

$$\dot{q}(t)=\begin{bmatrix}\dot{\eta}(t)\\ \dot{\epsilon}(t)\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-\epsilon(t)^T\\ \eta(t)\mathbf{I}+\epsilon(t)^*\end{bmatrix}[\omega(t)+r(t)] \quad (2)$$

where q, ω and r denote a unit quaternion, a pure quaternion–which represents the angular velocity in the body frame–and a disturbance acting upon the system, respectively. The term ϵ^* represents a skew-symmetric matrix that satisfies $\epsilon^*\omega=\epsilon\times\omega$. Note that, for all $\epsilon\in\mathbb{R}^3$, $\|\epsilon^*\|\leq\|\epsilon\|$ holds, where $\|.\|$ is the induced matrix norm of .

Since $\|q(t)\|=1$, for all $t\geq 0$, (1) implies that

$$|\eta(t)|\leq 1, \|\epsilon(t)\|\leq 1, \forall t\geq 0. \quad (3)$$

Let $J>0$ denote the inertia matrix of the rigid body and u a vector-sum of external torques—the actual control input. The rigid body satisfies Euler's rotational dynamics

$$J\dot{\omega}(t)=-\omega(t)\times J\omega(t)+u(t). \quad (4)$$

Throughout this paper, it is assumed that J is constant and known and that angular velocity measurements are instantaneously available. Nevertheless, attitude measurements are assumed to be available after an unknown, time-varying delay $d:\mathbb{R}_+\to\mathbb{R}_+$ which admits the existence of constants τ and ν such that

$$0\leq\tau\leq d(t)\leq\nu, \forall t\geq 0.$$

In this context, this paper addresses the problem of attitude and angular velocity tracking, where the objective is to design u such that q and ω asymptotically track a desired bounded reference trajectory, given by some q_d and ω_d. This suggests the definition of an error quaternion which accounts for the discrepancy between the desired attitude q_d and the actual one, q

$$q_e=q_d^{-1}q. \quad (5)$$

From (5) it follows that

$$\dot{q}_e(t)=\begin{bmatrix}\dot{\eta}_e(t)\\ \dot{\epsilon}_e(t)\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-\epsilon_e(t)^T\\ \eta_e(t)\mathbf{I}+\epsilon_e(t)^*\end{bmatrix}[\omega_e(t)+r(t)], \quad (6)$$

where $\omega_e(t)=\omega(t)-\bar{\omega}_d(t)$, $\bar{\omega}_d=q_e^{-1}\omega_d q_e$. Then, $\omega_e(t)$ satisfies[1]

$$J\dot{\omega}_e=-\omega\times J\omega+J\left(\omega_e\times\bar{\omega}_d-\dot{\bar{\omega}}_d\right)+u. \quad (7)$$

Thus, if with an appropriate choice of u the system (6)-(7) is asymptotically stable, the tracking error converges to zero and, consequently, the original system follows the desired trajectory. Nevertheless, the presence of a disturbance $r(t)$ acting upon $q_e(t)$ motivates obtaining conditions which, in addition to stability of system (6)-(8), guarantee some degree of perturbation attenuation. For this purpose, disturbance attenuation is considered in the $\mathscr{H}_\infty$ sense.

Definition 1: Given a scalar $\gamma>0$, $\mathscr{H}_\infty$ disturbance rejection performance is achieved with $\mathscr{H}_\infty$ norm bound γ if the following conditions hold

1) The closed-loop system (6)-(8) is asymptotically stable when $r(t)\equiv 0$;
2) The disturbance $r(t)$ is attenuated below a desired level in the sense of $\mathscr{H}_\infty$ with index γ: under null initial conditions, if $r(t)\in L_2[0,+\infty)$, $\|\epsilon_e(t)\|\leq\gamma\|r(t)\|$ holds.

Since J, $\bar{\omega}_d, \dot{\bar{\omega}}_d$ are precisely known and ω, ω_e can be measured, the nonlinear dynamics of (7) can be cancelled out by introducing a feedforward term to the control law. In addition, attitude and angular velocity proportional terms take care of actual stabilization, yielding the following control law

$$u=\omega\times J\omega-J\left(\omega_e\times\bar{\omega}_d-\dot{\bar{\omega}}_d\right)-\kappa_1\epsilon_e(t-d(t))-\kappa_2\omega_e, \quad (8)$$

where $\kappa_1,\kappa_2\in\mathbb{R}$ are the constant proportional and derivative gains, respectively.

The upcoming analysis will make use of the indicator function $\chi:[\tau,\nu]\to\{0,1\}$, given by

$$\chi(s):=\begin{cases}1, & s\in[\tau,\mu]\\ 0, & s\in(\mu,\nu]\end{cases}, \mu=\frac{\nu+\tau}{2}. \quad (9)$$

Lemmas

The following lemmas will support the derivation of the paper's main article.

