Proceedings Paper:

Magro, PPM, Kussaba, HTM, Figueredo, LFC orcid.org/0000-0002-0759-3000 et al. (1 more author) (2017) Dual quaternion-based bimodal global control for robust rigid body pose kinematic stabilization. In: Proceedings of the 2017 American Control Conference (ACC). American Control Conference (ACC 2017), 24-26 May 2017, Seattle, WA, USA. IEEE , pp. 1205-1210. ISBN 9781509059928

10.23919/ACC.2017.7963117

Dual quaternion-based bimodal global control for robust rigid body pose kinematic stabilization

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***Abstract*— A hybrid bimodal controller for rigid body pose stabilization within the group of unit norm dual-quaternions is proposed in this paper. Using two binary logic state variables, this hysteresis-based controller represents a middle term solution between the memoryless discontinuous controller and the fixed-width hysteretic one. The proposed strategy is novel within the dual-quaternions framework and addresses three common difficulties that appears in the literature of pose and attitude stabilization: global stability, robustness against chattering and against unwinding. The efficacy and performance of the proposed controller are illustrated with numerical examples.**

I. INTRODUCTION

In the study of aerospace and robotic systems, the Lie groups of rigid body motions SE(3) and its subgroup SO(3) of proper rotations arise naturally. Stemming from the seminal work of [1] about control theory on general Lie groups, much of the literature has been devoted to the control of systems defined on SO(3) and SE(3). Although it is usual to design controllers for these systems using matrices to represent elements of these Lie groups [2], [3], it has been noted by some authors that controllers designed using another type of representation, namely, the unit quaternions for SO(3) and the unit dual quaternions for SE(3), may have advantages regarding computational time and storage requirements [4], [5].

It is important to note that since in this cases the state space of a dynamical system is not the Euclidean space $\mathbb{R}^n$ but a general manifold, some difficulties to design a stabilizing controller to the system can arise. For instance, the topology of the manifold may be an obstacle to the existence of a global asymptotically stable equilibrium point in any continuous vector field defined on the manifold [6]. In particular, it is impossible to design a continuous feedback that globally stabilizes the attitude of a rigid body [6].

To avoid this topological obstruction in SO(3), one should resort to non-continuous feedback: this is what was done, for instance, in [7], [8]. As noted in [9], however, non-hybrid strategies are prone to chattering and are not robust to arbitrarily small measurement noise since it is impossible to use pure discontinuous state feedback to achieve robust global asymptotic stabilization of a disconnected set of points [10].

To tackle the problem of robust global attitude control, a quaternion-based hybrid controller with hysteretic memory was suggested in [9]. However, the cost for using the hysteretic controller is longer rotation trajectories for some initial attitudes leading to a higher average settling time or energy consumption. For satellites and other systems with limited energy, this problem is yet more critical [11].

The aforementioned problems also occur in the dual quaternion framework, as the Lie group of unit dual quaternions is a double cover for the Lie group of rigid body motions SE(3) [12], [13]. Moreover, in [13] it was verified that the lack of robustness in the context of dual quaternions is even more important, as the discontinuity of the controller not only affects the rotation of the rigid body, but may also degrade the trajectory of its translation. The problem of energy consumption also aggravates in this context, as the coupled translation and rotation movements consume more energy. Thus, to address the robust global stability problem of rigid bodies we propose a hybrid control law, called bimodal, that extends the hysteretic controller suggested by [13] and represents a compromise in terms of cost between the memoryless discontinuous controller and the hysteretic one.

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II. PRELIMINARY

A. Quaternion

The quaternion algebra is a four dimensional associative division algebra over $\mathbb{R}$ invented by Hamilton [14], which naturally extends the algebra of complex numbers. The elements 1, $\hat{i}$, $\hat{j}$, $\hat{k}$ are the basis of this algebra, satisfying

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -1$$

and the set of quaternions is defined as

$$\mathbb{H} \triangleq \left\{ \boldsymbol{q} = \eta + \mu_1\hat{i} + \mu_2\hat{j} + \mu_3\hat{k} \,:\, \eta, \mu_1, \mu_2, \mu_3 \in \mathbb{R} \right\}.$$

For ease of notation, it may be denoted as

$$\boldsymbol{q} = \eta + \boldsymbol{\mu}, \quad \text{with} \quad \boldsymbol{\mu} = \mu_1\hat{i} + \mu_2\hat{j} + \mu_3\hat{k}$$

In addition, it may be decomposed into a real component and an imaginary component: $\Re(\boldsymbol{q}) \triangleq \eta$ and $\Im(\boldsymbol{q}) \triangleq \boldsymbol{\mu}$ such that $\boldsymbol{q} = \Re(\boldsymbol{q}) + \Im(\boldsymbol{q})$. The quaternion conjugate is given by $\boldsymbol{q}^* \triangleq \Re(\boldsymbol{q}) - \Im(\boldsymbol{q})$.

