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Deformations and Smile: 100 years of D'Arcy Thompson's Book "On Growth and Form"

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Human interest in shape begins soon after birth. Babies start learning facial shape and facial expression from people around them. Perhaps the most important of these expressions is the smile, the psychophysiology of which seems universal across human cultures and, in its capacity as a signalling system, can be traced back to evolutionary origins among the primates of thirty million years ago.

Older children move on to the study of geometric shapes. We are taught the names of these objects, and we soon figure out that the round peg fits the round hole. Not long after that, we discover that material shapes can be transformed easily into different ones. In the same way a mouth is pulled into a smile, a rubber circle can be elongated to an ellipse, a square skewed to a parallelogram.

More than a century ago, D'Arcy Thompson – a Scottish biologist – made a similar discovery while studying the shapes of more complex, organic objects, such as fish and dogs. By applying an appropriate but simple transformation, he realised, the drawing of a creature could be deformed ("morphed", we might say today) to resemble another of the same genus.

These and other ideas eventually culminated in a book, *On Growth and Form*,[1] first published in 1917, which laid out the author's conviction "that growth and form in nature could be understood from physical and mathematical principles".[2] Writing in *Nature* in 2013, Phillip Ball set out Thompson's argument that "physical forces, not heredity, may govern biological form". Ball writes that Thompson's demonstration of the claim, that "no organic forms exist save such as are in conformity with physical and mathematical laws", encompasses a wide range of topics. "To name a few: the mathematical laws that relate growth, flight and locomotion to mass and size…; the shapes of cells, bubbles and soap films; geometrical compartmentalization and honeycombs; corals; banded minerals; the intricate shells of molluscs and of the minuscule protozoan radiolarians; antlers and horns; plant shapes; bone microstructure; skeletal mechanics; and the morphological comparison of species."[3]

We may be justified in saying that the book gives, for the first time, a comprehensive study of biological shapes with some mathematical formulation. Thompson considered many questions imaginatively and explored the mystery behind the development of plants and animals. He strove to find common patterns and structures in biological shapes, to see a unifying "principle". But while a review in *Nature* from 1917 described the book as "masterly", an editorial from last year suggested that the critical consensus perceived the book to be "a beautiful, though not practical, work".[2]

It would be hard to argue with that assessment, both then and now. But what the book lacked in practical applicability it more than made up for in inspirational thinking, and in setting out ideas that would be built on generations later by researchers in a variety of fields, including those working in statistical shape analysis.

Landmarks and features

Statistical shape analysis, as the name suggests, is the application of statistical methods to the study of shape. But what do we mean when we talk about "shape"? According to the Oxford Dictionary, shape is "the external form, contours, or outline of someone or something". The late statistician David Kendall defined shape as "all the geometrical information that remains when location, scale and rotational effects are removed from an object".[4] In other words, an object's shape is everything that does not change under the Euclidean similarity transformations – translation (moving from one point in space to another), scaling and rotation. That is, two objects have the same shape if they can be translated, rescaled and rotated to each other so that they match exactly.

In statistical shape analysis, we are interested in comparing objects with different shapes and in measuring shape changes. Sometimes we are also interested in retaining scale information (size) as well as the shape of the object, and so the joint analysis of size and shape (or form) is important too. But position and orientation continues to be ignored.

At the 2017 Royal Statistical Society Conference, Professor Adrian Bowman of the University of Glasgow outlined some of the contributions of this field, including the analysis of "variability in face shapes, modelling and quantifying asymmetry in individuals… and helping surgeons help patients who require facial reconstruction surgery" (bit.ly/2EavVwb), mainly for cleft lip.

Bowman previously described this work in more detail in a 2008 article for *Significance*, writing that: "Surgical repairs [for cleft lip] are very successful but it is possible that some unusual facial patterns may remain. It is helpful to have a quantitative assessment of this, not only to be able to assess the effectiveness of the surgery, but also to provide information that may be useful in later surgical planning, should that prove necessary."[5] These assessments are made based on a comparison of the lip or mouth shape of a patient against a control group.

