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## Reconstructing a pure state of a spin $s$ through three Stern-Gerlach measurements

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Consider a spin  $s$  prepared in a *pure* state. It is shown that, generically, the moduli of the  $(2s + 1)$  spin components along three directions in space determine the state unambiguously. These probabilities are accessible experimentally by means of a standard Stern-Gerlach apparatus. To reconstruct a pure state is therefore possible on the basis of  $3(2s + 1)$  measured intensities.

The reconstruction of a particle density-operator is possible in principle through repeated measurements on an ensemble of identically prepared systems [1,2]. Quantum states of vibrating molecules [3], of trapped ions [4], as well as the state of atoms in motion [5] have been reconstructed successfully in the laboratory. Similarly, quantum optical experiments [6] have been performed.

For a spin of length  $s$ , this question arises for states in a Hilbert space of finite dimension. There is an explicit expression for the density matrix  $\rho$  in terms of the moduli of spin components along  $(4s + 1)$  appropriate directions in space [7]. This number can be reduced to  $(2s + 1)$  upon adopting a different approach [8]. A standard Stern-Gerlach apparatus with variable orientation in space provides the corresponding probabilities in an experiment. Alternatively, a Wigner function defined on the discrete phase space associated with a finite-dimensional Hilbert space allows one to reconstruct quantum states [9]. This method has been adapted in [10] in order to determine a quantized electromagnetic mode of a cavity. Every proposed method of state reconstruction is bound to reflect on the link between the outcomes of a finite number of measurements obtained in an actual experiment and the mathematical probabilities which refer to *infinite* ensembles (see [11], for example).

Suppose now that the spin state to be reconstructed is known to be prepared in a *pure* state which is determined by less parameters than a mixed one. How to exploit this additional knowledge in the most efficient way? Reconstruction of pure states has been turned into a question as early as 1933 for a *particle* by Pauli [12] who did not provide an answer. One solution of the spin version of the problem [13] makes use of a *Feynman filter*. This is an advanced version of a Stern-Gerlach apparatus which is assumed to reveal the relative phases of the expansion coefficients of a pure spin state. Another approach relates expectation values of spin multipoles with the parameters which define the quantum state [14].

As shown in this letter, the pure state of a spin  $s$  is determined unambiguously if the *intensities* of the spin components are measured along *three* axes. Compared to the  $(2s + 1)$  axes required for a mixed state [8], the experimental effort to perform state reconstruction is thus reduced considerably for large spins. Further, this result

is satisfactory from a mathematical point of view since it generalizes an earlier result: the intensities along two *infinitesimally close* axes spanning a plane define a unique pure state when complemented by the expectation value of a spin component “out of plane” [15]. Effectively, this means to measure  $(2s + 1)$  probabilities along a third direction.

The states of a spin of magnitude  $s$  live in a Hilbert space  $\mathbf{H}^s$  of complex dimension  $(2s + 1)$ , which carries an irreducible representation of the group  $SU(2)$ . The components of the spin operator  $\vec{S} \equiv \hbar \vec{s}$  with standard commutation relations  $[s_x, s_y] = i s_z, \dots$  generate rotations about the corresponding axes. The standard basis of the space  $\mathbf{H}^s$  is given by the eigenvectors of the  $z$  component of the spin, denoted by  $|s, \mu_z\rangle$ ,  $-s \leq \mu_z \leq s$ . The transformation under the anti-unitary time reversal operator  $T$  fixes their phases,  $T|s, \mu_z\rangle = (-1)^{s-\mu_z} |s, -\mu_z\rangle$ . When expanded in the  $z$  basis ( $\mu_k \equiv \mu_z$ ),

$$|\psi\rangle = \sum_{\mu_k=-s}^s \psi_{\mu_k} |s, \mu_k\rangle, \quad k = x, y, z, \quad (1)$$

a pure state is seen to be determined by  $(2s + 1)$  complex coefficients  $\psi_{\mu_z} \equiv \langle s, \mu_z | \psi \rangle$ . If normalized, rays  $|\psi\rangle$  depend on  $4s$  real parameters. Two other bases of the space  $\mathbf{H}^s$  are used in Eq. (1): the sets  $\{|s, \mu_x\rangle\}$  and  $\{|s, \mu_y\rangle\}$  with  $-s \leq \mu_x, \mu_y \leq s$ , made up from the eigenvectors of the spin components  $s_x$  and  $s_y$ , respectively. Rotations about appropriate axes by an angle  $\pi/2$  map them to the  $z$  basis:

$$|s, \mu_z\rangle = e^{-i\pi s_y/2} |s, \mu_x\rangle = e^{i\pi s_x/2} |s, \mu_y\rangle. \quad (2)$$

