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RAMIFICATION OF THE EIGENCURVE AT CLASSICAL RM POINTS

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ABSTRACT. J.Bellaïche and M.Dimitrov have shown that the p-adic eigencurve is smooth but not étale over the weight space at p-regular theta series attached to a character of a real quadratic field F in which p splits. In this paper we prove the existence of an isomorphism between the subring fixed by the Atkin-Lehner involution of the completed local ring of the eigencurve at these points and an universal ring representing a pseudo-deformation problem. Additionally, we give also a precise criterion for which the ramification index is exactly 2. We finish this paper by proving the smoothness of the nearly ordinary and ordinary Hecke algebras for Hilbert modular forms over F at the overconvergent cuspidal Eisenstein points, being the base change lift for $\mathrm{GL}(2)_{/F}$ of these theta series. Our approach uses deformations and pseudo-deformations of reducible Galois representations.

1. INTRODUCTION

Let p be a prime number and $\mathcal{C}$ be the p-adic eigencurve of tame level N constructed using the Hecke operators U_p and $T_\ell, <\ell>$ for $\ell \nmid Np$. Recall that $\mathcal{C}$ is reduced and there exists a flat and locally finite morphism $\kappa : \mathcal{C} \to \mathcal{W}$, called the weight map, where $\mathcal{W}$ is the rigid space over $\mathbb{Q}_p$ representing homomorphisms $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{G}_m$. The eigencurve $\mathcal{C}$ was introduced by Coleman-Mazur in the case where the tame level is one (see [11]), and by Buzzard and Chenevier for any tame level (see [6] and [7] for more details).

By construction of $\mathcal{C}$, there exists a morphism $\mathbb{Z}[T_l, U_p]_{\ell\nmid Np} \to \mathcal{O}_{\mathcal{C}}^{rig}(\mathcal{C})$ such that we can see the elements of $\mathbb{Z}[T_l, U_p]_{\ell\nmid Np}$ as global sections of the sheaf $\mathcal{O}_{\mathcal{C}}^{rig}$, bounded by 1 on $\mathcal{C}$. Therefore, the canonical application "system of eigenvalues" $\mathcal{C}(\bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathbb{Z}[T_l, U_p]_{\ell\nmid Np}, \bar{\mathbb{Q}}_p)$ is injective, and induces a correspondence between the systems of eigenvalues for Hecke operators of normalised overconvergent modular eigenforms with Fourier coefficients in $\mathbb{C}_p$, of tame level N and of weight $k \in \mathcal{W}(\mathbb{C}_p)$, having nonzero U_p-eigenvalue and the set of $\mathbb{C}_p$-valued points of weight k on the eigencurve $\mathcal{C}$; moreover, since the image of $\mathbb{Z}[T_l, U_p]_{\ell\nmid Np}$ is relatively compact in $\mathcal{O}_{\mathcal{C}}^{rig}(\mathcal{C})$ and $\mathcal{O}_{\mathcal{C}}^{rig}(\mathcal{C})$ is

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reduced, there exists a pseudo-character $T : G_{\mathbb{Q},Np} \to \mathcal{O}_{\mathcal{C}}^{rig}(\mathcal{C})$ of dimension 2 such that $T(Frob_\ell) = T_\ell$.

The weight map $\mathcal{C} \to \mathcal{W}$ is étale at non-critical p-regular points corresponding to classical modular forms of weight ≥ 2. It follows from the semi-simplicity of the action of the Hecke algebra, the classicality criterion of overconvergent modular forms and the fact that the multiplicity of the operator U_p is exactly one (see [11, 7.6.2], [10], [21] and [27]). However, the étalness of the weight map can fail in weight one (For more details see [3], [9] and [16]).

The locus of $\mathcal{C}$ where $|U_p| = 1$ is open and closed in $\mathcal{C}$, is called the ordinary locus of $\mathcal{C}$ and denoted by $\mathcal{C}^{ord}$. The ordinary locus $\mathcal{C}^{ord}$ is isomorphic to the rigid space given by the maximal spectrum of the generic fiber of the universal p-ordinary Hecke algebra of tame level N generated by the Hecke operators T_ℓ for all primes $\ell \nmid Np$ and U_p.

Let $f(z) = \sum_{n\geq 1} a_n e^{2i\pi nz}$ be a cuspidal classical weight one newform corresponding to a point of $\mathcal{C}^{ord}$. According to a theorem of Deligne and Serre [14, Prop.4.1], there exists a continuous irreducible representation with finite image $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}})$ such that $\rho(\mathrm{Frob}_\ell) = a_\ell$ for all prime numbers $\ell \nmid Np$.

We fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $\imath_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, which determines an inclusion $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. Since the image of ρ is finite and f is ordinary at p, $\rho_{|G_{\mathbb{Q}_p}} = \psi_1 \oplus \psi_2$, where $\psi_1, \psi_2 : G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p^\times$ are characters and ψ_2 is unramified. We say that f is *regular* at p if and only if $\psi_1 \neq \psi_2$.

Let $\mathcal{T}$ be the completed local ring of $\mathcal{C}$ at f and Λ be the completed local ring of $\mathcal{W}$ at $\kappa(f)$. The weight map κ induces a finite flat local homomorphism $\kappa^\# : \Lambda \to \mathcal{T}$ of local reduced complete rings.

We denote by $\mathfrak{C}$ the category of complete noetherian local $\bar{\mathbb{Q}}_p$-algebras with residue field isomorphic to $\bar{\mathbb{Q}}_p$ and whose morphisms are local homomorphisms of $\bar{\mathbb{Q}}_p$-algebras. Under the assumption that f is p-regular, the functor of p-ordinary deformations of ρ is representable by an universal 2-tuple $(\mathcal{R}, \rho^{ord})$, where $\rho^{ord} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R})$ is the universal ordinary deformation of ρ (see [3, §2]). Under the assumption that ρ is p-regular, M.Dimitrov and J.Bellaïche obtained in [3] the following crucial results which will be often referred to in this paper.

Theorem (J.Bellaïche-M.Dimitrov [3])**.**

(i) *There exists an ordinary deformation* $\rho_{\mathcal{T}} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{T})$ *of* ρ *such that* $\mathrm{Tr}\,\rho_{\mathcal{T}}(\mathrm{Frob}_\ell) = T_\ell$ *when* $\ell \nmid Np$*, and the morphism* $\kappa^\# : \Lambda \to \mathcal{T}$ *sends the universal deformation of* $\det\rho$ *to* $\det\rho_{\mathcal{T}}$*.*

(ii) $\mathcal{R}$ *is a discrete valuation ring and the* p*-ordinary deformation* $\rho_{\mathcal{T}}$ *induces an isomorphism* $\mathcal{R} \simeq \mathcal{T}$*.*

(iii) *The morphism $\kappa^{\#} : \Lambda \to \mathcal{T}$ is ramified if and only if f has RM by a real quadratic field in which p splits.*

Let F be a quadratic real field in which p splits, $\epsilon_F : G_{\mathbb{Q}}/G_F \to \{-1, 1\}$ be the non trivial character and σ be a generator of $\mathrm{Gal}(F/\mathbb{Q})$. We say that f has RM by F if and only if $\rho \simeq \rho \otimes \epsilon_F$. According to [18, Prop.3.1], there exists a character $\phi : G_F \to \bar{\mathbb{Q}}_p^{\times}$ such that $\rho \simeq \mathrm{Ind}_F^{\mathbb{Q}} \phi$. The embedding ι_p singles out a place v of F above p; and denote by v^{σ} the other place above p. The hypothesis of f being p-regular implies that $\phi_{|G_{F_v}} \neq \phi^{\sigma}_{|G_{F_v}}$. Since p splits in F, it follows that $G_{F_v} = G_{\mathbb{Q}_p}$, $\phi_{|G_{F_v}} = \psi_1$ and $\phi^{\sigma}_{|G_{F_v}} = \psi_2$.

The map given by $\rho^{ord} \to \rho^{ord} \otimes \epsilon_F$ yields an automorphism $\tau : \mathcal{R} \to \mathcal{R}$. Denote by $\mathcal{R}_{\tau=1}$ the sub-ring of $\mathcal{R}$ fixed by τ.

In section §3, we introduce a local ring $\mathcal{R}^{ps}$ representing a pseudo-deformation functor of the reducible Galois representation $\rho_{|G_F}$ to the objects of the category $\mathfrak{C}$, with some local condition at p (i.e ordinary at v) and with invariant trace by the action of σ on G_F (see Definition 3.4). We write $\mathcal{R}^{ps}_{red}$ for the quotient of $\mathcal{R}^{ps}$ by its nilradical.

Theorem 1.1. *There exists an isomorphism $\mathcal{R}_{\tau=1} \simeq \mathcal{R}^{ps}_{red}$ and $\mathcal{R}^{ps}_{red}$ is a discrete valuation ring.*

Denote by $H \subset \bar{\mathbb{Q}}$ the number field fixed by $\ker(\mathrm{ad}\,\rho)$, $H_{\infty,v}$ (resp. $H_{\infty,v^{\sigma}}$) be the compositum of all $\mathbb{Z}_p$-extensions of H which are unramified outside v (resp. v^{σ}), H_{∞} be the compositum of $H_{\infty,v}$ and $H_{\infty,v^{\sigma}}$, L_{∞} be the maximal unramified abelian p-extension of H_{∞}, and X_{∞} be the Galois group $\mathrm{Gal}(L_{\infty}/H_{\infty})$. It is known that $\mathrm{Gal}(H_{\infty}/H) \simeq \mathbb{Z}_p^{2s}$ acts by conjugation on X_{∞} and that X_{∞} is a finitely generated $\mathbb{Z}_p[[\mathrm{Gal}(H_{\infty}/H)]]$-module (see [20]).

Theorem 1.2. *Let F'' be the maximal unramified extension of H contained in H_{∞} and L_0 be the subfield of L_{∞} such that $\mathrm{Gal}(L_0/H_{\infty})$ is the largest quotient of X_{∞} on which $\mathrm{Gal}(H_{\infty}/F)$ acts trivially. Assume that L_0 is an abelian extension of F'' or $\mathrm{Gal}(L_0/H_{\infty})$ is a finite group, then the ramification index e of $\mathcal{C}$ over $\mathcal{W}$ at f is exactly 2.*

When H is a biquadratic extension of $\mathbb{Q}$, the assumptions of the above Theorem are related to the semi-simplicity of some torsion Iwasawa Modules (see [26] for more details).

Our approach is inspired by the paper [9] of Cho-Vatsal and uses the results of paper [3]. More precisely, we prove in Lemma 2.4 that the ramification index of $\mathcal{R}_{\tau=1} \hookrightarrow \mathcal{R}$ is two. The key observation made in section §3 concerns that the ring $\mathcal{R}_{\tau=1}$ is isomorphic to $\mathcal{R}^{ps}_{red}$. Therefore, the ramification index of κ at f is two if, and only if $\mathcal{R}^{ps}_{red} \simeq \Lambda$. Hence, it is sufficient to prove that the relative tangent space of $\mathcal{R}^{ps}_{red}$ over Λ is trivial, which will be elaborated in Theorem 4.5.

Let $\bar{\rho} = \mathrm{Ind}_F^{\mathbb{Q}} \bar{\phi}$ denote the residual representation of ρ, where $\bar{\phi} : G_F \to \mathbb{F}_p^{\times}$ is a character and $\mathbb{F}_p$ is a finite field of characteristic p. Assume that ϕ is the Teichmuller lift of an unramified character $\bar{\phi}$ (in this case $F = \mathbb{Q}(\sqrt{N})$). We denote by $\mathfrak{m}$ the maximal ideal of the universal p-ordinary Hecke algebra $h_{\mathbb{Q}} = h_{\mathbb{Q}}(Np^{\infty})$ of tame level N determined by the representation $\bar{\rho}$, and by $h_F = h_F(p^{\infty})$ (resp. $h_F^{n.ord}$) the reduced p-ordinary (resp. p-nearly ordinary) Hecke algebra arising from cuspidal Hilbert modular forms of level p^{∞} for the real quadratic field F.

Langlands proved in [28] that any primitive elliptic cuspidal eigenform f_k belonging to $S_k(\Gamma_1(N), \epsilon_F)$ of weight $k \geq 2$ and of Neben type character ϵ_F has a base change lift $\tilde{f}_k$ for $\mathrm{GL}(2)_{/F}$. More precisely, $\tilde{f}_k$ is a primitive Hilbert modular eigenform for $\mathrm{GL}(2)_{/F}$ of weight k, level 1, with a trivial Neben type character and such that $L(\tilde{f}_k, s) = L(\rho_{f_k|G_F}, s)$, where ρ_{f_k} is the p-adic Galois representation attached to f_k (i.e $L(f_k, s) = L(\rho_{f_k}, s)$). Moreover, Hida constructed in [24, §2] an involution ω on $h_{\mathbb{Q},\mathfrak{m}}$, and following the work of Langlands, Doi, Hida and Ishii in the papers [28] and [18], there exists a base-change morphism:

$$\beta : h_F \to h_{\mathbb{Q}}.$$

The above-mentioned authors constructed also an action of $\Delta = \mathrm{Gal}(F/\mathbb{Q})$ on h_F given by $\sigma(T_q) = T_{q^{\sigma}}$. Let $\mathfrak{y}$ denote the inverse image of $\mathfrak{m}$ under this base-change map.

Doi, Hida, and Ishi were interested by the congruence relations between Hilbert modular forms, and their reflection in certain twisted adjoint L-values. This question led them to study the congruences between forms that arise via base-change from $\mathbb{Q}$, and those being intrinsic to F. Subsequently, they conjectured under suitable assumptions that

$$h_{F,\mathfrak{y}}/(\Delta - 1)h_{F,\mathfrak{y}} \simeq h_{\mathbb{Q},\mathfrak{m}}^{\omega=1},$$

where $h_{\mathbb{Q},\mathfrak{m}}^{\omega=1}$ is the fixed part of $h_{\mathbb{Q},\mathfrak{m}}$ by the involution w (see [18, 3.8]).

Since the dihedral representation ρ becomes reducible upon restriction to G_F, it follows from the properties of the base-change morphism β that the restriction of ρ to G_F is the Galois representation associated to an ordinary p-adic cuspidal weight one Hilbert Eisenstein series $E_1(\phi, \phi^{\sigma})$ of level 1 (see [18, §3.4]). The system of Hecke eigenvalues associated to $E_1(\phi, \phi^{\sigma})$ gives a height one prime ideal $\mathfrak{n} = \beta^{-1}(\mathfrak{p}_f)$ of h_F, where $\mathfrak{p}_f$ is the height one prime ideal of $h_{\mathbb{Q}}$ corresponding to the system of Hecke eigenvalues associated to f. Denote by $\mathfrak{n}^{n.ord}$ the height one prime ideal of the nearly ordinary Hecke algebra $h_F^{n.ord}$ given by the inverse image of $\mathfrak{n}$ via the natural surjection $h_F^{n.ord} \twoheadrightarrow h_F$; namely $\mathfrak{n}^{n.ord}$ is the closed point of $\mathrm{Spec}\, h_F^{n.ord}[1/p]$ associated to the system of Hecke eigenvalues of $E_1(\phi, \phi^{\sigma})$.

Let $\mathbb{T}^{ord}$ be the completed local ring for the étale topology of $\operatorname{Spec} h_F[1/p]$ at a geometric point (i.e $\bar{\mathbb{Q}}_p$-point) corresponding to $\mathfrak{n}$ (i.e $\mathbb{T}^{ord}$ is the completion of the strict local ring at $\mathfrak{n}$), and write $\mathbb{T}^{ord}_{\Delta}$ for the reduced quotient of $\mathbb{T}^{ord}$ by the radical of the ideal generated by elements of the form $\Delta(a) - a$.

Theorem 1.3. *The base-change morphism β induces an isomorphism of local rings $\beta_f : \mathbb{T}^{ord}_{\Delta} \simeq \mathcal{T}_+$, where $\mathcal{T}_+$ is the subring of $\mathcal{T}$ fixed by τ under the identification $\mathcal{R} \simeq \mathcal{T}$.*

Theorem 1.3 allows us to use the exact same arguments that are already given in the proof of Theorem [9, B] to deduce the following variant of the Conjecture [18, 3.8] without assuming that $\bar{\phi}^2_{|I_v} \neq 1$ as in [9, B].

Corollary 1.4. *Assume that $p > 2$ and that the following conditions hold for $\bar{\rho}$:*

(i) *The character $\bar{\phi}$ is everywhere unramified and $\bar{\phi}_{|G_{F_v}} \neq \bar{\phi}^{\sigma}_{|G_{F_v}}$.*
(ii) *The restriction of $\bar{\rho}$ to $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}))$ is absolutely irreducible.*

Then the image of the base-change morphism $\beta : h_F \to h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ has a finite index.

Theorem 1.5. *Assume that ϕ is unramified everywhere and $\phi(\mathrm{Frob}_v) \neq \phi^{\sigma}(\mathrm{Frob}_v)$, then:*

(i) *The affine scheme $\operatorname{Spec} h^{n.ord}_F$ is regular at the point $\mathfrak{n}^{n.ord}$ corresponding to the system of Hecke eigenvalues associated to $E_1(\phi, \phi^{\sigma})$.*
(ii) *The affine scheme $\operatorname{Spec} h_F$ is regular at the point $\mathfrak{n}$ corresponding to the system of Hecke eigenvalues associated to $E_1(\phi, \phi^{\sigma})$, and in this case $\mathbb{T}^{ord} \simeq \mathbb{T}^{ord}_{\Delta} \simeq \mathcal{T}_+$.*

Hida proved in [22] that an ordinary Hilbert cuspform of cohomological weight is a specialization of a unique, up to Galois conjugacy, primitive p-ordinary Hida family. Geometrically, this translates into the smoothness of the nearly ordinary Hecke algebra at the height one prime ideal corresponding to that cuspform. In fact, Hida proves even more, namely the nearly ordinary Hecke algebra being étale at that prime ideal over the Iwasawa algebra $\mathbb{Z}_p[[T_1, T_2, T_3]]$. On the other hand, the criterion for classicality of Hilbert overconvergent modular forms of [5] and [31] generalizes the result of Hida, and implies that the Hilbert eigenvarieties are étale over the weight space at the points corresponding to classical non-critical p-regular Hilbert cuspforms (see [1] for the construction of the Hilbert Eigenvarieties).

However, there are examples where the étaleness of the Hilbert eigenvarieties (resp. parallel Hilbert eigencurves) over the weight space fails in weight one. More precisely, while the Hilbert Eigenvariety is smooth at some classical weight one points with real multiplication, the parallel weight Hilbert Eigencurve is singular at those points, contrasting the famous Hida's control theorem (see [4] and [15]).

The purely quantitive question of how many Hida families specialize to a given classical p-stabilized weight one eigenform, can be reformulated geometrically as to describe the local structure of the ordinary locus of the Hilbert Eigenvarities at the corresponding point.