Lemma 1: Let $q\in\mathcal{S}^3$ be such that (6) holds. Then,

$$\|\dot{\epsilon}_e(t)\|^2\leq\frac{1}{4}\|\omega_e(t)+r(t)\|^2, \forall t\geq 0. \quad (10)$$

Proof: According to (6), from (3) it follows that

$$\begin{aligned}
\|\dot{\epsilon}_e(t)\|^2&=\dot{\epsilon}_e(t)^T\dot{\epsilon}_e(t)\\
&=\frac{1}{4}\left[(\eta_e(t)\mathbf{I}+\epsilon_e(t)^*)(\omega_e(t)+r(t))\right]^T\\
&\quad\times\left[(\eta_e(t)\mathbf{I}+\epsilon_e(t)^*)(\omega_e(t)+r(t))\right]\\
&=\frac{1}{4}\Big\{\eta_e(t)^2(\omega_e(t)+r(t))^T(\omega_e(t)+r(t))\\
&\quad+\left[\epsilon_e(t)^*(\omega_e(t)+r(t))\right]^T\left[\epsilon_e(t)^*(\omega_e(t)+r(t))\right]\Big\}\\
&=\frac{1}{4}\left[\eta_e(t)^2\|\omega_e(t)+r(t)\|^2+\|\epsilon_e(t)^*(\omega_e(t)+r(t))\|^2\right]\\
&\leq\frac{1}{4}\left(\eta_e(t)^2+\|\epsilon_e(t)^*\|^2\right)\|\omega_e(t)+r(t)\|^2\\
&\leq\frac{1}{4}\left(\eta_e(t)^2+\|\epsilon_e(t)\|^2\right)\|\omega_e(t)+r(t)\|^2\\
&\leq\frac{1}{4}\|\omega_e(t)+r(t)\|,^2
\end{aligned}$$

where it was used that $(\epsilon_e\times(\omega_e+r))\cdot(\omega_e+r)=0$ and $\left\|\epsilon_e(t)^*\right\|\leq\|\epsilon_e(t)\|$. ■

Lemma 2: Let $P\in\mathbb{S}^{n\times n}$ be a positive definite matrix. Then, for all vectors $x,y\in\mathbb{R}^n$ and all $\rho\in\mathbb{R}$, with $\rho>0$,

$$2x^TPy\leq\rho x^TPx+\frac{1}{\rho}y^TPy$$

holds.

Proof: Let $\tilde{\rho}$ be any given constant scalar. Since $P>0$, one obtains

$$\begin{aligned}
0&\leq\left(\tilde{\rho}x-\frac{1}{\tilde{\rho}}y\right)^TP\left(\tilde{\rho}x-\frac{1}{\tilde{\rho}}y\right)\\
&=\tilde{\rho}^2x^TPx-2x^TPy+\frac{1}{\tilde{\rho}^2}y^TPy,
\end{aligned}$$

[1] Throughout the paper, whenever time-dependency is clear, (t) is dropped in order to simplify notation.

and the result follows considering $\rho{=}\tilde{\rho}^2$. ∎

$\mathscr{H}_\infty$ ATTITUDE TRACKING CONTROL

This section presents conditions which guarantee stability of system (6)-(8) and disturbance rejection performance in the $\mathscr{H}_\infty$ sense according to Definition 1. Such conditions are cast as LMIs obtained from the following Lyapunov-Krasovskii function candidate

$$V\left(t\right){=}\sum_{i=1}^{4} V_i\left(t\right), \tag{11}$$

where

$$V_1{=}2\mathfrak{a}\left[\epsilon_e^T\epsilon_e+\left(1-\eta_e\right)^2\right]+\mathfrak{b}\omega_e^TJ\omega_e+2\mathfrak{c}\epsilon_e^TJ\omega_e, \tag{12}$$

$$V_2{=}\int_{t-\frac{\tau}{2}}^{t}\begin{bmatrix}\epsilon_e(s)\\ \epsilon_e\left(s-\frac{\tau}{2}\right)\end{bmatrix}^T M\begin{bmatrix}\epsilon_e(s)\\ \epsilon_e\left(s-\frac{\tau}{2}\right)\end{bmatrix}ds$$
$$+\int_{t-\mu}^{t-\tau}\begin{bmatrix}\epsilon_e(s)\\ \epsilon_e\left(s-\mu+\tau\right)\end{bmatrix}^T N\begin{bmatrix}\epsilon_e(s)\\ \epsilon_e\left(s-\mu+\tau\right)\end{bmatrix}ds, \tag{13}$$

$$V_3{=}\int_{-\tau}^{0}\int_{t+\beta}^{t}\tau\dot{\epsilon}_e(s)^T\mathfrak{r}\dot{\epsilon}_e(s)dsd\beta, \tag{14}$$

$$V_4{=}\int_{-\mu}^{-\tau}\int_{t+\beta}^{t}\left(\mu-\tau\right)\dot{\epsilon}_e(s)^T\mathfrak{s}\dot{\epsilon}_e(s)dsd\beta$$
$$+\int_{-\nu}^{-\mu}\int_{t+\beta}^{t}\left(\nu-\mu\right)\dot{\epsilon}_e(s)^T\mathfrak{t}\dot{\epsilon}_e(s)dsd\beta. \tag{15}$$