A measurement of the intensities  $\{|\langle s, \mu_z | \psi \rangle|^2\}$  does not fix a single state  $|\psi\rangle$  since the phases of the coefficients  $\psi_{\mu_z}$  remain undetermined. However:  
*a spin state  $|\psi\rangle \in \mathbf{H}^s$  is determined unambiguously if  $3(2s + 1)$  probabilities*

$$p(\mu_k) = |\psi_{\mu_k}|^2, \quad k = x, y, z, \quad (3)$$

*are measured with a Stern-Gerlach apparatus along three axes not in a plane. For some exceptional states of measure zero in Hilbert space  $\mathbf{H}^s$ , the probabilities  $p(\mu_k)$  might be compatible with a finite number of states.*

For simplicity, the proof is carried out for orthogonal axes, the generalization being straightforward. Measuring with respect to *two* axes provides  $2(2s+1)$  intensities which are usually compatible with a huge number of isolated states, in agreement with the result of [15]: the parameters fulfill nonlinear relations which may have multiple solutions. Enumerating the ensemble of possible “partner” states is complicated, so a distinctive third measurement is included from the very beginning.

It is useful to rephrase the statement at stake differently. According to (3) a state  $|\tilde{\psi}\rangle$  gives rise to the *same* intensities as does  $|\psi\rangle$  if its coefficients  $\tilde{\psi}_{\mu_k} = \langle s, \mu_k | \tilde{\psi} \rangle$  differ from  $\psi_{\mu_k}$  by phase factors only. Using (1) one writes thus

$$\sum_{\mu_k=-s}^s \psi_{\mu_k} e^{i\chi_k(\mu_k)} |s, \mu_k\rangle = \exp[i\chi_k(\mathbf{s}_k)] |\psi\rangle, \quad (4)$$

with three polynomials  $\chi_k(\mu)$  of order  $2s$  in  $\mu$  at most. From now on, the index  $k$  is understood to take the values  $x, y$ , and  $z$  throughout. The coefficients in (4) thus define three states  $|\psi_k\rangle = W_k^s |\psi\rangle$ , where  $W_k^s = \exp[i\chi_k(\mathbf{s}_k)]$  is a unitary operator diagonal in the  $k$  basis. Consequently, a state  $|\tilde{\psi}\rangle$  compatible with (3) exists if and only if there are nontrivial unitary operators  $W_k^s$  such that

$$W_x^s |\psi\rangle = W_y^s |\psi\rangle = W_z^s |\psi\rangle \equiv |\tilde{\psi}\rangle. \quad (5)$$

It will turn out that this relation is satisfied only if the operators  $W_k^s$  are multiples of the identity, implying that  $|\tilde{\psi}\rangle$  and  $|\psi\rangle$  represent the *same* ray in Hilbert space.

Before turning to the proof, the intensities  $p(\mu_k)$  in (3) are represented in a more compact way. Define three functions  $m_k(\alpha)$  of a complex variable  $\alpha \in \mathbf{C}$  by

$$m_k(\alpha) = \langle \psi | U_k^s(\alpha) | \psi \rangle \equiv \sum_{\mu_k=-s}^s e^{i\mu_k \alpha} p(\mu_k), \quad (6)$$

where the operator  $U_k^s(\alpha) = \exp(i\alpha \mathbf{s}_k)$  rotates a state  $|\psi\rangle$  about the  $k$  axis if  $\alpha \in \mathbf{R}$ . Eq. (6) is inverted easily using the orthogonality of the functions  $\exp[-i\mu_k \alpha]$  on the interval  $0 \leq \alpha < 2\pi$ .

