Classical and Quantum Super-integrability: From Lissajous Figures to Exact Solvability

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Abstract

The first part of this paper explains what super-integrability is and how it differs in the classical and quantum cases. This is illustrated with an elementary example of the resonant harmonic oscillator.

For Hamiltonians in "natural form", the kinetic energy has geometric origins and, in the flat and constant curvature cases, the large isometry group plays a vital role. We explain how to use the corresponding first integrals to build separable and super-integrable systems. We also show how to use the automorphisms of the symmetry algebra to help build the Poisson relations of the corresponding non-Abelian Poisson algebra.

Finally, we take both the classical and quantum Zernike system, recently discussed by Pogosyan, et al, and show how the algebraic structure of its super-integrability can be understood in this framework.

1 Introduction

We start this paper with a brief reminder of the definitions of complete and super-integrability.

A Hamiltonian system of n degrees of freedom, Hamiltonian H, is said to be *completely integrable in the Liouville sense* if we have n independent functions I_n, which are *in involution* (mutually Poisson commuting), with H being a function of these and typically just one of them. Whilst n is the maximal number of independent functions which can be *in involution*, it is possible to have further integrals of the Hamiltonian H, which necessarily generate a non-Abelian algebra of integrals of H. The maximal number of additional *independent* integrals is $n - 1$, since the "level surface" of $2n - 1$ integrals (meaning the intersection of individual level surfaces) is just the (unparameterised) integral curve. Well known elementary examples are the isotropic harmonic oscillator, the Kepler system and the Calogero-Moser system.

The quadratures of complete integrability are often achieved through the separation of variables of the Hamilton-Jacobi equation. The solution of a maximally super-integrable system can also be calculated purely algebraically (albeit implicitly), requiring just the solution of the equations $I_k = c_k$, $k = 1, \ldots, 2n - 1$. Maximally superintegrable systems have a number of interesting properties: they can be separable in more than one coordinate system; all bounded orbits are closed; they give rise to interesting Poisson algebras with polynomial Poisson relations.

The idea can be extended to *quantum integrable systems*, with first integrals replaced by commuting differential operators. If we are truly interested in the system as a physical quantum system, then we would require our operators to be Hermitian. However, the same ideas are applicable to the study of polynomial eigenfunctions of Laplace-Beltrami operators, so we will not impose this condition. In fact, algebraically, Poisson algebras and operator algebras are *formally* very similar and it is very straightforward to "quantise"

many classical, super-integrable systems. For some examples of superintegrable quantum systems it is possible to use the additional commuting operators to *explicitly* build sequences of eigenfunctions [1, 2, 3], as will be seen below.

There is a large literature on the classification and analysis of superintegrable systems (see the review [4]) and they naturally occur in many applications in physics (additional integrals being referred to as "hidden symmetries" [5]).

Section 2 is expository, using the well known example of resonant harmonic oscillators to explain the basics of super-integrability. It will be seen how the classical and quantum versions are more or less just two different representations of a single algebraic structure.

In Section 3 we consider Hamiltonians in "natural form" (the sum of kinetic and potential energies). The kinetic energy has geometric origins, and, in the flat and constant curvature cases, the existence of a large isometry group leads to the superintegrability of the geodesic equations. We emphasise the role of these isometries in the construction of separable potentials and in the choice of additional integrals for the superintegrable case. These ideas are illustrated through a 3D constant curvature case, introduced in [6], with the symmetry algebra and its *involutive automorphisms* playing an important role in constructing a $10-$dimensional Poisson algebra.

Section 4 is concerned with the classical and quantum Zernike system, whose super-integrability (and much more!) was recently studied in detail by Pogosyan, et al [7, 8]. My purpose here is show how the methods of Section 3 can be applied, giving a new perspective on their results. The quantum case is equivalent to case IX of the Krall-Sheffer classification of 2D second order, linear differential operators with polynomial eigenfunctions [9, 10].

2 Resonant Harmonic Oscillators

This section is expository, using a well known example to explain the basics of super-integrability. For this simple example, all information can be gleaned from the explicit solution, which can be found by elementary means. However, we can extract the same information from additional first integrals, which exist in the resonant case. This sledge hammer can then be applied to much tougher nuts, whose explicit solution is far from elementary.

As usual, the object of the classical game is the trajectory, whilst that of the quantum case is the spectrum and the eigenfunctions. However, most of the intermediate steps are *algebraically* identical, with one in terms of Poisson algebras and the other in operator algebras. In the classical case, the first integrals are used to construct an algebraic curve, which represents the familiar Lissajous figure. In the quantum case, the existence of commuting operators accounts for the degeneracy of an eigenvalue. The commuting operators act like "horizontal ladders" taking us through the eigenspace of a given eigenvalue.

2.1 The Classical Case

Consider the system with Hamiltonian

$$H = H_1 + H_2, \quad \text{where} \quad H_k = \frac{1}{2}(p_k^2 + \omega_k^2 q_k^2), \quad \text{with} \quad \{H_1, H\} = \{H_2, H\} = 0.$$

As shown by Jauch and Hill [11], the classical (not just the quantum) oscillator has ladder operators. By writing

$$p_k^2 + \omega_k^2 q_k^2 = (p_k + i\omega_k q_i)(p_k - i\omega_k q_k),$$

we find

$$\{p_k + i\omega_k q_k, p_k - i\omega_k q_k\} = 2i\omega_k \quad \Rightarrow \quad \{H_k, p_k \mp i\omega_k q_k\} = \pm i\omega_k(p_k \mp i\omega_k q_k). \tag{1}$$

Generically the ratio ω_1/ω_2 is irrational, so the trajectory performs *quasi-periodic motion*, filling the torus, defined by $H_i = \frac{1}{2}\omega_i^2 A_i^2$, where A_i are the initial amplitudes.

In the resonant case, with $\omega_1 = m$, $\omega_2 = n$ (integers), the solution is periodic and, projecting onto the (q_1, q_2) plane, the trajectory forms a Lissajous figure, which represents an algebraic curve in the plane.