The functional (11) contains some well-known terms which extract useful delay information allowing a less conservative analysis to be conducted [5], [6], [14]. First, however, the positiveness of (11) must be assessed. Since terms (13)-(15) consist of quadratic expressions, positiveness of the corresponding matrices and scalar variables suffice to guarantee positiveness. In the case of (12), positiveness conditions reveal themselves as follows. Suppose $\mathfrak{a}{>}0$ and $\mathfrak{b}{>}0$. According to Lemma 2,

$$\begin{aligned}V_1&{=}2\mathfrak{a}\left[\epsilon_e^T\epsilon_e+\left(1-\eta_e\right)^2\right]+\mathfrak{b}\omega_e^TJ\omega_e+2\mathfrak{c}\epsilon_e^TJ\omega_e\\ &\geq 2\mathfrak{a}\epsilon_e^T\epsilon_e+\mathfrak{b}\omega_e^TJ\omega_e+2\mathfrak{c}\epsilon_e^TJ\omega_e\\ &\geq\frac{2\mathfrak{a}}{\lambda_{max}\left(J\right)}\epsilon_e^TJ\epsilon_e+\mathfrak{b}\omega_e^TJ\omega_e-\mathfrak{c}\epsilon_e^TJ\epsilon_e-\mathfrak{c}\omega_e^TJ\omega_e\\ &=\left(\frac{2\mathfrak{a}}{\lambda_{max}\left(J\right)}-\mathfrak{c}\right)\epsilon_e^TJ\epsilon_e+\left(\mathfrak{b}-\mathfrak{c}\right)\omega_e^TJ\omega_e.\end{aligned}$$

Since the positiveness of the quadratic coefficients is enough to conclude positiveness, the following conditions guarantee (11) be positive

$$\mathfrak{a}{>}0, \mathfrak{b}{>}0, \mathfrak{b}{>}\mathfrak{c}, \mathfrak{r}{>}0, \mathfrak{s}{>}0, \mathfrak{t}{>}0,$$
$$2\mathfrak{a}{>}\lambda_{max}\left(J\right)\mathfrak{c}, M{=}\begin{bmatrix}M_{11} & M_{12}\\ * & M_{22}\end{bmatrix}{>}0, N{=}\begin{bmatrix}N_{11} & N_{12}\\ * & N_{22}\end{bmatrix}{>}0. \tag{16}$$

Theorem 1: For given scalars τ,ν such that $0{\leq}\tau{\leq}\nu$, and κ_1,κ_2, the system (6)-(8) is stable with disturbance rejection upper bound $\gamma{>}0$ if there exist scalars $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{r},\mathfrak{s},\mathfrak{t}$ and matrices M,N satisfying (16) as well as free-weighting matrices $\mathscr{F}_l$ such that the following LMIs hold

$$\bar{\Omega}+\Omega_l|_{\bar{\mathbb{D}}_l}+\mathscr{F}_l\mathscr{G}_l+\mathscr{G}_l^T\mathscr{F}_l^T{<}0; \tag{17}$$

for all $\bar{\mathbb{D}}_l{\in}\{0,1\}$ and $l{\in}\{1,2\}$, where

$$\bar{\Omega}{=}\begin{bmatrix}\Omega_{1,1} & \Omega_{1,2} & \Omega_{1,3} & 0 & 0 & \Omega_{1,6} & \Omega_{1,7} & 0 & \Omega_{1,9}\\ * & \Omega_{2,2} & \Omega_{2,3} & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & \Omega_{3,3} & \Omega_{3,4} & 0 & 0 & 0 & 0 & 0\\ * & * & * & \Omega_{4,4} & \Omega_{4,5} & 0 & 0 & 0 & 0\\ * & * & * & * & \Omega_{5,5} & 0 & 0 & 0 & 0\\ * & * & * & * & * & 0 & \Omega_{6,7} & 0 & 0\\ * & * & * & * & * & * & \Omega_{7,7} & 0 & \Omega_{7,9}\\ * & * & * & * & * & * & * & 0 & 0\\ * & * & * & * & * & * & * & * & \Omega_{9,9}\end{bmatrix}$$

with

$$\begin{aligned}&\Omega_{1,1}{=}M_{11}-\mathfrak{r}\mathbf{I}+\mathbf{I}, && \Omega_{3,4}{=}N_{12},\\ &\Omega_{1,2}{=}M_{12}, && \Omega_{4,4}{=}N_{22}-N_{11},\\ &\Omega_{1,3}{=}\mathfrak{r}\mathbf{I}, && \Omega_{4,5}{=}-N_{12},\\ &\Omega_{1,6}{=}-\mathfrak{c}\kappa_1\mathbf{I}, && \Omega_{5,5}{=}-N_{22},\\ &\Omega_{1,7}{=}\left(\mathfrak{a}-\mathfrak{c}\kappa_2\right)\mathbf{I}, && \Omega_{6,7}{=}-\mathfrak{b}\kappa_1\mathbf{I},\\ &\Omega_{1,9}{=}\mathfrak{a}\mathbf{I}, && \Omega_{7,7}{=}\mathfrak{c}JJ^T+\left(\mathfrak{m}{+}\mathfrak{c}{-}2\mathfrak{b}\kappa_2\right)\mathbf{I},\\ &\Omega_{2,2}{=}M_{22}-M_{11}, && \Omega_{7,9}{=}\left(\mathfrak{m}+\mathfrak{c}\right)\mathbf{I},\\ &\Omega_{2,3}{=}-M_{12}, && \Omega_{9,9}{=}\left(\mathfrak{m}+\mathfrak{c}\right)\mathbf{I}-\gamma^2\mathbf{I},\\ &\Omega_{3,3}{=}N_{11}-M_{22}-\mathfrak{r}\mathbf{I}, && \end{aligned}$$