Later, in a 2013 article, we (Mardia, Bookstein and Kent) explained how brain shape could be analysed in a similar way to study the neurological birth defects now known as the fetal alcohol spectrum of disorders.[6]

One approach to shape analysis involves the use of landmarks – points marked on an object that correspond to key features, such as the corner of a mouth or the peak of the curve of an upper lip. Since the 1980s, there have been many key developments in shape analysis that allow us to work directly on the Cartesian coordinates of these landmark points. The main idea of the modern approach to shape analysis is that, rather than working with single quantities derived from organisms (such as distances between landmarks), one works with the complete list of all the landmark coordinates together as a single coherent statistical entity. This approach is very much in the spirit of Thompson, who considered the geometric transformations of pictures of one species to another, rather than any derived quantities.

Comparison of related forms

Thompson's contribution to shape analysis can be found in the last chapter of his book. The chapter, "On the Theory of Transformations, or the Comparison of Related Forms", foreshadowed a range of biometric approaches to data that have come to fruition only now, a century later, when powerful modelling tools could finally catch up with his imagination.

The key points of his work that are relevant for shape analysis are (1) the use of the full geometrical information of the object, and (2) the use of transformations to understand shape change between two objects.

These two ideas are clearly demonstrated in the transformation sketches Thompson included in *On Growth and Form*. As shown in Figure 1, Thompson would begin by drawing a grid of vertical and horizontal lines over the picture of one organism in some orientation. Then, he would deform those vertical and horizontal grid lines over the picture of a second organism so that grid blocks that corresponded graphically would roughly correspond morphologically as well.

He starts with the simplest example related to two fish, which involves only a shearing transformation (again, see Figure 1).

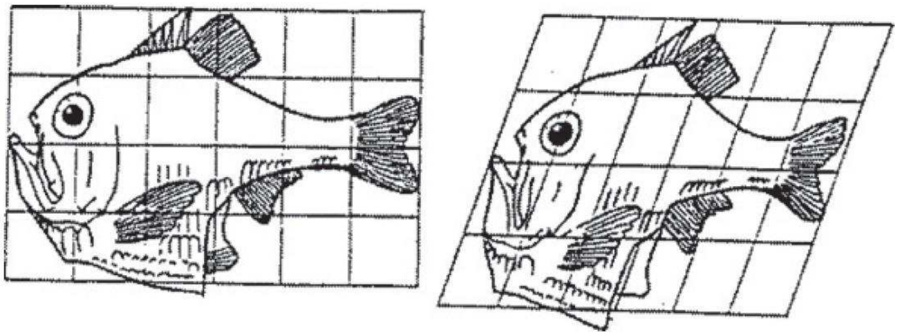


FIGURE 1 An example from Thompson's 1917 book, showing a species of fish, *Argyropelecus olfersi* (the source), geometrically transformed into another species, *Stemoptyx diaphana* (the target). A regular square grid pattern is drawn on the first fish and the grid is deformed to lie on the second fish, with corresponding biological parts located in the corresponding grid blocks.[1]

Another example of Thompson's method is his comparison of the porcupine fish, *Diodon*, to the sunfish, *Orthagoriscus mola*. His drawing is reproduced in Figure 2. Here, Thompson explained: "I have deformed [the *Diodon* grid's] vertical coordinates into a system of concentric circles, and its horizontal coordinates into a system of curves which, approximately and provisionally, are made to resemble a system of hyperbolas. The old outline, transferred in its integrity to the new network, appears as a manifest representation of the closely allied, but very different looking, sunfish *Orthagoriscus mola*. [This] particularly instructive case ... accounts, by one single integral transformation, for all the apparently separate and distinct external differences between the two fish."[1]

He goes further, to bring physics into it, writing that "[the morphologist] will enquire whether two different but more or less obviously related forms can be so analysed and interpreted that each may be shown to be a transformed representation of the other. This once demonstrated, it will be a comparatively easy task (in all probability) to postulate the direction and magnitude of the force capable of effecting the required transformation."[1]