In the resonant case, (1) implies

$$\{H, (p_1 \mp imq_1)^n\} = \pm imn(p_1 \mp imq_1)^n \quad \text{and} \quad \{H, (p_2 \mp inq_2)^m\} = \pm imn(p_2 \mp inq_2)^m,$$

so

$$K = (p_1 + miq_1)^n (p_2 - niq_2)^m = K_1 + iK_2$$

gives us a pair of *real* first integrals. For low values of m and n these are easy to write explicitly, but are generally very complicated as functions of q_k, p_k. However, the general structure in terms of ladder operators enables us to deduce the complete set of Poisson relations for H_1, H_2, K_1, K_2. We already have $\{H_1, H_2\} = 0$ and $H = H_1 + H_2$, but then

$$\{H_1, K_1\} = -\{H_2, K_1\} = mnK_2, \quad \{H_1, K_2\} = -\{H_2, K_2\} = -mnK_1,$$
$$\{K_1, K_2\} = 2mn(2H_1)^{n-1}(2H_2)^{m-1}(mH_1 - nH_2) \quad \text{and} \quad K_1^2 + K_2^2 = (2H_1)^n(2H_2)^m,$$

the latter showing that at most 3 of these integrals are independent. It is easy to see that H_1, H_2 and K_1 are functionally independent.

Lissajous Figures: The general solution of the equations of motion is given by $q_1 = A_1 \sin(mt + \delta_1)$, $q_2 = A_2 \sin(nt + \delta_2)$, with the constants $(A_1, A_2, \delta_1, \delta_2)$ being determined by initial conditions. A parametric plot then gives the Lissajous figure.

The values of the four integrals $H_i(0) = h_i$, $K_i(0) = k_i$, are fixed by the initial conditions and satisfy $k_1^2 + k_2^2 = (2h_1)^n(2h_2)^m$. We can then use three of these integrals to eliminate p_i and obtain an algebraic relation between q_1 and q_2. This gives a *non-parametric* representation of the Lissajous curve.

Example 2.1 (The Case $(m, n) = (1, 2)$) Here we have $H_1 = \frac{1}{2}(p_1^2 + q_1^2)$, $H_2 = \frac{1}{2}(p_2^2 + 4q_2^2)$ and

$$K_1 = p_1^2 p_2 - q_1^2 p_2 + 4q_1 q_2 p_1, \quad K_2 = 2q_1 p_1 p_2 - 2q_2 p_1^2 + 2q_1^2 q_2. \tag{2}$$

The solution $q_1 = \sin t, q_2 = 2\sin(2t), p_1 = \cos t, p_2 = 4\cos(2t)$, with $q_1(0) = q_2(0) = 0, p_1(0) = 1, p_2(0) = 4$, corresponds to Figure 1(a). The first integrals have values $H_1 = \frac{1}{2}$, $H_2 = 8$, $K_1 = 4$, $K_2 = 0$, so

$$p_1^2 = 1 - q_1^2, \ p_2^2 = 4(4 - q_2^2), \ q_1 p_1 p_2 = q_2(p_1^2 - q_1^2) = q_2(1 - 2q_1^2) \quad \Rightarrow \quad q_2^2 = 16q_1^2(1 - q_1^2),$$

giving the algebraic curve corresponding to Figure 1(a).

(a) Symmetric Figure of Eight   (b) Degenerate Parabolic Figure

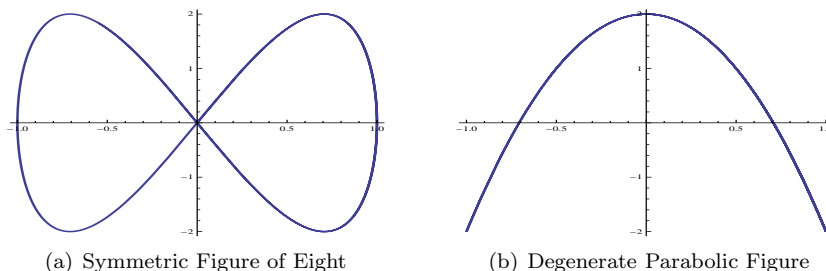Figure 1: Lissajous Figures with $(m, n) = (1, 2)$

The solution $q_1 = \sin t, q_2 = 2\cos(2t), p_1 = \cos t, p_2 = -4\sin(2t)$, with $q_1(0) = 0, q_2(0) = 2, p_1(0) = 1, p_2(0) = 0$, corresponds to Figure 1(b). The first integrals have values $H_1 = \frac{1}{2}$, $H_2 = 8$, $K_1 = 0$, $K_2 = -4$, which leads to the degenerate form: $q_2^2 = 4(1 - 2q_1^2)^2$. Clearly, Figure 1(b) corresponds to $q_2 = 2(1 - 2q_1^2)$.

2.2 The Quantum Case

The quantum version of the harmonic oscillator is algebraically very similar. We consider the eigenvalue problem

$$L\psi \equiv (\partial_x^2 + \partial_y^2 - \omega_1^2 x^2 - \omega_2^2 y^2)\psi = \lambda\psi,$$

with ladder operators $A_x^\pm = \mp\partial_x + \omega_1 x$, $A_y^\pm = \mp\partial_y + \omega_2 y$, satisfying

$$[L, A_x^\pm] = \mp 2\omega_1 A_x^\pm, \quad [L, A_y^\pm] = \mp 2\omega_2 A_y^\pm, \quad [A_x^\pm, A_y^\pm] = 0.$$

The ground state $\psi_{00} = e^{-\frac{1}{2}(\omega_1 x^2 + \omega_2 y^2)}$ satisfies

$$A_x^- \psi_{00} = A_y^- \psi_{00} = 0, \quad L\psi_{00} = -(\omega_1 + \omega_2)\psi_{00},$$

and we use the raising operators to define an infinite triangular array of eigenfunctions:

$$\psi_{ij} = (A_x^+)^i (A_y^+)^j \psi_{00}, \quad \text{with eigenvalues } \lambda_{ij} = -(2i+1)\omega_1 - (2j+1)\omega_2.$$

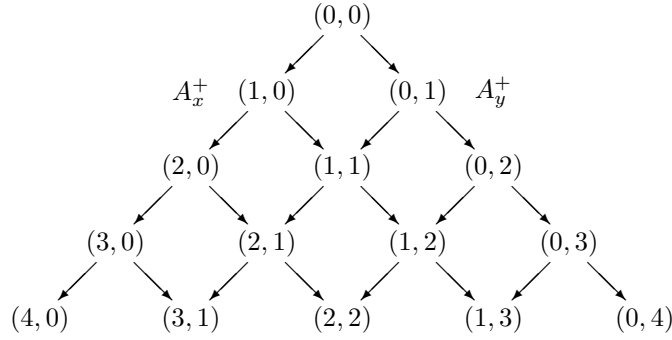Since the ladders in the x−direction commute with those in the y−direction, it doesn't matter which order

Figure 2: The action of A_x^+ and A_y^+ on the array $\psi_{(jk)}$

we operate with A_x^+ and A_y^+. Each order corresponds to a different path through the array depicted in Figure 2. We can also move in the negative direction $A_x^- \psi_{jk} = 2i\omega_1 \psi_{j-1\,k}$, $A_y^- \psi_{jk} = 2i\omega_2 \psi_{j\,k-1}$.