$$\mathfrak{m}{=}\frac{1}{4}\left[\tau^2\mathfrak{r}+\left(\mu-\tau\right)^2\mathfrak{s}+\left(\eta-\mu\right)^2\mathfrak{t}\right],$$
$$\Omega_1|_{\bar{\mathbb{D}}_1}=-\mathfrak{t}\left(\mathbb{J}_4-\mathbb{J}_5\right)^T\left(\mathbb{J}_4-\mathbb{J}_5\right)-\mathfrak{s}\mathbb{J}_8^T\mathbb{J}_8,$$
$$\Omega_2|_{\bar{\mathbb{D}}_2}=-\mathfrak{s}\left(\mathbb{J}_3-\mathbb{J}_4\right)^T\left(\mathbb{J}_3-\mathbb{J}_4\right)-\mathfrak{t}\mathbb{J}_8^T\mathbb{J}_8,$$

and $\mathscr{G}_l{=}\left[\bar{\mathscr{G}_l}\;\mathscr{G}_{l\bar{\mathbb{D}}_l}\;0\right]$

$$\bar{\mathscr{G}_1}=\begin{bmatrix}0\,0\,{-\mathbf{I}}\,0\,0\;\mathbf{I}\;0\\ 0\,0\;0\;\mathbf{I}\,0\,{-\mathbf{I}}\,0\end{bmatrix},\quad \mathscr{G}_{1\bar{\mathbb{D}}_1}=\begin{bmatrix}\bar{\mathbb{D}}_1\mathbf{I}\\ \left(1-\bar{\mathbb{D}}_1\right)\mathbf{I}\end{bmatrix},$$
$$\bar{\mathscr{G}_1}=\begin{bmatrix}0\,{-\mathbf{I}}\,0\,0\,0\;\mathbf{I}\;0\\ 0\;0\;\mathbf{I}\,0\,0\,{-\mathbf{I}}\,0\end{bmatrix},\quad \mathscr{G}_{2\bar{\mathbb{D}}_2}=\begin{bmatrix}\bar{\mathbb{D}}_2\mathbf{I}\\ \left(1-\bar{\mathbb{D}}_2\right)\mathbf{I}\end{bmatrix}, \tag{18}$$

where $\mathbb{J}_k{\in}\mathbb{R}^{3\times 27}$, $k{\in}\{1,\cdots,9\}$, are block entry matrices with nine elements whose k-th element is the identity and all the others are null, e.g., $\mathbb{J}_9{=}\left[0\,0\,0\,0\,0\,0\,0\,0\,\mathbf{I}\right]$.

Proof: Supposing (11) is valid, it is left to show that the derivative of (11) along the trajectories of $q_e\left(t\right)$ is negative definite. Then, $\dot{V}{=}\sum_{i=1}^{4}\dot{V}_i$ with

$$\dot{V}_1{=}-4\mathfrak{a}\dot{\eta}_e+\frac{d}{dt}\left(\mathfrak{b}\omega_e^TJ\omega_e\right)+\frac{d}{dt}\left(2\mathfrak{c}\epsilon_e^TJ\omega_e\right), \tag{19}$$

$$\begin{aligned}\dot{V}_2{=}&\begin{bmatrix}\epsilon_e(t)\\ \epsilon_e\left(t-\frac{\tau}{2}\right)\end{bmatrix}^T\begin{bmatrix}M_{11} & M_{12}\\ * & M_{22}\end{bmatrix}\begin{bmatrix}\epsilon_e(t)\\ \epsilon_e\left(t-\frac{\tau}{2}\right)\end{bmatrix}\\ &-\begin{bmatrix}\epsilon_e\left(t-\frac{\tau}{2}\right)\\ \epsilon_e\left(t-\tau\right)\end{bmatrix}^T\begin{bmatrix}M_{11} & M_{12}\\ * & M_{22}\end{bmatrix}\begin{bmatrix}\epsilon_e\left(t-\frac{\tau}{2}\right)\\ \epsilon_e\left(t-\tau\right)\end{bmatrix}\\ &+\begin{bmatrix}\epsilon_e(t-\tau)\\ \epsilon_e\left(t-\mu\right)\end{bmatrix}^T\begin{bmatrix}N_{11} & N_{12}\\ * & N_{22}\end{bmatrix}\begin{bmatrix}\epsilon_e(t-\tau)\\ \epsilon_e\left(t-\mu\right)\end{bmatrix}\\ &-\begin{bmatrix}\epsilon_e(t-\mu)\\ \epsilon_e\left(t-\nu\right)\end{bmatrix}^T\begin{bmatrix}N_{11} & N_{12}\\ * & N_{22}\end{bmatrix}\begin{bmatrix}\epsilon_e(t-\mu)\\ \epsilon_e\left(t-\nu\right)\end{bmatrix},\end{aligned} \tag{20}$$

$$\dot{V}_3(t){=}\tau^2\dot{\epsilon}_e(t)^T\mathfrak{r}\dot{\epsilon}_e(t)-\tau\int_{t-\tau}^{t}\dot{\epsilon}_e\left(r\right)^T\mathfrak{r}\mathbf{I}\dot{\epsilon}_e\left(r\right)dr, \tag{21}$$