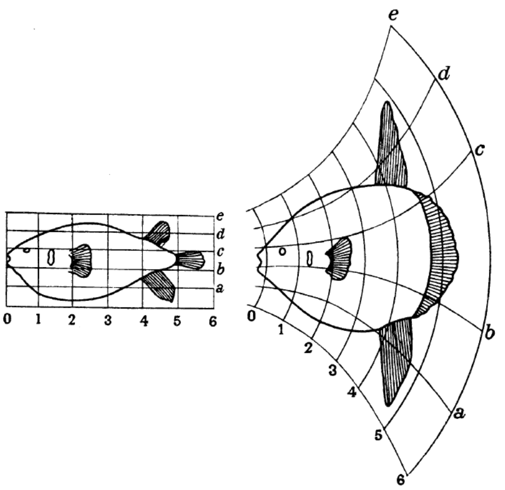


FIGURE 2 A second example from Thompson's 1917 book, geometrically transforming one species of fish, *Diodon* (the source), into another species, *Orthagoriscus mola* (the target).[1]

Thompson called his transformations "Cartesian transformations", by which he meant mapping the Cartesian *axes* of a grid over one object onto a different set of axes drawn over another. These transformations sometimes enable a biologist to describe the size and shape change between the two species, albeit in a rather subjective way. Thompson's figures were drawn by hand and were not always very accurate. In fact, writing in 1958, the biologist Sir Peter Brian Medawar commented: "The reason why D'Arcy's method has been so little used in practice (only I and one or two others have tried to develop it at all) is because it is analytically unwieldy."[7]

Medawar died in 1987. Two years later, Fred Bookstein (a co-author of this article) pioneered a new, more powerful approach to deformation, known as the Pairs of Thin-Plate Splines (PTPS) deformation, which has since replaced Thompson's Cartesian transformations of grid lines by properly defined mathematical transformations of the Cartesian coordinates themselves.[8] This last approach has been successfully applied to a different array of shapes and objects, including one of the earliest shapes we humans relate with after birth: the human smile.

Smile trajectories

Smiling is one of the most fundamental modes of interpersonal communication, and the smile has been the subject of experimental study ever since Charles Darwin's 1872 text, *The Expression of the Emotions in Man and Animals*.[9] This great treatise devotes many pages to the contemplation of the smile and its origins, with Darwin speculating that perhaps "the habit of uttering loud reiterated sounds from a sense of pleasure first led to the retraction of the corners of the mouth". Only later did the sciences of anatomy and physiology extend to the analysis of the modes by which this evocative facial expression is produced, specifically, the elementary muscle actions.

We can conduct shape analysis on the human smile by first characterising a mouth using ten landmarks, as shown in Figure 3. Some landmark points are well-defined, others are interpolated, but all are located on what is known anatomically as the vermilion border: the sharp boundary between the vermilion-colored lips and the less colored adjacent skin.

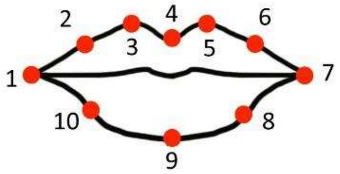


FIGURE 3 Ten landmarks on the lips.

In a pilot study of the smiles of a control group of human adults, 14 subjects were asked to produce a “full smile”, showing their teeth but not separating the jaws.[10] The subjects were trained to make the smile in a specific way in order to reduce variation between subjects. A 3D motion capture camera system, called Di4D, was used for facial surface landmarking. The camera recorded the trajectory of the smile from resting to full smile and back again. The number of time points was normalised to 101 frames of video for each subject, leading to comparable trajectories of the movement of the ten landmarks, which were then projected into two dimensions for a parsimonious analysis.

The smile trajectories have three key time points: at the start, which is the rest frame; at half-smile, which is a frame near the median at which the smile changes its direction in shape space; and at full smile, which is the frame representing maximum smile (in the sense of being farthest from the rest position). The mouth action involved can be decomposed into two components – one that can be described as the “uniform” smile component (the opening of the mouth), the other the “bending component” (the widening of the mouth) – and can be related to two different facial muscle actions, as will we later explain. Also, “the opening of mouth” corresponds to stretching in a vertical direction relative to the horizontal direction, whereas “the widening of the mouth” corresponds to the middle of the mouth going downward and the corners going upward.

We first describe our finding through schematic deformation, in Figures 4-6, and then give real examples with deformations in Figure 7. Figure 4 shows the grids over the uniform components of the full smile. There is no bending of the mouth, but there is possible shear from one smile into another.