Furthermore, we have $\psi_{jk} = \psi_{00} H_j(x) H_k(y)$, where H_k are just the Hermite polynomials (but with coefficients depending upon ω_1 or ω_2). Once again everything we need is obtained by elementary calculations.

The eigenvalues λ_{ij} are distinct, if and only if ω_1 and ω_2 are not commensurate.

When $(\omega_1, \omega_2) = (m, n)$, we can build two commuting operators:

$$[L, (A_x^\pm)^n] = \mp 2mn(A_x^\pm)^n, \quad [L, (A_y^\pm)^m] = \mp 2mn(A_y^\pm)^m \quad \Rightarrow \quad \begin{cases} \left[L, (A_y^+)^m (A_x^-)^n\right] = 0, \\ \left[L, (A_x^+)^n (A_y^-)^m\right] = 0. \end{cases}$$

These play the role of connecting eigenfunctions with the *same* eigenvalue, so are directly related to *degeneracy.*

Example 2.2 (The Isotropic Oscillator) When $\omega_1 = \omega_2 = 1$, we have $\lambda_{ij} = -2(i+j+1)$, so, for each integer $\ell \geq 0$, we have $\ell + 1$ pairs (i, j) satisfying $i + j = \ell$, corresponding to eigenfunctions with eigenvalue $\lambda_{ij} = -2(\ell+1)$. These lie on the horizontal levels in Figure 2. The operators

$$A_y^+ A_x^- = xy + y\partial_x - x\partial_y - \partial_x\partial_y \quad \text{and} \quad A_x^+ A_y^- = xy - y\partial_x + x\partial_y - \partial_x\partial_y$$

move us, respectively, right and left across the horizontal levels. The usual commuting operators for this case are just

$$K_1 = \frac{1}{2}(A_x^- A_y^+ - A_x^+ A_y^-) = y\partial_x - x\partial_y, \quad K_2 = -\frac{1}{2}(A_x^- A_y^+ + A_x^+ A_y^-) = \partial_x \partial_y - xy.$$

Example 2.3 (The Case $(m,n) = (1,2)$) In this case, $\lambda_{ij} = -(2(i+2j)+3)$, so, starting on the left edge, with $(i,j) = (m,0)$, the eigenfunctions with eigenvalue $\lambda_{m\,0} = -(2m+3)$, satisfy $i+2j = m$. For $m = 2\ell$ or $m = 2\ell + 1$, we have $\ell + 1$ solutions. The operators $(A_y^+ A_x^-)^2$ and $(A_x^+)^2 A_y^-$ move us respectively right and left in the direction of the line $i+2j = m$. For example, the first of these takes the point (i,j) to $(i-2, j+1)$ (or, more precisely, $\psi_{ij} \mapsto \psi_{i-2\,j+1}$).

Operators which look more like the integrals of (2) are the even and odd parts

$$K_1 = \frac{1}{2}((A_x^+)^2 A_y^- + (A_x^-)^2 A_y^+) = 2y\partial_x^2 - 2x\partial_x\partial_y - \partial_y + 2x^2 y,$$

$$K_2 = \frac{1}{2}((A_x^+)^2 A_y^- - (A_x^-)^2 A_y^+) = \partial_x^2 \partial_y - 4xy\partial_x + x^2 \partial_y - 2y,$$

which satisfy

$$[K_1, L_1] = 4K_2, \quad [K_1, L_2] = -4K_2, \quad [K_2, L_1] = 4K_1, \quad [K_2, L_2] = -4K_1,$$
$$[K_1, K_2] = 4L_1 L_2 - 2L_1^2 - 6, \quad K_1^2 - K_2^2 + L_1^2 L_2 = 8L_1 - 3L_2.$$

Remark 2.1 (Separability) *For any values of ω_i, this system is clearly separable in Cartesian coordinates. However, as is a feature of super-integrability, each of the above resonant cases is separable in a second coordinate system. The isotropic case is separable in polar coordinates, whilst the case $(m,n) = (1,2)$ is separable in parabolic coordinates.*

3 Motion on Flat and Constant Curvature Manifolds

In this section we consider Hamiltonians in "natural form" (the sum of kinetic and potential energies). A non-degenerate kinetic energy is associated with (pseudo-)Riemannian metric, so we first consider the geodesic equations on such manifolds. We are particularly interested in the flat and constant curvature manifolds, in which case the geodesic motion is completely integrable, and even maximally super-integrable. We then consider the problem of adding potentials in such a way that the Hamiltonian system is still completely integrable. Since we restrict ourselves to integrals which are *quadratic* in momenta, we naturally build *separable systems*, depending on arbitrary functions. We then ask if it is possible to restrict these arbitrary functions in order to build *super-integrable* systems.

3.1 The Geodesic Equations

For a manifold with coordinates $(q_1, \ldots, q_n)$, metric coefficients g_{ij}, with inverse g^{ij}, the geodesic equations are Hamiltonian, with *kinetic energy*

$$H = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j, \quad \text{where} \quad p_i = \sum_k g_{ik} \dot{q}_k. \tag{3}$$

For a metric with isometries, the infinitesimal generators (Killing vectors) give rise to first integrals, which are *linear* in momenta (Noether constants). When the space is either flat or constant curvature, it possesses the maximal group of isometries, which is of dimension $\frac{1}{2}n(n+1)$. In this case, (3) is actually the second order *Casimir* function of the symmetry algebra (see [12]).

Remark 3.1 (Maximally Super-Integrable) *Notice that since $\frac{1}{2}n(n+1) - (2n+1) = \frac{1}{2}(n-1)(n-2)$, the geodesic equations on such spaces are always maximally super-integrable and that, when $n \geq 3$, the Noether constants cannot be functionally independent (the Killing vectors are, of course, linearly independent). The standard example is the Euclidean algebra in 3D, for which $\mathbf{p} \cdot \mathbf{L} = 0$.*

3.1.1 An Explicit Constant Curvature Metric

As a working example, we now consider the Hamiltonian

$$H = q_1^2(p_1^2 - p_2^2 - p_3^2), \tag{4}$$

corresponding to the diagonal upper index metric diag $\left(q_1^2, -q_1^2, -q_1^2\right)$, which has constant curvature, with isometry algebra (see [6])

$$
\begin{aligned}
e_1 &= p_2, & h_1 &= -2(q_1p_1 + q_2p_2 + q_3p_3), & f_1 &= -2q_1q_2p_1 + (q_3^2 - q_1^2 - q_2^2)p_2 - 2q_2q_3p_3, \\
e_2 &= p_3, & h_2 &= 2(q_2p_3 - q_3p_2), & f_2 &= -4q_3(q_1p_1 + q_2p_2) - 2(q_1^2 - q_2^2 + q_3^2)p_3,
\end{aligned}
\tag{5}
$$

with Poisson relations given in Table 1. We see that it has rank 2. In fact, it is easy to see that this is isomorphic to $\mathbf{so}(1,3)$.