The multiplication of two quaternions $\boldsymbol{q}_1 = \eta_1 + \boldsymbol{\mu}_1$ and $\boldsymbol{q}_2 = \eta_2 + \boldsymbol{\mu}_2$ is given by

$$\boldsymbol{q}_1\boldsymbol{q}_2 = (\eta_1\eta_2 - \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2) + (\eta_1\boldsymbol{\mu}_2 + \eta_2\boldsymbol{\mu}_1 + \boldsymbol{\mu}_1 \times \boldsymbol{\mu}_2).$$

Pure imaginary quaternions are given by the set

$$\mathbb{H}_0 \triangleq \{\boldsymbol{q} \in \mathbb{H} \,:\, \Re(\boldsymbol{q}) = 0\}$$

which are very convenient to represent vectors of $\mathbb{R}^3$.

The quaternion norm is defined as $\|\boldsymbol{q}\| \triangleq \sqrt{\boldsymbol{q}\boldsymbol{q}^*}$. Unit quaternions are defined as the quaternions that lie in the subset

$$\mathcal{S}^3 \triangleq \{\boldsymbol{q} \in \mathbb{H} \,:\, \|\boldsymbol{q}\| = \mathbf{1}\}, \quad \mathbf{1} = 1 + 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}.$$

The set $\mathcal{S}^3$ forms, under multiplication, the Lie group Spin(3), whose identity element is $\mathbf{1}$ and group inverse is given by the quaternion conjugate $\boldsymbol{q}^*$. As the unit quaternions $\boldsymbol{q}$ and $-\boldsymbol{q}$ represent the same rotation, the unit quaternion group double covers the rotation group SO(3).

B. Dual Quaternions

Similarly to how the quaternion algebra was introduced to address rotations in the three-dimensional space, the dual quaternion algebra was introduced by Clifford [15] and Study [16] to describe rigid body movements. This algebra is constituted by the set

$$\underline{\mathbb{H}} \triangleq \{\boldsymbol{q} + \varepsilon \boldsymbol{q}' \,:\, \boldsymbol{q}, \boldsymbol{q}' \in \mathbb{H}\},$$

where $\boldsymbol{q}$ and $\boldsymbol{q}'$ are called the primary part and the dual part of the dual quaternion and ε is called the dual unit which is nilpotent—that is, $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. Given $\underline{\boldsymbol{q}} = \eta + \boldsymbol{\mu} + \varepsilon(\eta' + \boldsymbol{\mu}')$, we define $\Re(\underline{\boldsymbol{q}}) \triangleq \eta + \varepsilon\eta'$ and $\Im(\underline{\boldsymbol{q}}) \triangleq \boldsymbol{\mu} + \varepsilon\boldsymbol{\mu}'$, such that $\underline{\boldsymbol{q}} = \Re(\underline{\boldsymbol{q}}) + \varepsilon\Im(\underline{\boldsymbol{q}})$. The dual quaternion conjugate is $\underline{\boldsymbol{q}}^* \triangleq \Re(\underline{\boldsymbol{q}}) - \varepsilon\Im(\underline{\boldsymbol{q}})$.

The multiplication of two dual quaternions $\underline{\boldsymbol{q}}_1 = \boldsymbol{q}_1 + \varepsilon\boldsymbol{q}_1'$ and $\underline{\boldsymbol{q}}_2 = \boldsymbol{q}_2 + \varepsilon\boldsymbol{q}_2'$ is given by

$$\underline{\boldsymbol{q}}_1\underline{\boldsymbol{q}}_2 = \boldsymbol{q}_1\boldsymbol{q}_2 + \varepsilon(\boldsymbol{q}_1\boldsymbol{q}_2' + \boldsymbol{q}_1'\boldsymbol{q}_2).$$

The subset of dual quaternions

$$\underline{\boldsymbol{S}} = \{\boldsymbol{q} + \varepsilon\boldsymbol{q}' \in \underline{\mathbb{H}} \,:\, \|\boldsymbol{q}\| = \mathbf{1}, \boldsymbol{q}\boldsymbol{q}'^* + \boldsymbol{q}'\boldsymbol{q}^* = \mathbf{0}\} \qquad (1)$$

forms a Lie group [17] called unit dual quaternions group, whose identity is $\underline{\mathbf{1}} = \mathbf{1} + \varepsilon\mathbf{0}$, $\mathbf{0} = 0 + 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ and group inverse is the dual quaternion conjugate.