Now, let $\mathbb{T}^{n.ord}$ be the completed local ring for the étale topology of $\operatorname{Spec} h_F^{n.ord}$ at a geometric point (i.e $\bar{\mathbb{Q}}_p$-point) corresponding to $\mathfrak{n}^{n.ord}$ and $\mathfrak{F}$ (resp. $\mathfrak{F}^{ord}$) be any nearly ordinary (resp. cuspidal ordinary of parallel weight) p-adic family which specializes to the ordinary p-adic cuspidal Eisenstein series $E_1(\phi, \phi^\sigma)$ in weight one. It follows from Theorem 1.5 that $\mathfrak{F}$ (resp. $\mathfrak{F}^{ord}$) is unique up to a Galois conjugation, since there is only one irreducible component of $\operatorname{Spec} h_F^{n.ord}$ (resp. $\operatorname{Spec} h_F$) specializing to the point $\mathfrak{n}^{n.ord}$ (resp. $\mathfrak{n}$), and it follows from the fact that $\mathbb{T}^{n.ord}$ and $\mathbb{T}^{ord}$ are regular rings (hence integral domains). Moreover, $\mathfrak{F}^{ord}$ is the base change lift of a p-ordinary Hida Family passing through f.

In the following, the main ideas behind the proof of Theorem 1.5 will be explained :

First, we construct in Proposition 6.3 a p-nearly ordinary deformation

$$\rho_{\mathbb{T}^{n.ord}} : G_F \to \mathrm{GL}_2(\mathbb{T}^{n.ord})$$

of a reducible but indecomposable representation $\tilde{\rho}$ with trace $\phi + \phi^\sigma$ (this construction is inspired by [2]).

Subsequently, we introduce a deformation problem $\mathcal{D}^{n.ord}$ of $\tilde{\rho}$ with some local conditions at p; as such, $\mathcal{D}^{n.ord}$ is representable by $\mathcal{R}^{n.ord}$ which surjects to the local ring $\mathbb{T}^{n.ord}$ of dimension 3. The computation of the tangent space $t_{\mathcal{D}}^{n.ord}$ of $\mathcal{D}^{n.ord}$ represents an important part of the proof and shows using Galois cohomology that $t_{\mathcal{D}}^{n.ord}$ is of dimension 3 (see Theorem 6.8). Hence, the surjection $\mathcal{R}^{n.ord} \twoheadrightarrow \mathbb{T}^{n.ord}$ is an isomorphism of complete local regular rings of dimension 3.

Finally, a direct computation shows that the tangent space of the p-ordinary quotient $\mathbb{T}^{ord}$ of $\mathbb{T}^{n.ord}$ is of dimension one, and hence $\mathbb{T}^{ord}$ a discrete valuation ring.

Remark.

(i) Suppose that the residual representation $\bar{\rho}$ of ρ satisfies the assumptions of the theorems of Taylor-Wiles [36] and [38], $\bar{\phi}^2_{|I_v} \neq 1$ and $p \geq 3$, then Cho-Vatsal showed under theses additional assumptions Theorem 1.1.

(ii) H.Darmon, A.Lauder and V.Rotger stated in [13] a formula for the q-expansion of a generalised overconvergent form $f^\dagger$ in the generalized space associated to f (which is not classical). The coefficients of the generalised eigenform $f^\dagger$ are expressed as p-adic logarithms of algebraic numbers.

(iii) S.Cho provided in [8, §7] several examples of the ramification index e of $\mathcal{C}$ over $\mathcal{W}$ at f being exactly 2. More precisely, Cho presented examples where $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ is unramified over the Iwasawa algebra $\mathbb{Z}_p[[T]]$.

(iv) Dimitrov and Ghate provided in [16, §7.3] several examples emphasising that $\mathcal{T}$ is of rank two over Λ. As such, the index e is also 2 in their examples.

(v) Pilloni gave in [30] a geometric definition of overconvergent modular forms of any p-adic weight and reconstructs the eigencurve $\mathcal{C}$ without using the Eisenstein family.

Notation. If L is a number field and S the places of L above Np, we denote by $G_{L,S}$ the Galois group of the maximal extension of L unramified except at the places belonging to S and at infinite *places*.

Throughout this paper, $\mathcal{O}$ will denote the ring of integers of a p-adic field containing the image of the character ϕ.

Let $\mathbb{F}_p$ denote the residue field of $\mathcal{O}$.

Let $\mathrm{CNL}_{\mathcal{O}}$ denote the category of complete, local, Noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}_p$, and whose morphisms are the local morphisms of local rings inducing the identity on their residue fields.

For any commutative local ring A, write M_A for the free A-module $A \oplus A$, and $\mathfrak{m}_A$ for the maximal ideal of A.

Let $\Lambda_{\mathcal{O}}$ denote the Iwasawa algebra $\mathcal{O}[[T]]$.

If W is a representation of G and $\{G_i\}_{i\in I}$ are subgroups of G, we will write:

$\mathrm{H}^i(G,W)_{G_i} = \ker\left(\mathrm{H}^i(G,W) \to \underset{i\in I}{\oplus} \mathrm{H}^i(G_i,W)\right)$.

Let H be a normal subgroup of G, then we denote by $\mathrm{H}^i(H,W)^{G/H}$ the elements of $\mathrm{H}^i(H,W)$ which are invariant under the action of G/H.

We assume throughout this paper that p splits into two places v, v^σ of F, and let $\mathfrak{p}$ (resp. $\mathfrak{p}^\sigma$) denote the prime ideal over p of the ring of integers of F corresponding to the place v (resp. v^σ).

Let Δ be the Galois group of the real quadratic extension $F/\mathbb{Q}$.

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2. Preliminaries and some properties of $\mathcal{R}$ and $\mathcal{R}_{\tau=1}$

For A any local ring with maximal ideal $\mathfrak{m}_A$ and belonging to the category $\mathfrak{C}$, let $\mathcal{D}(A)$ be the set of strict equivalence classes of representations $\rho_A : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ such that $\rho_A \mod \mathfrak{m}_A = \rho$ and which are ordinary at p in the sense that

$$(\rho_A)_{|G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \psi'_A & * \\ 0 & \psi''_A \end{pmatrix},$$

where ψ''_A is an unramified character lifting ψ_2. According to Schlesinger's criteria the functor $\mathcal{D}$ is representable by $(\mathcal{R}, \rho^{ord})$ (see [3, §2]) and denotes by $t_{\mathcal{D}}$ its tangent space.

2.1. **Some properties of ρ^{ord} and the ring $\mathcal{R}_{\tau=1}$.** Let $H \subset \bar{\mathbb{Q}}$ be the number field fixed by $\ker(\mathrm{ad}\,\rho)$ and G be the Galois group of the finite Galois extension $H/\mathbb{Q}$. Since the projective image of ρ is dihedral, G contains elements of order 2 and with non trivial restriction to F; with a slight abuse of notation we will denote one of them by σ. Let (e_1, e_2) be a basis in which $\rho_{|G_F} = \phi \oplus \phi^{\sigma}$, by rescaling this basis one can assume that $\rho(\sigma) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(\bar{\mathbb{Q}})$

We will exhibit a suitable basis of the free $\mathcal{R}$-module $M_{\mathcal{R}}$, where the diagonal entries of the realization of ρ^{ord} in this basis depend only on the trace of $\mathrm{Tr}\,\rho^{ord}$. The existence of this basis will be crucial to define the functor of p-ordinary pseudo-deformations in section §3, since the line of $M_{\mathcal{R}}$ which is stable under the action of $G_{\mathbb{Q}_p}$ is not necessarily stable under the action of the complexe conjugation c.

Lemma 2.1. *Let γ_0 be a fixed element of G_{F_v}, which lifts* Frob_v *($\iota_p : G_{F_v} \xrightarrow{\simeq} G_{\mathbb{Q}_p}$) and satisfies $\phi(\gamma_0) \neq \phi^{\sigma}(\gamma_0)$, then there exists a basis $\mathfrak{B}^{ord}_{\mathcal{R}}$ of $M_{\mathcal{R}}$, such that $\rho^{ord}(\gamma_0) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ and $\rho^{ord}_{|G_{F_v}} = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ in this basis.*

Proof. Let K be the field of fractions of $\mathcal{R}$ ($\mathcal{R}$ is a discrete valuation ring). Since $\mathcal{R}$ is Henselian (even complete) and $\phi(\gamma) \neq \phi^{\sigma}(\gamma_0)$, there exists a basis of $M_{\mathcal{R}} \otimes K$ such that $\rho^{ord} \otimes K(\gamma_0) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ and $\rho^{ord}_{|G_{F_v}} \otimes K = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. Moreover, $\mathcal{R}$ is a discrete valuation ring, hence we can rescale this basis in the aim to get a basis of $M_{\mathcal{R}}$ fulfilling the desired conditions. ∎

Remark 2.2. *Since $\phi(\gamma_0) \neq \phi^{\sigma}(\gamma_0)$, any other basis satisfying the same assumptions of Lemma 2.1 is obtained by conjugating the chosen basis by a diagonal matrix. Such conjugation does not change $a(g), d(g)$ and the product $b(g).c(g)$, where $\rho^{ord}(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$.*

As ρ is dihedral, $N(\rho \otimes \epsilon_F)N = \rho$, where $N = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in (e_1, e_2).

Definition 2.3. *Let $g \to \begin{pmatrix} \tilde{a}(g) & \tilde{b}(g) \\ \tilde{c}(g) & \tilde{d}(g) \end{pmatrix}$ be the realization of ρ^{ord} in a basis $\mathfrak{B}^{ord}_{\mathcal{R}}$ satisfying the assumption of Lemma 2.1. Consider the automorphism $\tilde{N}$ of* $\mathrm{End}_{\mathcal{R}}(M_{\mathcal{R}})$ *given by*

$\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ in the basis $\mathfrak{B}_{\mathcal{R}}^{ord}$, then the map $\rho^{ord} \to \tilde{N}(\rho^{ord} \otimes \epsilon_F)\tilde{N}$ induces an automorphism $\mathfrak{t}$ of the deformation functor $\mathcal{D}$, hence an automorphism $\tau : \mathcal{R} \to \mathcal{R}$, with $\tau^2 = 1$.

Since $\operatorname{Tr}\mathfrak{t}(\rho^{ord}) = \operatorname{Tr}(\rho^{ord} \otimes \epsilon_F)$, a theorem of Nyssen [29] and Rouquier [32] implies that the deformation $\mathfrak{t}(\rho^{ord})$ is isomorphic to $\rho^{ord} \otimes \epsilon_F$. Therefore, the involution τ is independent of the choice of a basis of $M_{\mathcal{R}}$ in which $\tilde{N} = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$.

Let A be a ring in the category $\mathfrak{C}$. Then any deformation $\varphi_A : G_{\mathbb{Q},Np} \to A^\times$ of $\det(\rho^{ord})$ is equivalent to a continuous homomorphism $h : G_{\mathbb{Q},Np} \to 1 + \mathfrak{m}_A$. Using the class field theory, we obtain an isomorphism $\operatorname{Hom}(G^{ab}_{\mathbb{Q}_{Np}}, 1+\mathfrak{m}_A) \simeq \operatorname{Hom}((\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times, 1+\mathfrak{m}_A) = \operatorname{Hom}(1 + q\mathbb{Z}_p, 1 + \mathfrak{m}_A)$, where $q = p$ if $p > 2$, and $q = 4$ if $p = 2$.

Since $1 + \mathfrak{m}_A$ does not contain elements of finite order and $\Lambda \simeq \bar{\mathbb{Q}}_p[[1 + q\mathbb{Z}_p]]$, any deformation of $\det\rho$ to the ring A is obtained via a unique morphism $\Lambda \to A$. By an abuse of notation, we will write $\kappa^\# : \Lambda \to \mathcal{R}$ for the morphism induced by the deformation $\det\rho^{ord}$ of $\det\rho$ (i.e we identify $\mathcal{R}$ and $\mathcal{T}$).

Lemma 2.4.

(i) *The involution τ is an automorphism of Λ-algebras.*

(ii) *Let $\mathcal{R}_{\tau=1}$ denote the subring of $\mathcal{R}$ fixed by τ, then $\mathcal{R}_{\tau=1}$ is an object of the category $\mathfrak{C}$ and has Krull dimension equal to one.*

(iii) *$\mathcal{R}_{\tau=1}$ is a discrete valuation ring.*

(iv) *Let L denote the field of fractions of $\mathcal{R}_{\tau=1}$ and recall that K is the field of fractions of $\mathcal{R}$, then L is equal to the set of elements of K fixed by τ.*

(v) *The involution $\tau : \mathcal{R} \to \mathcal{R}$ is not trivial and the injection $\iota : \mathcal{R}_{\tau=1} \to \mathcal{R}$ has ramification index equal to 2.*

Proof.

(i) Since $\det(\rho^{ord}) = \det(\tilde{N}(\rho^{ord} \otimes \epsilon_F)\tilde{N})$, $\tau \circ \kappa^\# = \kappa^\#$.

(ii) Since $\kappa^\# : \Lambda \to \mathcal{T}$ is a finite flat homomorphism and $\mathcal{R} \simeq \mathcal{T}$, $\mathcal{R}_{\tau=1}$ is finite over Λ. The fact that Λ is a Henselian ring of dimension one (even complete) implies that $\mathcal{R}_{\tau=1}$ is a finite product of local rings with Krull dimension equal to one. However, the ring $\mathcal{R}_{\tau=1}$ is a domain ($\mathcal{R}_{\tau=1} \subset \mathcal{R}$), so $\mathcal{R}_{\tau=1}$ is a complete local ring of dimension one.

(iii) Since $\mathcal{R}_{\tau=1}$ is a local domain, Noetherian and has Krull dimension equal to one, it is sufficient to show that it is integrally closed. Let α be any element of the field of fractions of $\mathcal{R}_{\tau=1}$ such that α is integral over $\mathcal{R}_{\tau=1}$; write $\alpha = x/y$, where $x \in \mathcal{R}_{\tau=1}$ and $y \in \mathcal{R}_{\tau=1} - \{0\}$. Since $\mathcal{R}_{\tau=1}$ is a subring of $\mathcal{R}$, α is integral over $\mathcal{R}$, and it follows that $\alpha \in \mathcal{R}$ (as $\mathcal{R}$ is integrally closed). However, $\tau(\alpha) = \tau(x)/\tau(y) = x/y = \alpha$, hence $\tau(\alpha) = \alpha$ and $\alpha \in \mathcal{R}_{\tau=1}$.

(iv) Let $a \in K$ and assume that $\tau(a) = a$. Since $\mathcal{R}$ is a valuation ring, $a \in \mathcal{R}$ or $a^{-1} \in \mathcal{R}$, so $a \in \mathcal{R}_{\tau=1}$ or $a^{-1} \in \mathcal{R}_{\tau=1}$, hence $a \in L$.

(v) Assume that τ is trivial, then $\rho^{ord} \simeq \rho^{ord} \otimes \epsilon_F$. According to [18, Prop.3.1], $\rho^{ord} \simeq \mathrm{Ind}_F^{\mathbb{Q}} \phi^{ord}$, where $\phi^{ord} : G_F \to \mathcal{R}^\times$ a character. Since $\mathcal{R} \simeq \mathcal{T}$, ρ^{ord} is a representation associated to a primitive Hida family containing f (i.e corresponding to the unique irreducible component of $\mathrm{Spec}\,\mathcal{T}$). Thus, ρ^{ord} is a dihedral representation with real multiplication by F. Therefore, any specialization to weight $k \geq 2$ of a Hida family passing through f is a classical modular form of weight $k \geq 2$ having a real multiplication by F. However, it is well known that there is no RM modular forms of weight ≥ 2, resulting in a contradiction. Therefore, τ is not trivial. Since $K = L^{\tau=1}$ and $\tau^2 = 1$, L/K is an extension of degree two. ■

In the following proposition, we will compute the valuation of any generator of the ideal of reducibility of $\rho^{ord}_{|G_F}$ (i.e the ideal generated by $\{\tilde{b}(g)\tilde{c}(g') \mid g, g' \in G_F\}$).

Proposition 2.5. *Let $g \to \left(\begin{smallmatrix} \tilde{a}(g) & \tilde{b}(g) \\ \tilde{c}(g) & \tilde{d}(g) \end{smallmatrix}\right)$ be the realization of the universal deformation ρ^{ord} in the basis $\mathfrak{B}^{ord}_{\mathcal{R}}$ which lifts (e_1, e_2), $\nu_{\mathcal{R}} : \mathcal{R} \to \mathbb{N} \cup \{\infty\}$ be the discrete valuation of $\mathcal{R}$, and w_0 (resp. w_0^σ) be the place of H over v (resp. v^σ) singles out by ι_p, then:*

(i) *There exist elements g_0, h_0 of G_H such that the order of both $\tilde{b}(g_0)$ and $\tilde{c}(h_0)$ in $\mathcal{R}$ is one, and the image of $G_{H_{w_0^\sigma}}$ under $\tilde{b}$ is contained in $\mathfrak{m}^2_{\mathcal{R}}$.*

(ii) *One always has $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(F, \phi^\sigma/\phi)_{G_{F_v}} = 1$.*

Proof. (i) Note that $t_{\mathcal{D}}$ is also the tangent space of the local ring $\mathcal{R}$ representing $\mathcal{D}$.

Since p splits in F (i.e $G_{\mathbb{Q}_p} = G_{F_v}$), [3, Prop.2.3] implies the following isomorphism :

$$t_{\mathcal{D}} = \ker\left(\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{ad}\,\rho) \to \mathrm{H}^1(G_{\mathbb{Q}_p}, \phi/\phi^\sigma) \oplus \mathrm{H}^1(I_p, \bar{\mathbb{Q}}_p)\right) \qquad (1)$$

We have the following decomposition of $\mathrm{ad}\,\rho$:
$\mathrm{ad}\,\rho \simeq 1 \oplus \epsilon_F \oplus \mathrm{Ind}_F^{\mathbb{Q}}(\phi/\phi^\sigma)$, given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ and inducing the following decomposition:
$$\mathrm{H}^1(G_F, \mathrm{ad}\,\rho) \simeq \mathrm{H}^1(G_F, \phi/\phi) \oplus \mathrm{H}^1(G_F, \phi/\phi^\sigma) \oplus \mathrm{H}^1(G_F, \phi^\sigma/\phi) \oplus \mathrm{H}^1(G_F, \phi^\sigma/\phi^\sigma) \qquad (2)$$
given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to (a, b, c, d)$, where the action of $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ exchanges a, d and b, c.

After applying restriction-inflation exact sequence to the isomorphism (1), we deduce from the relation (2) and [3, Prop.4.2] that $\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{ad}\,\rho) = \mathrm{H}^1(G_F, \mathrm{ad}\,\rho)^{\mathrm{Gal}(F/\mathbb{Q})}$, and $a = d = 0$, $b = c^\sigma$, $c \in \mathrm{H}^1(F, \phi^\sigma/\phi)_{G_{F_v}}$ if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in t_{\mathcal{D}} \subset \mathrm{H}^1(G_F, \mathrm{ad}\,\rho)^{\mathrm{Gal}(F/\mathbb{Q})}$. According to [3, Theorem 2.2], $\dim t_{\mathcal{D}} = 1$, so c is not trivial and the same holds for b (since $b = c^\sigma$).

On the other hand, $c = b^\sigma$, so $b \in \mathrm{H}^1(F, \phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$. The restriction-inflation exact sequence yields $b_{|G_H} \in \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\mathrm{Gal}(H/F)}_{G_{H_{w_0^\sigma}}}$, where $\mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\mathrm{Gal}(H/F)}_{G_{H_{w_0^\sigma}}}$ is the subspace of

$\mathrm{H}^1(H, \bar{\mathbb{Q}}_p)$ given by the homomorphisms which are unramified at w_0^σ and invariant under the action of $\mathrm{Gal}(H/F)$.