Table 1: The 6−dimensional isometry algebra of (4)

	e_1	h_1	f_1	e_2	h_2	f_2
e_1	0	$2e_1$	$-h_1$	0	$-2e_2$	$-2h_2$
h_1	$-2e_1$	0	$2f_1$	$-2e_2$	0	$2f_2$
f_1	h_1	$-2f_1$	0	$-h_2$	$-f_2$	0
e_2	0	$2e_2$	h_2	0	$2e_1$	$-2h_1$
h_2	$2e_2$	0	f_2	$-2e_1$	0	$-4f_1$
f_2	$2h_2$	$-2f_2$	0	$2h_1$	$4f_1$	0

The quadratic Casimir of this algebra is proportional to H:

$$H = e_1f_1 + \frac{1}{4}h_1^2 + \frac{1}{4}(2e_2f_2 - h_2^2).$$

There is a second independent (fourth order) Casimir element of the abstract algebra, but in this representation it is a perfect square and *zero*, giving the quadratic constraint $e_1f_2 + h_1h_2 - 2f_1e_2 = 0$. This algebra has the following useful pair of involutive automorphisms:

$$
\begin{aligned}
\iota_1 : (e_1, h_1, f_1, e_2, h_2, f_2) &\mapsto \left(f_1, -h_1, e_1, -\frac{1}{2}f_2, -h_2, -2e_2\right), \\
\iota_{23} : (e_1, h_1, f_1, e_2, h_2, f_2) &\mapsto \left(e_2, h_1, \frac{1}{2}f_2, e_1, -h_2, 2f_1\right).
\end{aligned}
$$

The involution ι_{23} just corresponds to the interchange $(q_2, p_2) \leftrightarrow (q_3, p_3)$, so is clearly a canonical transformation. The involution ι_1 is also canonical, with generating function $S_1 = \frac{q_1P_1 - q_2P_2 + q_3P_3}{q_1^2 - q_2^2 - q_3^2}$. *These will be important later.*

3.2 Adding Potentials: Separability

We start with a kinetic energy H_0 corresponding to a flat or constant curvature metric and use its isometry algebra to build commuting quadratic (in momenta) functions. We can then add potential functions. This will be explained in the context of the specific 3D Hamiltonian (4), but the idea is easily applied to any metric with enough symmetries.

Starting with $H = H_0 + h(\mathbf{q})$, with kinetic energy H_0 of (4), we seek two functions F_1 and F_2, such that H, F_1, F_2 are in involution:

$$\{H, F_1\} = \{H, F_2\} = \{F_1, F_2\} = 0.$$

In our quadratic case, such functions will be the sum of two homogeneous parts, $F_i = F_i^{(2)} + F_i^{(0)}$, and

$$\{H, F_i\} = 0 \quad \Rightarrow \quad \{H_0, F_i^{(2)}\} = 0 \quad \text{and} \quad \{H_0, F_i^{(0)}\} + \{h, F_i^{(2)}\} = 0.$$

The first of these means that the coefficients of $p_i p_j$ in $F_i^{(2)}$ define a second order Killing tensor of the metric corresponding to H_0. When this metric is constant curvature, all Killing tensors are built as tensor products of Killing vectors (see [12]). In the Poisson representation, this just means that $F_i^{(2)}$ is some quadratic form of the elements of the isometry algebra. Since this algebra is of rank 2, any isometry K will commute with exactly one other element $\bar{K}$. Since we require $\{F_1^{(2)}, F_2^{(2)}\} = 0$, we choose these quadratic parts to be independent quadratic forms of some pair $K, \bar{K}$. For simplicity, we can choose our pairs to be one of e_1, e_2, or h_1, h_2 or f_1, f_2.

The choice of quadratic integrals means that our systems will be separable. The calculation of separable potentials is standard (in principle!) and it is well known that in the standard orthogonal coordinate systems, with separable kinetic energies, we can add potentials which depend upon a number of arbitrary functions of a single variable [13]. If a complete (possessing n parameters) solution of the Hamilton-Jacobi equation is found, then, by Jacobi's theorem, these parameters, when written in terms of the canonical variables, are quadratic (in momenta) first integrals of H. The problem has also been posed in the "opposite" direction: given a pair of Poisson commuting, homogeneously quadratic integrals (in two degrees of freedom) what sort of potentials can be added, whilst maintaining commutativity? This is a classical problem (see Whittaker [14], chapter 12, section 152) and leads to the Bertrand-Darboux equation for the potential [15, 16].

The Commuting Pairs e_1, e_2 and f_1, f_2

These two cases are connected by the action of the automorphism ι_1. It should be emphasised that without the knowledge of the underlying isometry algebra, these two cases would be *independent*, with the f_1, f_2 case presenting a considerably more difficult calculation!

Starting with the pair e_1, e_2, we consider the two quadratic integrals

$$F_1 = e_1^2 + g_1(\mathbf{q}) = p_2^2 + g_1(\mathbf{q}), \quad F_2 = e_2^2 + g_2(\mathbf{q}) = p_3^2 + g_2(\mathbf{q}). \tag{6a}$$

The simple form of e_1 and e_2 means that we are already in separation coordinates, leading to

$$h = -q_1^2(\varphi_1(q_2) + \varphi_2(q_3)) + \varphi_3(q_1), \quad g_1 = \varphi_1(q_2), \quad g_2 = \varphi_2(q_3). \tag{6b}$$

The much more difficult case to calculate, involving f_1 and f_2, is simply obtained by using the automorphism ι_1, which preserves H_0, whilst mapping $e_1^2 \mapsto f_1^2$ and $e_2^2 \mapsto \frac{1}{4}f_2^2$. Recall that this involution is realised by the *canonical transformation*, generated by

$$S_1 = \frac{q_1 P_1 - q_2 P_2 + q_3 P_3}{q_1^2 - q_2^2 - q_3^2} \quad \Rightarrow \quad Q_1 = \frac{q_1}{q_1^2 - q_2^2 - q_3^2}, \ Q_2 = \frac{-q_2}{q_1^2 - q_2^2 - q_3^2}, \ Q_3 = \frac{q_3}{q_1^2 - q_2^2 - q_3^2}.$$