An arbitrary rigid displacement characterized by a rotation $\boldsymbol{q} \in$ Spin(3), followed by a translation $\boldsymbol{p} \in \mathbb{H}_0$, with $\boldsymbol{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$, is represented by the unit dual quaternion [12], [18]

$$\underline{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon\frac{1}{2}\boldsymbol{q}\boldsymbol{p}.$$

As the displacement $\underline{\boldsymbol{q}}$ is equally described by $-\underline{\boldsymbol{q}}$, the unit dual quaternions group double covers SE(3).

C. Rigid Motion Description

Using Hamilton convention [19], let $\boldsymbol{q}$ represent the rigid-body attitude $R \in SO(3)$, defined as the relative rotation of a body-fixed frame to a reference frame. The quaternion kinematic equation is

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q}\boldsymbol{\omega}, \qquad (2)$$

where $\boldsymbol{\omega} \in \mathbb{H}_0$ is the angular velocity expressed in the body frame [18].

Similarly, the unit dual quaternion $\underline{\boldsymbol{q}}$ describe the coupled attitude and position. The kinematic equation of a rigid body motion is given by [18]

$$\dot{\underline{\boldsymbol{q}}} = \frac{1}{2}\underline{\boldsymbol{q}}\underline{\boldsymbol{\omega}}, \qquad (3)$$

where $\underline{\boldsymbol{\omega}}$ is called twist and is given by

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\left[\dot{\boldsymbol{p}} + \boldsymbol{\omega} \times \boldsymbol{p}\right] \qquad (4)$$

and $\boldsymbol{p}$ is the translation expressed in the body frame.

Let $\underline{\boldsymbol{q}} \triangleq \boldsymbol{q} + \varepsilon\boldsymbol{q}'$ and $\underline{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega} + \varepsilon\boldsymbol{\omega}'$. It is straightforward to notice that (3) embodies both equation (2) and $\dot{\boldsymbol{p}} = \boldsymbol{\omega}' - \boldsymbol{\omega} \times \boldsymbol{p}$.

III. HYBRID POSE CONTROL

The problem of robust and global pose stabilization of rigid-bodies is not simple. Firstly, there is no continuous feedback controller capable of globally asymptotically stabilizing an equilibrium point on the manifold of the unit dual quaternion group $\underline{\boldsymbol{S}}$ [13].

Secondly, $\underline{\boldsymbol{S}}$ double covers SE(3), that is, $\underline{\boldsymbol{q}}$ and $-\underline{\boldsymbol{q}}$ corresponds to the same pose in SE(3), and this leads, when a continuous dual quaternion based controller is used, to a phenomenon similar to “unwinding” in SO(3) [6]: the body may start at rest arbitrarily close to the desired final pose and yet travel to the farther stable point before coming to rest.

Lastly, even using a (memoryless) discontinuous state feedback, it is impossible to achieve robust global asymptotic stabilization of a disconnected set of points resulted from the double covering of the SE(3)[9], [10].

There are few works on unwinding avoidance in the context of pose stabilization using unit dual quaternions [12], [20], [21], [22]. All of them are based on a discontinuous feedback approach and are prone to chattering for initial conditions arbitrarily close to the discontinuity.

Inspired on the hysteresis-based hybrid control of [9] applied only to attitude control stabilization, [13] extended it to render both coupled kinematics—attitude and translation—stable.

According to [9], there is a price to pay for robust global asymptotic stabilization of attitude using the hysteretic controller—a region in the state space where the hybrid control law pulls the rigid body in the direction of a longer rotation. The pose controller suggested by [13] inherits the same behavior. We propose a hybrid control law, called bimodal, devised to reduce this price. Actually, the bimodal control halves the hysteresis width in certain situations and is a middle term solution between the hysteretic hybrid control and the discontinuous control (equivalent to the hysteretic control with zero-width hysteresis). This control may be especially useful in applications which use low-cost sensors and requires larger hysteresis width due to attitude measurement noise magnitude. For such applications, the standard deviation in attitude error may reach 10° [23].

A. Hybrid Hysteretic Controller

The hysteretic controller strategy for plant (3), suggested by [13], uses only one state variable $h \in X_c \triangleq \{-1, 1\}$ that determines the rotation direction so the system is regulated either to $-\mathbf{1}$ or $\mathbf{1}$ (see Fig. 1).