Let $\rho_\epsilon \in \mathcal{D}(\bar{\mathbb{Q}}_p[\epsilon])$ be the deformation of ρ induced by the composition of ρ^{ord} with the canonical projection $\mathcal{R} \twoheadrightarrow \mathcal{R}/\mathfrak{m}_{\mathcal{R}}^2 \simeq \bar{\mathbb{Q}}_p[\epsilon]$. Therefore, $\rho_\epsilon(g) = (1+\epsilon\rho_1(g))\rho(g)$, where the cohomology class of the cocycle $\rho_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a generator of $t_{\mathcal{D}}$. Let $g \to \left(\begin{smallmatrix} a'(g) & b'(g) \\ c'(g) & d'(g) \end{smallmatrix}\right)$ be the realization of ρ_ϵ by a matrix. Since $\rho_{|G_F}$ is diagonal, $b_{|G_H} \neq 0$, $c_{|G_H} \neq 0$ and $b_{|G_{H_{w_0^\sigma}}} = 0$, then $b'_{|G_H} \neq 0$, $c'_{|G_H} \neq 0$ and $b'_{|G_{H_{w_0^\sigma}}} = 0$, hence $\tilde{b}_{|G_H} \neq 0$, $\tilde{c}_{|G_H} \neq 0$ modulo $\mathfrak{m}_{\mathcal{R}}^2$, and we have also $\tilde{b}_{|G_{H_{w_0^\sigma}}} = 0$ modulo $\mathfrak{m}_{\mathcal{R}}^2$ (since $G_H = \ker(\mathrm{ad}\,\rho)$).

(ii) It is a direct result of the isomorphism $t_{\mathcal{D}} \simeq \mathrm{H}^1(F, \phi^\sigma/\phi)_{G_{F_v}}$ and [3, Theorem 2.2] (i.e $\dim_{\bar{\mathbb{Q}}_p} t_{\mathcal{D}} = 1$). ■

2.2. **Criterion to extend a G_F-representation to $G_{\mathbb{Q}}$.** In this subsection, we give a sufficient condition for extending a representation $\rho_K : G_F \to \mathrm{GL}_2(K)$ to all $G_{\mathbb{Q}}$, which will be crucial in the proof of Theorem 1.1.

Definition 2.6. *Let K be a ring and $\rho_K : G_F \to \mathrm{GL}_n(K)$ be a representation. Write $\rho_K^t(g)$ for $\rho_K(tgt^{-1})$, where t is an element of $G_{\mathbb{Q}}$ with a non trivial restriction to F.*

Consider the following condition on ρ_K:

(C) For each $t \in G_{\mathbb{Q}}$, there exists $r(t) \in GL_n(K)$ such that $\rho_K = r(t)^{-1}\rho_K^t r(t)$.

Proposition 2.7. *Let $\rho_K : G_F \to \mathrm{GL}_n(K)$ be a representation, where K is a ring. Assume that the only matrices in $M_n(K)$ that commute with the image of ρ_K are the scalar matrices, and ρ_K satisfies the condition (C). Then:*

(i) *If $G_{\mathbb{Q}} = G_F \sqcup G_F.t$ for a fixed $t \in G_{\mathbb{Q}}$, r can be selected, guaranteeing that the following conditions are satisfied : $\forall h \in G_F$, $r(ht) = \rho_K(h)r(t)$ and $r(h) = \rho_K(h)$.*

(ii) *The function $\varrho : G_{\mathbb{Q}} \times G_{\mathbb{Q}} \to K^\times$ defined by $\varrho(t', t) = r(t')r(t)r^{-1}(t't)$ is an element of $\mathrm{H}^2(G_{\mathbb{Q}}, K^\times)$ for the trivial action of $G_{\mathbb{Q}}$. Moverover, ϱ factors through $\Delta = \mathrm{Gal}(F/\mathbb{Q})$.*

(iii) *If the cohomology class of $\varrho \in \mathrm{H}^2(\Delta, K^\times)$ vanishes, then there exists a representation $r : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(K)$ extending ρ_K, and if r' is another extension of ρ_K, then $r' = r \otimes \epsilon_F$.*

Proof. See [25, A 1.1]. ■

Corollary 2.8.

(i) *Let $\rho_K : G_F \to \mathrm{GL}_n(K)$ be a representation, where K is a field. If ρ_K satisfies the hypothesis of Proposition 2.7, there exists a finite extension L/K and a representation $\rho_L : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(L)$ extending ρ_K.*

(ii) *Let A be a ring in the category $\mathfrak{C}$ and $\psi_A : G_F \to A^\times$ be a character invariant under the action of $G_\mathbb{Q}$. Then there exists a character $\psi'_A : G_\mathbb{Q} \to A^\times$ extending ψ_A.*

Proof. (i) We have a functorial isomorphism $\mathrm{H}^2(\Delta, K^\times) \simeq K^\times/(K^\times)^2$. Choose an element $x \in K^\times$ corresponding to the cohomology class of $[\varrho]$ in $\mathrm{H}^2(\Delta, K^\times)$. Let L be a finite extension of K containing $\sqrt{x}$, then the cohomology class of $[\varrho]$ in $\mathrm{H}^2(\Delta, L^\times)$ vanishes. Hence, we may conclude by Proposition 2.7.

(ii) The residue field of A is $\bar{\mathbb{Q}}_p$ and it is algebraically closed. Consequently, Hensel lemma's implies that the group $\mathrm{H}^2(\Delta, A^\times) = A^\times/(A^\times)^2$ is trivial, and as such the desired result follows from Proposition 2.7. ■

3. Pseudo-deformation and the ring $\mathcal{R}^{ps}$

3.1. **Pseudo-Character and pseudo-representation.** The first occurrence of pseudo-representation appeared in the work of Wiles (see [37], pp $563 - 564$ for details), but his definition requires the presence of a complex conjugation c, which forces the pseudo-representation to depend only its trace. In our case, the complex conjugation c will be replaced by γ_0 which is a fixed lift of Frob_v to G_{F_v}. In Lemma 3.3, we illustrate through the presence of γ_0 how a pseudo-representation depends only on its trace and determinant.

Definition 3.1. *Let A be a commutative ring and γ_0 be a fixed lift of Frob_v to G_{F_v} such that $\phi(\gamma_0) \neq \phi^\sigma(\gamma_0)$.*

Let $\tilde{a}, \tilde{d} : G_F \to A$, $\tilde{x} : G_F \times G_F \to A$ be a three continuous functions satisfying the following conditions: For all $g, h, t, s, w, n \in G_F$, the following applies :

1) $\tilde{a}(st) = \tilde{a}(s).\tilde{a}(t) + \tilde{x}(s,t)$
2) $\tilde{d}(st) = \tilde{d}(s).\tilde{d}(t) + \tilde{x}(t,s)$
3) $\tilde{x}(s,t).\tilde{x}(w,n) = \tilde{x}(s,n).\tilde{x}(w,t)$
4) $\tilde{x}(st, wn) = \tilde{a}.(s).\tilde{a}(n).\tilde{x}(t,w) + \tilde{a}(n).\tilde{d}(t).\tilde{x}(s,w) + \tilde{a}(s).\tilde{d}(w).\tilde{x}(t,n) + \tilde{d}(t).\tilde{d}(w).\tilde{x}(s,n)$
5) $\tilde{a}(1) = \tilde{d}(1) = 1$ *and* $\tilde{x}(h,1) = \tilde{x}(1,g) = 0$.
6) $\tilde{x}(\gamma_0, g) = \tilde{x}(h, \gamma_0) = 0$.

We say that $\pi_A = (\tilde{a}, \tilde{d}, \tilde{x})$ is a pseudo-representation (see [37, 2.2.3] *for more details). The trace and determinant of π_A are the functions* $\mathrm{Tr}(\pi_A)(g) = \tilde{a}(g) + \tilde{d}(g)$, *and* $\det \pi_A(g) = \tilde{a}(g)\tilde{d}(g) - \tilde{x}(g,g)$.

Let $\pi = (\phi, \phi^\sigma, 0)$ be the pseudo-representation associated to the representation $\rho_{|G_F}$.

Definition 3.2. *Let A be a ring in $\mathfrak{C}$ and $\pi_A = (\tilde{a}_A, \tilde{d}_A, \tilde{x}_A)$ be a continuous pseudo-representation in A, we say that π_A is a pseudo-deformation if and only if π_A mod $\mathfrak{m}_A = \pi$.*

Meantime, the work [34] is a reference for pseudo-deformations.

Lemma 3.3.

(i) *Let A be a ring in $\mathfrak{C}$, and $\pi_A = (\tilde{a}_A, \tilde{d}_A, \tilde{x}_A)$ be a pseudo-deformation, then π_A depends only on $\operatorname{Tr}\pi_A$ and $\det\pi_A$ by the following formula:*

$$\begin{aligned}\tilde{a}_A(g) &= \frac{\operatorname{Tr}\pi_A(\gamma_0 g) - \lambda_2 \operatorname{Tr}\pi_A(g)}{\lambda_1 - \lambda_2}\\ \tilde{d}_A(g) &= \frac{\operatorname{Tr}\pi_A(\gamma_0 g) - \lambda_1 \operatorname{Tr}\pi_A(g)}{\lambda_2 - \lambda_1}\end{aligned} \tag{1}$$

where $\lambda_1 = \tilde{a}(\gamma_0)$ and $\lambda_2 = \tilde{d}(\gamma_0)$ are the unique roots of the polynomial

$$X^2 - \operatorname{Tr}\pi_A(\gamma_0)X + \det\pi_A(\gamma_0). \tag{2}$$

(ii) *If A is a domain, then π_A depends only on its trace (i.e $\det\pi_A$ depends on $\operatorname{Tr}\pi_A$).*

Proof. (i) Since $\tilde{x}(\gamma_0, \gamma_0) = 0$, $\det\pi_A(\gamma_0) = \tilde{a}(\gamma_0)\tilde{d}(\gamma_0)$, then $\tilde{a}(\gamma_0)$ and $\tilde{d}(\gamma_0)$ are solutions of (2). By assumption $\phi(\gamma_0) \neq \phi^\sigma(\gamma_0)$, so Hensel lemma's implies that $\tilde{a}(\gamma_0)$ and $\tilde{d}(\gamma_0)$ are the unique solution of (2).

Finally, the relation (1) follows directly from relations defining pseudo-deformation.

(ii) Let K be the fraction field of A and $\bar{K}$ its algebraic closure. The function $\operatorname{Tr}\pi_A : G_F \to K$ is a pseudo-character. According to [35, Theorem.1.1], there exists a unique semi-simple Galois representation $\rho_K : G_F \to \mathrm{GL}_2(\bar{K})$ such that $\operatorname{Tr}\rho_K = \operatorname{Tr}\pi_A$ and $\det\rho_K = \det\pi_A$. ∎

3.2. **Ordinary Pseudo-deformation.** In this sub-section, we will define a sub-functor of the pseudo-deformation functor of π, which is representable by a ring $\mathcal{R}^{ps}$ belonging to the objects of the category $\mathfrak{C}$.

Definition 3.4. *Let $\mathfrak{G} : \mathfrak{C} \to \mathrm{SETS}$ be the functor of all pseudo-deformations $\pi_A = (\tilde{a}_A, \tilde{d}_A, \tilde{x}_A)$ of π which satisfy the following conditions:*

(i) *For all $h \in G_{F_v}$, $h' \in G_F$, $\tilde{x}_A(h', h) = 0$.*
(ii) *$\tilde{d}_A(g) = 1$ if $g \in I_v$.*
(iii) *$\operatorname{Tr}\pi_A(t^{-1}gt) = \operatorname{Tr}\pi_A(g)$ for each t in $G_{\mathbb{Q}}$ and $g \in G_F$.*

Proposition 3.5.

(i) *Let $\pi'_\epsilon = (a', d', x')$ be an element of $\mathfrak{G}(\bar{\mathbb{Q}}_p[\epsilon])$, then for any h in G_F, $\frac{x'(h,.)}{\phi^\sigma(h)\phi(.)}$ (resp. $\frac{x'(.,h)}{\phi^\sigma(.)\phi(h)}$) is an element of $\mathrm{Z}^1(F, \phi/\phi^\sigma)$ (resp. $\mathrm{Z}^1(F, \phi^\sigma/\phi)$).*
(ii) *The functor $\mathfrak{G}$ is representable by $(\mathcal{R}^{ps}, \pi^{ps})$.*
(iii) *The determinant $\det\pi^{ps}$ is invariant under the action of σ.*

Proof. (i) It results from the defining properties of a pseudo-deformation.

(ii) The functor $\mathfrak{G}$ satisfies Schlesinger's criteria, the only non-trivial point is the finiteness of the dimension of the tangent space $t_{\mathfrak{G}}$ of $\mathfrak{G}$, and this follows from [34, Lemma.2.10] and the fact that $\mathrm{H}^1(F, \phi/\phi^\sigma)$ has a finite dimension.

(iii) A direct computation shows that

$$\operatorname{Tr}\pi^{ps}(g^2) = (\operatorname{Tr}\pi^{ps}(g))^2 - 2\det\pi^{ps}(g),$$

so the assertion follows from the fact that $\forall t \in G_{\mathbb{Q}}, \forall g \in G_F$, $\operatorname{Tr}\pi^{ps}(t^{-1}gt) = \operatorname{Tr}\pi^{ps}(g)$. ■

Lemma 3.6. *There exists a natural morphism $\Lambda \longrightarrow \mathcal{R}^{ps}$ induced by the deformation* $\det\pi^{ps}$ *of* $\det\pi$.

Proof. According to (iii) of Lemma 3.5 and Corollary 2.8, we can extend $\det\pi^{ps}$ into a character $\varphi : G^{ab}_{\mathbb{Q},Np} \to (\mathcal{R}^{ps})^\times$ and we choose one whose reduction modulo $\mathfrak{m}_{\mathcal{R}^{ps}}$ is equal to $\det\rho$. Therefore, there exists a unique morphism $\Lambda \longrightarrow \mathcal{R}^{ps}$ which sends the universal deformation of $\det\rho$ to φ.

■

3.3. Proof of the isomorphism $\mathcal{R}^{ps}_{red} \simeq \mathcal{R}_{\tau=1}$.

Lemma 3.7. *Let $g \to \begin{pmatrix} \tilde{a}_g & \tilde{b}_g \\ \tilde{c}_g & \tilde{d}_g \end{pmatrix}$ be the realization of ρ^{ord} in a basis $\mathfrak{B}^{ord}_{\mathcal{R}} = \{v_1, v_2\}$ (see Lemma 2.1), then:*

(i) *The 3-tuple $\pi_{\mathcal{R}_{\tau=1}} = (\tilde{a}_{|G_F}, \tilde{d}_{|G_F}, \tilde{b}_{|G_F}\tilde{c}_{|G_F})$ is a pseudo-deformation of π.*

(ii) *There exists a unique local homomorphism $g : \mathcal{R}^{ps} \to \mathcal{R}_{\tau=1}$ inducing the pseudo-deformation $\pi_{\mathcal{R}_{\tau=1}}$.*

Proof. (i) It is a direct result of the relations defining a pseudo-representation.

(ii) Since the representation $\rho^{ord}_{|G_F}$ is ordinary at G_{F_v}, there exists a unique morphism $g : \mathcal{R}^{ps} \to \mathcal{R}$ such that $g \circ \pi^{ps} = \pi_{\mathcal{R}_{\tau=1}}$.

Moreover, the action of τ on $\operatorname{Tr}\rho^{ord}$ (resp. on $\det\rho^{ord}$) is given by $\operatorname{Tr}\rho^{ord} \to \operatorname{Tr}\rho^{ord}\otimes\epsilon_F$ (resp. $\det\rho^{ord} \to \det\rho^{ord} \otimes \epsilon_F$), so τ acts trivially on $\operatorname{Tr}\rho^{ord}_{|G_F}$ (resp. on $\det\rho^{ord}$).

Since $\mathcal{R}_{\tau=1}$ is henselian (even complete), $\phi(\gamma_0) \neq \phi^\sigma(\gamma_0)$ and $\operatorname{Tr}\rho^{ord}(\gamma_0)$, $\det\rho^{ord}(\gamma_0)$ are elements of the ring $\mathcal{R}_{\tau=1}$ ($\gamma_0 \in G_{F_v} \subset G_F$), the eigenvalues λ_1 and λ_2 of $\rho^{ord}(\gamma_0)$ are in $\mathcal{R}_{\tau=1}$.

On the other hand, a direct computation shows that $\tilde{a}(g) = \frac{\operatorname{Tr}\rho^{ord}(\gamma_0 g) - \lambda_2 \operatorname{Tr}\rho^{ord}(g)}{\lambda_1 - \lambda_2}$, $\tilde{d}(g) = \frac{\operatorname{Tr}\rho^{ord}(\gamma_0 g) - \lambda_1 \operatorname{Tr}\rho^{ord}(g)}{\lambda_2 - \lambda_1}$ and $\tilde{a}(gh) = \tilde{a}(g)\tilde{a}(h) + \tilde{x}(g,h)$. Therefore, $\tau(\tilde{a}_{|G_F}) = \tilde{a}_{|G_F}$, $\tau(\tilde{d}_{|G_F}) = \tilde{d}_{|G_F}$ and $\tau(\tilde{b}_{|G_F}.\tilde{c}_{|G_F}) = \tilde{b}_{|G_F}.\tilde{c}_{|G_F}$. Therefore, g factors through $\mathcal{R}_{\tau=1}$. ■

Lemma 3.8.

The morphism $g : \mathcal{R}^{ps} \to \mathcal{R}_{\tau=1}$ is surjective.

Proof. According to Lemma 2.5, there exist g_0, h_0 in G_F such that the order of both $\tilde{b}(g_0)$ and $\tilde{c}(h_0)$ in $\mathcal{R}$ is one, so $\tilde{x}(g_0,h_0) = \tilde{b}(g_0)\tilde{c}(h_0)$ is of order 2 in $\mathcal{R}$. However, $\mathcal{R}_{\tau=1}$ is a discrete valuation ring and the injection $\iota : \mathcal{R}_{\tau=1} \hookrightarrow \mathcal{R}$ is ramified with a ramification index equal to 2, so $\tilde{b}(g_0)\tilde{c}(h_0) = \tilde{x}(g_0,h_0)$ has order one in $\mathcal{R}_{\tau=1}$. On the other hand, since $\mathcal{R}^{ps}$ is the universal ring representing the functor $\mathfrak{G}$, $\tilde{x}(g_0,h_0)$ is contained in the image of the maximal ideal of $\mathcal{R}^{ps}$ under the morphism g.

Let $\mathcal{B}$ be the image of the morphism g, then $\mathcal{B}$ is a sub-algebra of $\mathcal{R}_{\tau=1}$. Let ν_τ denote the discrete valuation of the ring $\mathcal{R}_{\tau=1}$ and $\mathfrak{m}_{\mathcal{B}}$ denote the maximal ideal of $\mathcal{B}$. The discussion further above implies that $\mathfrak{m}_{\mathcal{B}}$ contains an uniformizing element of $\mathcal{R}_{\tau=1}$. Write $\mathfrak{a}$ for the ideal $\mathfrak{m}_{\mathcal{B}}\mathcal{R}_{\tau=1}$, so $\mathfrak{a} = \mathfrak{m}_{\mathcal{R}_{\tau=1}}$ since $\mathfrak{m}_{\mathcal{B}}$ contains an uniformizing element of $\mathcal{R}_{\tau=1}$.