$$\dot{V}_4\left(t\right){=}\dot{V}_{4(0)}\left(t\right)+\dot{V}_{4(I)}\left(t\right),$$
$$\dot{V}_{4(0)}\left(t\right){=}\left(\mu-\tau\right)^2\dot{\epsilon}_e\left(t\right)^T\mathfrak{s}\dot{\epsilon}_e\left(t\right)+\left(\nu-\mu\right)^2\dot{\epsilon}_e\left(t\right)^T\mathfrak{t}\dot{\epsilon}_e\left(t\right), \tag{22}$$

$$\dot{V}_{4(I)}(t){=}-\left(\mu-\tau\right)\int_{t-\mu}^{t-\tau}\dot{\epsilon}_e\left(r\right)^T\mathfrak{s}\mathbf{I}\dot{\epsilon}_e\left(r\right)dr$$
$$-\left(\nu-\mu\right)\int_{t-\eta}^{t-\mu}\dot{\epsilon}_e\left(r\right)^T\mathfrak{t}\mathbf{I}\dot{\epsilon}_e\left(r\right)dr, \tag{23}$$

where it was used the fact that $\epsilon_e^T\epsilon_e + (1-\eta_e)^2 = 2-2\eta_e$ to obtain $\dot{V}_1$.

Using (6)-(7) and Lemma 2, and considering control law (8) one obtains

$$\frac{d}{dt}\left(\mathfrak{b}\omega_e^T J\omega_e\right) = -2\mathfrak{b}\kappa_1\epsilon_e\left(t-d(t)\right)^T\omega_e - 2\mathfrak{b}\kappa_2\omega_e^T\omega_e$$

$$\begin{aligned}\frac{d}{dt}\left(2\mathfrak{c}\epsilon_e^T J\omega_e\right) &= 2\mathfrak{c}\omega_e^T J\dot{\epsilon}_e + 2\mathfrak{c}\epsilon_e^T\left[-\kappa_1\epsilon_e\left(t-d(t)\right) - \kappa_2\omega_e\right]\\ &\leq \mathfrak{c}\left[\omega_e^T\left(JJ^T\right)\omega_e + \dot{\epsilon}_e^T\dot{\epsilon}_e\right] - 2\mathfrak{c}\kappa_1\epsilon_e^T\epsilon_e\left(t-d(t)\right)\\ &\quad - 2\mathfrak{c}\kappa_2\epsilon_e^T\omega_e\\ &\leq -2\mathfrak{c}\kappa_1\epsilon_e^T\epsilon_e\left(t-d(t)\right) - 2\mathfrak{c}\kappa_2\epsilon_e^T\omega_e\\ &\quad + \mathfrak{c}\omega_e^T\left(JJ^T + \mathbf{I}\right)\omega_e + 2\mathfrak{c}\omega_e^T r + \mathfrak{c}r^T r\end{aligned}$$

and it follows that

$$\begin{aligned}\dot{V}_1 \leq & -2\mathfrak{c}\kappa_1\epsilon_e^T\epsilon_e\left(t-d(t)\right) + \epsilon_e^T\left(2\mathfrak{a}\mathbf{I} - 2\mathfrak{c}\kappa_2\mathbf{I}\right)\omega_e + 2\mathfrak{a}\epsilon_e^T r\\ & - 2\mathfrak{b}\kappa_1\epsilon_e\left(t-d(t)\right)^T\omega_e + \omega_e^T\left[\mathfrak{c}\left(JJ^T\right) + \left(\mathfrak{c}-2\mathfrak{b}\kappa_2\right)\mathbf{I}\right]\omega_e\\ & + 2\mathfrak{c}\omega_e^T r + \mathfrak{c}r^T r.\end{aligned} \quad (24)$$

The analysis of terms (21)-(22) rely on the use of Lemma 1 and Jensen's Lemma [7]

$$\begin{aligned}\dot{V}_3 \leq & \tau^2\dot{\epsilon}_e(t)^T\mathfrak{r}\dot{\epsilon}_e(t) - \left[\int_{t-\tau}^{t}\dot{\epsilon}_e(r)\,dr\right]^T\mathfrak{r}\mathbf{I}\left[\int_{t-\tau}^{t}\dot{\epsilon}_e(r)\,dr\right]\\ \leq & \frac{\tau^2\mathfrak{r}}{4}\left(\omega_e^T\omega_e + 2\omega_e^T r + r^T r\right)\\ & + \begin{bmatrix}\epsilon_e(t)\\ \epsilon_e(t-\tau)\end{bmatrix}^T\begin{bmatrix}-\mathfrak{r}\mathbf{I} & \mathfrak{r}\mathbf{I}\\ \mathfrak{r}\mathbf{I} & -\mathfrak{r}\mathbf{I}\end{bmatrix}\begin{bmatrix}\epsilon_e(t)\\ \epsilon_e(t-\tau)\end{bmatrix},\end{aligned} \quad (25)$$

$$\dot{V}_{4(0)} \leq \left[\frac{(\mu-\tau)^2\mathfrak{s} + (\eta-\mu)^2\mathfrak{t}}{4}\right]\left(\omega_e^T\omega_e + 2\omega_e^T r + r^T r\right). \quad (26)$$