The state of the system is represented by $x_1 = (\underline{\boldsymbol{q}}, h) \in X_1 \triangleq \underline{\boldsymbol{S}} \times X_c$. The controller is given by the feedback law

$$\underline{\boldsymbol{\omega}} \triangleq -k_1 h \boldsymbol{\mu} - \varepsilon k_2 \eta \boldsymbol{\mu}', \tag{5}$$

where $k_1, k_2 > 0$ are the control gains and the dynamics[1] of h is defined by

$$\begin{aligned} \dot{h} &= 0 & x_1 \in C_1 &\triangleq \{x_1 \in X_1 : h\eta \geq -\delta\}, \\ h^+ &\in \overline{\text{sgn}}\,(\eta) & x_1 \in D_1 &\triangleq \{x_1 \in X_1 : h\eta \leq -\delta\}, \end{aligned} \tag{6}$$

where h^+ is the value associated to h just after the state transition and

$$\overline{\text{sgn}}\,(\eta) = \begin{cases} \{1\}, & \eta > 0, \\ \{-1\}, & \eta < 0, \\ \{-1, 1\}, & \eta = 0. \end{cases}$$

The parameter $\delta \in (0, 1)$ represents the hysteresis half-width and provides robustness against chattering caused by noise in the output measurement. Note that, as commented in Section II-C, the primary part of (3) equals (2). As a consequence, the rotation evolves as the control suggested by [9]. When $h\eta$ gets negative, the feedback determines that the body rotates in the longer rotation direction until a safe distance is achieved to prevent chattering, i.e., until $h\eta \leq -\delta$.

The closed-loop hybrid system, denoted as $\overline{\mathcal{H}}_1$, is formed of equations (3), (5) and (6).

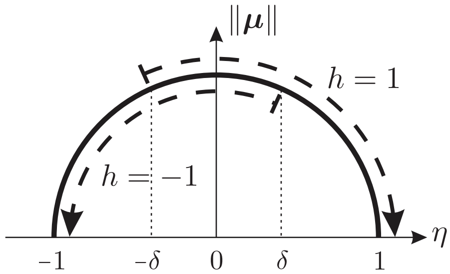

Fig. 1. State space representation of the hysteretic controller (with one state variable h). Arrows indicate the direction of the rotation so the attitude is regulated to $\mathbf{1}$ or $-\mathbf{1}$.

B. Hybrid Bimodal Controller

The proposed bimodal controller strategy uses two state variables $(h, m) \in X_c \times X_c$ as shown in Fig. 2. The state h determines the rotation direction as in the hysteretic controller. The state m is introduced in order to adapt the hysteresis width $\delta_a \in \{\delta/2, \delta\}$ of the on-off control for state h in such a way that the width gets shorter whenever the attitude is relatively far from the chattering prone region $(\eta = 0)$.

[1]Along the text, the dynamics representations follows the hybrid systems framework of [24].

Let the state of the system be represented by $x_2 = (\underline{\boldsymbol{q}}, h, m) \in X_2 \triangleq \underline{\boldsymbol{S}} \times X_c \times X_c$. The bimodal controller is given by the feedback law (5) and the dynamics of h and m are defined by

$$\left.\begin{aligned} \dot{h} &= 0 \\ \dot{m} &= 0 \end{aligned}\right\} \quad x_2 \in C_2, \qquad \left.\begin{aligned} h^+ &\in \overline{\text{sgn}}\,(\eta - h\delta/2) \\ m^+ &\in h\,\overline{\text{sgn}}\,(\eta - h\delta/2) \end{aligned}\right\} \quad x_2 \in D_2, \tag{7}$$

$$\begin{aligned} C_2 &\triangleq \{x_2 \in X_2 : (h\eta \geq -\delta) \text{ and} \\ &(m = -1 \text{ or } h\eta \geq -\delta/2) \text{ and } (m = 1 \text{ or } h\eta \leq 3\delta/2)\}, \\ D_2 &\triangleq \{x_2 \in X_2 : (h\eta \leq -\delta) \text{ or} \\ &(m = 1 \text{ and } h\eta \leq -\delta/2) \text{ or } (m = -1 \text{ and } h\eta \geq 3\delta/2)\}, \end{aligned}$$

where m^+ and h^+ are values associated to m and h, respectively, just after state transition. Note that $C_2 = \overline{X_2 \setminus D_2}$.

The closed-loop hybrid system, denoted as $\overline{\mathcal{H}}_2$, is formed of equations (3), (5) and (7).