The proof showing that the data (3) are sufficient for state reconstruction is divided into five steps. (i) A  $2^{2s}$  dimensional “parent” space  $\mathcal{H}^s$  is introduced which contains the Hilbert space  $\mathbf{H}^s$  of the spin  $s$  as a subspace. (ii) To each state  $|\psi\rangle \in \mathbf{H}^s$  an equivalence class of product states  $\{|\Psi\rangle \in \mathcal{H}^s\}$  is associated. (iii) A natural definition of *generic* states emerges for *product* states in  $\mathcal{H}^s$  and, *a fortiori*, in  $\mathbf{H}^s$ . (iv) An appropriate set of expectation values of the parent states  $|\Psi\rangle$  fixes them uniquely. (v) Finally, it is shown that all states  $|\tilde{\psi}\rangle$  satisfying (5) have parents in the *same* equivalence class as the original  $|\psi\rangle$ . Consequently, the (generic) state  $|\psi\rangle$  is the only one giving rise to the intensities (3).

(i) The  $2^{2s}$  dimensional “parent” space  $\mathcal{H}^s$  of  $\mathbf{H}^s$  is obtained from tensoring  $2s$  copies of the Hilbert space  $\mathbf{C}^2$  of a spin  $1/2$ :

$$\mathcal{H}^s = \bigotimes_{r=1}^{2s} \mathbf{C}_r^2. \quad (7)$$

A basis of  $\mathbf{C}^2$  is given by the eigenstates  $|\sigma\rangle \equiv |s = 1/2, \mu_3 = \sigma/2\rangle, \sigma = \pm 1$ , of the third component of the spin  $1/2$ :  $\sigma_3 |\sigma\rangle = \sigma |\sigma\rangle$ . This choice induces a basis of  $\mathcal{H}^s$  formed by all product states

$$|\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} |\sigma_r\rangle. \quad (8)$$

The parent space  $\mathcal{H}^s$  decomposes into a subspace  $\mathcal{H}_{\text{sym}}^s$  and its complement,

$$\mathcal{H}^s = \mathcal{H}_{\text{sym}}^s \oplus (\mathcal{H}_{\text{sym}}^s)^\perp, \quad (9)$$

where  $\mathcal{H}_{\text{sym}}^s$  is spanned by the  $(2s+1)$  states obtained from completely symmetrizing those in (8):

$$|s, \mu_3\rangle = \mathcal{S}_{2s} |\{\sigma_r\}\rangle \equiv N_{\mu_3}^s \sum_{\{\sigma_r\}} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu_3) |\{\sigma_r\}\rangle, \quad (10)$$

where  $-s \leq \mu_3 \leq s$ , using a symmetrizer of  $2s$  objects,  $\mathcal{S}_{2s}$ , and the normalization factor  $N_{\mu_3}^s = ((s - \mu_3)!(s + \mu_3)!/(2s)!)^{1/2}$ . The space  $\mathcal{H}_{\text{sym}}^s$  is important here because it carries a  $(2s+1)$  dimensional irreducible representation of the group of rotations,  $SU(2)$ , obtained upon reducing the product representation [16]

$$\mathcal{U} |\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} \sum_{\sigma'_r=\pm 1} |\sigma'_r\rangle \langle \sigma'_r | u_r | \sigma_r \rangle, \quad (11)$$

where  $u_r$  is the  $r$ -th copy of a rotation  $u \in SU(2)$  of the fundamental representation acting on  $\mathbf{C}^2$ , and  $\mathcal{U}$  is an operator defined on  $\mathcal{H}^s$ . Since Hilbert spaces of the same dimension are isomorphic,  $\mathcal{H}_{\text{sym}}^s$  and  $\mathbf{H}^s$  will be identified from now on.

(ii) There is a one-to-one relation between states  $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$  and equivalence classes of *product* states  $|\Psi\rangle \in \mathcal{H}^s$ :

$$|\Psi\rangle \equiv |\{\Psi^r\}\rangle = \bigotimes_{r=1}^{2s} \left( \sum_{\sigma_r} \Psi_{\sigma_r}^r |\sigma_r\rangle \right). \quad (12)$$

The equivalence relation  $\sim$  is defined as follows: the projection of a state  $|\Psi\rangle$  in (12) onto a basis state  $|s, \mu_3\rangle \in \mathcal{H}_{\text{sym}}^s$  must equal the corresponding expansion coefficient of  $|\psi\rangle$  in the  $z$  basis, i.e. ,

$$\langle s, \mu_3 | \Psi \rangle = N_\psi \langle s, \mu_z | \psi \rangle, \quad -s \leq \mu_3 = \mu_z \leq s, \quad (13)$$

and the factor  $N_\psi > 0$  may depend on the state  $|\psi\rangle$  under consideration but *not* on the index  $\mu_z$ . Thus,  $|\Psi\rangle \sim |\Psi'\rangle$

means that for a fixed  $|\psi\rangle$ , the Eqs. (13) hold for both product states,  $|\Psi\rangle \sim |\Psi'\rangle$ . The association of spin states  $|\psi\rangle$  with product or “parent” states  $|\Psi\rangle$  is essential for the following.

