



UNIVERSITY OF LEEDS

This is a repository copy of *Smooth compactness of  $f$ -minimal hypersurfaces with bounded  $f$ -index*.

White Rose Research Online URL for this paper:  
<http://eprints.whiterose.ac.uk/132570/>

Version: Accepted Version

---

**Article:**

Barbosa, E, Sharp, B [orcid.org/0000-0002-7238-4993](https://orcid.org/0000-0002-7238-4993) and Wei, Y (2017) Smooth compactness of  $f$ -minimal hypersurfaces with bounded  $f$ -index. *Proceedings of the American Mathematical Society*, 145 (11). pp. 4945-4961. ISSN 0002-9939

<https://doi.org/10.1090/proc/13628>

---

© 2017 American Mathematical Society. This is an author produced version of a paper published in the *Proceedings of the American Mathematical Society*. Uploaded in accordance with the publisher's self-archiving policy.

**Reuse**

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.



[eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk)  
<https://eprints.whiterose.ac.uk/>

# SMOOTH COMPACTNESS OF $f$ -MINIMAL HYPERSURFACES WITH BOUNDED $f$ -INDEX

EZEQUIEL BARBOSA, BEN SHARP, AND YONG WEI

ABSTRACT. Let  $(M^{n+1}, g, e^{-f}d\mu)$  be a complete smooth metric measure space with  $2 \leq n \leq 6$  and Bakry-Émery Ricci curvature bounded below by a positive constant. We prove a smooth compactness theorem for the space of complete embedded  $f$ -minimal hypersurfaces in  $M$  with uniform upper bounds on  $f$ -index and weighted volume. As a corollary, we obtain a smooth compactness theorem for the space of embedded self-shrinkers in  $\mathbb{R}^{n+1}$  with  $2 \leq n \leq 6$ . We also prove some estimates on the  $f$ -index of  $f$ -minimal hypersurfaces.

## 1. INTRODUCTION

Let  $(M^{n+1}, g)$  be a complete smooth Riemannian manifold and  $f$  be a smooth function on  $M$ . We denote by  $\bar{\nabla}$  the Levi-Civita connection on  $M$  with respect to  $g$ . The Bakry-Émery Ricci curvature of  $M$  is defined by

$$Ric_f = Ric + \bar{\nabla}^2 f, \tag{1.1}$$

which is an important generalization of the Ricci curvature. The equation  $Ric_f = \kappa g$  for some constant  $\kappa$  is just the gradient Ricci soliton equation, which plays an important role in the singularity analysis of Ricci flow (see [4]). Denote by  $d\mu$  the volume form on  $M$  with respect to  $g$ , then  $(M^{n+1}, g, e^{-f}d\mu)$  is often called a smooth metric measure space. We refer the interested readers to [40] for further motivation and examples of smooth metric measure spaces.

Suppose that  $\Sigma$  is a hypersurface in  $M$ . Let  $x \in \Sigma$  and  $\nu$  be the unit normal vector at  $x$ . The mean curvature  $H$  of  $\Sigma$  is defined as  $H = \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \nu, e_i \rangle$  for an orthonormal basis  $\{e_i\}_{i=1}^n \subset T\Sigma$ . Define the  $f$ -mean curvature at  $x \in \Sigma$  by

$$H_f = H - \langle \bar{\nabla} f, \nu \rangle. \tag{1.2}$$

Then  $\Sigma$  is called an  $f$ -minimal hypersurface in  $M$  if its  $f$ -mean curvature  $H_f$  vanishes everywhere. The most well known example of smooth metric measure space is the Gaussian soliton:  $(\mathbb{R}^{n+1}, g_0, e^{-\frac{1}{4}|x|^2}d\mu)$  whose Bakry-Émery Ricci curvature satisfies  $Ric_f = \frac{1}{2}g_0$ , where  $g_0$  is the standard Euclidean metric on  $\mathbb{R}^{n+1}$ . The  $f$ -minimal hypersurface in the Gaussian soliton is the self-shrinker  $\Sigma^n \subset \mathbb{R}^{n+1}$  which satisfies

$$H = \frac{1}{2} \langle x, \nu \rangle. \tag{1.3}$$

Self-shrinkers play an important role in the mean curvature flow, as they correspond to the self-similar solutions to mean curvature flow, and also describe all possible blow ups at a given singularity. The simplest examples of self-shrinkers in  $\mathbb{R}^{n+1}$  are the flat hyperplanes  $\mathbb{R}^n$  through the origin, the spheres centred at the origin  $\mathbb{S}^n(\sqrt{2n})$  and the cylinders  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ,  $1 \leq k \leq n-1$ .

---

*Date:* November 23, 2016.

*2010 Mathematics Subject Classification.* 53C42, 53C21.

*Key words and phrases.* Compactness,  $f$ -minimal, self-shrinker, smooth metric measure space, index.

The first author was supported by FAPEMIG and CNPq grants; The second author was supported by André Neves' ERC and Leverhulme trust grants; The third author was supported by Jason D Lotay's EPSRC grant.