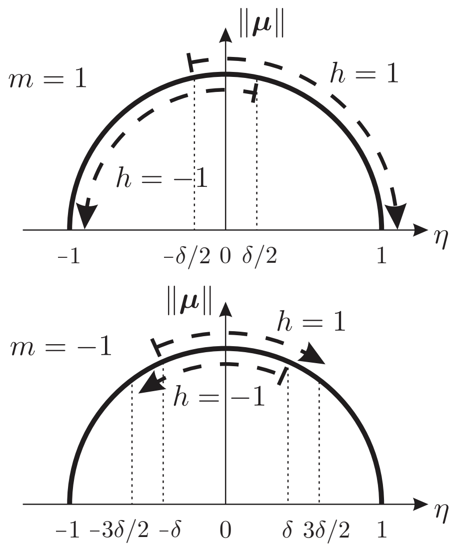

Fig. 2. State space representation of the bimodal controller (with two state variables, h and m). Arrows indicate the direction of the rotation so the attitude is regulated to $\mathbf{1}$ or $-\mathbf{1}$.

IV. STABILITY ANALYSIS

In this section, we prove that the proposed hybrid bimodal control globally asymptotically stabilizes the pose of a rigid body even in the presence of measurement noise.

Theorem 4.1: Let $\delta \in (0, 1)$ and $k_1, k_2 > 0$. The compact set A_2 defined below (8), is globally asymptotically stable for the closed-loop hybrid system $\overline{\mathcal{H}}_2$.

$$A_2 = \left\{x_2 \in X_2 : \underline{\boldsymbol{q}} = h\underline{\mathbf{1}}, m = 1\right\}. \tag{8}$$

Proof: For easy presentation, let us first consider $\delta \in (0, 2/3]$. Let $\underline{\boldsymbol{q}} \triangleq \eta + \boldsymbol{\mu} + \varepsilon(\eta' + \boldsymbol{\mu}')$ and $V : X_2 \to \mathbb{R}$,

$$V(x_2) = 2(1 - h\eta) + \|\boldsymbol{p}\|^2/4. \tag{9}$$

As $m = 1$ whenever $\underline{\boldsymbol{q}} = \pm\underline{\mathbf{1}}$ and as $\boldsymbol{p} = \mathbf{0}$ if and only if $\eta' = 0$ and $\boldsymbol{\mu}' = \mathbf{0}$, we have that $V(x_2) > 0$ for $x_2 \in$

$X_2 \setminus A_2$ and $V(x_2) = 0$ for $x_2 \in A_2$. Hence function V is positive definite on X_2 with respect to A_2.

The time derivative $\dot{V}$ of V is given by

$$\dot{V}(x_2) = -2h\dot{\eta} + \boldsymbol{p} \cdot \dot{\boldsymbol{p}}/2 \quad (10)$$
$$= -h^2 k_1 \|\boldsymbol{\mu}\|^2 - k_2 \eta \boldsymbol{p} \cdot \boldsymbol{\mu}'/2 \quad (11)$$
$$= -k_1 \|\boldsymbol{\mu}\|^2 - k_2 \eta (\boldsymbol{q}^* \boldsymbol{q}') \cdot \boldsymbol{\mu}' \quad (12)$$
$$= -k_1 \|\boldsymbol{\mu}\|^2 - k_2 \eta^2 (\eta'^2 + \|\boldsymbol{\mu}'\|^2) \quad (13)$$

So, $\dot{V}_2$ is negative definite on X_2 with respect to A_2. Besides, observing that the time derivative of $\|\boldsymbol{p}\|^2$ is lower than or equal to zero, we can conclude that the distance of the body along time always decreases, except when $\eta = 0$.

Along jumps, when $x_2 \in D_2$, since $\underline{\boldsymbol{q}}^+ = \underline{\boldsymbol{q}}$,

$$\Delta V(x_2) = V(x_2^+) - V(x_2) = -2\eta(h^+ - h).$$

Let $D_2 = D_{2a} \cup D_{2b} \cup D_{2c}$, where

$$D_{2a} \triangleq \{x_2 \in X_2 : h\eta \leq -\delta\}, \quad (14)$$
$$D_{2b} \triangleq \{x_2 \in X_2 : m = 1 \text{ and } h\eta \leq -\delta/2\}, \quad (15)$$
$$D_{2c} \triangleq \{x_2 \in X_2 : m = -1 \text{ and } h\eta \geq 3\delta/2\}. \quad (16)$$

Thus,

$$\Delta V(x_2) = \begin{cases} \leq -4\delta_a, & x_2 \in D_{2a} \cup D_{2b}, \\ 0, & x_2 \in D_{2c}, \end{cases}$$

where $\delta_a = \delta$ for $x_2 \in D_{2a} \setminus D_{2b}$ and $\delta_a = \delta/2$ for $x_2 \in D_{2b}$.