Nevertheless, the analysis of (23) is more fruitful when split considering different delay intervals. Take two equally-spaced subintervals $[\tau,\mu]$ and $(\mu,\nu]$. At this point, indicator function comes into play, allowing one to rewrite (23) explicitly in terms of those two subinterval scenarios. Indeed, defining

$$\mathbb{S}_1 := \left\{d(t)\in\mathbb{R}_+ : \chi = 1\right\}, \mathbb{S}_2 := \left\{d(t)\in\mathbb{R}_+ : \chi = 0\right\},$$

results in

$$\dot{V}_4 = \dot{V}_{4(0)} + \dot{V}_{4(\mathbb{S}_1)} + \dot{V}_{4(\mathbb{S}_2)},$$

with $\dot{V}_{4(\mathbb{S}_1)} = \chi\dot{V}_{4(I)}$ and $\dot{V}_{4(\mathbb{S}_2)} = (1-\chi)\dot{V}_{4(I)}$. Consider the first scenario. Since $d(t)\in[\tau,\mu]$, then $\chi=1$ implies $\dot{V}_{4(\mathbb{S}_2)} \equiv 0$, and $\dot{V}_{4(\mathbb{S}_1)}$ can be conveniently rewritten to use convex analysis

$$\begin{aligned}\dot{V}_{4(\mathbb{S}_1)} = -\chi\Bigg[&(\mu-\tau)\int_{t-\mu}^{t-d(t)}\dot{\epsilon}_e(r)^T\mathfrak{s}\dot{\epsilon}_e(r)\,dr + (\mu-\tau)\\ &\times\int_{t-d(t)}^{t-\tau}\dot{\epsilon}_e(r)^T\mathfrak{s}\dot{\epsilon}_e(r)\,dr + (\nu-\mu)\int_{t-\nu}^{t-\mu}\dot{\epsilon}_e(r)^T\mathfrak{t}\dot{\epsilon}_e(r)\,dr\Bigg].\end{aligned} \quad (27)$$

Let $\xi_{11}(t) := \frac{\mu-\tau}{d(t)-\tau}\int_{t-d(t)}^{t-\tau}\dot{\epsilon}_e(r)\,dr$ and $\xi_{12}(t) := \frac{\mu-\tau}{\mu-d(t)}\int_{t-\mu}^{t-d(t)}\dot{\epsilon}_e(r)\,dr$. Joining (26) with (27) yields an expression which can be bounded by Jensen's inequality [7]

$$\begin{aligned}\dot{V}_4 \leq & \left[\frac{(\mu-\tau)^2\mathfrak{s} + (\eta-\mu)^2\mathfrak{t}}{4}\right]\left(\omega_e^T\omega_e + 2\omega_e^T r + r^T r\right)\\ & - \chi\Big\{\xi_{11}^T\left(\mathbb{D}_1\mathfrak{s}\mathbf{I}\right)\xi_{11} + \xi_{12}^T\left(1-\mathbb{D}_1\right)\mathfrak{s}\mathbf{I}\xi_{12}\\ & + \left[\epsilon_e(t-\mu) - \epsilon_e(t-\nu)\right]^T\mathfrak{t}\mathbf{I}\left[\epsilon_e(t-\mu) - \epsilon_e(t-\nu)\right]\Big\},\end{aligned} \quad (28)$$

where the introduction of $\mathbb{D}_1(t) := \frac{d(t)-\tau}{\mu-\tau}\in[0,1]$ exposes the convexity of $\dot{V}_4$ in relation to $d(t)\in\mathbb{S}_1$. Thus, $\dot{V}_4$ attains its maximum at the edges of $\mathbb{D}_1$—0 or 1.

Merging inequalities (20), (24), (25) and (28) yields LMI conditions $\tilde{\Omega}$ such that $\dot{V}_{\mathbb{S}_1} \leq \tilde{\zeta}_1^T\tilde{\Omega}\tilde{\zeta}_1$, where $\tilde{\zeta}_1^T = \left[\epsilon_e^T\ \epsilon_e\left(t-\frac{\tau}{2}\right)^T\ \epsilon_e(t-\tau)^T\ \epsilon_e(t-\mu)^T\ \epsilon_e(t-\nu)^T\ \epsilon_e(t-d(t))^T\ \omega_e^T\ \xi_{11}^T\ \xi_{12}^T\ r^T\right]$. Since at each extrema of $\mathbb{D}_1$ either ξ_{11} or ξ_{12} will be weighted by a zero matrix, the corresponding null row and column can be eliminated and a new vector ζ_1 not containing the state multiplied by zero—ξ_{11} or ξ_{12}. Now, let $\mathscr{G}_l = \left[\bar{\mathscr{G}}_l\ \mathscr{G}_{l\bar{\mathbb{D}}_l}\ 0\right]$, with $\bar{\mathscr{G}}_1$ and $\mathscr{G}_{1\bar{\mathbb{D}}_1}$ defined according to (18), such that $\zeta_1^T\mathscr{G}_1 = 0$.