According to Lemma 3.6, the ring $\mathcal{R}^{ps}$ has a natural structure of Λ-algebra. Since $\det \pi_{\mathcal{R}_{\tau=1}} = g \circ \det \pi^{ps}$, g is a morphism of Λ-algebras. Moreover, $\mathcal{R}_{\tau=1}$ is a finite Λ-module, thus the morphism $g : \mathcal{R}^{ps} \to \mathcal{R}_{\tau=1}$ is finite.

Now, apply Nakayama's lemma to the $\mathcal{R}^{ps}$-module $\mathcal{R}_{\tau=1}$, and it will become apparent that 1 is a generator of $\mathcal{R}_{\tau=1}$ as $\mathcal{R}^{ps}$-module. Hence, the morphism g is surjective.

■

Proof of theorem 1.1. We will show that the morphism $g : \mathcal{R}^{ps} \to \mathcal{R}_{\tau=1}$ rises to an isomorphism $\mathcal{R}^{ps}/\mathfrak{N} \simeq \mathcal{R}_{\tau=1}$, where $\mathfrak{N}$ is the radical of $\mathcal{R}^{ps}$. Let $\mathfrak{L}$ denote the kernel of the morphism g; since g is surjective (see Proposition 3.8), the statement is equivalent to $\mathfrak{L} \subset \mathfrak{N}$, meaning that $\operatorname{Spec} \mathcal{R}_{\tau=1} = \operatorname{Spec} \mathcal{R}^{ps}$.

Let $\mathfrak{P}$ be a prime ideal of $\mathcal{R}^{ps}$, and $\pi'' : \mathcal{R}^{ps} \twoheadrightarrow \mathcal{R}^{ps}/\mathfrak{P}$ be the canonical surjection. Let K denote the field of fractions of $\mathcal{R}^{ps}/\mathfrak{P}$ and $\pi_{\mathfrak{P}} = (\tilde{a}_{\mathfrak{P}}, \tilde{d}_{\mathfrak{P}}, \tilde{x}_{\mathfrak{P}})$ denote the pseudo-deformation obtained by the composition $\pi'' \circ \pi^{ps}$.

If $\tilde{x}_{\mathfrak{P}} = 0$, then $\rho_K(g) = \begin{pmatrix} \tilde{a}_{\mathfrak{P}}(g) & 0 \\ 0 & \tilde{d}_{\mathfrak{P}}(g) \end{pmatrix}$ is the unique semi-simple representation associated to $\pi_{\mathfrak{P}}$.

By assumption $\operatorname{Tr}(\rho_K) = \operatorname{Tr}(\rho_K^\sigma)$, so $\tilde{a}_{\mathfrak{P}}^\sigma = \tilde{d}_{\mathfrak{P}}$ (since the action of σ exchanges ϕ et ϕ^σ and $\phi \neq \phi^\sigma$). In these terms, $\operatorname{Ind}_F^{\mathbb{Q}} \tilde{a}_{\mathfrak{P}}$ is a representation extending ρ_K to $G_{\mathbb{Q}}$.

If there exist $g_1, h_1 \in G_F$ such that $\tilde{x}_{\mathfrak{P}}(g_1,h_1) \neq 0$, [37, Prop.2.2.1] implies the existence of a Galois representation $\rho_K : g \to \begin{pmatrix} \tilde{a}_{\mathfrak{P}}(g) & \tilde{x}_{\mathfrak{P}}(g,h_1)/\tilde{x}_{\mathfrak{P}}(g_1,h_1) \\ \tilde{x}_{\mathfrak{P}}(g_1,g) & \tilde{d}_{\mathfrak{P}}(g) \end{pmatrix}$ with $\operatorname{Tr} \rho_K = \operatorname{Tr} \pi_{\mathfrak{P}}$.

As $\rho_K(\gamma_0)$ is diagonal with distinct eigenvalues and $\tilde{x}_{\mathfrak{P}}(g_1,h_1) \neq 0$ implies that ρ_K is absolutely-irreducible. Moreover, $\operatorname{Tr} \pi_{\mathfrak{B}}$ is invariant under the action of σ (i.e $\operatorname{Tr} \pi_{\mathfrak{B}} = \operatorname{Tr} \rho_K = \operatorname{Tr} \rho_K^\sigma$), so [35, Theorem.1] yields an isomorphism $\rho_K \otimes \overline{K} \simeq \rho_K^\sigma \otimes \overline{K}$.

Therefore, there exists $r(\sigma) \in \mathrm{GL}_2(L')$, where L' is finite extension of K such that $r(\sigma)\rho_K r^{-1}(\sigma) = \rho_K^\sigma$. Thus, the representation ρ_K satisfies the hypothesis of Corollary

2.8, and hence there exist a finite extension L/L' and a representation $\rho_L : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(L)$ extending ρ_K.

Let $\mathcal{A}$ be the integral closure of $\mathcal{R}^{ps}/\mathfrak{P}$ in L. Since $\mathcal{R}^{ps}/\mathfrak{P}$ is a local Nagata ring (even complete), $\mathcal{A}$ is finite over $\mathcal{R}^{ps}/\mathfrak{P}$; by using samilar arguments to those already used to proof (ii) of Lemma 2.4, we may deduce that $\mathcal{A} \in \mathfrak{C}$.

On the other hand, $\mathrm{Tr}\,\rho_L(\sigma^2) = \mathrm{Tr}\,\rho_L(\sigma))^2 - 2\det\rho_L(\sigma)$, so $\mathrm{Tr}(\rho_L(G_{\mathbb{Q}})) \subset \mathcal{A}$. Thus, $\mathrm{Tr}\,\rho_L : G_{\mathbb{Q}} \to \mathcal{A}$ is a pseudo-character such that the restriction to G_F of its reduction modulo $\mathfrak{m}_{\mathcal{A}}$ is equal to $\mathrm{Tr}\,\rho_{|G_F}$.

According to Proposition 2.7, the restriction of ρ to G_F extends uniquely to $G_{\mathbb{Q}}$ since $\rho \simeq \rho \otimes \epsilon_F$, hence [35, Theorem.1] implies that the reduction of the pseudo-character $\mathrm{Tr}\,\rho_L$ modulo $\mathfrak{m}_A$ is equal to $\mathrm{Tr}(\rho)$.

According to a theorem of Nyssen [29] and Rouquier [32], there exists a deformation $\rho_{\mathcal{A}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{A})$ of ρ, such that $\mathrm{Tr}\,\rho_{\mathcal{A}} = \mathrm{Tr}\,\rho_L$. In addition, we have $G_{F_v} = G_{\mathbb{Q}_p}$ (since p splits in F) and by construction $(\rho_K)_{|G_{\mathbb{Q}_p}} \simeq (\rho_{\mathcal{A}} \otimes L)_{|G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \psi_1' & * \\ 0 & \psi_2' \end{pmatrix}$, where $\psi_2' : G_{\mathbb{Q}_p} \to \mathcal{A}^{\times}$ is an unramified character lifting $\phi^{\sigma}_{|G_{F_v}}$ (i.e $\psi_2' = (\tilde{d}_{\mathfrak{P}})_{|G_{\mathbb{Q}_p}}$); therefore, by using similar arguments to those already used to proof [3, Prop.5.1], we deduce that the representation $\rho_{\mathcal{A}}$ is ordinary at p.

Thus, there exists a unique morphism $h : \mathcal{R} \to \mathcal{A}$ inducing $\rho_{\mathcal{A}}$.

$$\begin{array}{ccccc} \mathcal{R}^{ps} & \xrightarrow{\ g\ } & \mathcal{R}_{\tau=1} & \xhookrightarrow{\ \iota\ } & \mathcal{R} \\ \downarrow{\scriptstyle \pi''} & & & \swarrow{\scriptstyle h} & \\ \mathcal{R}^{ps}/\mathfrak{P} & \xhookrightarrow{} & \mathcal{A} & & \end{array}$$

The morphisms $h \circ \iota \circ g$ and π'' induce two pseudo-deformations of π with the same trace and determinant. Now, thanks to Lemma 3.3 we know that a pseudo-deformation depends only on its trace and determinant, so $h \circ \iota \circ g = \pi''$. Therefore, the diagram above is commutative and implies immediately the inclusion $\mathfrak{L} \subset \mathfrak{P}$. Finally, we may conclude that the ideal $\mathfrak{L}$ is included in the radical of $\mathcal{R}^{ps}$.

4. Proof of the main Theorem 1.2

Recall that $H \subset \bar{\mathbb{Q}}$ is the number field fixed by $\ker(\mathrm{ad}\,\rho)$, $H_{\infty,v}$ (resp. H_{∞,v^σ}) is the compositum of all $\mathbb{Z}_p$-extensions of H which are unramified outside v (resp. v^σ), H_∞ is the compositum of $H_{\infty,v}$ and H_{∞,v^σ}, L_∞ is the maximal unramified abelian p-extension of H_∞, and X_∞ is the Galois group $\mathrm{Gal}(L_\infty/H_\infty)$. The Galois group $\mathrm{Gal}(H_\infty/H) \simeq \mathbb{Z}_p^{2s}$ acts by conjugation on X_∞ and Greenberg proves that X_∞ is a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(H_\infty/H)]]$-module (see [20]).

Let F'' be the maximal unramified extension of H contained in H_∞ and L_0 be the subfield of L_∞ such that $\mathrm{Gal}(L_0/H_\infty)$ is the largest quotient of X_∞ on which $\mathrm{Gal}(H_\infty/F)$ acts trivially.

Hypothesis. Assume that $\mathrm{Gal}(L_0/F'')$ is abelian. (**G**)

In this section, we prove that $\mathcal{R}_{\tau=1}$ is isomorphic to Λ when (**G**) holds, and it is equivalent to prove that the tangent space of $\mathcal{R}_{\tau=1}/(\mathfrak{m}_\Lambda, \mathfrak{m}^2_{\mathcal{R}_{\tau=1}})$ is trivial when (**G**) holds.

4.1. **Tangent space of $\mathcal{R}_{\tau=1}$.** Denote by $t_{\mathcal{R}_{\tau=1}}$ the tangent space of $\mathcal{R}_{\tau=1}$, since $\mathcal{R}_{\tau=1}$ is a discrete valuation ring (see Lemma 2.4), the dimension of $t_{\mathcal{R}_{\tau=1}}$ is one.

Write $t'_{\mathcal{R}_{\tau=1}}$ for the sub-space of $t_{\mathcal{R}_{\tau=1}}$ of pseudo-deformations with determinant equal to $\det\pi = \det\rho_{|G_F}$. It follows from theorem 1.1 that $t'_{\mathcal{R}_{\tau=1}} \hookrightarrow t_{\mathcal{R}_{\tau=1}} \hookrightarrow \mathfrak{G}(\bar{\mathbb{Q}}_p[\epsilon])$. One can see that the tangent space of $\mathcal{R}_{\tau=1}/(\mathfrak{m}_\Lambda, \mathfrak{m}^2_{\mathcal{R}_{\tau=1}})$ is isomorphic to $t'_{\mathcal{R}_{\tau=1}}$.

In the following lemma, we introduce a representation $\rho_{\tau=1} : G_F \to \mathrm{GL}_2(\mathcal{R}_{\tau=1})$ which is conjugate to $\rho^{ord}_{|G_F}$ by a matrix with coefficients in the field of fractions of $\mathcal{R}$ and such that $\mathrm{Tr}\,\rho_{\tau=1} = \pi_{\mathcal{R}_{\tau=1}}$. The introduction $\rho_{\tau=1}$ is necessary in order to produce a non trivial extension in $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)$.

Lemma 4.1.

(i) *There exists a representation $\rho_{\tau=1} : G_F \to \mathrm{GL}_2(\mathcal{R}_{\tau=1})$ such that the pseudo-representation associated to $\rho_{\tau=1}$ is $\pi_{\mathcal{R}_{\tau=1}}$.*

(ii) *The residual representation of $\rho_{\tau=1}$ modulo $\mathfrak{m}_{\mathcal{R}_{\tau=1}}$ has the following form $\tilde{\rho}(g) = \begin{pmatrix} \phi & \eta \\ 0 & \phi^\sigma \end{pmatrix}$, where η/ϕ^σ is a non trivial element of $\mathrm{H}^1(F, \phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$.*

(iii) *There exists a basis (e'_1, e'_2) of $M_{\bar{\mathbb{Q}}_p}$ such that $\tilde{\rho}_{|G_{F_{v^\sigma}}}$ splits in this basis. Moreover, $\rho_{\tau=1}$ is ordinary at v^σ and the line stabilized by $G_{F_{v^\sigma}}$ lifts e'_2.*

Proof. (i) According to Lemma 2.5, there exist $g_0, h_0 \in G_H$ such that the order of both $\tilde{b}(g_0)$ and $\tilde{c}(h_0)$ in $\mathcal{R}$ is one. By [37, Prop.2.2.1], $\rho_{\tau=1}(g) = \begin{pmatrix} \tilde{a}(g) & \tilde{x}(g,h_0)/\tilde{x}(g_0,h_0) \\ \tilde{x}(g_0,g) & \tilde{d}(g) \end{pmatrix}$ is a representation of G_F. Since $\tilde{b}(G_F) \subset \mathfrak{m}_\mathcal{R}$ and the order of $\tilde{b}(g_0)$ in $\mathcal{R}$ is one, the order of $\frac{\tilde{x}(g,h_0)}{\tilde{x}(g_0,h_0)} = \frac{\tilde{b}(g)}{\tilde{b}(g_0)}$ in $\mathrm{Frac}(\mathcal{R})$ is non-negative. Hence, $\frac{\tilde{x}(g,h_0)}{\tilde{x}(g_0,h_0)} = \frac{\tilde{b}(g)}{\tilde{b}(g_0)}$ is an element of $\mathcal{R}$.

However, $\frac{\tilde{x}(g,h_0)}{\tilde{x}(g_0,h_0)}$ is invariant by τ, so that it belongs to $\mathcal{R}_{\tau=1}$.

(ii) Since for all $g \in G_F$, $\tilde{x}(g_0, g) \in \mathfrak{m}_{\mathcal{R}_{\tau=1}}$, the residual representation of $\rho_{\tau=1}$ has the following form $g \to \begin{pmatrix} \phi(g) & \eta(g) \\ 0 & \phi^\sigma(g) \end{pmatrix}$, where η/ϕ^σ is a non trivial element of $\mathrm{H}^1(F, \phi/\phi^\sigma)$. Proposition 2.5 implies that $\tilde{b}(G_{H_{w_0^\sigma}}) \subset \mathfrak{m}^2_\mathcal{R}$. Thus, for all g in $G_{H_{w_0^\sigma}}$, $\frac{\tilde{x}(g,h_0)}{\tilde{x}(g_0,h_0)} = \frac{\tilde{b}(g)}{\tilde{b}(g_0)} \in \mathfrak{m}_\mathcal{R}$.

Moreover, $\frac{\tilde{x}(g,h_0)}{\tilde{x}(g_0,h_0)} = \frac{\tilde{b}(g)}{\tilde{b}(g_0)}$ is invariant by τ, so that it belongs to $\mathfrak{m}_{\mathcal{R}_{\tau=1}}$. Hence, $\eta/\phi^\sigma_{|G_{H_{w_0^\sigma}}} = 0$, so $\eta/\phi^\sigma_{|G_H} \in \mathrm{H}^1(H, \bar{\mathbb{Q}}_p)^{\mathrm{Gal}(H/F)}_{G_{H_{w_0^\sigma}}}$.

On the other hand, the restriction inflation exact-sequence yields the following isomorphism $\mathrm{H}^1(H,\bar{\mathbb{Q}}_p)^{\mathrm{Gal}(H/F)}_{G_{H_{w_0^\sigma}}} \simeq \mathrm{H}^1(F,\phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$, hence $\eta/\phi^\sigma \in \mathrm{H}^1(F,\phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$.

(iii) Observe that $\rho_{\tau=1}$ is conjugate to $\rho^{ord}_{|G_F}$ by the matrix $\left(\begin{smallmatrix} 1/\tilde{b}(g_0) & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so the representation $\rho_{\tau=1}\otimes K$ is ordinary at v^σ.

Since the representation $\tilde{\rho}_{|G_{F_{v^\sigma}}}$ splits ($\eta/\phi^\sigma \in \mathrm{H}^1(F,\phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$), $\mathcal{R}_{\tau=1}$ contains the eigenvalues of $\rho_{\tau=1}(\sigma^{-1}\gamma_0\sigma)$ and $\rho_{\tau=1}\otimes L$ is ordinary at v^σ, then by using similar arguments to those already used to proof [3, Prop.5.1] we deduce that $\rho_{\tau=1}$ is ordinary at v^σ. ■

Lemma 4.2. *Let $\pi_\epsilon = (\tilde{a}_\epsilon, \tilde{d}_\epsilon, \epsilon\tilde{x}_\epsilon)$ be an element of $t_{\mathcal{R}_{\tau=1}}$ and w be a place of H above v^σ, then :*

(i) *For any g in G_F, the restriction of the function $h \to \tilde{x}_\epsilon(h,g)$ to the decomposition group G_{H_w} is trivial.*
(ii) *The function $\tilde{x}_\epsilon(.,*)$ is trivial when one of its components belongs to* $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$.

Proof. (i) Let g be any element of G_F and w be any place of H above v^σ, then (*iii*) of Proposition 4.1 implies that $\tilde{x}(h,g) \in \mathfrak{m}^2_{\mathcal{R}_{\tau=1}}$ when $h \in G_{H_w}$ (since $\eta_{|G_{H_w}} = 0$). Hence, the function $h \to \tilde{x}_\epsilon(h,g)$ is necessarily trivial on the decomposition group G_{H_w}.

(ii) Let M_v (resp. M_{v^σ}) be the maximal abelian unramified outside v (resp. v^σ) pro-p extension of H. By class field theory, $H_{\infty,v}$ (resp. H_{∞,v^σ}) is the fixed field by the torsion part of $\mathrm{Gal}(M_v/H)$ (resp. $\mathrm{Gal}(M_{v^\sigma}/H)$). Since $\tilde{x}_\epsilon(.,*)$ is bilinear on $G_H \times G_H$, the assertion follows immediately from the fact that any homomorphism of $\mathrm{Hom}(G_H,\bar{\mathbb{Q}}_p)$ unramified outside v (resp. v^σ) factors through $\mathrm{Gal}(H_{\infty,v}/H)$ (resp. $\mathrm{Gal}(H_{\infty,v^\sigma}/H)$).
■

The purpose of the following two lemmas is to explain the ordinariness of the elements of $t_{\mathcal{R}_{\tau=1}}$ at all prime places of H lying over v^σ and v.

Lemma 4.3. *Let $\alpha : \mathcal{R}_{\tau=1} \twoheadrightarrow \mathcal{R}_{\tau=1}/\mathfrak{m}^2_{\mathcal{R}_{\tau=1}}$ be the canonical projection, $\pi'_\epsilon = (a',d',x')$ be the pseudo-deformation obtained by the composition $\alpha\circ\pi_{\mathcal{R}_{\tau=1}}$, w' be a place of H above v^σ, and $I_{w'}$ be the inertia group at the place w', then for any h' in $I_{w'} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$, $a'(h') = 1$.*

Proof. Let ρ^τ_ϵ be the representation obtained by the composition $\alpha \circ \rho_{\tau=1}$ and $\rho^\tau_\epsilon(g) = \left(\begin{smallmatrix} a'(g) & b'(g) \\ c'(g) & d'(g) \end{smallmatrix}\right)$ be the realization of ρ^τ_ϵ in a basis (u_1,u_2) of $M_{\bar{\mathbb{Q}}_p[\epsilon]}$.