From Theorem 7.6 of [25], it follows that the compact set A_2 is stable since $\Delta V(x_2) \leq 0$ and $\dot{V}(x_2) < 0$ for all $x_2 \in X_2$.

To conclude that the set A_2 is globally asymptotically stable, it is necessary to apply Theorem 4.7 of [25] to prove that the set A_2 is the largest invariant set in $W = W_1 \cup W_2$, where $W_1 \triangleq \{x_2 \in C_2 : \dot{V}(x_2) = 0\}$ and $W_2 \triangleq \Delta V^{-1}(0) \cap G_2(\Delta V^{-1}(0))$, $G_2(x_2) \triangleq x_2^+$. It follows that $W_1 = A_2$, $\Delta V^{-1}(0) = D_{2c}$ and $G_2(\Delta V^{-1}(0)) = \{x_2 \in X_2 : m = 1 \text{ and } h\eta \geq 3\delta/2\}$. Thus, $W_2 = \emptyset$, $W = A_2$ and any solution $x_2(t)$ approaches the largest invariant set A_2.

This controller restricts parameter δ to $(0, 2/3]$. For the case $\delta \in (2/3, 1)$, the system still behaves as proposed until the first jump. Afterward, it will behave as the hysteretic controller, since m will remain fixed thereafter. ∎

Following we will show that the analysis of either the presence of Zeno solutions (infinite number of jumps in a finite amount of time) or chattering are only related to the rotation.

The rotation evolution follows the primary part of (3). As pointed out in Section II-C, it follows the same kinematic equation for quaternions (2). Substituting (5) into (2),

$$\dot{\boldsymbol{q}} = \frac{1}{2}(\eta + \boldsymbol{\mu})(-k_1 h \boldsymbol{\mu})$$
$$= \frac{1}{2}(k_1 h \|\boldsymbol{\mu}\|^2 - k_1 h \eta \boldsymbol{\mu}).$$

Note that $\dot{\boldsymbol{q}}$ depends only on $\boldsymbol{q}$ and the dynamics of h. On the other hand, the dynamics of h and m depend only on the body rotation (η). Hence, we conclude not only that the rotation is independent of the translation but also that jumps on state variables h and m depend only on the rotation evolution.

The proof that no Zeno solutions occur even when "outer perturbations"—that includes both measurement and modeling errors [26], [9] are taken into account—is similar to the proofs of Theorem 5.3 and Theorem 5.4 of [9] and will not be proved here.

A. Chattering Analysis

Due to noise present in measurements, chattering is possible to occur when jumps map the state back into the jump set, i.e., when $G_2(D_2) \cap D_2 \neq \emptyset$, $G_2(x_2) = x_2^+$. As the number of discrete states is higher than one, h and m, the immediate consecutive jumps must also be analyzed to make sure the following states are mapped to the jump set again. Considering that the output $\boldsymbol{q}$ is corrupted by noise of maximum magnitude α, the verification should be concentrated on intersections $G_2^\alpha(D_2^\alpha) \cap D_2^\alpha$, $G_2^\alpha(G_2^\alpha(D_2^\alpha) \cap D_2^\alpha) \cap D_2^\alpha$, and so on until a loop or an empty set is achieved, where G_2^α and D_2^α are the sets G_2 and D_2, respectively, expanded to accommodate noise of maximum magnitude α as exemplified in [26, Example 5.3].

Theorem 4.2: Let $\alpha > 0$, $\delta > 2\alpha$, $\delta \in (0, 1)$. Then, either $G_2^\alpha(D_2^\alpha) \cap D_2^\alpha = \emptyset$, or $G_2^\alpha(G_2^\alpha(D_2^\alpha) \cap D_2^\alpha) \cap D_2^\alpha = \emptyset$ for system $\overline{\mathcal{H}}_2$.

This proof is not presented here due to space restrictions. The theorem affirms that after two jumps, at most, the state is mapped outside the jump set and no loop (chattering) occurs.

V. NUMERICAL SIMULATIONS

This section presents simulation[2] results to compare performance among the discontinuous controller, the hysteretic controller, and the proposed bimodal controller. To this aim, two different scenarios considering an initial pose defined in a region near 180° away from the desired attitude have been depicted whereby the different behavior is expected.

To maintain fairness, all simulated controllers have been implemented with the same control gains $k_1 = 1$ and $k_2 = 1$. The initial state of the hysteretic controller has been set to $h(0) = 1$ and the ones of the bimodal controller were set to $h(0) = 1$, $m(0) = 1$. The hysteresis parameter defined both for the hysteretic and bimodal controllers was set to $\delta = 0.4$. Please note that by setting the hysteresis parameter to $\delta = 0$ yields a discontinuous control law.