At this point, Finsler's Lemma [14] can be invoked by considering a free-weighting matrix $\mathscr{F}_1$, such that $\zeta_1^T\Omega\zeta_1 < 0$ if, and only if, $\hat{\Omega} = \Omega + \mathscr{F}_1\mathscr{G}_1 + \mathscr{G}_1^T\mathscr{F}_1^T < 0$. Thus, this equivalence maintains the convexity of $\hat{\Omega}$ with relation to $\mathbb{D}_1$—$\hat{\Omega}$ attains its maximum at either $\bar{\mathbb{D}}_1 = 0$ or 1. Therefore, if (17) holds, $\dot{V}_{\mathbb{S}_1} < 0$ and the system (6)-(7) is stable for the first delay scenario.

The second delay scenario is amenable to considerations similar to those made in order to prove stability for the first scenario. Thus, if conditions (17) from Theorem 1 are fulfilled, $\dot{V}_{\mathbb{S}_2}$ is negative definite and, consequently, regardless the scenario of delay considered—$\mathbb{S}_1$ or $\mathbb{S}_2$—$\dot{V}$ is negative definite. Therefore, the system (6)-(7) is asymptotically stable, fulfilling condition 1 of Definition 1.

In addition, if (17) is satisfied for the conditions presented in Theorem 17, then

$$\dot{V} + \epsilon_e^T\epsilon_e - \gamma^2 r^T r < 0 \quad (29)$$

must hold. From the definite-positiveness of V and the definite-negativeness of $\dot{V}$, it follows from (29) that

$$\int_0^{+\infty}\epsilon_e^T\epsilon_e < \gamma^2\int_0^{+\infty} r^T r,$$

if null initial conditions are assumed. Thus, condition 2 of Definition 1 is fulfilled since

$$\left\|\epsilon_e(t)\right\|_2^2 < \gamma^2\left\|r(t)\right\|_2^2, \forall t\geq 0.$$

■

SIMULATION RESULTS

This section assesses the effectiveness of the proposed quaternion-based H_∞ controller under different time-delays, disturbances, and system characteristics.

First, we demonstrate the attitude and angular velocity stabilization–that is, stabilizing the system to a desired constant attitude, $q_d = \mathbf{1}$–subjected to time-varying delays and disturbances. To this aim, a simulated scenario is proposed considering the cube satellite introduced by [11] with inertia matrix given by

$$J = 10^{-2}\begin{bmatrix}4.65 & -0.07 & 0.04\\ 0.07 & 4.86 & -0.21\\ 0.04 & -0.21 & 4.82\end{bmatrix}, \quad (30)$$

attitude feedback time delays varying within $[0, 100]$ ms, and exogenous disturbance behavior described by $r(t) = r_1(t)$

TABLE I: Exogenous disturbance profiles $r_1(t)$ and $r_2(t)$.

$r_1(t)$	$\underbrace{0\leq t[s]\leq 10}_{0.3\sin(1.15t)}$	$\underbrace{10\leq t[s]\leq 20}_{0.012}$	$\underbrace{20\leq t[s]\leq 30}_{0.3\sin(1.15t)}$	$\underbrace{30\leq t[s]}_{\mathcal{N}(0,0.035)}$
$r_2(t)$	$\underbrace{0\leq t[s]\leq 12}_{0.05\sin(3t)+\mathcal{N}(0,0.025)}$	$\underbrace{12\leq t[s]\leq 24}_{\mathcal{N}(0,0.045)}$	$\underbrace{24\leq t[s]\leq 32}_{0.05\sin(3t)}$	$\underbrace{32\leq t[s]}_{0.015}$

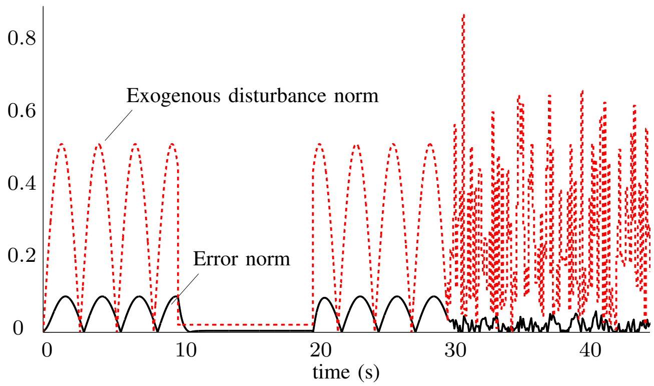

Fig. 1: Norm of the attitude stabilization error $\epsilon_e(t)$ compared to the exogeneous disturbance $r_1(t)$

from Table I. The control gains from Theorem 1 are set to $k_1{=}5$ and $k_2{=}1$. The simulated attitude error is shown and compared against the exogenous disturbance in Figure 1. It is clear that the proposed controller succeeded in reducing the disturbance influence upon the system attitude, whereas maintaining its stability. Indeed, the disturbance to attitude error attenuation numerically attained from the simulation, $\gamma_{sim}{=}0.20$, is smaller than the upper bound for the H_∞ norm provided by Theorem 1, $\gamma_{thm}{=}1.01$.