This gives

$$\begin{aligned} F_1 &= f_1^2 + g_3(\mathbf{q}) = (-2q_1 q_2 p_1 + (q_3^2 - q_1^2 - q_2^2)p_2 - 2q_2 q_3 p_3)^2 + g_3(\mathbf{q}), \\ F_2 &= \frac{1}{4}f_2^2 + g_4(\mathbf{q}) = (2q_3(q_1 p_1 + q_2 p_2) + (q_1^2 - q_2^2 + q_3^2)p_3)^2 + g_4(\mathbf{q}), \end{aligned} \tag{7a}$$

where

$$\begin{aligned} g_3 &= \varphi_1\left(\frac{-q_2}{q_1^2 - q_2^2 - q_3^2}\right), \quad g_4 = \varphi_2\left(\frac{q_3}{q_1^2 - q_2^2 - q_3^2}\right), \\ h &= \varphi_3\left(\frac{q_1}{q_1^2 - q_2^2 - q_3^2}\right) - \frac{q_1^2\left(\varphi_1\left(\frac{-q_2}{q_1^2 - q_2^2 - q_3^2}\right) + \varphi_2\left(\frac{q_3}{q_1^2 - q_2^2 - q_3^2}\right)\right)}{(q_1^2 - q_2^2 - q_3^2)^2}. \end{aligned} \tag{7b}$$

3.3 Adding More Potentials: Super-integrability

We now suppose we have an involutive system H, F_1, F_2, as above. For super-integrability, we must add two more functions F_3, F_4, satisfying $\{H, F_3\} = \{H, F_4\} = 0$, but with the (now) *given* H. The functions should be chosen to be *functionally independent*, so the Jacobian matrix

$$\frac{\partial(H, F_i)}{\partial \mathbf{x}}, \quad \text{where} \quad \mathbf{x} = (q_1, \ldots, p_3),$$

has <u>maximal rank</u> 5, since in this case, the level surface $\mathcal{S} = \{\mathbf{x} : H = c_0, F_i = c_i\}_{i=1}^4$, has dimension <u>one</u>, so represents an (unparameterised) trajectory of the dynamical system. In the case of the resonant oscillator of Example 2.1, this enabled us to derive the algebraic curve for a Lissajous figure, but generally, we cannot reduce to such explicit formulae.

The case with $F_1 = e_1^2 + g_1, F_2 = e_2^2 + g_2, F_3 = f_1^2 + g_3, F_4 = \frac{1}{4}f_2^2 + g_4$

Here we start with the formulae of (6) and seek a pair of additional functions $F_3 = f_1^2 + g_3(\mathbf{q})$ and $F_4 = \frac{1}{4}f_2^2 + g_4(\mathbf{q})$, satisfying $\{H, F_3\} = \{H, F_4\} = 0$. This restricts our potentials from depending upon 2 *arbitrary functions* to 2 *arbitrary parameters*:

$$h = q_1^2 \left(\frac{k_1}{q_2^2} + \frac{k_2}{q_3^2} \right), \quad g_1 = -\frac{k_1}{q_2^2}, \quad g_2 = -\frac{k_2}{q_3^2}, \quad g_3 = -\frac{k_1(q_2^2 + q_3^2 - q_1^2)^2}{q_2^2}, \quad g_4 = -\frac{k_2(q_2^2 + q_3^2 - q_1^2)^2}{q_3^2}.$$

In this case, we also find that h_1 is a first integral. We therefore have 6 first integrals $(H, F_1, F_2, F_3, F_4, h_1)$ (with Jacobian of rank 5) and find

$$\{F_1, F_2\} = \{F_3, F_4\} = 0 \quad \text{and} \quad \{F_i, h_1\} = \lambda_i F_i, \quad i = 1, \ldots, 4, \quad \text{where} \quad \lambda = (4, 4, -4, -4).$$

Under the action of the involutions ι_1 and ι_{23}, we have

$$\iota_1 : (H, F_1, F_2, F_3, F_4, h_1, k_1, k_2) \quad \mapsto \quad (H, F_3, F_4, F_1, F_2, -h_1, k_1, k_2),$$
$$\iota_{23} : (H, F_1, F_2, F_3, F_4, h_1, k_1, k_2) \quad \mapsto \quad (H, F_2, F_1, F_4, F_3, h_1, k_2, k_1).$$

The Poisson brackets $\{F_1, F_3\}, \{F_1, F_4\}, \{F_2, F_3\}, \{F_2, F_4\}$ are all cubic in momenta, but cannot be written as linear combinations of $\{h_1 F_i, h_1 H, h_1^3\}_{i=1}^4$, so we need to introduce new elements. Since the involutions are canonical, $\{F_1, F_3\}$ and $\{F_2, F_4\}$ (respectively $\{F_1, F_4\}$ and $\{F_2, F_3\}$) are related through the involutions. Since $\{F_1, F_3\}$ and $\{F_2, F_4\}$ factorise, we can define two new <u>quadratic</u> elements F_5, F_6 through the relations

$$\{F_1, F_3\} = h_1(h_1^2 - 4H - 4F_5 - 4k_1), \quad \{F_2, F_4\} = h_1(h_1^2 - 4H - 4F_6 - 4k_2),$$

with $F_5 \leftrightarrow F_6$ under ι_{23}. We define F_7, F_9 by the equations

$$\{F_1, F_6\} = 2h_1 F_1 + 4F_7, \quad \{F_3, F_6\} = -2h_1 F_3 + 4F_9,$$

related by $F_7 \leftrightarrow F_9$ under ι_1.

The function F_8 is defined by the second of the following equations

$$\{F_1, F_4\} + \{F_2, F_3\} = 8h_1 \left(H + F_5 + F_6 - \frac{1}{4} h_1^2 \right), \quad \{F_1, F_4\} - \{F_2, F_3\} = 16F_8,$$

after which, we find that $\{F_5, F_6\} = 4F_8$.