Moreover, to illustrate the robustness of the proposed controller and the performance of all three controllers, additional measured noise have been included to the value of $\boldsymbol{q}$ $(\boldsymbol{q}_m)$ and was calculated as follows: $\boldsymbol{q}_m = (\boldsymbol{q} + b\hat{\boldsymbol{e}}) / \|\boldsymbol{q} + b\hat{\boldsymbol{e}}\|$, $\hat{\boldsymbol{e}} = \boldsymbol{e} / \|\boldsymbol{e}\|$, where each component of $\boldsymbol{e} \in \mathbb{R}^4$ was chosen from a Gaussian distribution of zero mean and unitary standard deviation and $b \in \mathbb{R}$ was chosen from a uniform distribution on the interval $[0, 0.2]$.

[2] All simulations have been performed in MATLAB ambient, using ordinary differential equation solver with variable integration step (ode45) restricted to a maximum step of 1 ms.

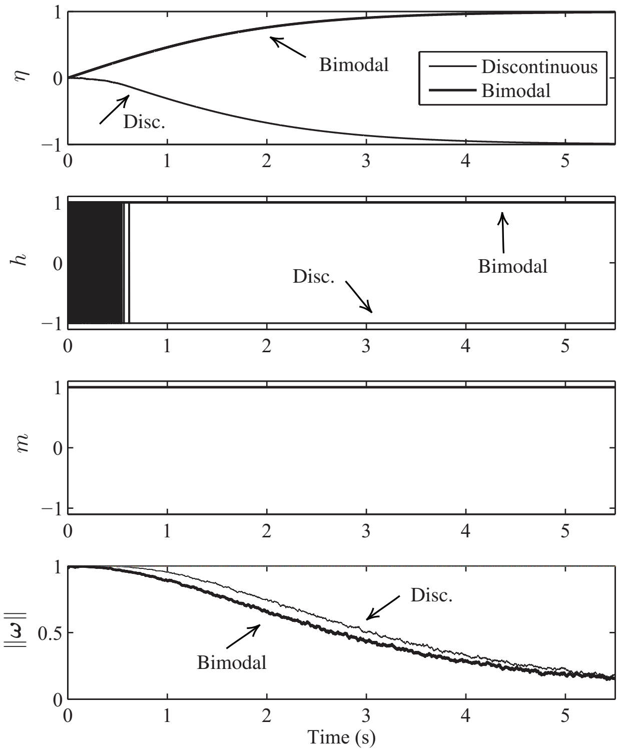

Fig. 3. Rotation comparison between the discontinuous and bimodal controllers.

In the first scenario, the performance of the proposed bimodal controller is investigated against perturbations on the measurement signal and compared to the discontinuous controller. The initial pose in this case was set to $\boldsymbol{q}(0) = 0 + (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})/\sqrt{14}$ and $\boldsymbol{p}(0) = -0.24\hat{\imath} + 1.76\hat{\jmath} + 6.2\hat{k}$. Figs. 3 and 4 illustrate the results from both controllers. Clearly, the chattering phenomenon occurs solely when using the discontinuous control law whereby the resulting controller takes more than 0.5 s to set the final equilibrium point (in this case to $-\mathbf{1}$)—in other words, it takes a considerable amount of time to travel away from its discontinuity at $\eta = 0$. The translation $\boldsymbol{p}$ was also affected. During the period of chattering, the system got stuck around the initial conditions resulting in a convergence lag. The proposed bimodal controller, on the other hand, presents a robust response as expected for both rotation and translation convergence.

The last scenario compares the state evolution between the hysteretic and the bimodal controller. To investigate the liability of the controllers to being pulled to the direction of longer rotation, the initial conditions were $\boldsymbol{q}(0) = -0.2 + \sqrt{1 - 0.2^2}(1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})/\sqrt{14}$ and $\boldsymbol{p}(0) = -0.24\hat{\imath} + 1.76\hat{\jmath} + 6.2\hat{k}$. The consequence of such initial conditions is that it belongs to the hysteresis region from the hysteric controller and therefore the result from such controller travels to the further antipodal equilibrium. As shown in Figs. 5 and 6, the hysteretic and bimodal controllers made the rigid body take a different direction of rotation from the beginning. Regarding the energy spent, if we take the area below the graph of the angular velocity norm, $\|\boldsymbol{\omega}\|$, it is possible to affirm that the bimodal controller spent less energy.