In order to determine the class of states satisfying Eq. (13) for a prescribed vector  $|\psi\rangle$  (with definite phase), multiply by the factor  $1/N_\mu^s$ , by powers  $(-z)^{\mu+s}$  and sum all terms. The right-hand-side then defines an analytic function

$$f_R(z) = N_\psi \sum_{\mu=-s}^s \frac{(-z)^{\mu+s}}{N_\mu^s} \psi_\mu \propto \prod_{r=1}^{2s} (z_r - z), \quad (14)$$

specified by the location of its  $2s$  zeroes  $z_r$  in the complex plane. The left-hand-side yields a second analytic function of  $z$ ,

$$\begin{aligned} f_L(z) &= \sum_{\mu=-s}^s \sum_{\{\sigma_r\}} (-z)^{\mu+s} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu) \Psi_{\sigma_1}^1 \dots \Psi_{\sigma_{2s}}^{2s} \\ &\equiv \prod_{r=1}^{2s} (\Psi_-^r - z \Psi_+^r), \quad \Psi_\pm^r \equiv \Psi_{\pm 1}^r. \end{aligned} \quad (15)$$

The  $(2s+1)$  equations (13) are satisfied if  $f_L(z)$  and  $f_R(z)$  coincide. Being two polynomials of degree  $2s$ , this requires them to have identical zeroes,

$$\frac{\Psi_-^r}{\Psi_+^r} = z_r, \quad r = 1, \dots, 2s; \quad (16)$$

in addition,  $f_L(0) = f_R(0)$  must hold. Due to the normalization  $\langle \Psi^r | \Psi^r \rangle = |\Psi_+^r|^2 + |\Psi_-^r|^2 = 1$ , one can write

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{e^{i\kappa_r}}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix}, \quad \kappa_r \in [0, 2\pi). \quad (17)$$

Thus, there are  $2s$  undetermined phase factors  $e^{i\kappa_r}$  with a product equal to 1 (remember that  $|\psi\rangle$  denotes a *vector*). However, the overall ambiguity is even larger: when comparing the zeroes of the functions  $f_L(z)$  and  $f_R(z)$ , there is no rule which would indicate what order to choose when writing down the product state  $|\{\Psi^r\}\rangle$ . In other words, the equivalence class of states defined by (13) consists of all states with coefficients (17) distributed in any order over the  $2s$  spinors in (12). All these states are parents of the same  $|\psi\rangle$  since they satisfy Eq. (13).

A given product state  $|\Psi\rangle$  with components

$$\langle \{\sigma_r\} | \Psi \rangle = \Psi_{\{\sigma_r\}} = \prod_{r=1}^{2s} \Psi_{\sigma_r}^r, \quad (18)$$

has a unique “daughter”  $|\psi\rangle$  to be read off directly. Upon parametrizing each factor  $|\Psi^r\rangle$  by a complex number  $z_r$ ,

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{1}{\sqrt{1+|z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix}, \quad (19)$$

one sees that the ensemble  $\{z_r\} \equiv (z_1, \dots, z_{2s})$  (*no* order implied) defines the daughter  $|\psi\rangle$  completely while a maximum of  $(2s)!$  different parent states  $|\Psi\rangle$  is associated with a given set  $\{z_r\}$ .

(*iii*) Suppose that three ensembles of  $2s$  real numbers each,  $\{x_r\}$ ,  $\{y_r\}$ , and  $\{|z_r|\} \equiv (|z_1|, \dots, |z_{2s}|)$  with  $z_r = x_r + iy_r$  are given in disorder. If one is able to construct the disordered ensemble of  $2s$  complex numbers  $\{z_r = x_r + iy_r\}$  upon using the  $2s$  conditions  $|z_r|^2 = x_r^2 + y_r^2$ , the equivalence class with representative  $|\Psi\rangle$  is called *generic*. In other words, it must be possible to combine unambiguously real and imaginary parts into complex numbers  $z_r$ . In this spirit, a daughter  $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$  will be called *generic* if it has generic parents  $|\Psi\rangle$ . The procedure does not work if equalities such as  $x_r = \pm y_{r'}, r \neq r'$  exist; hence *exceptional* states have measure zero.