In this way we build a $10-$dimensional Poisson algebra. The action of the two involutions is then given by:

	H	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	h_1	k_1	k_2
$\iota_1 :$	H	F_3	F_4	F_1	F_2	F_5	F_6	F_9	F_8	F_7	$-h_1$	k_1	k_2
$\iota_{23} :$	H	F_2	F_1	F_4	F_3	F_6	F_5	$-F_7$	$-F_8$	$-F_9$	h_1	k_2	k_1

The action of ι_{23} on $\{F_1, F_6\}$ and $\{F_3, F_6\}$ then gives

$$\{F_2, F_5\} = 2h_1 F_2 - 4F_7, \quad \{F_4, F_5\} = -2h_1 F_4 - 4F_9.$$

This phenomenon of connecting four different Poisson relations through the involutions is depicted in Figure 3(a), where we define $P_{ij} = \{F_i, F_j\}$. Sometimes only <u>two</u> relations are connected, such as with P_{13} and
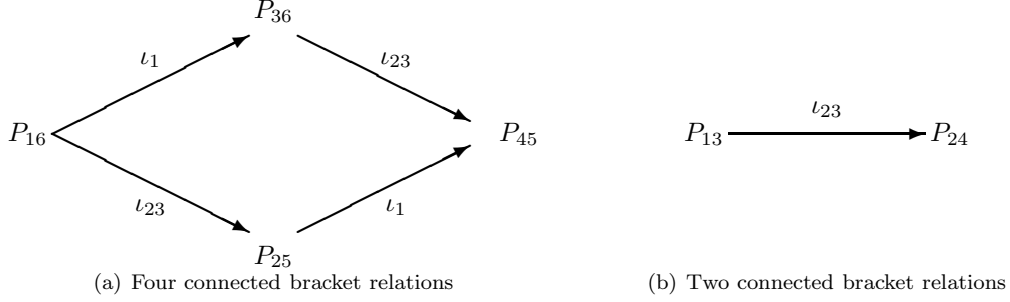

(a) Four connected bracket relations   (b) Two connected bracket relations

Figure 3: Bracket relations connected through ι_1 and ι_{23}

P_{24}, or even just <u>one</u>, such as with P_{79}, because of invariance properties. These relations mean that when we find one entry P_{ij}, we automatically have up to 3 others.

Remark 3.2 (Nontrivial Relations) *Out of our 11 functions, only 5 can be functionally independent. The Jacobi identity, applied to our 10 functions F_i and h_1, give us the necessary, nontrivial relations.*

The Trajectory as a Level Curve

Consider the level curve, defined by $h_1 = c_0$ and $\{F_i = c_i\}_{i=1}^4$. By omitting either F_4 or F_3, we can eliminate p_i in two ways, to get

$$(c_1\eta^2 - c_0^2 q_2^2)^2 - 2(2k_1 c_0^2 + c_1 c_3)\eta^2 + c_3(c_3 - 2c_0^2 q_2^2) = 0, \quad (c_2\eta^2 - c_0^2 q_3^2)^2 - 2(2k_2 c_0^2 + c_2 c_4)\eta^2 + c_4(c_4 - 2c_0^2 q_3^2) = 0,$$

where $\eta = q_1^2 - q_2^2 - q_3^2$. These are related through the action of ι_{23}. The special case of $c_3 = c_4 = 0$ gives

$$c_1\eta^2 - c_0^2 q_2^2 = \pm 2\sqrt{k_1} c_0 \eta, \qquad c_2\eta^2 - c_0^2 q_3^2 = \pm 2\sqrt{k_2} c_0 \eta, \tag{8a}$$

where the same sign must be chosen to preserve invariance under ι_{23}. We then obtain

$$\eta = \pm \frac{c_0(c_1 q_3^2 - c_2 q_2^2)}{2(c_2\sqrt{k_1} - c_1\sqrt{k_2})}, \tag{8b}$$

which, when substituted back into (8a), gives a quartic expression in the (q_2, q_3) plane, with (8b) giving an expression for q_1^2, thus defining a curve:

$$(c_2 q_2^2 - c_1 q_3^2)^2 = 4(c_2\sqrt{k_1} - c_1\sqrt{k_2})(\sqrt{k_1} q_3^2 - \sqrt{k_2} q_2^2), \qquad q_1^2 = q_2^2 + q_3^2 \pm \frac{c_0(c_1 q_3^2 - c_2 q_2^2)}{2(c_2\sqrt{k_1} - c_1\sqrt{k_2})}.$$

4   The Classical and Quantum Zernike System

In this section, we apply the techniques of Section 3 to the classical Zernike system [7] and then consider the algebraic aspects of its quantisation [8] in the framework of Section 2.2. Of course, we mainly just reproduce formulae which can be found in [7, 8], but we obtain them in a different way. In fact, these two approaches are exactly as discussed in Section 3.2, in the context of the Bertrand-Darboux equation. Pogosyan, et al,

start from separability and, as guaranteed by Jacobi's Theorem, construct constants of the motion, whilst here, we use the symmetries of the kinetic energy of the Zernike system to build commuting integrals and separable potentials. Of course, Pogosyan, et al, do a lot more, but our purpose here is just to discuss the algebraic aspects of the super-integrability. In the quantum case, we point out that, when written in terms of Cartesian coordinates (x, y), the Zernike system is just Case IX of the Krall-Sheffer classification [9], so can use some of the results of [10].

4.1   The Classical Zernike System

A classical form of the Zernike system is given in [7], which, in polar coordinates, takes the form

$$H = (1 + ar^2)p_r^2 + \frac{p_\phi^2}{r^2} + 2brp_r. \tag{9a}$$

First we remove the awkward linear (in p_r) term by applying a gauge transformation:

$$H = (1 + ar^2)\left(p_r + \frac{br}{1 + ar^2}\right)^2 + \frac{p_\phi^2}{r^2} - \frac{b^2r^2}{1 + ar^2},$$

so, redefining momenta by $\hat{p}_r = p_r + \frac{br}{1+ar^2}$, $\hat{p}_\phi = p_\phi$ (which are still canonical to (r, ϕ)), and dropping "hats", we obtain the Zernike system in "natural" form:

$$H = (1 + ar^2)p_r^2 + \frac{p_\phi^2}{r^2} - \frac{b^2r^2}{1 + ar^2}. \tag{9b}$$

The kinetic energy defines a constant curvature metric, with scalar curvature $R = -2a$. This metric therefore has a 3D isometry algebra, which is easily calculated to be

$$k_1 = \frac{\sqrt{1 + ar^2}}{r}\left(r\cos\phi\, p_r - \sin\phi\, p_\phi\right), \quad k_2 = \frac{\sqrt{1 + ar^2}}{r}\left(r\sin\phi\, p_r - \cos\phi\, p_\phi\right), \quad k_3 = p_\phi, \tag{10a}$$

satisfying

$$\{k_1, k_2\} = -ak_3, \quad \{k_1, k_3\} = -k_2, \quad \{k_2, k_3\} = k_1. \tag{10b}$$

The kinetic energy is just the quadratic Casimir function: $H_0 = k_1^2 + k_2^2 - ak_3^2$.