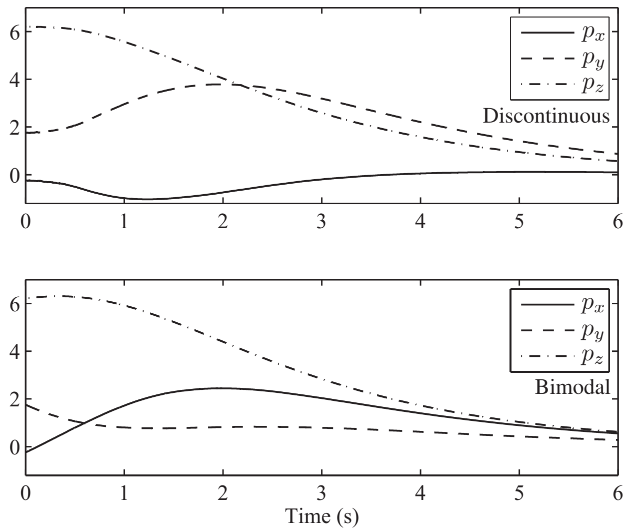

Fig. 4. Evolution of the translation components of $\boldsymbol{p} = p_x\hat{\imath} + p_x\hat{\jmath} + p_x\hat{k}$ for the discontinuous and bimodal controllers.

VI. CONCLUSIONS

This work presented a novel control strategy for robust global rigid body kinematic stabilization using a dual quaternion framework. To address the topological obstruction to global stability inherent to any rigid body representation—which renders the unwinding phenomenon in the case of unit quaternions and unit dual quaternions—this paper exploited an hybrid control technique based on hysteresis, which ensures solution without chattering, in addition to introducing a novel state memory variable that reduces the liability of having the solution trajectory travel to the farther antipodal equilibrium.

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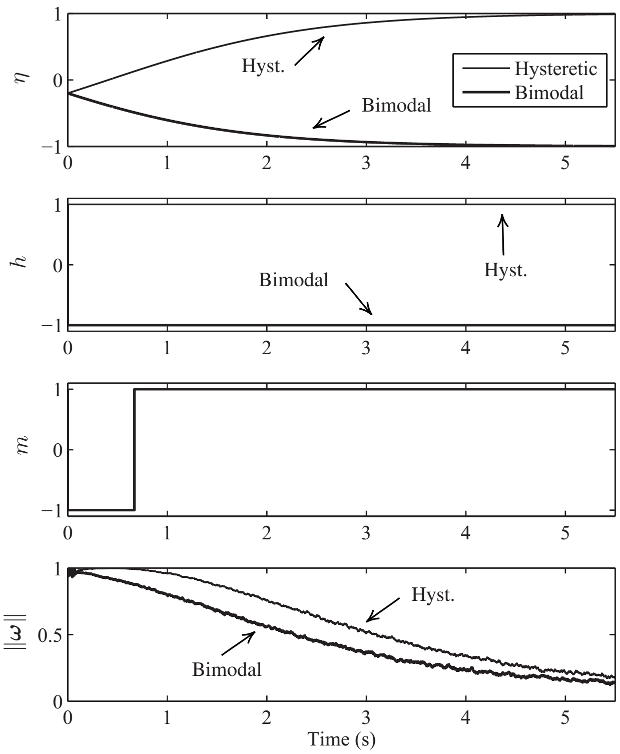

Fig. 5. Rotation comparison between the hysteretic and bimodal controllers.

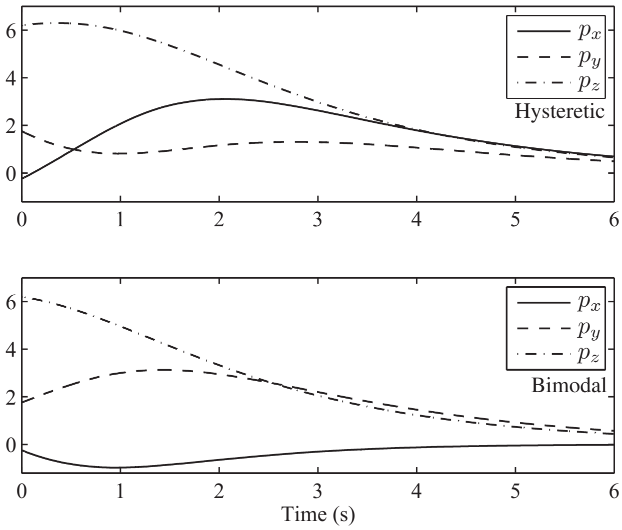

Fig. 6. Evolution of the translation components of $\boldsymbol{p} = p_x\hat{i} + p_x\hat{j} + p_x\hat{k}$ for the hysteretic and bimodal controllers.

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