We have $b'(g) = \alpha(\tilde{x}(g,h_0)/\tilde{x}(g_0,h_0))$ and $x'(g_0,g) = c'(g)$.

On the other hand, as a result of (iii) of Lemma 4.1, $\tilde{\rho}_{|G_{F_{v^\sigma}}} = \phi \oplus \phi^\sigma$ in the basis (e'_1,e'_2) of $M_{\bar{\mathbb{Q}}_p}$, and that ρ^τ_ϵ is ordinary at v^σ in a basis (u_1,v_2) of $M_{\bar{\mathbb{Q}}_p}[\epsilon]$ lifting (e'_1,e'_2).

Let h be an element of $I_{w_0^\sigma} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ and $\begin{pmatrix} a''(h) & b''(h) \\ c''(h) & d''(h) \end{pmatrix}$ be the realization of ρ_ϵ^τ in the basis (u_1, v_2), then $a''(h) = 1$ and $b''(h) = 0$. According to Lemma 4.2, we state $c'(h) = 0$, and hence a direct computation shows that $a''(h) = a'(h) = 1$.

Now, if w' is another place above v^σ such that $g(w_0^\sigma) = w'$ for $g \in \mathrm{Gal}(H/F)$, then the assertion follows by using a similar argument for the basis $(u_1, (\rho_\epsilon^\tau)^{-1}(g)v_2)$.

■

Lemma 4.4. *Let w be a place of H above v and $\pi'_\epsilon = (a', d', x')$ be an element of $t_{\mathcal{R}_{\tau=1}}$, then for any g in $\mathrm{Gal}(H_\infty/F)$ and h' in $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$, $d'(gh'g^{-1}) = d'(h')$ and d' is trivial on $I_w \cap \mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$, where I_w is the inertia group at the place w.*

Proof. (i) Let h denote the element $gh'g^{-1}$. Since $x'(.,.)$ is trivial when one of its component belongs to $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ (see Lemma 4.2), we obtain:

$$d'(h) = d'(gh'g^{-1}) = d'(g)d'(h'g^{-1}) + x'(h'g^{-1}, g) \tag{3}$$

$$= d'(g)d'(h')d'(g^{-1}) + \phi(h')x'(g^{-1}, g) \tag{4}$$

A direct computation shows that $d'(gg^{-1}) = 1 = d'(g)d'(g^{-1}) + x'(g^{-1}, g)$ and $\phi(h') = \phi^\sigma(h')$. Hence, $d'(h) = d'(h')(1 - x'(g^{-1}, g)) + \phi(h')x'(g^{-1}, g) = d'(h')$.

As the Galois group $\mathrm{Gal}(H/F)$ acts transitively on the places of H above v, the assertion stems directly from the above discussion and the fact that $d'_{|I_{w_0}} = 1$.

■

4.2. **Tangent space of $\mathcal{R}_{\tau=1}/\mathfrak{m}_\Lambda$ and proof of Theorem 1.2.**

Let $\pi_\epsilon = (\tilde{a}_\epsilon, \tilde{d}_\epsilon, \tilde{x}_\epsilon)$ be the pseudo-deformation induced by the canonical projection $\pi' : \mathcal{R}_{\tau=1} \twoheadrightarrow \mathcal{R}_{\tau=1}/(\mathfrak{m}_\Lambda, \mathfrak{m}^2_{\mathcal{R}_{\tau=1}})$.

We have seen in Lemma 4.2 that $\tilde{x}_\epsilon$ is trivial when one of its component belongs to $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$; so on $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ the pseudo-deformation π_ϵ is equal to $(\tilde{a}_\epsilon, \tilde{d}_\epsilon, 0)$, where $\tilde{a}_\epsilon$, $\tilde{d}_\epsilon$ are characters on $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$. Let N_∞ denote the splitting field over $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ of $\tilde{a}_\epsilon \oplus \tilde{d}_\epsilon$.

Theorem 4.5. *Let $\pi_\epsilon = (\tilde{a}_\epsilon, \tilde{d}_\epsilon, \tilde{x}_\epsilon)$ be the pseudo-deformation induced by the projection $\pi' : \mathcal{R}_{\tau=1} \twoheadrightarrow \mathcal{R}_{\tau=1}/(\mathfrak{m}_\Lambda, \mathfrak{m}^2_{\mathcal{R}_{\tau=1}})$, then :*

(i) *N_∞ is an unramified abelian p-extension of H_∞ and the action by conjugation of $\mathrm{Gal}(H_\infty/F)$ on $\mathrm{Gal}(N_\infty/H_\infty)$ is trivial.*

(ii) *Assume that $(\mathbf{G})$ holds, then the pseudo-deformation $\pi_\epsilon = (\tilde{a}_\epsilon, \tilde{d}_\epsilon, \tilde{x}_\epsilon)$ is trivial.*

(iii) *Assume that the rank of the finite type $\mathbb{Z}_p$-module $\mathrm{Gal}(L_0/H_\infty)$ is zero (i.e $\mathrm{Gal}(L_0/H_\infty)$ is a finite group), then the pseudo-deformation $\pi_\epsilon = (\tilde{a}_\epsilon, \tilde{d}_\epsilon, \tilde{x}_\epsilon)$ is trivial.*

(iv) *Assume that* (**G**) *holds or* $\mathrm{Gal}(L_0/H_\infty)$ *is a finite group, then the morphism* $\kappa^{\#} : \Lambda \to \mathcal{R}_{\tau=1}$ *is an isomorphism and the ramification index* e *of* $\mathcal{C}$ *over* $\mathcal{W}$ *at* f *is exactly* 2.

Proof. (i) Let g be an element of $\mathrm{Gal}(H_\infty/F)$ and h be an element of $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$. Since $\det \pi_\epsilon = \det \pi$ and $\tilde{x}_\epsilon$ is trivial when one of its component belongs to $\mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$, Lemma 4.4 implies that $\tilde{a}_\epsilon(ghg^{-1}) = \tilde{a}_\epsilon(h)$ and $\tilde{d}_\epsilon(ghg^{-1}) = \tilde{d}_\epsilon(h)$. Hence the action of the Galois group $\mathrm{Gal}(H_\infty/F)$ on $\mathrm{Gal}(N_\infty/H_\infty)$ is also trivial.

Since $\det \pi_\epsilon = \det \pi$, it follows from lemmas 4.3, 4.4 that the restriction of both $\tilde{a}_\epsilon$ and $\tilde{d}_\epsilon$ to $I_w \cap \mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ is necessarily trivial, where w is any place of H above p. Thus, the algebraic extension N_∞/H_∞ is unramified at the primes above p.

In addition, [3, Prop.7.1] implies that the image of $I_\ell \cap \mathrm{Gal}(\bar{\mathbb{Q}}/H_\infty)$ by $\tilde{a}_\epsilon$ is finite (so trivial), where $\ell \neq p$ is a prime number. Therefore, the extension N_∞/H_∞ is everywhere unramified.

(ii) Since the abelian p-extension N_∞/H_∞ is everywhere unramified, N_∞ is a subfield of L_∞ and since $\mathrm{Gal}(H_\infty/F)$ acts trivially on $\mathrm{Gal}(N_\infty/H_\infty)$, N_∞ is contained in the subfield L_0. Moreover, by assumption L_0 is an abelian extension of F'', hence N_∞ is an abelian extension of F''.

It follows that $(\pi_\epsilon)_{|\,\mathrm{Gal}(\bar{\mathbb{Q}}/F'')}$ factors through $\mathrm{Gal}(N_\infty/F'')$ which is abelian group. Thus, $\tilde{a}_\epsilon(gh) = \tilde{a}_\epsilon(hg)$ and implying that $\tilde{x}_\epsilon$ is symmetric bilinear and trivial if one of its components belongs to any inertia group I_w (w is any place of H above p).

Since the Galois group $\mathrm{Gal}(H_\infty/F'')$ can be expressed as the product of all its inertia subgroups for the places of H above p, the function $\tilde{x}_\epsilon$ is necessarily trivial on $\mathrm{Gal}(H_\infty/F'') \times \mathrm{Gal}(H_\infty/F'')$.

In addition, the number field F'' is a finite abelian extension of H, then $\tilde{x}_\epsilon$ is trivial on $G_H \times G_H$. If the pseudo-deformation π_ϵ is not trivial, then π_ϵ is a generator of the tangent space of $\mathcal{R}_{\tau=1}$ (since the tangent space of a discrete valuation ring is always of dimension one). However, this contradicts the fact that $\tilde{x}_\epsilon$ defines a nonzero bilinear map of $G_H \times G_H$ (see (i) of Proposition 2.5), since there exist two elements g_0 and h_0 such that $\tilde{x}_\epsilon(g_0, h_0)$ is non zero ($\tilde{x}(g_0, h_0)$ has order 1 in the discrete valuation associated to $\mathcal{R}_{\tau=1}$). Hence, π_ϵ is necessarily trivial and the assertion follows immediately.

(iii) By assumption and referring to the discussion above, N_∞ is a finite extension of H_∞, so $N_\infty = H_\infty$ (since $\bar{\mathbb{Q}}_p$ is a torsion-free group). Therefore, we can conclude using a similar argument as above.

(iv) Since the tangent space of $\mathcal{R}_{\tau=1}/(\mathfrak{m}_\Lambda)$ is trivial, the local homomorphism $\kappa^{\#} : \Lambda \to \mathcal{R}_{\tau=1}$ is unramified. On the other hand, the local homomorphism $\kappa^{\#} : \Lambda \to \mathcal{R}_{\tau=1}$ is flat, and hence it is an étale morphism between complete local rings having the same residue field, then it is necessarily an isomorphism. ∎

5. Pseudo-deformations of $\bar{\rho}$ and base-change $F/\mathbb{Q}$

Let $h_{\mathbb{Q}}$ be the p-ordinary Hecke algebra of tame level N constructed by Hida in [21], $\mathfrak{p}_f$ be the closed point of $\operatorname{Spec} h_{\mathbb{Q}}[1/p]$ corresponding to the system of eigenvalues for Hecke operators associated to f.

Denote by $h_{\mathbb{Q},\mathfrak{p}_f}$ the completed local ring for the étale topology of Spec $h_{\mathbb{Q}}[1/p]$ at a geometric point corresponding to $\mathfrak{p}_f$. Let $h'_{\mathbb{Q}}$ be the sub-algebra of $h_{\mathbb{Q}}$ generated by the Hecke operators U_p, T_ℓ and $<\ell>$ for primes ℓ not dividing Np.

Proposition 5.1. *There exists an isomorphism between $\mathcal{T}$ and $h_{\mathbb{Q},\mathfrak{p}_f}$.*

Proof. The weight one form f corresponds to a point $x \in \mathcal{C}^{ord,0}$, where $\mathcal{C}^{ord,0}$ is the cuspidal locus of the ordinary locus of $\mathcal{C}^{ord}$ ($\mathcal{C}^{ord,0}$ is a Zariski closed subset of $\mathcal{C}^{ord}$). It is known that $h'_{\mathbb{Q}}$ is an integral model of $\mathcal{C}^{ord,0}$ (i.e $\mathcal{C}^{ord,0} = \operatorname{Spm} h'_{\mathbb{Q}}[1/p]$). Denote by $h'_{\mathbb{Q},\mathfrak{p}_f}$ for the completed local ring for the étale topology of Spec $h'_{\mathbb{Q}}$ at a geometric point corresponding to $\mathfrak{p}_f \cap h'_{\mathbb{Q}}$. Hence, the results of [17, §7] and [3, Prop.7.2] imply that there exists an isomorphism $h'_{\mathbb{Q},\mathfrak{p}_f} \simeq \mathcal{T}$ and an isomorphism $h'_{\mathbb{Q},\mathfrak{p}_f} \simeq h_{\mathbb{Q},\mathfrak{p}_f}$.

■

Remark 5.2. *If A is a Noetherian complete local ring, then A is a Nagata ring and hence any localization of A is also a Nagata ring. Moreover, the completion of a reduced Noetherian local Nagata ring with respect to its maximal ideal is always reduced. On the other hand, if A is reduced (resp. Nagata), then the strict henselization A^{sh} of A is reduced (resp. Nagata).*

Hence, $h_{\mathbb{Q},\mathfrak{p}_f}$, $h'_{\mathbb{Q},\mathfrak{p}_f}$, $\mathbb{T}^{n.ord}$ and $\mathbb{T}^{ord}$ are reduced local rings.

Proof of Theorem 1.3. The representation ρ associated to f is dihedral, so the involution ω fixes the height one primes $\mathfrak{p}_f$ of $h_{\mathbb{Q},\mathfrak{m}}$ associated to f. In addition, after the identification $\mathcal{R} \simeq \mathcal{T}$, the action of ω on $\mathcal{T}$ coincides with the involution τ (see [18, §3] and [24, §2]).

There exists a pseudo-character $\mathrm{Ps}_{h_{\mathbb{Q}}} : G_{\mathbb{Q},Np} \to h_{\mathbb{Q}}$ such that $\mathrm{Ps}_{h_{\mathbb{Q}}}(\mathrm{Frob}_\ell) = T_\ell$ for all primes $\ell \nmid Np$ (see [21]). Let $\mathfrak{q} \nmid Np$ be a prime ideal of $\mathcal{O}_F$, then the base-change morphism $\beta : h_F \to h_{\mathbb{Q}}$ sends the Hecke operator $T_{\mathfrak{q}}$ to $\mathrm{Ps}_{h_{\mathbb{Q}}}(\mathrm{Frob}_{\mathfrak{q}})$.

Let $\mathfrak{n}$ denote the height one prime ideal $\beta^{-1}(\mathfrak{p}_f)$ of h_F, so the morphism β induces a morphism of complete local rings $\beta_f : \mathbb{T}^{ord} \to \mathcal{T}$ and the values of β_f are in $\mathcal{T}_+$, where $\mathcal{T}_+$ is the subring of $\mathcal{T}$ fixed by τ.

On the other hand, there exists a pseudo-character $\mathrm{Ps}_{h_F} : G_F \to h_F$ of dimension two such that $\mathrm{Ps}_{h_F}(\mathrm{Frob}_{\mathfrak{q}}) = T_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q} \nmid p$ of $\mathcal{O}_F$ (see [23]). Let

$$\mathrm{Ps}_{\mathbb{T}^{ord}} : G_F \to \mathbb{T}^{ord}$$

be the pseudo-character given the composition of Ps_{h_F} with the localization homomorphism $h_F \to \mathbb{T}^{ord}$. It is apparent that $\mathrm{Ps}_{\mathbb{T}^{ord}}$ lifts the pseudo-character $\phi + \phi^\sigma$ and $\beta_f(\mathrm{Ps}_{\mathbb{T}^{ord}}) = \mathrm{Tr}(\rho_{\mathcal{T}})_{|G_F}$ (since $\beta(\mathrm{Ps}_{h_F}) = (\mathrm{Ps}_{h_{\mathbb{Q}}})_{|G_F}$).

Let S be the total quotient ring of the reduced local ring $\mathbb{T}^{ord}$ ($\mathbb{T}^{ord} \subset S$), then $S = \prod \mathbb{T}^{ord}_{\mathfrak{p}_i}$, where $\mathfrak{p}_i$ runs over the set of minimal prime ideals of $\mathbb{T}^{ord}$, and it is known that each $\mathfrak{p}_i$ corresponds to a Hida family passing through $E_1(\phi, \phi^\sigma)$ (since $\mathbb{T}^{ord}$ is a noetherian ring, $\mathbb{T}^{ord}$ has a finite number of minimal prime ideals).

A result of Wiles [37] indicates the existence a unique semi-simple Galois representation $\rho_S : G_F \to \mathrm{GL}_2(S)$ ordinary at v and v^σ, and such that $\mathrm{Tr}(\rho_S) = \mathrm{Ps}_{\mathbb{T}^{ord}}$. Since $\phi(\gamma_0) \neq \phi^\sigma(\gamma_0)$, Hensel lemma's implies that the eigenvalues of $\rho_S(\gamma_0)$ are distincts (they belong to $\mathbb{T}^{ord}$). Thus, we can find a basis $\mathfrak{B}_S$ of M_S in which $\rho_S(\gamma_0)$ is diagonal and $(\rho_S)_{|G_{F_v}}$ is upper triangular with an unramified quotient.

In fact, Lemma 3.3 implies that the coefficients of the matrix of the realization of ρ_S in the basis $\mathfrak{B}_S$ rise to an ordinary pseudo-deformation $\pi_{\mathbb{T}^{ord}} = (a, d, bc) : G_F \to \mathbb{T}^{ord}$ of π.

Note that the action of Δ fixes $\mathfrak{n}$ and denote by $\pi_{\mathbb{T}^{ord}_\Delta}$ the push-forward of $\pi_{\mathbb{T}^{ord}}$ via the canonical surjection $\mathbb{T}^{ord} \twoheadrightarrow \mathbb{T}^{ord}_\Delta$.

Subsequently, the trace of $\pi_{\mathbb{T}^{ord}_\Delta}$ is invariant by the action of Δ and $\pi_{\mathbb{T}^{ord}_\Delta}$ is a point of $\mathfrak{G}(\mathbb{T}^{ord}_\Delta)$. Thus, there exists a unique morphism $h : \mathcal{R}^{ps}_{red} \to \mathbb{T}^{ord}_\Delta$ inducing the pseudo-deformation $\pi_{\mathbb{T}^{ord}_\Delta}$.

By construction, we have $h(\mathrm{Tr}\,\pi^{ps}(\mathrm{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}}$ for $\mathfrak{q} \nmid p$, so the homomorphism h is surjective since the topological generator $\{T_{\mathfrak{q}}\}_{\mathfrak{q}\nmid p}$, $U_{\mathfrak{p}}$ and $U_{\mathfrak{p}^\sigma}$ over Λ of $\mathbb{T}^{ord}_\Delta$ are in the image of h (the fact that $\phi_{|G_{F_v}} \neq \phi^\sigma_{|G_{F_v}}$ implies that $U_{\mathfrak{p}}, U_{\mathfrak{p}^\sigma} \in \mathrm{im}\, h$).

According to Theorem 1.1, we have the isomorphisms

$$\mathcal{T}_+ \simeq \mathcal{R}_{\tau=1} \simeq \mathcal{R}^{ps}_{red}.$$

Moreover, according to Lemma 3.3, $\mathcal{R}^{ps}$ is topologically generated over Λ by $\mathrm{Tr}\,\pi^{ps}(g)$, where g runs over the elements of G_F. Therefore, the morphism $\beta_f : \mathbb{T}^{ord} \to \mathcal{T}_+$ is surjective (since the morphism β_f sends $T_{\mathfrak{q}}$ to $\mathrm{Tr}\,\rho_{\mathcal{T}}(\mathrm{Frob}_{\mathfrak{q}})$).

Since the trace of $(\rho_{\mathcal{T}})_{|G_F}$ is invariant by the action of σ, β_f factors through $\mathbb{T}^{ord}_\Delta$, so the Krull dimension of $\mathbb{T}^{ord}_\Delta$ is ≥ 1. In addition, the Krull dimension of the Hecke algebra h_F is two, hence $\mathbb{T}^{ord}$ is of dimension 1 and $\mathbb{T}^{ord}_\Delta$ is also of dimension 1.