(*iv*) It is shown now that the expectation values of rotations  $\mathcal{U}_k(\alpha)$  about the axes  $x, y$ , and  $z$ , fix generic product states  $|\Psi\rangle = |\{\Psi^r\}\rangle$  up to a permutation of the factors  $|\Psi^r\rangle$  and an overall phase factor. A generic  $|\Psi\rangle \in \mathcal{H}^s$  leads to three expectation values

$$M_k(\alpha) = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \equiv \prod_{r=1}^{2s} \langle \Psi^r | u_k^r(\alpha) | \Psi^r \rangle, \quad (20)$$

where  $u_k(\alpha) = \mathbf{1} \cos(\alpha/2) + \boldsymbol{\sigma}_k \sin(\alpha/2)$  represents a rotation about axis  $k$  in  $\mathbf{C}^2$ . Using the parametrization of Eq. (19), the functions  $M_k(\alpha)$  defined in (20) read explicitly

$$M_x(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2ix_r \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21a)$$

$$M_y(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + 2iy_r \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21b)$$

$$M_z(\alpha) = \prod_{r=1}^{2s} \frac{\cos(\alpha/2) + i(1 - |z_r|^2) \sin(\alpha/2)}{1 + |z_r|^2}, \quad (21c)$$

where again  $z_r = x_r + iy_r$ . Denote by  $|\tilde{\Psi}\rangle \equiv |\{\tilde{\Psi}^r\}\rangle$  another product state with expectations  $\tilde{M}_k(\alpha)$ :

$$\tilde{M}_k(\alpha) = \langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle \equiv \prod_{r=1}^{2s} \langle \tilde{\Psi}^r | u_k^r(\alpha) | \tilde{\Psi}^r \rangle. \quad (22)$$

Upon describing the state  $|\tilde{\Psi}\rangle$  by the sequence  $\{\tilde{z}_r\}$ , the three functions  $\tilde{M}_k(\alpha)$  are given by Eqs. 21) after replacing each  $z_r$  by  $\tilde{z}_r$ . It is shown now that the conditions

$$\langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle, \quad k = x, y, z. \quad (23)$$

necessitate  $|\tilde{\Psi}\rangle \sim |\Psi\rangle$ . Being analytic in the complex  $\alpha$  plane, the functions  $M_k(\alpha)$  and  $\widetilde{M}_k(\alpha)$  are equal if they have same zeroes. The equality  $\widetilde{M}_z(\alpha) = M_z(\alpha)$  requires  $|\tilde{z}_r| = |z_r|$ . The condition  $\widetilde{M}_x(\alpha) = M_x(\alpha)$  in turn implies  $\tilde{x}_r = x_r$ ; finally,  $\tilde{y}_r = y_r$  follows from  $\widetilde{M}_y(\alpha) = M_y(\alpha)$ . However, this procedure determines the ensembles  $\{x_r\}$ ,  $\{y_r\}$ , and  $\{|z_r|\}$  *without* any order of its members. Nevertheless, one can reconstruct the ensemble  $\{z_r\}$  (*no* order implied) according to (iv) if  $|\Psi\rangle$  is *generic* providing thus a *unique* equivalence class. For exceptional states, the  $2s$  complex numbers cannot be reconstructed unambiguously since they might allow for parents contained in different equivalence classes.

(v) The results (i) to (iv) imply that the probabilities  $p(\mu_k)$  for three directions  $x$ ,  $y$ , and  $z$  as given in Eq. (3) determine a generic state  $|\psi\rangle$  unambiguously. According to Eq. (5), a state  $|\tilde{\psi}\rangle$  gives rise to the same probabilities as does  $|\psi\rangle$  if one has

$$|\psi_x\rangle = |\psi_y\rangle = |\psi_z\rangle = |\tilde{\psi}\rangle. \quad (24)$$

For parent states  $|\Psi_k\rangle$  of  $|\psi_k\rangle$  this relation says that

$$|\Psi_x\rangle \sim |\Psi_y\rangle \sim |\Psi_z\rangle \sim |\tilde{\Psi}\rangle. \quad (25)$$

