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MAKING SENSE OF THE TEACHING OF CALCULUS FROM A COMMOGNITIVE PERSPECTIVE

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Examining the discourse through the lens of commognition theory allowed an investigation of how teachers of mathematics teach elementary calculus. Analysing the *teachers' word use and narratives provided insights into the specialisation of the* mathematical language used in the discourse. Analysing the visual mediators, routines and meta-rules used in the classroom discourse, but more importantly, how and when they were used, explained the modes of mediation used in teaching elementary calculus.

INTRODUCTION

This paper reports on a discursive analysis of mathematical discourse on elementary calculus through the lens of the commognitive framework (Sfard, 2008). Given the microscopic nature of commognitive analyses and the word count limitations, one case out of nine was selected for this paper. Thus, this case study is part of a more extensive (doctoral) study, which seeks to research how teachers of mathematics teach elementary calculus in England. Elementary calculus is part of school (post-16) or college mathematics curriculum in the United Kingdom (UK) and many other countries. The object of enquiry is how mathematics teachers teach the derivative. The unit of analysis is the discourse of the teacher, primarily, though the classroom discourse is also considered in as far as it provides the social context of the teacher's discourse. It is a discursive analysis, therefore, a qualitative study. In the following sections, a brief introduction to the commognitive framework is outlined, followed by the methodology explaining the discursive approach to data analysis. This is then followed by a discussion of selected findings, and finally, a summary of conclusions and implications.

THE COMMOGNITIVE FRAMEWORK

Sfard (2008) presents the commognitive framework for the study of (mathematical) thinking. Commognition is a term founded by Sfard, which conceptualises thinking as a 'form of communication' with oneself. Thus, cognition plus communication constitute commognition (p.570). Thinking is construed as individualisation of interpersonal communication. Thus, thinking processes and interpersonal communication are facets of the same phenomena. Discourse is the core unit of analysis. Discourse can either be non-specialised discourses – 'colloquial discourse' or 'literate discourses' (Sfard, 2008, p.299) which are artefact-mediated mainly by symbolic tools designed specifically for communication. Mathematics is regarded as a

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form of discourse, which is characterised by four commognitive constructs: word use; visual mediators; endorsed narratives; and routines.

Word use refers merely to the kinds of words used in the discourse. Narratives are utterances in the discourse, thus, made up of words, any written or spoken text used to construct or endorse other narratives. In literate mathematical discourse, endorsed narratives include processes such as defining, estimating, abstracting, conjecturing, proving and generalising (Sfard, 2007, 2008). Visual mediators are visible objects, including symbolic artefacts such as formulae, graphs, drawings and diagrams that are created and used to enhance mathematical communication. According to Sfard (2007), visual mediators are the means through “which participants of a discourse identify the object of their talk and coordinate their communication” (p. 571). Noticing and categorising visual mediators is important in commognitive analyses. Routines are the “well defined repetitive patterns” (Sfard, 2007, p.572) in teachers’ actions in classroom discourse. Didactical and mathematical routines can be noticed in the use of mathematical words, visual mediators and narratives, i.e., can be observed in the processes of “creating and substantiating narratives” (p.572) about say, differentiation. Routines are the meta-rules that govern when and how these visual mediators and narratives are used. Meta-rules, if formulated, take the form of meta-level narratives – “propositions about the discourse rather than its objects” (Sfard, 2007, p.573).

The commognitive framework allows for the study of the discursive developments of individual students and the discursive practices of the teacher.

METHODOLOGY

Data collection and participants

Nine teachers of mathematics (and their classes) took part in the study. However, this paper reports on data sets from one of the participant teachers. Peter is a male teacher of mathematics in a college who had been teaching post-16 mathematics for more than three decades. He has a first-class honours degree in mathematics and a Post-Graduate Certificate in Education (PGCE), both from the UK. It was mainly because of his long teaching experience why Peter was chosen for this study.

Data sets for the case study include two audio-recorded interviews with the teacher and one video-recorded lesson observation in which the teacher discussed tangents, gradients and differentiation. The teacher was interviewed first, prior to teaching the observed lesson on calculus and secondly, after teaching the lesson. The lesson observation video data and the interviews audio data were transcribed with respect to the participants’ utterances and actions. The primary focus of the study is the teacher’s utterances and actions.