It follows from Theorem 1.1 that the tangent space of $\mathcal{R}^{ps}_{red}$ is of dimension 1, and since $\mathbb{T}^{ord}_\Delta$ is equidimensional of dimension 1, the surjection $h : \mathcal{R}^{ps}_{red} \twoheadrightarrow \mathbb{T}^{ord}_\Delta$ is necessarily an isomorphism of regular local rings of dimension 1. ■

Let $\mathcal{O}$ be the ring of integers of a p-adic field containing the image of ϕ. After an extension of scalars, one can assume that the p-ordinary Hecke algebra $h_{\mathbb{Q},\mathfrak{m}}$ contains $\mathcal{O}$, and hence $h_{\mathbb{Q},\mathfrak{m}}$ is an obejct of the category $\mathrm{CNL}_{\mathcal{O}}$.

Assume until the end of this section that :

(i) $p > 2$ and the restriction of $\bar{\rho}$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}))$ is absolutely irreducible.

(ii) There exists an element $\gamma_0 \in G_{F_v}$ such that $\bar{\phi}(\gamma_0) \neq \bar{\phi}^{\sigma}(\gamma_0)$.

(iii) The character $\bar{\phi}$ is everywhere unramified.

Thus, we are able to use the results of Taylor-Wiles [38] to claim that the p-ordinary Hecke algebra $h_{\mathbb{Q},\mathfrak{m}}$ is isomorphic to an universal ring R^{ord}, representing the p-ordinary minimally ramified deformations of $\bar{\rho}$ to the objects of $\mathrm{CNL}_{\mathcal{O}}$.

Definition 5.3. *Let A be a ring in $\mathrm{CNL}_{\mathcal{O}}$, Σ be the set of primes of F lying over p, $\tilde{a}, \tilde{d} : G_{F,\Sigma} \to A$ and $\tilde{x} : G_{F,\Sigma} \times G_{F,\Sigma} \to A$ be continuous functions forming a pseudo-representation. We say that π_A is a pseudo-deformation of $\bar{\pi} = (\bar{\phi}, \bar{\phi}^{\sigma}, 0)$ if and only if π_A mod $\mathfrak{m}_A = \bar{\pi}$. Let $\mathfrak{G}_{\mathcal{O}} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ denote the functor of all pseudo-deformations $\pi_A = (\tilde{a}_A, \tilde{d}_A, \tilde{x}_A)$ of $\bar{\pi}$ which satisfy the following conditions:*

(i) *For all $h \in G_{F_v}$, $h' \in G_{F,\Sigma}$, $\tilde{x}_A(h', h) = 0$.*

(ii) *$\tilde{d}_A(g) = 1$ if $g \in I_v$.*

(iii) *$\mathrm{Tr}\, \pi_A(t^{-1}gt) = \mathrm{Tr}\, \pi_A(g)$ for each t in $G_{\mathbb{Q}}$ and $g \in G_{F,\Sigma}$.*

Lemma 5.4. *One always has:*

(i) *Let A be an object of $\mathrm{CNL}_{\mathcal{O}}$, and $\pi_A = (\tilde{a}_A, \tilde{d}_A, \tilde{x}_A)$ be a pseudo-deformation of $\bar{\pi}$, then π_A depends only on the trace $\mathrm{Tr}\, \pi_A = \tilde{a}(g) + \tilde{d}(g)$ and the determinant $\det \pi_A = \tilde{a}(g)\tilde{d}(g) - \tilde{x}(g, g)$ as follow:*

$$\begin{aligned} \tilde{a}_A(g) &= \frac{\mathrm{Tr}\, \pi_A(\gamma_0 g) - \lambda_2 \,\mathrm{Tr}\, \pi_A(g)}{\lambda_1 - \lambda_2} \\ \tilde{d}_A(g) &= \frac{\mathrm{Tr}\, \pi_A(\gamma_0 g) - \lambda_1 \,\mathrm{Tr}\, \pi_A(g)}{\lambda_2 - \lambda_1} \end{aligned} \tag{5}$$

where $\lambda_1 = \tilde{a}(\gamma_0)$ and $\lambda_2 = \tilde{d}(\gamma_0)$ are the unique roots of the polynomial $X^2 - \mathrm{Tr}\, \pi_A(\gamma_0)X + \det \pi_A(\gamma_0)$.

(ii) *The functor $\mathfrak{G}_{\mathcal{O}}$ is representable by $(R^{ps}, \pi_{R^{ps}})$.*

Proof.

i) The same proof as in Lemma 3.3.

ii) The functor $\mathfrak{G}_{\mathcal{O}}$ satisfies Schlesinger's criteria, the only non-trivial point is the finiteness of the dimension of the tangent space of $\mathfrak{G}_{\mathcal{O}}$, and this is provided by the same argument of [34, Lemma 2.10] (since $\mathrm{H}^1(G_{F,\Sigma}, \bar{\phi}/\bar{\phi}^{\sigma})$ has a finite dimension).

■

Hensel lemma's implies that there exists a basis $\mathfrak{B}_{R^{ord}}$ of $M_{R^{ord}}$ such that the universal p-ordinary deformation satisfies the following properties :

$$\rho_{R^{ord}}(\gamma_0) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right), \text{ and } (\rho_{R^{ord}})_{|G_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \text{ in this basis.}$$

Therefore, by using similar arguments to those already applied to prove Lemma 3.7, there exists a morphism $\alpha : R^{ps} \to h_{\mathbb{Q},\mathfrak{m}}$ which factors through $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ and inducing the pseudo-deformation of $\bar{\pi}$ associated to $(\rho_{R^{ord}})_{|G_F}$ in the basis $\mathfrak{B}_{R^{ord}}$.

The local ring $\mathcal{R}^{ps}$ is isomorphic to the completed local ring for the étale toplogy of $\operatorname{Spec} R^{ps}$ at a $\bar{\mathbb{Q}}_p$-point corresponding to the pseudo-deformation π of $\bar{\pi}$.

Remark 5.5. *It follows directly from Lemma 5.4 that R^{ps} is generated over the Iwasawa algebra $\Lambda_{\mathcal{O}} \simeq \mathcal{O}[[T]]$ by the Trace of the universal pseudo-deformation (see* [37]*, p564 for more details).*

Now, Theorem 1.3 and the exact same arguments that are already used to proof [9, 3.10], we deduce that the morphism $\alpha : R^{ps} \to h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ is unramified at non maximal prime ideals. Hence we obtain the following corollary without assuming that $\bar{\phi}^2_{|I_v} \neq 1$ as in Theorem [9, B].

Corollary 5.6. *Assume that the following conditions holds for $\bar{\rho}$:*

(i) *The character $\bar{\phi}$ is everywhere unramified.*
(ii) *$\bar{\rho}$ is p-distinguished and the restriction of $\bar{\rho}$ to $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}))$ is absolutely irreducible.*

Then the image of the base-change morphism $\beta : h_F \to h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ has a finite index, and the image of the morphism $\alpha : R^{ps} \to h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ is contained in $\operatorname{im}\beta$ and has also a finite index in $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$.

6. Deformation of a reducible Galois representation and proof of theorem 1.5

The Hecke algebra h_F is reduced, since it specializes to level 1 Hecke algebras (which are reduced) for an infinitely weights $k \geq 3$ (see [24, p.279] for more details).

Lemma 6.1. *The ring $\mathbb{T}^{n.ord}$ is equidimensional of dimension* 3*.*

Proof. Since the reduced nearly ordinary Hecke algebra $h_F^{n.ord}$ is a finite torsion-free module over the Iwasawa algebra of three variables $\Lambda_{\mathcal{O}}^{n.ord} = \mathcal{O}[[T_1, T_2, T_3]]$ (see p.119 of [23]), every irreducible component of $\operatorname{Spec} h_F^{n.ord}$ has Krull dimension equal to 4. Thus, $\mathbb{T}^{n.ord}$ is an equidimensional ring of dimension 3. ■

Let A be an object of the category $\mathfrak{C}$ and $\rho_A : G_F \to \mathrm{GL}_2(A)$ be a deformation of $\tilde{\rho}$, then we state that ρ_A is a nearly-ordinary deformation at p, if

$$(\rho_A)_{|G_{F_v}} \simeq \begin{pmatrix} \psi'_{v,A} & * \\ 0 & \psi''_{v,A} \end{pmatrix} \text{ and } (\rho_A)_{|G_{F_{v^\sigma}}} \simeq \begin{pmatrix} \psi''_{v^\sigma,A} & 0 \\ * & \psi'_{v^\sigma,A} \end{pmatrix},$$

where $\psi''_{v,A}$ is a character lifting $\phi^\sigma_{|G_{F_v}}$ and $\psi''_{v^\sigma,A}$ is a character lifting $\phi_{|G_{F_{v^\sigma}}}$. Moreover, if $\psi''_{v,A}$ and $\psi''_{v^\sigma,A}$ are unramified, then we say that ρ_A is ordinary at p.

Definition 6.2. *Let $\mathcal{D}^{n.ord} : \mathfrak{C} \to \mathrm{SETS}$ be the functor of strict equivalence classes of deformation of $\tilde{\rho} = \begin{pmatrix} \phi & \eta \\ 0 & \phi^\sigma \end{pmatrix}$ which are nearly ordinary at p, and let $\mathcal{D}^{ord}$ be the subfunctor of $\mathcal{D}^{n.ord}$ of deformations which are ordinary at p.*

Since $\tilde{\rho}$ is not semi-simple and $\phi(\mathrm{Frob}_v) \neq \phi^\sigma(\mathrm{Frob}_v)$, Schlesinger's criterions imply that $\mathcal{D}^{n.ord}$ (resp. $\mathcal{D}^{ord}$) is representable by $(\mathcal{R}^{n.ord}, \rho_{\mathcal{R}^{n.ord}})$ (resp. $(\mathcal{R}^{ord}, \rho_{\mathcal{R}^{ord}})$). The determinant $\det \rho_{\mathcal{R}^{ord}}$ is a deformation of the determinant $\det \pi$, so $\mathcal{R}^{ord}$ is endowed naturally with a structure of Λ-algebra (since the quadratic real field F has a unique $\mathbb{Z}_p$-extension).

6.1. **Nearly ordinary deformation of a reducible representation.**

There exits a pseudo-character $\mathrm{Ps}_{h_F^{n.ord}} : G_F \to h_F^{n.ord}$ of dimension two such that for all prime ideals $\mathfrak{q} \nmid p$ of $\mathcal{O}_F$, $\mathrm{Ps}_{h_F^{n.ord}}(\mathrm{Frob}_\mathfrak{q})$ is the Hecke operator $T_\mathfrak{q}$. $\mathrm{Ps}_{h_F^{n.ord}}$ is the trace of a representation of dimension 2 with coefficients in the total quotient ring of $h_F^{n.ord}$ (see [23] for more details). Let $\mathrm{Ps}_{\mathbb{T}^{n,ord}} : G_F \to \mathbb{T}^{n,ord}$ be the pseudo-character of dimension 2 obtained by composing $\mathrm{Ps}_{h_F^{n.ord}}$ with the localization morphism $h_F^{n,ord} \to \mathbb{T}^{n,ord}$. It appears that $\mathrm{Ps}_{\mathbb{T}^{n,ord}}$ lifts the pseudo-character $\mathrm{Tr}\,\tilde{\rho} = \phi \oplus \phi^\sigma$.

Let $Q(\mathbb{T}^{n,ord}) := \prod S'_i$ be the total quotient ring of the reduced noethrian ring $\mathbb{T}^{n,ord}$($\mathbb{T}^{n,ord} \subset Q(\mathbb{T}^{n,ord})$), so $Q(\mathbb{T}^{n,ord}) = \prod \mathbb{T}^{n,ord}_{\mathfrak{F}_i}$, where $\mathfrak{F}_i$ runs over the minimal prime ideals of $\mathbb{T}^{n,ord}$. It is known that each $\mathfrak{F}_i$ corresponds to a nearly ordinary p-adic family passing through the weight one form $E_1(\phi, \phi^\sigma)$.

Moreover, there exists a unique semi-simple Galois representation

$$\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(Q(\mathbb{T}^{n,ord}))$$

satisfying $\mathrm{Tr}(\rho_{Q(\mathbb{T}^{n,ord})}) = \mathrm{Ps}_{\mathbb{T}^{n,ord}}$.

Since $U_\mathfrak{p}(E_1(\phi, \phi^\sigma)) = \phi^\sigma(\mathrm{Frob}_\mathfrak{p}).E_1(\phi, \phi^\sigma)$ and $U_{\mathfrak{p}^\sigma}(E_1(\phi, \phi^\sigma)) = \phi(\mathrm{Frob}_{\mathfrak{p}^\sigma}).E_1(\phi, \phi^\sigma)$ (see Lemma 4.1), it follows from the results of Hida ([23]) that $(\rho_{Q(\mathbb{T}^{n,ord})})_{|G_{F_v}}$ (resp. $(\rho_{Q(\mathbb{T}^{n,ord})})_{|G_{F_v^\sigma}}$) is the extension a character $\psi''_{\mathbb{T}^{n,ord},v}$ (resp. $\psi''_{\mathbb{T}^{n,ord},v^\sigma}$) lifting $\phi^\sigma_{|G_{F_v}}$ (resp. $\phi_{|G_{F_{v^\sigma}}}$) by a character $\psi'_{\mathbb{T}^{n,ord},v}$ (resp. $\psi'_{\mathbb{T}^{n,ord},v^\sigma}$).

Let $\gamma_0' \in G_{F_{v^\sigma}}$ such that $\phi(\gamma_0') \neq \phi^\sigma(\gamma_0')$. Hensel lemma's implies that the eigenvalues of $\rho_{Q(\mathbb{T}^{n,ord})}(\gamma_0')$ are distinct and belong in $\mathbb{T}^{n,ord}$. Hence there exists a basis (e_1'', e_2'') of $M_{Q(\mathbb{T}^{n,ord})}$ such that $\rho^{ord}_{Q(\mathbb{T}^{n,ord})}(\gamma_0') = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ and $(\rho_{Q(\mathbb{T}^{n,ord})})_{|G_{F_{v^\sigma}}} = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$ in this basis.

Let a, b, c, d be the coefficients of the realization of $\rho_{Q(\mathbb{T}^{n,ord})}$ by matrix in the basis (e_1'', e_2'') of $M_{Q(\mathbb{T}^{n,ord})}$, B and C be the $\mathbb{T}^{n,ord}$-sub modules of $Q(\mathbb{T}^{n,ord})$ generated respectively by the coefficients $b(g)$ and $c(g')$, where g and g' run over the elements of G_F.

Let $\mathfrak{m}_{\mathbb{T}^{n,ord}}$ be the maximal ideal of $\mathbb{T}^{n,ord}$ and $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}}$ be the subspace of $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)$ given by the extensions of ϕ^σ by ϕ which are trivial at $G_{F_{v^\sigma}}$.

The following proposition is a generalization of [2, Prop.2].

Proposition 6.3. *One always has :*

(i) $\mathrm{Hom}_{\mathbb{T}^{n,ord}}(B, \bar{\mathbb{Q}}_p)$ *injects* $\mathbb{T}^{n,ord}$*-linearly in* $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}}$.

(ii) B *is an* $\mathbb{T}^{n,ord}$*-module of finite type and the annihilator of* B *is zero.*

Proof.

i) Since $\mathbb{T}^{n,ord}$ is a complete local ring and $\phi(\gamma_0') \neq \phi^\sigma(\gamma_0')$, $a(\gamma_0')$ and $d(\gamma_0')$ are the unique roots of the polynomial $X^2 - \mathrm{Tr}\,\rho_{Q(\mathbb{T}^{n,ord})}(\gamma_0')X + \det \rho_{Q(\mathbb{T}^{n,ord})}(\gamma_0')$. Hence, $a(\gamma_0')$ and $d(\gamma_0')$ belong to $\mathbb{T}^{n,ord}$. Thus, as in Lemma 3.3, the coefficients a, d and $b(g).c(g')$ can be obtained exclusively from the trace $\mathrm{Ps}_{\mathbb{T}^{n,ord}}$ and the determinant $\det \rho_{Q(\mathbb{T}^{n,ord})}$. Moreover, the reduction of $\mathrm{Ps}_{\mathbb{T}^{n,ord}}$ is $\phi + \phi^\sigma$. Hence, $(a, d, bc) : G_F \to \mathbb{T}^{n,ord}$ is a pseudo-deformation of $\pi = (\phi, \phi^\sigma, 0)$, and $a - \phi$, $d - \phi^\sigma$ and $b(g)c(g')$ are in $\mathfrak{m}_{\mathbb{T}^{n,ord}}$.

Denote by $\bar{b}$ the image of b in $\bar{B} = B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B$. We have a group homomorphism:

$$G \to \left(\begin{smallmatrix} \bar{\mathbb{Q}}_p & \bar{B} \\ 0 & \bar{\mathbb{Q}}_p \end{smallmatrix}\right), \text{ given by } g \to \left(\begin{smallmatrix} \phi & \bar{b}(g) \\ 0 & \phi^\sigma \end{smallmatrix}\right)$$

Since the restriction of b to $G_{F_{v^\sigma}}$ is trivial in our basis, we obtain a morphism

$$j : \mathrm{Hom}_{\mathbb{T}^{n,ord}}(B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B, \bar{\mathbb{Q}}_p) \to \mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}}$$

which associates to a homomorphism $f : B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B \to \bar{\mathbb{Q}}_p$ the cohomology class of the cocycle $g \to f(\bar{b}(g))$ (since $b(g)c(g') \in \mathfrak{m}_{\mathbb{T}^{n,ord}}$). The choice of the basis (e_1'', e_2'') of $M_{Q(\mathbb{T}^{n,ord})}$ implies that the cocycle $g \to f(\bar{b}(g))$ is trivial on $G_{F_{v^\sigma}}$.

Subsequently, we will prove that j is injective. First of all, a direct computation demonstrates that

$$\bar{b}(\gamma_0' g \gamma_0'^{-1} g^{-1}) = \frac{\bar{b}(g)}{\phi^\sigma(g)}\left(\frac{\phi(\gamma_0')}{\phi^\sigma(\gamma_0')} - 1\right)$$

and implies that $B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B$ is generated over $\mathbb{T}^{n,ord}$ by the elements $\bar{b}(g)$, when g runs over $G_H = \ker \phi/\phi^\sigma$ ($\gamma_0' g \gamma_0'^{-1} g^{-1} \in G_H$ since H/F is cyclic).

Now, let f be an element of $\mathrm{Hom}_{\mathbb{T}^{n,ord}}(B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B, \bar{\mathbb{Q}}_p)$ such that $f(\bar{b})$ is equal to zero in $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}}$, then $f(\bar{b})$ is a coboundary and the restriction of $f(\bar{b})$ to G_H is trivial (since H is the splitting field of ϕ/ϕ^σ). However, $B/\mathfrak{m}_{\mathbb{T}^{n,ord}}B$ is generated by $\{\bar{b}(g), g \in G_H\}$, therefore f is necessarily trivial.

ii) Since the representation $\rho_{Q(\mathbb{T}^{n,ord})}$ is semi-simple, Lemma [2, 4] implies that B is a finite type $\mathbb{T}^{n,ord}$-module.