Adding a potential of the form $h(r)$ means that $k_3 = p_\phi$ is a first integral, so the system is Liouville integrable. In 2D we just need one additional integral for maximal superintegrability. For example, with $H = H_0 + h(r)$, we have $F_1 = k_3$ and may choose a quadratic integral $F_2 = k_1k_2 + f(r, \phi)$, with $\{H, F_2\} = 0$, leading to

$$h(r) = \frac{b_1 + b_2r^2}{1 + ar^2}, \qquad f(r, \phi) = \frac{(b_2 - ab_1)r^2\sin 2\phi}{2(1 + ar^2)}, \tag{11}$$

so, choosing $b_1 = 0, b_2 = -b^2$, we obtain (9b). Whilst $\{H, F_1\} = \{H, F_2\} = 0$, we must introduce a new quadratic element, defined by

$$F_3 = \{F_1, F_2\} = k_1^2 - k_2^2 - \{f(r, \phi), k_3\} = k_1^2 - k_2^2 + \frac{b^2r^2\cos 2\phi}{1 + ar^2},$$

where the expression for the quadratic part is derived by using the Poisson bracket relations (10b). The remaining Poisson brackets are

$$\{F_1, F_3\} = -4F_2, \quad \{F_2, F_3\} = -F_1(2aH + 2a^2F_1^2 - 4b^2),$$

and these 4 functions satisfy the algebraic relation

$$(H + aF_1^2)^2 - F_3^2 - 4F_2^2 + 4b^2F_1^2 = 0.$$

Once again, the leading order quartic part is easily found by looking at the expressions for F_i in terms of k_i, leaving only the "quadratic correction" to be determined.

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Cartesian Coordinates (x, y) **and the original gauge:** To compare results with those of [7] we first change to Cartesian coordinate:

$$F_1 = xp_y - yp_x, \quad F_2 = (1 + a(x^2 + y^2))p_x p_y - \frac{b^2 xy}{1 + a(x^2 + y^2)}, \quad F_3 = (1 + a(x^2 + y^2))(p_y^2 - p_x^2) - \frac{b^2(y^2 - x^2)}{1 + a(x^2 + y^2)}.$$

Replacing the hats over the above momenta, we use the inverse gauge transformation

$$p_r = \hat{p}_r - \frac{br}{1 + ar^2}, \quad p_\phi = \hat{p}_\phi \quad \Rightarrow \quad p_x = \hat{p}_x - \frac{bx}{1 + a(x^2 + y^2)}, \quad p_y = \hat{p}_y - \frac{by}{1 + a(x^2 + y^2)},$$

to obtain $F_1 = xp_y - yp_x$ and

$$F_2 = (1 + a(x^2 + y^2))p_x p_y - b(yp_x + xp_y), \quad F_3 = (1 + a(x^2 + y^2))(p_y^2 - p_x^2) + 2b(xp_x - yp_y),$$

which should be compared, respectively, with J_1, J_3 and J_2 (equations (73), (76) and (74) of [7]).

Remark 4.1 *The integrals I_2 and I_3 (equations (59) and (65) of [7]) are similarly related to k_2^2 and k_1^2. The choice of separable coordinate corresponds, in each case, to the simultaneous diagonalisation of the pair of commuting quadratic integrals, such as (H, F_2) or (H, F_3) above.*

4.2   The Quantum Zernike System

For a given metric g_{ij}, we have the classical kinetic energy and its quantum analogue (the Laplace-Beltrami operator):

$$H^{(2)} = \frac{1}{2} \sum_{i,j=1}^n g^{ij} p_i p_j \quad \text{and} \quad L_b f = \sum_{i,j=1}^n g^{ij} \nabla_i \nabla_j f = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial q^i} \right),$$

where g is the determinant of the matrix g_{ij}. For a metric with isometries, the infinitesimal generators (Killing vectors) are just first order differential operators which commute with the Laplace-Beltrami operator L_b. When the space is either flat or constant curvature, it possesses the maximal group of isometries, which is of dimension $\frac{1}{2}n(n+1)$. In this case, L_b is proportional to the second order *Casimir* function of the symmetry algebra (see [12]).

Given such a classical Hamiltonian, each Noether constant, being of degree one in momenta, generates a Hamiltonian vector field whose first n components define a vector field on the configuration space, with coordinates q_i. This is the corresponding Killing vector. In the case of the classical Zernike system (9), the Noether constants k_i give us three Killing vectors X_i:

$$X_1 = \frac{\sqrt{1 + ar^2}}{r} \left(r \cos\phi \, \partial_r - \sin\phi \, \partial_\phi \right), \quad X_2 = \frac{\sqrt{1 + ar^2}}{r} \left(r \sin\phi \, \partial_r - \cos\phi \, \partial_\phi \right), \quad X_3 = \partial_\phi, \qquad (12a)$$

which, through $[X_i, X_j] = -X_{\{k_i, k_j\}}$, obey the commutation relations

$$[X_1, X_2] = aX_3, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1, \qquad (12b)$$

with

$$L_b = X_1^2 + X_2^2 - aX_3^2 = (1 + ar^2)\partial_r^2 + \left(\frac{1}{r} + 2ar \right) \partial_r + \frac{1}{r^2} \partial_\phi^2. \qquad (12c)$$

Notice that the second order parts of the Laplace-Beltrami operator can be read from the metric coefficients, but that first order terms can arise (as in this case). The *quantum Zernike Hamiltonian* is [8]

$$L = L_b + (b - a)r\partial_r = (1 + ar^2)\partial_r^2 + \left(\frac{1}{r} + (a + b)r \right) \partial_r + \frac{1}{r^2} \partial_\phi^2. \qquad (12d)$$

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We could again gauge away this additional first order term to write this as $L = L_b + h(r)$, but it is more convenient to remain in this gauge.