Method of analysis

The analysis uses a priori characterisation of discourse comprising the four main commognitive constructs of word use, narratives, visual mediators and endorsed routines (Sfard, 2008). For the analysis of word use, the extent to which the teacher uses specialised mathematical terminology in his mathematical discourse is examined. This focuses on the teachers' literate and colloquial word use in differential calculus. The analysis explores the visual mediators incorporated in the discourse and examines how the discourse makes use of the multiple mathematical, visual mediators. A key focus is an analysis of the transitions between different visual mediators, signified by the presence of both, verbal and visual realisations - words or symbols that 'function as nouns' (Sfard, 2008, p.155). For the investigation of routines, the analysis focuses on the meta-rules with respect to analyses of word use, visual mediators, and endorsed narratives in terms of how and when they are used (Sfard, 2008). For the analysis of narratives, attention is given to both written and spoken text about definitions, proofs, and facts related to differentiation. The focus is on the meta-level narratives that were particularly pertinent to the teacher's word use, visual mediators, and routines within the mathematical discourse. The meta-rules are important in the analyses of narratives as they regulate practices when the participants generate and substantiate mathematical meaning (Güçler, 2013).

FINDINGS AND DISCUSSION

In the analysis below, I discuss three findings of the study: the teacher's approach to introducing the 'derivative'; inconsistency in the teacher's use of calculus words; and ambiguity with calculus symbolism. The excerpts and the numbering of the utterances, as presented in the discussion, are all extracted from the original transcripts of the data sets.

How do teachers introduce the idea of differentiation?

In the pre-lesson interview with the teacher, Peter explains the necessity and importance for promoting conceptual and 'relational' understanding, i.e., "knowing both what to do and why" (Skemp, 1976, p.20, italics mine) of differentiation. Talking about his approach to introducing differentiation, Peter said: [interview transcript]:

46 Teacher: I want them to have at least a feel of what we are trying to do, what differentiation means rather than just state that, right, when you start with x^2 you get $2x$. Right, they will get it, but what does it mean? I just want them to have a feel of what it actually means.

48 Teacher: I don't see how you can start saying, right, $y = x^2, \frac{dy}{dx} = 2x$ ... you know... I certainly won't be using the $\frac{dy}{dx}$. I don't think… I certainly ... I mean I can't believe I will be using that notation today. If I do, I haven't planned to anyway.

The word 'certainly' is used twice in [48]. The teacher's view, as expressed here, is that it would be inappropriate to use the $\frac{dy}{dx}$ notation in the first lesson on differentiation. Notice the teacher's didactical objective in [46], what it (differentiation) means is repeated at least three times in [46] alone. The word 'what' is repeated four times and the word 'mean' three times.

Orton's studies (Orton 1983_a and 1983_b) found that students had 'instrumental' knowledge' (Skemp, 1976) of differentiation; they could carry out the standard calculations/rules in differentiation very well, but they did not have adequate knowledge of where the standard rules come from. They lacked the relational understanding of how and why the methods or rules work. Peter explains the need for the substantiation of the narrative: If $y = x^2$, then $\frac{dy}{dx} = 2x$ rather than starting with the standard rules for differentiation. This belief may explain his approach to introducing differentiation in the lessons that followed the interview.

The teacher gave out the graph of $y = x^2$ with the following instructions [lesson transcript]:

26 Teacher: Now I want you to locate the point on the graph where x equals one. Can you locate the point $x = 1$? y will also be one as well, and I want you to

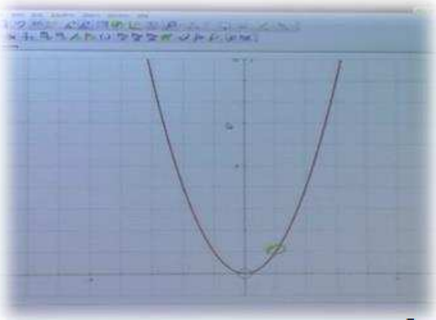

Fig.1: The graph of $y = x^2$

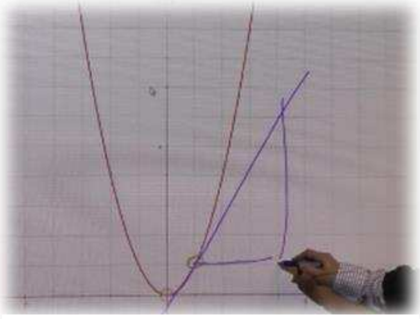

Fig.2: Drawing tangents

draw with a ruler the tangent. I want the tangent to be as long as you like, a straight line. You're doing this by eye, by no other way, by eye.

30 Teacher: I want you to imagine you're traveling around this curve… $x = 1$ is about there, isn't it? Make your line long and bold. Now I want you to measure the gradient of that line.

The mathematical object of the discourse is differentiation. The teacher's approach is to construct the definition of derivative by exemplar (Viirman, 2013), using the visual mediators [Fig.1 & Fig.2] of the graph of $y = x^2$. Substantiation of differentiation is done through estimating the derivative using tangents [Fig.2], rather than starting with the standard rules for differentiation.