This implies that the mean values  $\langle\Psi_k|U_x(\alpha)|\Psi_k\rangle$  of the operator  $U_x(\alpha) = \otimes_r \exp[i\alpha\sigma_x/2]$  are equal for  $k = x, y, z$ : as products they are invariant under a permutation of their factors. This also holds for expectation values of the operators  $U_y(\alpha)$  and  $U_z(\alpha)$ . Write the parent states  $|\Psi_k\rangle$  in the form  $\mathcal{W}_k|\Psi\rangle$  with operators  $\mathcal{W}_k(\{\alpha_{k,r}\}) = \otimes_r \exp[i\alpha_{k,r}\sigma_k/2]$  defined on the parent space  $\mathcal{H}^s$  such that they have  $W_k^s$  as component acting in  $\mathcal{H}_{\text{sym}}^s$ . Contrary to the rotations  $U_k(\alpha)$  which depend linearly on the generators  $s_k$ , the operators  $W_k^s$  are *non-linear* functions  $\chi(s_k)$  of them, Eq. (4). Therefore, the operators  $\mathcal{W}_k(\{\alpha_{k,r}\})$  depend on a set of  $2s$  *different* angles  $\{\alpha_{k,r}\}$ . Using (25) one concludes

$$\begin{aligned} \langle\tilde{\Psi}|U_k|\tilde{\Psi}\rangle &= \langle\Psi_k|U_k|\Psi_k\rangle \\ &= \langle\Psi|\mathcal{W}_k^\dagger U_k \mathcal{W}_k|\Psi\rangle = \langle\Psi|U_k|\Psi\rangle. \end{aligned} \quad (26)$$

The third equality follows because  $\mathcal{W}_k$  and  $U_k$  do commute, both being functions of  $s_k$  only. Eq. (26) comes down to saying that the functions  $M_k(\alpha)$  and  $\widetilde{M}_k(\alpha)$  coincide for all  $k$  and  $\alpha$ . One concludes thus with (iv) that the state  $|\tilde{\Psi}\rangle$ , a parent of  $|\tilde{\psi}\rangle$ , is necessarily a member of the *same* equivalence class as the parent  $|\Psi\rangle$  of  $|\psi\rangle$ . In other words, the application of the operators  $\mathcal{W}_k$  on a parent  $|\Psi\rangle$  does not map it into another equivalence class. In the generic case, there is thus no state different from  $|\psi\rangle$  with the same data (3) what was to be shown.

The reasoning (i) to (v) remains valid if one measures the intensities along directions characterized by unit vectors  $\mathbf{n}_\zeta, \mathbf{n}_\eta$ , and  $\mathbf{n}_\xi$  instead of three orthogonal axes.

These vectors must be linearly independent, that is, they have to span a *volume* in space:  $\mathbf{n}_\zeta \cdot \mathbf{n}_\eta \times \mathbf{n}_\xi \neq 0$ .

As a matter of fact, it is not excluded that the set of data (3) be also sufficient to determine exceptional states unambiguously. Suppose that the numbers  $\{z_r\}$  are associated with a parent state  $|\Psi\rangle$  and  $\{z'_r\}$  with another one,  $|\Psi'\rangle$ , where both sets of complex numbers are obtained from the ensembles  $\{x_r\}$  and  $\{y_r\}$  through  $|z_r|^2 = x_r^2 + y_r^2$ . This does not necessarily imply the existence of an independent  $|\psi'\rangle \neq |\psi\rangle$  since it is the basic conditions  $|\psi'_{\mu k}| = |\psi_{\mu k}|$  which must be satisfied. Explicit calculations for low values of spin  $s$  show that this happens only if  $\psi'_{\mu k} = \psi_{\mu k}^*$ , resulting in  $\langle\psi|(\mathbf{s}_y)^{2n+1}|\psi\rangle \equiv 0$  for all integers  $n$ . In any case one expects every non-genericity to vanish if the spatial directions involved are slightly modified.

To sum up, state reconstruction is possible if based on the  $3(2s+1)$  moduli of the spin components with respect to three directions in space not all in the same plane. Compared to a constructive method using  $(2s+1)^2$  real numbers, the non-constructive method presented here requires that considerably less parameters be determined experimentally, namely  $3(2s+1)$ .

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