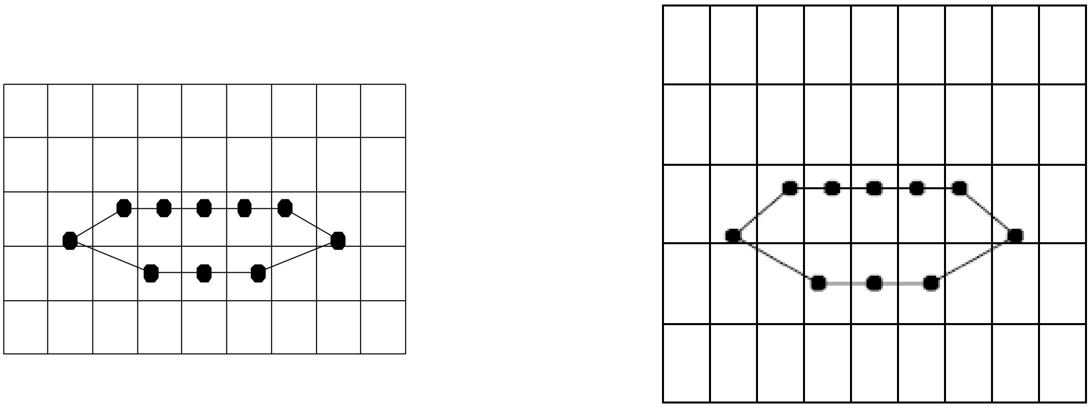


FIGURE 4 (a) Square grid of smile start (the source) on the left, and (b) the uniform or unwarped component of full smile (the target) on the right. Note that the landmark points 2-6 and 8-10 of Figure 3 are shown along straight lines in the figures above to simplify our graphical presentation.

Going beyond this uniform scaling, we can allow non-linear deformation of the kind demonstrated by Thompson (as seen in Figure 2). A regular square grid is drawn over the first figure, and at each point where two lines on the grid meet, the corresponding position in the second figure is calculated using a formula. The junction points are joined with lines in the same order as in the first figure to give a deformed grid over the second figure. The resulting interpolant produces transformation grids that bend as little as possible using Bookstein's PTPS approach. When we think of each square in the deformation as being deformed into a quadrilateral, the spline function minimises the local variation of these small quadrilaterals with respect to their neighbours.[4]

Figure 5 shows the bending component on a smile from start to "half-smile" where the uniform component is absent.

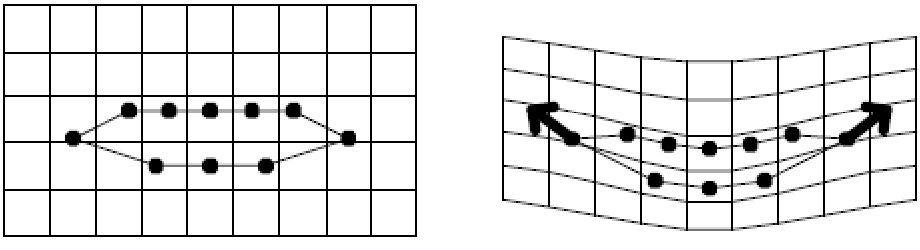
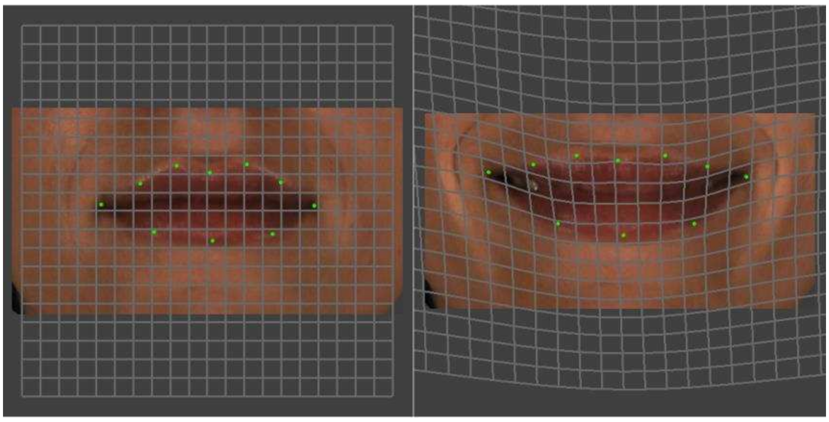


FIGURE 5. (a) Square grid of the smile start (source) as in Figure 4a and (b) the bending component in half-smile. Heavy arrows are the graphically shifted vectors showing the displacements responsible for this shape change.