Inconsistency in word use

I look at two phrases that Peter used repeatedly in his discourse: 'gradient of a curve' and 'tangent'.

The gradient of a curve: Routines are the meta-rules governing the repetitive discursive actions of participants of the discourse (Sfard, 2008). To identify any well-defined didactical practices or repetitive patterns in the teacher's actions, i.e., routines, it is important to identify the object of the discourse (Nardi et al., 2014, p.185), the 'discursive objects' (Sfard, 2008, p.166). Here is Peter introducing the mathematical object of his lesson, [lesson transcript]:

4 Teacher: And I want to pose a problem to you, and the problem is this… [Teacher writing on the white board – “The gradient of a curve”].

5 Teacher: What do we mean by that? That's my first question to you. Now we all know, I hope what is meant by the gradient of a line.

7 Teacher: How do you measure the gradient of a line then? How do you measure the gradient?

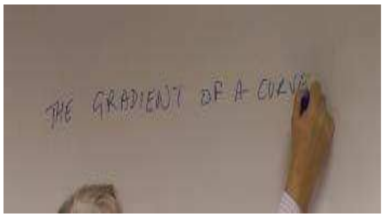

Fig.3: The title of the lesson

9 Teacher: Right, so my question to you is what do we mean by the gradient of a curve?

The teacher’s narrative ‘gradient of a curve’, which is signified both verbally [4] and visually [Fig.3], is inconsistent with literate mathematical discourse. An endorsed narrative describes ‘the gradient of a curve *at a point’*. The teacher’s narrative ‘the gradient of a curve’ is, therefore, colloquial discourse. But what is this discursive object here framed as ‘the gradient of a curve’? Notice, the teacher begins by asking the ‘what’ gradient question and he did not get a satisfactory answer from the students; he changed the question to the ‘how’ to measure the gradient of a line, and then asked about the ‘what’ gradient of a curve. The questioning suggests that by knowing ‘how to’ measure the gradient of a line (operational), that would lead the students to knowing ‘what is’ the ‘gradient (object) of a curve’; it doesn’t say ‘at a point’. What the teacher refers to, in the discourse, as ‘gradient of a curve’, is indeed, the gradient function or derived function.

Tangent: To understand the routine for constructing the object of the derivative, it is important to observe and analyse the processes of creating and substantiating narratives (Sfard, 2007). Together with the use of visual mediators, the analysis of narratives would enable us to identify the types and characteristics of the routine procedures. Using the graph of the function $y = x^2$, the teacher talks about the tangent:

20 Teacher: Is there anywhere on that curve where you definitely, already know its gradient?

21 Student: $x - axis$

22 Teacher: Good, would you all accept that the x-axis is a tangent to the curve? What is the gradient of the $x - axis$?

23 Student: Zero

24 Teacher: Zero. A tangent, you did this in mechanics, is sort of the direction in which you are instantaneously traveling.

25 Teacher: The direction in which you're going there [Teacher pointing at the graph on the board] is the instantaneous direction, the tangent of the curve.

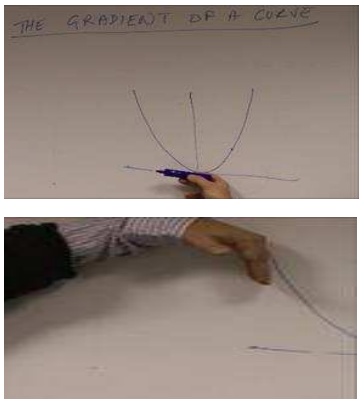

Fig.4: Teacher’s sketch diagrams

In [20] - [25] the teacher constructs a definition of the tangent by exemplar (Viirman, 2013) by illustrating its key properties with a specific example of the visual mediator, the graph of the function $y = x^2$, and such routines are characteristic of, and prevalent in mathematical discourse. However, notice that the teacher describes 'instantaneous direction' as 'the tangent of the curve' [25]. Although objectified, treating the mathematical concept of direction as a mathematical object, the narrative is inconsistent with literate mathematical discourse. An endorsed narrative describes direction as the slope or gradient of the tangent. Thus, the narrative 'the tangent of the curve' here should be substantiated to mean 'the gradient of the tangent'.