Furthermore, to illustrate the effectiveness of the proposed criterion in the more challenging problem of tracking a desired attitude and angular velocity, we set a different simulation scenario with a desired angular acceleration described by

$$\dot{w}_d = \begin{cases} 0.3\sin(1.25t)\,, & \text{if} \quad t_0 \leq t \leq 15 \text{ s;} \\ 0.01, & \text{if } 15 \text{ s} \leq t \leq 20 \text{ s;} \\ 0.15\sin(10t)\,, & \text{if } 20 \text{ s} \leq t \leq 30 \text{ s;} \\ 0.06\sin(4t)\,, & \text{if } 30 \text{ s} \leq t; \end{cases}$$

The initial values for the desired attitude and desired angular velocity are respectively given by $q_d(t_0){=}[0.298 \quad -0.536 \quad 0.318 \quad 0.723]$ and $w_d(t_0){=}[0 \quad 0.1 \quad 0.05]$, whereas the initial configuration for the system is given by $q(t_0){=}[1 \;\; 0 \;\; 0 \;\; 0]$ and $w(t_0){=}[0 \;\; 0 \;\; 0]$. The system inertia matrix is assumed to be the same from the previous scenario (30), but the time-delay configuration is now given by $d(t){\in}[0,150]$ ms and the disturbance behavior described by $r(t){=}r_2(t)$ from Table I. We also consider different control gains, $k_1{=}10$ and $k_2{=}1$.

The simulated attitude tracking error ($q_e{=}q_d^{-1}q$) is shown and compared against the exogenous disturbance in Figure 2. Despite the large exogenous disturbance over the closed-loop system, it is easy to see that the proposed controller successfully tracks the desired attitude, as shown in Figure 3. The numerical calculation for the disturbance attenuation

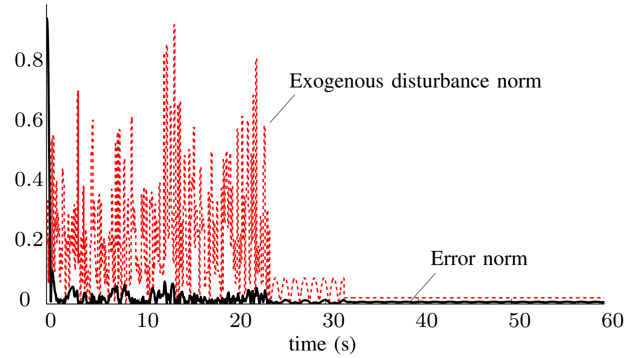

Fig. 2: Norm of the attitude tracking error $\epsilon_e(t)$ compared to the exogeneous disturbance $r_2(t)$ (red dashed line)

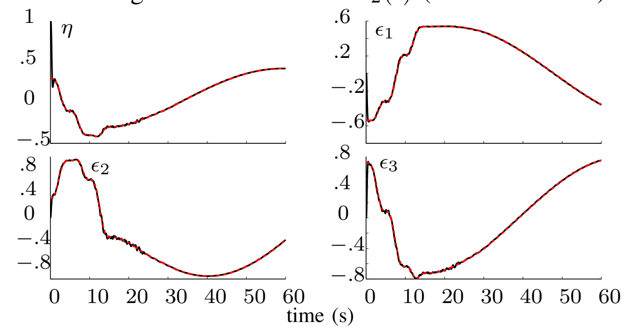

Fig. 3: Quaternion elements $q{=}[\eta \quad \epsilon_1 \quad \epsilon_2 \quad \epsilon_3]$ (black solid line) compared to the time-varying desired quaternion (red dashed line) over time subjected to disturbance $r_2(t)$

from the simulation yields $\gamma_{sim}{=}0.37$, which is smaller than the upper bound provided by Theorem 1, $\gamma_{thm}{=}1.25$. Lastly, Figure 3 shows the angular velocity tracking, that is, the system angular velocity over time compared to the desired velocity. From Figures 2,3 and Table I, it is easy to see that the angular velocity error is directly influenced by the exogenous disturbance, that is, the velocity error is larger for $t{<}24$ s and it is reduced afterwards—which coincides with the start of the last profiles for $r_2(t)$.

CONCLUSION

This work addressed the problem of rigid body attitude and angular velocity tracking subjected to exogenous disturbances and unknown time-varying delays in the attitude

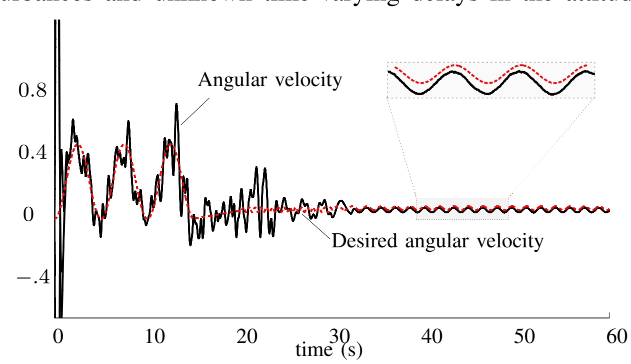

Fig. 4: Plot shows the angular velocity tracking subjected to disturbance $r_2(t)$. Angular body velocity (black solid line) compared to the time-varying desired angular velocity (red dashed line) over time

feedback loop. The result, based on the exploitation of the unit quaternion manifold characteristics, enlarges the applicability of attitude control theory to more realistic scenarios and conditions. Sufficient conditions guaranteeing attitude tracking and $\mathscr{H}_\infty$ disturbance rejection were presented in the form of LMIs, which enable the conditions to be readily tested. The proposed controller was simulated to illustrate its effectiveness on tracking desired attitude and angular velocity trajectories regardless of large exogenous disturbances, and initial orientation and angular velocities.

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