The construction of commuting operators is no different from finding first integrals in the classical case, except that we symmetrise our expressions where necessary:

$$I_1 = X_3, \quad I_2 = X_1 X_2 + X_2 X_1 + A(r, \phi)\partial_r + B(r, \phi)\partial_\phi.$$

The condition $[L, I_2] = 0$ determines the coefficients to be $A(r, \phi) = (b-a)r \sin 2\phi, \; B(r, \phi) = (b-a)r \cos 2\phi.$

4.2.1 Cartesian Coordinates and Krall-Sheffer Polynomials

Changing to Cartesian coordinates (x, y), and setting $a = -1$, we find

$$L = (1 - x^2)\partial_x^2 - 2xy\partial_x\partial_y + (1 - y^2)\partial_y^2 + (b-1)(x\partial_x + y\partial_y), \tag{13}$$

which is the Krall-Sheffer operator of type IX [9, 10]. Krall and Sheffer classified second order differential operators in 2D with polynomial eigenfunctions, generalising the classical results in the theory of orthogonal polynomials in 1D. Our two commuting operators now take the form

$$I_1 = x\partial_y - y\partial_x, \quad I_2 = 2(1 - x^2 - y^2)\partial_x\partial_y + b(y\partial_x + x\partial_y).$$

These generate an operator algebra which is very similar to the Poisson algebra of the classical case:

$$[I_1, I_2] = 2I_3 = 2((1 - x^2 - y^2)(\partial_x^2 - \partial_y^2) + b(x\partial_x - y\partial_y)),$$
$$[I_1, I_3] = -2I_2, \quad [I_2, I_3] = 2I_1(2L - 2I_1^2 - b(b+2)),$$

satisfying the constraint

$$(L - I_1^2)^2 - I_2^2 - I_3^2 + (b^2 + 2b - 4)I_1^2 - 2(b+2)L = 0.$$

Consider *monic* polynomial eigenfunctions of L:

$$P_{mn} = x^m y^n + \text{lower order terms}, \quad \text{for} \quad m + n = N. \tag{14}$$

The eigenvalue for P_{mn} is easily calculated from the leading order term to be $\lambda_{m+n} = (m + n)(b - m - n)$, with the notation reflecting the fact that it is the *same* eigenvalue for all (m, n) with $m + n = N$. This means that in Figure 4, all polynomials on a horizontal line have the same eigenvalue. This degeneracy reflects the existence of the commuting operators, which do not change the eigenvalue.
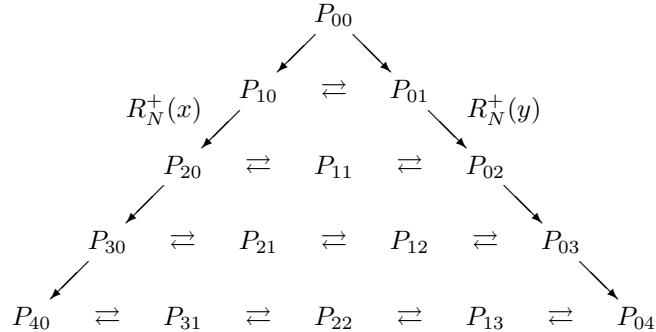

Figure 4: The triangular lattice of polynomials P_{mn} with $P_{00} = 1$. Horizontal arrows denote the action of J_m^+ (right) and J_n^- (left). The operators $R_N^+(x)$ and $R_N^+(y)$ represent the edge ladder operators.

If we act on P_{mn} with I_1 and I_2, then, looking at the leading order terms, we find

$$I_1 P_{mn} = nP_{m+1\,n-1} - mP_{m-1\,n+1}, \quad I_2 P_{mn} = n(b-2m)P_{m+1\,n-1} + m(b-2n)P_{m-1\,n+1},$$

so the operators

$$\begin{aligned}
J_m^+ &= I_2 - (b-2m)I_1 = 2(mx\partial_y + (b-m)y\partial_x + (1-x^2-y^2)\partial_x\partial_y), \\
J_n^- &= I_2 + (b-2n)I_1 = 2(ny\partial_x + (b-n)x\partial_y + (1-x^2-y^2)\partial_x\partial_y),
\end{aligned} \tag{15a}$$

move us to right or left across the horizontal levels:

$$J_m^+ P_{mn} = 2m(b-m-n)P_{m-1\,n+1}, \qquad J_n^- P_{mn} = 2n(b-m-n)P_{m+1\,n-1}. \tag{15b}$$

Notice that any eigenfunction of (13), which is independent of y, just satisfies the 1D Gegenbauer equation,

$$L_x P_{N0} = ((1-x^2)\partial_x^2 + (b-1)x\partial_x)P_{N0} = \lambda_N P_{N0}, \quad \text{with} \quad \lambda_N = N(b-N),$$

for which we have a standard raising operator:

$$R_N^+ = \frac{1}{b-2N}((1-x^2)\partial_x + (b-N)x), \quad \text{with} \quad [L_x, R_N^+] = \frac{2x}{2N-b}(L_x - \lambda_N) + (\lambda_{N+1} - \lambda_N)R_N^+,$$

generating Gegenbauer polynomials on the left edge of Figure 4, with J_m^+ taking us to the polynomials P_{mn}, for which $m+n = N$. Notice that $J_0^+ P_{0N} = 0$, since $m = 0$.

Remark 4.2 *The operator (13) is symmetric in x and y, so the polynomials on the right edge of Figure 4 are also Gegenbauer polynomials, but this time, in the variable y. We could, therefore, have started on the right edge and used J_n^- to move to the left, with $J_0^- P_{N0} = 0$.*

5   Conclusions

Maximally super integrable systems are distinguished by their number of first integrals or commuting operators. In this paper, the emphasis has been on the *role* of these in constructing trajectories or eigenfunctions. In the classical case, the level set of the $2n-1$ first integrals is a curve, representing the unparameterised trajectory. In the quantum case, the existence of commuting operators is responsible for the degeneracy of eigenvalues. The additional eigenfunctions are connected through the action of the commuting operators. The emphasis throughout has been that symmetries of the kinetic energy have a major influence on the structure of completely/super-integrable potentials and their corresponding Poisson algebras. As seen above, the Poisson algebras in 2D are very simple, but in higher dimensions (even in 3D), can be very complicated.

The term "quantum integrable" is not only used to describe *physical, quantum* systems, but also any (large enough) family of commuting differential operators. In particular, there is a theory of orthogonal polynomials in $2-$dimensions (and higher) and an important subclass of these are *eigenfunctions* of some sort of differential operator.

In [9], Krall and Sheffer classified second order, linear differential operators in 2D with polynomial eigenfunctions. These Krall-Sheffer polynomials are 2D generalisations of the classical orthogonal polynomials, so much of the standard machinery can be derived. In [10] we construct $3-$level recurrence relations (depending upon at most 9 points) for each of the 9 cases, as well as raising operators in *both* directions (parallel to, but not on, the edges), and discuss the reduction of these to $1-$dimensional cases on the edges. In particular, for case IX (the Zernike case) we construct $4-$point, $3-$level recurrence relations, raising operators in *both* directions, as well as a *generating function*.

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