Ambiguity with calculus symbolism

In calculus discourse, symbolic artefacts, such as the $\frac{dy}{dx}$, are integral to the thinking and communication process (Sfard, 2007). Apparent in the teacher's discourse are visual mediators: written words, graphs [Fig.1 & Fig.2], deictic language [25] and gesturing [Fig.4] and symbolism. However, there is some ambiguity in the teacher's use of calculus symbolism. Here is one of the teacher-student dialogue from the lesson [lesson transcript]:

85 Teacher:	So, let's make a note of this, [writing on the board] If $f(x)$ is x^3, it means $f'(x)$ is $3x^2$.
86 Student:	What is that dash mean?
87 Teacher:	It means the derivative, the gradient function. That's the notation I have used here.
88 Student:	What does the derivative mean?
89 Teacher:	It means the gradient function, the gradient of the curve is $2x$, of x^2. It's not a constant, is it?
90 Student:	No
91 Teacher:	The gradient, a constant?
92 Student:	No
93 Teacher:	It's a function of x
95 Teacher:	We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

The use of symbolic signifier f' 'f-dash' in [86 - 87] by the teacher, poses some challenges for the students. The question in [86] suggests that the student is having some difficulties with symbolic realisations, which seems to be exacerbated by the teacher's response with specialised calculus vocabulary [88]. In substantiating the narrative, the teacher switches between visual and vocal signifiers, from symbols [85] to specialised words – derivative, gradient function [87]. However, these specialised calculus words added to the student's difficulty with calculus – the meaning of the derivative [88]. The teacher reiterates his earlier narrative [87] in [95], linking the words 'derivative' and 'gradient function'. Notice that the teacher's routine is to construct a definition of derivative by exemplar (Viirman, 2013). Thus, by illustrating the properties [89 – 94] of the object of the discourse with a specific example. However, note the inconsistency in word use of gradient in [91-93], the teacher's utterances in [91] and [93] are in fact contradictory; the gradient is indeed a constant! The word derivative could refer to the derivative function (a function) or the derivative at a point (a constant). This dualism was not substantiated in the lesson; it was not made explicit for the student. A commognitive study with calculus students by Park (2013), found that 'most students did not appreciate the derivative at a point as a number and the derivative function as a function' (p.624). In calculus discourse, such ambiguity is compounded by calculus symbolism.

The teacher's use of visual signifier f' draws upon historically established mathematical discourse in calculus symbolism (Sfard, 2008). Symbolic mediators such as $\frac{dy}{dx}$ or $f'(x)$ have a dual role. On the one hand, $f'(x)$ can be an objectified narrative for 'the derivative of $f(x)$', and an operational narrative for 'the process of differentiation' on the other. Such a symbolic signifier is what Gray & Tall (1991) described as 'procept' (Tall, 1992b, p.4). In the mind of a literate mathematician, a procept can evoke either a process or a concept, and it all happens subconsciously (Tall, 1992b). The term procept refers to "the amalgam of process and concept in which process and product are represented by the same symbolism" (Tall, 1992b, p.4). The "duality (as process or concept), flexibility (using whichever is appropriate at the time) and ambiguity (not always making it explicit which we are using)" (p.4) in the use of calculus procepts presents challenges for many students. Calculus symbolism and vocabulary has been found to present challenges for both students and teachers (Tall, 1992a). Given the flexibility and the duality of use of such procepts, it is essential that teachers make it explicit enough for students to develop the necessary flexible thinking and understanding to be able to deal with the possible ambiguity of use (Tall, 1992b).

CONCLUSION AND IMPLICATIONS

Mathematical discourse on calculus involves specialised mathematical language and visual mediators. In calculus, symbolic realisations are an important aspect of visual mediation, so are graphical representations. For example, notice that the graph [in 20-23] is used, not as a mere auxiliary means for conveying a pre-existing thought, but as a way of communicating. Thus, visual mediators are integral to commognition, i.e., the thinking and communication process in the discourse, contrary to the common understanding of tool use.

The classroom discourse involves multiple visual mediators, but more importantly, the didactical routines show evidence of constant shifts between different signifiers or modes of mediation. We see shifts between symbolic signifiers (e.g.f') and specialised mathematical words (derivative); shifts between verbal mediation (e.g. use of deictic language) and visual mediation (e.g. the graph, the teacher gesturing), in [25] for instance. For the teaching of calculus, Tall (1992a) argues for the need for versatile transitions between representations, graphics, numerics and symbolics (p.9). Such representations resemble Sfard's 'realisations' (p.154) of the signifiers which could be spoken words or written words or visual symbols. Nardi et al. (2014) explain the importance of symbolic realisations in mathematical discourse, that symbolic mediation brings 'generative power' (Sfard, 2008, p.159) and 'powerful manipulative ability' (Tall. 1992a, p.9) of the discourse.

There is also evidence of some inconsistency in the teacher's use of calculus words, and some ambiguity in the use of calculus symbolism in the classroom discourse. This suggests that difficulties with calculus persist, both for students and teachers alike. Therefore, mathematics teachers and educators should always pay particular attention to the specialised vocabulary and symbolism in the calculus discourse.

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