The pseudo-character $\mathrm{Ps}_{h_F^{n.ord}}$ rise to a totally odd representation

$$\rho_{h_F^{n.ord}} : G_F \to \mathrm{GL}_2(Q(h_F^{n.ord})),$$

where $Q(h_F^{n.ord})$ is the total fraction field of $h_F^{n.ord}$. We have $Q(h_F^{n.ord}) = \prod \mathfrak{I}_i$, where $\mathfrak{I}_i$ runs over the fields given by the localization of $h_F^{n.ord}$ at the minimal prime ideals of $h_F^{n.ord}$ (each $\mathfrak{I}_i$ corresponds to a nearly ordinary Hida family). There exists a basis of $M_{Q(h_F^{n.ord})}$ in which $\rho_{h_F^{n.ord}}(c) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$; and let a', b', c', d' be the entries of the realization of $\rho_{h_F^{n.ord}}$ by a matrix in this basis. The functions a', d' and $b'c'$ depend only on the trace $\mathrm{Ps}_{h_F^{n.ord}}$ and the determinant $\det \rho_{h_F^{n.ord}}$, and the values the functions a', d' and $b'c'$ are in $h_F^{n.ord}$.

Since the non critical classical cuspidal Hilbert modular forms are Zariski dense on each irreducible component of $\mathrm{Spec}\, h_F^{n.ord}$, for each field $\mathfrak{I}_i$ there exist g_i, g_i' in G_F, such that the image by projection of $b'(g_i)c'(g_i')$ is not trivial in $\mathfrak{I}_i$. Thus, all the representations $\rho_{S_i'}$ given by composing $\rho_{Q(\mathbb{T}^{n,ord})}$ with the projections $\prod \mathbb{T}^{n,ord}_{\mathfrak{F}_j} \to S_i' = \mathbb{T}^{n,ord}_{\mathfrak{F}_i}$ are absolutely irreducible, so the image of B is each S_i' is non zero. Hence, we may conclude that the annihilator of B in $\mathbb{T}^{n,ord}$ is zero. ■

Corollary 6.4. *One always has:*

(i) *The $\mathbb{T}^{n,ord}$-module B is free of rank one and there exists an adapted basis (e_1'', e_2'') of $M_{Q(\mathbb{T}^{n,ord})}$ such that B is generated over $\mathbb{T}^{n,ord}$ by 1.*

(ii) *In the basis (e_1'', e_2''), the realization $\rho_{Q(\mathbb{T}^{n,ord})}(\gamma_0')$ is diagonal and the representation $\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(\mathbb{T}^{n,ord})$ is a nearly ordinary deformation of $\tilde{\rho}$.*

Proof.

i) Since $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}} \simeq \mathrm{H}^1(F, \phi/\phi^\sigma)_{G_{F_{v^\sigma}}}$, Propositions 2.5 (or [4, Prop.5.1]) and 6.3 attest that the dimension of $\mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}}$ is one and $\dim_{\bar{\mathbb{Q}}_p} B \otimes \bar{\mathbb{Q}}_p \leq 1$.

Since we proved in Proposition 6.3 that B is a non zero finite type $\mathbb{T}^{n,ord}$-module, Nakayama's lemma implies that B is a monogenic $\mathbb{T}^{n,ord}$-module.

Moreover, the fact that the annihilator of B in $\mathbb{T}^{n,ord}$ is zero yields that B is a free $\mathbb{T}^{n,ord}$-module of rank one. Thus, by rescaling the basis (e_1'', e_2''), the representation $\rho_{Q(\mathbb{T}^{n,ord})}$ takes values in $\mathrm{GL}_2(\mathbb{T}^{n,ord})$.

ii) Since any representation isomorphic to an extension of ϕ^σ by ϕ trivial on $G_{F_{v^\sigma}}$ is necessarily isomorphic to $\tilde{\rho}$ (i.e $\dim_{\bar{\mathbb{Q}}_p} \mathrm{Ext}^1_{\bar{\mathbb{Q}}_p[G_F]}(\phi^\sigma, \phi)_{G_{F_{v^\sigma}}} = 1$), (i) implies that $\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(\mathbb{T}^{n,ord})$ is a deformation of $\tilde{\rho}$, and by construction $\rho_{Q(\mathbb{T}^{n,ord})}$ is nearly ordinary at v^σ.

On the other hand, the deformation $\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(Q(\mathbb{T}^{n,ord}))$ is nearly ordinary at v and $\phi(\mathrm{Frob}_v) \neq \phi^\sigma(\mathrm{Frob}_v)$. Thus, by using the exact same arguments that are already applied to prove [3, Prop 5.1], we deduce that $\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(\mathbb{T}^{n,ord})$ is ordinary at v.

■

6.2. **Tangent space of $\mathcal{D}^{n.ord}$.**

Let $t_{\mathcal{D}^{n.ord}}$ (resp. $t_{\mathcal{D}^{ord}}$) denote the tangent space of $\mathcal{D}^{n.ord}$ (resp. $\mathcal{D}^{ord}$). The choice of the basis (e'_1, e'_2) of $M_{\bar{\mathbb{Q}}_p}$ defined in Lemma 4.1 identifies $\mathrm{End}_{\bar{\mathbb{Q}}_p}(M_{\bar{\mathbb{Q}}_p})$ with $M_2(\bar{\mathbb{Q}}_p)$. Since $\tilde{\rho}_{|G_{F_{v^\sigma}}}$ splits completely in the basis (e'_1, e'_2), we obtain the following decomposition of $\bar{\mathbb{Q}}_p[G_{F_{v^\sigma}}]$-modules

$$\begin{aligned} (\mathrm{ad}\,\tilde{\rho})_{|G_{F_{v^\sigma}}} &= \bar{\mathbb{Q}}_p \oplus \phi/\phi^\sigma \oplus \phi^\sigma/\phi \oplus \bar{\mathbb{Q}}_p \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\longrightarrow (a, b, c, d) \end{aligned} \tag{6}$$

Let $W_{\tilde{\rho}}$ be the subspace of $\mathrm{ad}\,\tilde{\rho}$ given by the following elements

$$W_{\tilde{\rho}} = \{g \in \mathrm{End}_{\bar{\mathbb{Q}}_p}(M_{\bar{\mathbb{Q}}_p}) \mid g(e_1) \subset (e_1)\}.$$

By composing the restriction morphism $\mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \to \mathrm{H}^1(F_{v^\sigma}, \mathrm{ad}\,\tilde{\rho})$ and the morphism $b^* : \mathrm{H}^1(F_{v^\sigma}, \mathrm{ad}\,\tilde{\rho}) \to \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma)$ (obtained by functoriality from (6)), we obtain the natural map:

$$\mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \xrightarrow{B^*} \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma) \tag{7}$$

Let $P = \bar{\mathbb{Q}}_p[\phi^\sigma/\phi]$ be the $\bar{\mathbb{Q}}_p[G_F]$-module of dimension one over $\bar{\mathbb{Q}}_p$ and on which G_F acts by ϕ^σ/ϕ. Since $\tilde{\rho}$ is reducible, $W_{\tilde{\rho}}$ is preserved by the action of $\mathrm{ad}\,\tilde{\rho}$ and we have a natural G_F-equivariant map given by the quotient of $\mathrm{ad}\,\tilde{\rho}$ by $W_{\tilde{\rho}}$:

$$\begin{aligned} \mathrm{ad}\,\tilde{\rho} &\xrightarrow{C} \bar{\mathbb{Q}}_p[\phi^\sigma/\phi] \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto c \end{aligned} \tag{8}$$

Let $\mathfrak{r} : \mathrm{H}^1(F, \phi^\sigma/\phi) \to \mathrm{H}^1(F_v, \phi^\sigma/\phi)$ denote the natural morphism given by the restriction of the cocycles to G_{F_v}, and $C^* : \mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \xrightarrow{C^*} \mathrm{H}^1(F, \phi^\sigma/\phi)$ be the morphism obtained by functoriality from (8). By using a standard argument of the deformation theory, we achieve the following result.

Lemma 6.5. *We have the following isomorphism:*

$$t_{\mathcal{D}^{n.ord}} = \ker\left(\mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \overset{(\mathfrak{r}\circ C^*, B^*)}{\longrightarrow} (\mathrm{H}^1(F_v, \phi^\sigma/\phi) \oplus \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma))\right)$$

We have an exact sequence of $\bar{\mathbb{Q}}_p[G_F]$-modules :

$$0 \to W_{\tilde{\rho}} \to \mathrm{ad}\,\tilde{\rho} \to P \to 0 \tag{9}$$

Since $\phi^\sigma/\phi \neq 1$, $\mathrm{H}^0(G_F, P) = \{0\}$, we have the following long exact sequence of groups cohomology:

$$0 \to \mathrm{H}^1(F, W_{\tilde{\rho}}) \to \mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \to \mathrm{H}^1(F, P) \to \mathrm{H}^2(F, W_{\tilde{\rho}}) \tag{10}$$

We will show that $\mathrm{H}^2(F, W_{\tilde{\rho}})$ is trivial. First, we start by computing the dimension of $\mathrm{H}^1(F, W_{\tilde{\rho}})$ in order to use the Global Euler characteristic formula to deduce that $\mathrm{H}^2(F, W_{\tilde{\rho}})$ vanishes.

Under the identification $\mathrm{End}_{\bar{\mathbb{Q}}_p}(M_{\bar{\mathbb{Q}}_p}) \simeq M_2(\bar{\mathbb{Q}}_p)$, $W_{\tilde{\rho}}$ is the subspace of the upper triangular matrices of $M_2(\bar{\mathbb{Q}}_p)$. Since $\tilde{\rho}$ is reducible, the space

$$W^0_{\tilde{\rho}} := \{g \in \mathrm{End}_{\bar{\mathbb{Q}}_p}(M_{\bar{\mathbb{Q}}_p}) \mid g(e_1) = 0, g(e_2) \subset (e_1)\} \subset W_{\tilde{\rho}}$$

is stable by the action of G_F, and the adjoint action on this sub-space is given by ϕ/ϕ^σ.

Under the identification $\mathrm{End}_{\bar{\mathbb{Q}}_p}(M_{\bar{\mathbb{Q}}_p}) \simeq M_2(\bar{\mathbb{Q}}_p)$, $W^0_{\tilde{\rho}}$ is the subspace of $M_2(\bar{\mathbb{Q}}_p)$ given by the strict upper triangular matrices, and it is isomorphic to $\bar{\mathbb{Q}}_p[\phi/\phi^\sigma]$ as $\bar{\mathbb{Q}}_p[G_F]$-module.

Therefore, we obtain the following exact sequence of $\bar{\mathbb{Q}}_p[G_F]$-modules:

$$0 \to \bar{\mathbb{Q}}_p[\phi/\phi^\sigma] \to W_{\tilde{\rho}} \to \bar{\mathbb{Q}}_p^2 \to 0$$

Hence, there exists a long exact cohomology sequence:

$$0 \to \mathrm{H}^0(F, W_{\tilde{\rho}}) \to \mathrm{H}^0(F, \bar{\mathbb{Q}}_p^2) \xrightarrow{\delta} \mathrm{H}^1(F, \phi/\phi^\sigma) \to \mathrm{H}^1(F, W_{\tilde{\rho}}) \to \mathrm{H}^1(F, \bar{\mathbb{Q}}_p^2) \to \mathrm{H}^2(F, \phi/\phi^\sigma) \tag{11}$$

Lemma 6.6.

(i) *The cohomology group* $\mathrm{H}^2(F, \phi/\phi^\sigma)$ *is trivial.*
(ii) *One always has* $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(F, W_{\tilde{\rho}}) = 3$.

Proof. It follows from Global Euler characteristic formula that :

$$\begin{aligned}&\dim \mathrm{H}^0(F,\phi/\phi^\sigma)-\dim \mathrm{H}^1(F,\phi/\phi^\sigma)+\dim \mathrm{H}^2(F,\phi/\phi^\sigma)\\&\qquad=\sum_{v|\infty}\dim(\bar{\mathbb{Q}}_p)^{G_{F_v}}-[F:\mathbb{Q}]\end{aligned}\tag{12}$$

Since ϕ/ϕ^σ is a totally odd character, the relation above yields that:

$$-\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,\phi/\phi^\sigma)+\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^2(F,\phi/\phi^\sigma)=-2$$

It follows from (ii) of [4, Prop.5.2] that $\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,\phi/\phi^\sigma)=2$, and as such $\mathrm{H}^2(F,\phi/\phi^\sigma)$ is trivial. Finally, F is a real quadratic field, so F has a unique $\mathbb{Z}_p$-extension and $\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,\bar{\mathbb{Q}}_p^2)=2.\dim_{\bar{\mathbb{Q}}_p}\mathrm{Hom}(G_F,\bar{\mathbb{Q}}_p)=2$. Thus, the long exact sequence (11) implies that $\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,W_{\tilde{\rho}})=3$. ■

Corollary 6.7.

(i) *The cohomology group* $\mathrm{H}^2(F,W_{\tilde{\rho}})$ *is trivial.*
(ii) *There exists an exact sequence*

$$0\to \mathrm{H}^1(F,W_{\tilde{\rho}})\to \mathrm{H}^1(F,\mathrm{ad}\,\tilde{\rho})\xrightarrow{C^*}\mathrm{H}^1(F,\phi^\sigma/\phi)\to 0$$

Proof. i) It follows from Global Euler characteristic formula that:

$$\begin{aligned}&\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^0(F,W_{\tilde{\rho}})-\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,W_{\tilde{\rho}})+\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^2(F,W_{\tilde{\rho}})\\&\qquad=\sum_{v|\infty}\dim_{\bar{\mathbb{Q}}_p}(W_{\tilde{\rho}})^{G_{F_v}}-[F:\mathbb{Q}]\dim_{\bar{\mathbb{Q}}_p}W_{\tilde{\rho}}\end{aligned}\tag{13}$$

Thus, the assertion results directly from the fact that $\tilde{\rho}$ is a totally odd representation and $\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(F,W_{\tilde{\rho}})=3$.

ii) Since $\mathrm{H}^2(F,W_{\tilde{\rho}})=0$, the longue exact sequence (10) is unobstructed.

■

Theorem 6.8. *One always has* $\dim_{\bar{\mathbb{Q}}_p}t_{\mathcal{D}^{n.ord}}\leq 3$ *and* $\dim_{\bar{\mathbb{Q}}_p}t_{\mathcal{D}^{ord}}\leq 1$.

Proof. The proposition 6.6 and the long exact sequence (11) generate the following exact sequence :

$$\mathrm{H}^0(F,\bar{\mathbb{Q}}_p^2)\xrightarrow{\delta}\mathrm{H}^1(F,\phi/\phi^\sigma)\to\mathrm{H}^1(F,W_{\tilde{\rho}})\xrightarrow{\mathrm{i}}\mathrm{H}^1(F,\bar{\mathbb{Q}}_p^2)\to 0\tag{14}$$

A direct computation shows that the image of δ is of dimension one over $\bar{\mathbb{Q}}_p$.

Now, we will add the local conditions at v and v^σ arising from nearly ordinary deformations to (14):

(15)
$$\begin{array}{ccccccccc} \mathrm{H}^0(F,\bar{\mathbb{Q}}_p^2) & \xrightarrow{\delta} & \mathrm{H}^1(F,\phi/\phi^\sigma) & \xrightarrow{i} & \mathrm{H}^1(F,W_{\tilde{\rho}}) & \xrightarrow{\mathrm{i}} & \mathrm{H}^1(F,\bar{\mathbb{Q}}_p^2) & \longrightarrow & 0 \\ & & \downarrow \mathfrak{r}' & & \downarrow B^* & & & & \\ & & \mathrm{H}^1(F_{v^\sigma},\phi/\phi^\sigma) & \xrightarrow{=} & \mathrm{H}^1(F_{v^\sigma},\phi/\phi^\sigma), & & & & \end{array}$$

where $\mathfrak{r}'$ is the map given by restriction of the cocycles to $G_{F_{v^\sigma}}$.

First, we will prove that the composition of B^* with i is not trivial by proceeding by absurd :

Let ρ_1 be a cocycle representing a cohomology class of $\mathrm{H}^1(F,W_{\tilde{\rho}})$ lying in the image of i. Subsequently, we can modify ρ_1 by a coboundary in the aim that $\rho_1(g)=\left(\begin{smallmatrix}0 & b\\ 0 & 0\end{smallmatrix}\right)$. The function $b\to b(g)$ is a cocycle and its cohomology class belongs to $\mathrm{H}^1(F,\phi/\phi^\sigma)$. Suppose that cohomology class of $\left(\begin{smallmatrix}0 & b\\ 0 & 0\end{smallmatrix}\right)$ is non trivial (i.e $\left(\begin{smallmatrix}0 & b\\ 0 & 0\end{smallmatrix}\right)$ isn't a coboundary) and belonging to $\ker(\mathrm{H}^1(F,W_{\tilde{\rho}}) \xrightarrow{B^*} \mathrm{H}^1(F_{v^\sigma},\phi/\phi^\sigma))$; following this scenario, b can be modified by a coboundary so that $b=\lambda\eta/\phi^\sigma$, where $\lambda\in\bar{\mathbb{Q}}_p^\times$ (see Lemma 4.1). A direct computation demonstrates that the cocycle $\rho_1(g)$ is the coboundary given by

$$g\to\tilde{\rho}(g)A\tilde{\rho}(g)^{-1}-A \text{ , where } A:=\left(\begin{smallmatrix}-\lambda & 0\\ 0 & 0\end{smallmatrix}\right).$$

As a consequence, there is a contradiction since we assumed that ρ_1 is not a coboundary. Therefore, we obtain that

$$\dim_{\bar{\mathbb{Q}}_p}\ker(\mathrm{H}^1(F,W_{\tilde{\rho}}) \xrightarrow{B^*} \mathrm{H}^1(F_{v^\sigma},\phi/\phi^\sigma))=2.$$

The exact sequence presented below follows from Corollary 6.7, Lemma 6.5 and the above discussion :

$$0\to(\ker(\mathrm{H}^1(F,W_{\tilde{\rho}}) \xrightarrow{B^*} \mathrm{H}^1(F_{v^\sigma},\phi/\phi^\sigma))) \xrightarrow{i} t_{\mathcal{D}^{n.ord}} \xrightarrow{C^*} \mathrm{H}^1(F,\phi^\sigma/\phi)_{G_{F_v}} \tag{16}$$

Since $\dim\mathrm{H}^1(F,\phi^\sigma/\phi)_{G_{F_v}}=1$, it follows from (16) that $\dim_{\bar{\mathbb{Q}}_p} t_{\mathcal{D}^{n.ord}}\leq 3$.