Finally, the full smile that we asked pilot study subjects to produce is in fact a combination of the two deformations, both opening and widening of the mouth, as we show in Figure 6.

FIGURE 6 (a) Square grid of the smile start (source) as in Figure 4a. The full smile is the deformation (target) from Figure 4a to the sum of the displacement from Figure 4a to Figure 5b and the displacement from Figure 5a to Figure 5b.

In Figure 7, these deformations are overlaid on the smile of a human subject. In the research paper that explains this pilot study in more detail,[10] we show that the lip-opening component of the open-lip smile mainly corresponds to the relaxation of the orbicularis oris muscle (uniform), while the widening of the mouth (lateral and posterior displacement of the corners of the mouth) is due to the action of the zygomaticus major muscle (coming from bending energy). This analysis forms part of our project to understand the underlying features in a normal human smile, with the goal of quantifying “normality". The eventual aim is to use this work in clinical applications to judge the success of reconstructive surgery, such as the repair of a cleft lip.[4] The list of other applications, extant or potential, of this deformation-driven statistical toolkit is lengthy. For example, in 2001, Glasbey and Mardia applied a cost function similar to that of our thin-plate splines to the rectification of images derived from geographical and ecological monitoring via remote sensing.[11]



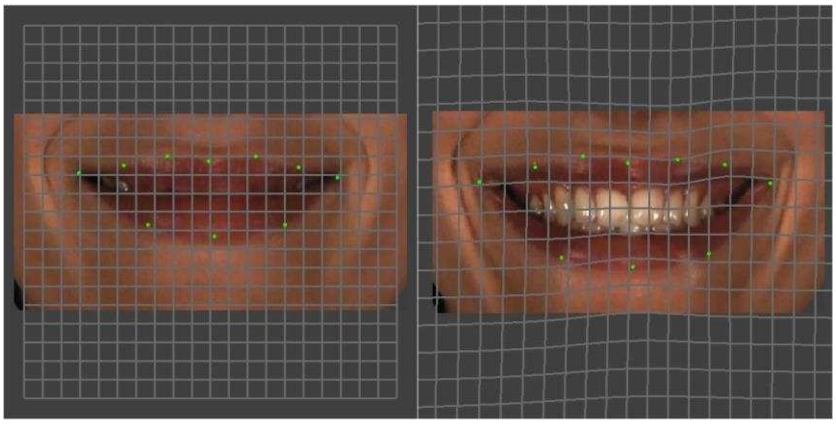


FIGURE 7 Grid deformation on a subject with the frames extracted from the trajectory, going from start to half-smile; half-smile to full smile. Top figures: source to target (first-half of the smile); the bottom figures: source to target second half of the smile.

Catching up to Thompson

Modern statistical tools for studying deformation have been developed by a variety of authors across various disciplines including biology, computer vision, engineering and statistics.[4] These are a major part of statistical shape analysis, which has grown into a rich branch of mathematics. In particular, the deformation tools like the one discussed here provide today's researchers with the machinery to study shape in a way that was impossible in Thompson's time.

Indeed, Thompson was aware of the limitations he faced. As he wrote (1942 edition, Epilogue p.1097), "…while I have sought to shew the naturalist how a few mathematical concepts and dynamical principles may help and guide him, I have tried to shew the mathematician a field for his labour - a field which few have entered and no man has explored. Here may be found homely problems, such as often tax the highest skill of the mathematician, and reward his ingenuity all the more for their trivial associations and outward semblance of simplicity. …

"That I am no skilled mathematician I have had little need to confess. I am advanced in these enquiries no farther than the threshold; but something of the use and beauty of mathematics I think I am able to understand."

Thompson's work on deformation, even though it relied on the wrong mathematical strategy, nevertheless highlighted the issue of deformation that was eventually tackled by statistical shape analysis. Since its publication, his book has been credited with sparking the interest of "scientists, artists and thinkers as diverse as Alan Turing, C.H. Waddington, Claude Lévi Strauss, Norbert Wiener, Henry Moore and Mies van der Rohe" (bit.ly/2E8nu27).

Those working in statistical shape analysis should celebrate the work of this unique and original thinker, who took the first bold steps to explore biological shapes in a more formal mathematical way.

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