To compute the dimension of $t_{\mathcal{D}^{ord}}$, the extra conditions of ordinariness at p need to be added to $t_{\mathcal{D}^{n.ord}}$, which appear in the filtration $W_{\tilde{\rho}}$ as follow:

We have a natural map of $\bar{\mathbb{Q}}_p[G_{F_{v^\sigma}}]$-modules (see (6))

$$\begin{aligned} \mathrm{ad}\,\tilde{\rho} &\longrightarrow \bar{\mathbb{Q}}_p \\ \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) &\longrightarrow a \end{aligned} \tag{17}$$

and inducing by functoriality a map

$$A^*:\mathrm{H}^1(F,\mathrm{ad}\,\tilde{\rho})\to\mathrm{Hom}(G_{F_{v^\sigma}},\bar{\mathbb{Q}}_p).$$

The following inclusion results

$$t_{\mathcal{D}^{ord}} \subset W = \ker\left(\mathrm{H}^1(F, \mathrm{ad}\,\tilde{\rho}) \overset{(\mathfrak{r}\circ C^*, B^*, A^*)}{\longrightarrow} (\mathrm{H}^1(F_v, \phi^\sigma/\phi) \oplus \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma) \oplus \mathrm{Hom}(G_{F_{v^\sigma}}, \bar{\mathbb{Q}}_p))\right)$$

Let W_0 denote $\ker(\mathrm{H}^1(F, W_{\tilde{\rho}}) \overset{(B^*, A^*)}{\longrightarrow} \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma) \oplus \mathrm{Hom}(G_{F_{v^\sigma}}, \bar{\mathbb{Q}}_p))$; the exact sequence below emerges

$$0 \to W_0 \overset{i}{\to} W \overset{C^*}{\longrightarrow} \mathrm{H}^1(F, \phi^\sigma/\phi)_{G_{F_v}} \tag{18}$$

Therefore, the isomorphism $\ker(\mathrm{H}^1(F, W_{\tilde{\rho}}) \overset{B^*}{\to} \mathrm{H}^1(F_{v^\sigma}, \phi/\phi^\sigma)) \simeq \mathrm{H}^1(F, \bar{\mathbb{Q}}_p^2)$ (coming from the above discussion) implies that W_0 is of dimension one over $\bar{\mathbb{Q}}_p$ and $\dim_{\bar{\mathbb{Q}}_p} W \leq 2$.

Any cocycle $\rho_1 \in W_0$ satisfies the condition of ordinariness at p is necessarily an homomorphism in $\mathrm{H}^1(F, \bar{\mathbb{Q}}_p)$, which is unramified at v, so trivial (since F is a real quadratic extension of $\mathbb{Q}$, F has a unique $\mathbb{Z}_p$-extension). Thus, the exact sequence (18) yields that $\dim_{\bar{\mathbb{Q}}_p} t_{\mathcal{D}^{ord}} = \dim_{\bar{\mathbb{Q}}_p} W - 1 \leq 1$.

■

Proof of the theorem 1.5.

The p-nearly ordinary deformation

$$\rho_{Q(\mathbb{T}^{n,ord})} : G_F \to \mathrm{GL}_2(\mathbb{T}^{n,ord})$$

of $\tilde{\rho}$ yields a canonical morphism :

$$\mathcal{R}^{n.ord} \to \mathbb{T}^{n.ord} \tag{19}$$

Let $\mathfrak{n}_1 := \mathfrak{n}^{n.ord} \cap \Lambda_{\mathcal{O}}^{n.ord}$ and $\widehat{\Lambda}_{(1)}^{n.ord}$ be the completed local ring for the étale topology of $\mathrm{Spec}\,\Lambda_{\mathcal{O}}^{n.ord}$ at a geometric point corresponding to $\mathfrak{n}_1$. Since $h_F^{n.ord}$ is a torsion-free $\Lambda_{\mathcal{O}}^{n.ord}$-module of finite type, we gain (after localization) a finite torsion-free morphism $w : \widehat{\Lambda}_{(1)}^{n.ord} \to \mathbb{T}^{n.ord}$. On the other hand, the local ring $\mathbb{T}^{ord}$ is endowed naturally with structure of Λ-algebra originating from the finite flat morphism $\Lambda_{\mathcal{O}} \to h_F$ (see [23]).

The ring $\mathcal{R}^{n.ord}$ has a canonical structure of $\widehat{\Lambda}_{(1)}^{n.ord}$-algebra (see [4, §6.2]), and the morphism (19) is a morphism of $\widehat{\Lambda}_{(1)}^{n.ord}$-algebras.

Moreover, the ring $\mathcal{R}^{ord}_{\det\rho} := \mathcal{R}^{n.ord}/\mathfrak{m}_{\widehat{\Lambda}_{(1)}^{n.ord}}\mathcal{R}^{n.ord}$ represents the largest p-ordinary quotient of $\mathcal{R}^{n.ord}$ of determinant equal to $\det\tilde{\rho}$ (see [4, §6.2]).

Proposition 6.9.

(i) *The morphism (19) yields an isomorphism of regular rings* $\mathcal{R}^{n.ord} \simeq \mathbb{T}^{n.ord}$.

(ii) *There exists an isomorphism between local regular rings* $\mathcal{R}^{ord} \simeq \mathbb{T}^{ord}$.

(iii) *There exists an isomorphism* $\mathcal{R}^{ord} \simeq \mathcal{R}_{\tau=1}$.

(iv) *There exists an isomorphism* $\mathcal{R}^{ord}_{\det\rho} \simeq \mathbb{T}^{ord}/\mathfrak{m}_{\Lambda}\mathbb{T}^{ord} \simeq \mathbb{T}^{n.ord}/\mathfrak{m}_{\widehat{\Lambda}^{n.ord}_{(1)}}\mathbb{T}^{n.ord}$.

Proof.

i) First of all, it needs to be demonstrate that the morphism (19) is surjective. By construction, the Hecke algebra $\mathbb{T}^{n.ord}$ is generated of $\widehat{\Lambda}^{n.ord}_{(1)}$ by the Hecke operators $T_{\mathfrak{q}}$ with $\mathfrak{q} \nmid p$, $U_{\mathfrak{p}}$ and $U_{\mathfrak{p}^{\sigma}}$. The morphism (19) sends the trace of $\rho_{\mathcal{R}^{n.ord}}(\mathrm{Frob}_{\mathfrak{q}})$ to $T_{\mathfrak{q}}$ when $\mathfrak{q} \nmid p$. Otherwise, the restriction of $\rho_{\mathcal{R}^{n.ord}}$ to $G_{F_{\mathfrak{p}_i}}$ for all primes $\mathfrak{p}_i \mid p$ of F is an extension of the character $\psi''_{i,\mathcal{R}^{n.ord}}$ by the character $\psi'_{i,\mathcal{R}^{n.ord}}$, where the image of the character $\psi''_{i,\mathcal{R}^{n.ord}}$ in $\mathbb{T}^{n.ord}$ is just the character $\delta_{\mathfrak{p}_i}$ which sends $[y, F_{\mathfrak{p}_i}]$ on the Hecke operator $T(y)$, where $[., F_{\mathfrak{p}_i}] : \widehat{F^{\times}_{\mathfrak{p}_i}} \to G^{ab}_{F_{\mathfrak{p}_i}}$ is the Artin symbol. Thus, $U_{\mathfrak{p}_i} = [\pi_{\mathfrak{p}_i}, F_{\mathfrak{p}_i}]$ in the image of the morphism (19) for some uniformizing parameter $\pi_{\mathfrak{p}_i}$ of the local field $F_{\mathfrak{p}_i}$. Hence, the morphism (19) is surjective and the Krull dimension of $\mathcal{R}^{n.ord}$ is at least 3 (since the Krull dimension of $\mathbb{T}^{n.ord}$ is 3).

Finally, Proposition 6.8 implies that $\mathcal{R}^{n.ord}$ is a regular ring of dimension 3, because the Krull dimension of a local ring is less or equal to the dimension of its tangent space. Therefore, the surjection (19) is necessarily an isomorphism of regular local rings of dimension 3 (since the Krull dimension of $\mathbb{T}^{n.ord}$ is 3).

ii) It derives from (i) and the relation [4, (20)] that $\mathcal{R}^{ord} \simeq \mathbb{T}^{ord}$. On the other hand, Theorem 6.8 implies that the dimension of $\mathfrak{m}_{\mathcal{R}^{ord}}/\mathfrak{m}^2_{\mathcal{R}^{ord}}$ is one over $\bar{\mathbb{Q}}_p$. Moreover, the Krull dimension of $\mathbb{T}^{ord}$ is equal to one and the tangent space of $\mathbb{T}^{ord}$ is of dimension one, hence $\mathbb{T}^{ord}$ is a regular local ring of dimension one.

iii) The deformation $\rho_{\tau=1}$ of $\tilde{\rho}$ (see lemma 4.1) induces by functoriality an homomorphism $\mathcal{R}^{ord} \to \mathcal{R}_{\tau=1}$. Since $\mathcal{R}_{\tau=1}$ is generated over Λ by the trace of $\rho_{\tau=1}$ ($\mathcal{R}^{ps}_{red} \simeq \mathcal{R}_{\tau=1}$), this homomorphism $\mathcal{R}^{ord} \to \mathcal{R}_{\tau=1}$ is necessarily surjective. Finally, both $\mathcal{R}^{ord}$ and $\mathcal{R}_{\tau=1}$ are discrete valuation rings, then this surjection rises to an isomorphism.

iv) It follows from i), ii) and the relations of the section [4, §6.2].

■

Let $S^{\dagger}_1(1, \mathrm{Id})_{/F}$ denote the space of p-ordinary p-adic cuspidal Hilbert modular forms over F of weight 1, tame level 1, of trivial Nybentypus character and with coefficients in $\bar{\mathbb{Q}}_p$; and let $S^{\dagger}_1(1, \mathrm{Id})_{/F}[[E_1(\phi, \phi^{\sigma})]]$ be the generalised eigenspace attached to $E_1(\phi, \phi^{\sigma})$ inside $S^{\dagger}_1(1, \mathrm{Id})_{/F}$. By construction of the universal p-ordinary Hecke algebra h_F and the Hida duality between cuspidal p-adic modular forms and Hecke algebras, the following isomorphism is a generalization of [12, Prop.1.1] :

$$\mathrm{Hom}_{\bar{\mathbb{Q}}_p}(\mathbb{T}^{ord}/\mathfrak{m}_{\Lambda}\mathbb{T}^{ord}, \bar{\mathbb{Q}}_p) \simeq S^{\dagger}_1(1, \mathrm{Id})_{/F}[[E_1(\phi, \phi^{\sigma})]].$$

We have the following consequence of Proposition 6.9, summarizing the overall results of this paper.

Corollary 6.10. *Assume that ϕ is unramified everywhere and $\phi(\mathrm{Frob}_v) \neq \phi^\sigma(\mathrm{Frob}_v)$, then the following conditions are equivalents:*

(i) $\mathbb{T}^{n.ord}$ *is étale over* $\widehat{\Lambda}^{n.ord}_{(1)}$.
(ii) $\mathbb{T}^{ord}$ *is étale over* Λ.
(iii) $\mathcal{T}_+$ *is étale over* Λ.
(iv) *The ramification index* e *of* $\mathcal{C}$ *over* $\mathcal{W}$ *is exactly* 2.
(v) *The* $\bar{\mathbb{Q}}_p$*-vector space* $S_1^\dagger(1,\mathrm{Id})_{/F}[[E_1(\phi,\phi^\sigma)]]$ *is of dimension one and it is generated by* $E_1(\phi,\phi^\sigma)$.

Remark 6.11.

In case hypothesis (**G**) *holds, the equivalence of the above corollary holds as well, and every overconvergent form of* $S_1^\dagger(1,\mathrm{Id})_{/F}[[E_1(\phi,\phi^\sigma)]]$ *is necessarily classical.*

7. EXAMPLES WHERE THE RAMIFICATION INDEX e OF $\mathcal{C}$ OVER $\mathcal{W}$ AT f IS 2

Cho, Dimitrov and Ghate provided several examples for Hida families $\mathcal{F}$ containing a classical RM cuspform and such that the field generated by the coefficients of $\mathcal{F}$ is a quadratic extension of the fraction field of the Iwasawa algebra $\Lambda_{\mathcal{O}}$. Thus, we have several numerical examples for which the ramification index e of $\mathcal{C}$ over $\mathcal{W}$ at f is 2.

7.1. **Examples provided by Dimitrov-Ghate** [16, §7.3]**.**

Denote by $\mathbb{T}^{\mathrm{new}}_{N,\bar{\rho}}$ the N-New-quotient of $h_{\mathbb{Q},\mathfrak{m}}$ acting on the space of $\Lambda_{\mathcal{O}}$-adic ordinary cuspforms of tame level N which are N-New. Dimitrov and Ghate studied in [16, §7.3] the Hida families specializing to classical RM forms, and they give a couple of examples for which the rank of $\mathbb{T}^{\mathrm{new}}_{N,\bar{\rho}}$ over the Iwasawa algebra $\Lambda_{\mathcal{O}}$ is two. In this case, if $\mathcal{F}$ denote a p-adic Hida family specializing to the classical RM form f, then the field generated by the coefficients of $\mathcal{F}$ is obtained by adjoining to $\mathrm{Frac}(\Lambda_{\mathcal{O}})$ a square-root of an element in $\Lambda_{\mathcal{O}}$.

Their method of computation is based on the study of the specializations in weights two or more; specifically, they showed that the p-adic completions of the Hecke fields of modular forms f_k for the first few weights k are all quadratic extensions of $\mathbb{Q}_p$ (see Table 1 and Table 2 of [16, §7.3]).

7.2. **Examples provided by Cho** [8, §7]**.**

The method of computation of S.Cho in [8, §7] includes the study of the unramifiedness specializations of $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ of higher weight in the aim to prove that $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}} \simeq \Lambda_{\mathcal{O}}$ in many examples.

Let H_k be the Hecke algebra over $\mathbb{Q}$ for the space of cusp forms of weight k, Nybentypus character ϵ_F and level N, H_k^+ be the maximal real sub-algebra of H_k and, moreovoer, D_+ be the discriminant of H_k^+.

A direct computation illustrates that the Atkin-Lehner involution acts on H_k as the complex conjugation. Therefore, when $p \nmid D_+$, the specialization of $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ at the weight k is unramified over $\mathcal{O}$, and hence $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}} \simeq \Lambda_{\mathcal{O}}$ by [19, Prop.8].

Thus, it is sufficient to detect examples such that the specialization of $h^{\omega=1}_{\mathbb{Q},\mathfrak{m}}$ at higher weight k is unramified over $\mathcal{O}$, and Cho checked this unramifiedness from the discriminant table from [18, Table 1].

References

[1] F. Andreatta, A. Iovita et V. Pilloni, *On overconvergent Hilbert modular cusp forms.* Astérisque No. **382** (2016), 163-193.

[2] J. Bellaïche and G. Chenevier, *Lissite de la courbe de Hecke de* GL(2) *aux points Eisenstein critiques*, J. Inst. Math. Jussieu **5** (2006), 333-349.

[3] J. Bellaïche and M. Dimitrov, *On the eigencurve at classical weight one points*, Duke Math. J. **165** (2016), no. 2, 245-266.

[4] A. Betina, *Les Variétés de Hecke-Hilbert aux points classiques de poids un*, to appears in J. Théor. Nombres Bordeaux.

[5] S. Bijakowski, *Classicité de formes modulaires de Hilbert*, "Arithmetique p-adique des formes de Hilbert", Asterisque **382** (2016), 49-71.

[6] K. Buzzard, *Eigenvarieties*, in L-functions and Galois representations, vol. **320** of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2007, pp. 59-120.

[7] G. Chenevier, *Familles p-adiques de formes automorphes pour* GL_n, J. Reine Angew. Math. **570** (2004), pp. 143-217.

[8] S. Cho, *Deformations of induced Galois representations*, Thesis (Ph.D.)-University of California, Los Angeles. 1999. 57 pp.

[9] S. Cho and V. Vatsal, *Deformations of induced Galois representations*, J. Reine Angew. Math. **556** (2003), pp. 79-98.

[10] R. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124**, (1996), p.215-241.

[11] R. Coleman and B. Mazur, *The Eigencurve*, in Galois representations in arithmetic algebraic geometry (Durham, 1996), vol. **254** of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1998, pp. 1–113.

[12] H. Darmon, A. Lauder and V. Rotger, *Stark points and p-adic iterated integrals attached to modular forms of weight one*, Forum of Mathematics, Pi (2015), Vol. **3**, e8, 95 pages.

[13] H. Darmon, A. Lauder and V. Rotger, *Overconvergent generalised eigenforms of weight one and class fields of real quadratic field*, Advances in Mathematics **283** (2015) 130-142.

[14] P. Deligne and J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4), **7** (1974), pp. 507-530.

[15] S. Deo, *On the eigenvariety of Hilbert modular form at classical parallel weight one point with dihedral projective image*, Trans. Amer. Math. Soc. 370 (2018), no. **6**, 3885-3912.

[16] M. Dimitrov and E. Ghate, *On classical weight one forms in Hida families*, J. Théor. Nombres Bordeaux, **24** (2012), no. 3, 669-690.

[17] M. Dimitrov, *On the local structure of ordinary Hecke algebras at classical weight one points* . In Automorphic Forms and Galois Representations vol. 2 (Durham, 2011), 1-16, London Math. Soc. Lecture Note Series **415**, Cambridge Univ. Press, (2014).

[18] K Doi and H. Hida and H. Ishii, *Discriminant of Hecke fields and twisted adjoint L-values for* GL(2). Invent. Math. **134** (1998), no. 3, 547-577.

[19] F. Gouvea, *On the Ordinary Hecke Algebra*, J. Number Theory **41** (1992), no. 2, 178-198.

[20] R. Greenberg, *The Iwasawa invariant of* Γ*-extensions of a fixed number field*. Amer. J. Math. **95** (1973), 204-214.

[21] H. Hida, *Galois representations into* $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. **85** (1986), no. 3, 545-613.

[22] H. Hida, *On nearly ordinary Hecke algebras for* GL(2) *over totally real fields*, Adv. Studies in Pure Math., **17** (1989), 139-169.

[23] H. Hida, *Nearly ordinary Hecke algebra and Galois representation of several variables*, Proc. JAMI Inaugural Conference, Supplement to Amer. J. Math., (1989), 115-134.

[24] H. Hida, *Global quadratic units and Hecke algebras*, Documenta Math. **3** (1998), 273-285.

[25] H. Hida, *On Selmer groups of adjoint modular Galois representations*, Number Theory, Paris, 1993-94, LMS Lecture notes series, **235** (1996), 89-132.

[26] H. Kisilevsky. *Some non-semi-simple Iwasawa modules*, Compositio Math. **49** (1983), no. 3, 399-404.

[27] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153** (2003), no. 2, 373-454.

[28] R. Langlands, *Base change for* GL_2, Annals of Math. Studies **96**, Princeton University Press, 1980.

[29] L. Nyssen, *Pseudo-représentations*, Math. Ann., **306** (1996), pp. 257–283.

[30] V. Pilloni, *Formes modulaires surconvergentes*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 1, 219-239.

[31] V. Pilloni et B. Stroh, *Surconvergence et classicité : le cas Hilbert*, J. Ramanujan Math. Soc. **32** (2017), no. 4, 355-396.

[32] R. Rouquier, *Caractérisation des caractères et pseudo-caractères*, J. Algebra, **180** (1996), pp. 571-586.

[33] J.P. Serre, *Classes des corps cyclotomiques (d'après K.Iwasawa)*, Séminaire Bourbaki, Vol. 5, Exp. No. 174, 83-93, Soc. Math. France, Paris, 1995.

[34] C.M. Skinner, A. Wiles, *Residually reducible representations and modular forms*, Inst. Hautes Études Sci. Publ. Math. No. 89 (1999), 5-126 (2000).

[35] R. Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281-332.

[36] R. Taylor, A. Wiles, *Ring theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553-572.

[37] A. Wiles, *On ordinary* λ*-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529-573.

[38] A. Wiles, *Modular elliptic curves and Fermat's last theorem.* Ann. of Math. (2) **141** (1995), no. 3, 443-551.

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