Angenent [1] has constructed a compact embedded self-shrinker in  $\mathbb{R}^3$  of genus one, which is often called the Angenent torus. Recently, Brendle [2] proved that the only compact embedded self-shrinker in  $\mathbb{R}^3$  of genus zero is the round sphere  $\mathbb{S}^2(2)$ . See [17, 18, 26, 33, 35] for further examples of self-shrinkers.

In 2012, Colding-Minicozzi [8] proved the following smooth compactness theorem for the space of embedded self-shrinkers.

**Theorem 1.1** ([8]). *Given an integer  $\gamma \geq 0$  and a constant  $0 < \Lambda < \infty$ , the space of smooth complete embedded self-shrinkers  $\Sigma \subset \mathbb{R}^3$  with*

- *genus at most  $\gamma$ ,*
- *$\partial\Sigma = \emptyset$ ,*
- *$Area(B_R(x_0) \cap \Sigma) \leq \Lambda R^2$  for all  $x_0 \in \mathbb{R}^3$  and all  $R > 0$*

*is compact in the sense that any sequence in this class contains a subsequence that converges in the  $C^m$  topology on compact subsets for any  $m \geq 2$ .*

The surfaces  $\Sigma$  in the above theorem are each assumed to be homeomorphic to some closed surface  $S_\Gamma$  of genus  $\Gamma \leq \gamma$  with a finite number of disks removed  $\Sigma \cong S_\Gamma \setminus \{D_1, \dots, D_k\}$  - thus each surface has a finite number of ends. There is no uniform control on  $k$ , or equivalently the number of ends so the global topology of the limiting surface is *a-priori* not known.

The third condition in Theorem 1.1 means that the self-shrinkers have at most Euclidean volume growth. Generally, we say that a self-shrinker  $x : \Sigma \rightarrow \mathbb{R}^{n+1}$  has at most Euclidean volume growth if  $Area(B_R(x_0) \cap \Sigma) \leq \Lambda R^n$  where  $B_R(x_0)$  is the Euclidean ball of radius  $R$  centred at  $x_0 \in \mathbb{R}^{n+1}$ . If the self-shrinker arises as the singularity of mean curvature flow, Huisken's monotonicity formula implies that it has at most Euclidean volume growth, see [9, Corollary 2.13]; furthermore Lu Wang [41] proved that the entire graphical self-shrinker is a hyperplane and thus has at most Euclidean volume growth; and finally Ding-Xin [16] then proved that any properly-immersed self-shrinker has at most Euclidean volume growth. Since self-shrinkers model singularities of mean curvature flow, Theorem 1.1 can be viewed as a compactness result for the space of all singularities. In particular, Theorem 1.1 plays an important role in understanding generic mean curvature flow in [9].

Later, Ding-Xin [16] proved a related compactness theorem for the space of self-shrinkers in  $\mathbb{R}^3$ . After that, based on the observation that a self-shrinker is a special example of an  $f$ -minimal surface, the third author and H. Li [30] proved a compactness theorem for the space of closed  $f$ -minimal surfaces in a closed three dimensional smooth metric measure space with positive Ricci curvature, generalizing the classical compactness theorem of closed minimal surfaces in a closed three manifold with positive Ricci curvature by Choi and Schoen [10]. The results in [8, 16, 30] were later generalised by Cheng-Mejia-Zhou [13, 14] to complete  $f$ -minimal surfaces in complete 3-dimensional smooth metric measure spaces. On the other hand the second author [38] proved a smooth compactness result for closed embedded minimal hypersurfaces with bounded index and volume in a closed Riemannian manifold  $M^{n+1}$  with positive Ricci curvature and  $2 \leq n \leq 6$ , generalising Choi-Schoen's [10] compactness theorem to higher dimensions.

**Theorem 1.2** ([38]). *Let  $(M^{n+1}, g)$  be a closed Riemannian manifold with  $Ric > 0$  and  $2 \leq n \leq 6$ . Given an integer  $I$  and a constant  $0 < \Lambda < \infty$ , the space of smooth closed embedded minimal hypersurfaces  $\Sigma \subset M^{n+1}$  satisfying  $Vol(\Sigma) \leq \Lambda$  and  $index(\Sigma) \leq I$  is compact in the smooth topology.*

Note that Ejiri-Micallef [20] proved that for closed minimal surface  $\Sigma$  of genus  $\gamma$  in a Riemannian manifold  $M^m$ , the area index satisfies

$$ind(\Sigma) \leq C(m, M)(Area(\Sigma) + \gamma - 1), \quad (1.4)$$

where  $C(m, M)$  is a constant depending on  $m, M$ . The main interest in the above result is that no assumption is made on the second fundamental form of  $\Sigma \subset M^m$ . Moreover, when  $n = 2$  the area is uniformly bounded from above by the genus and the lower Ricci-bound by an estimate of Choi-Wang [11]. Thus Theorem 1.2 is a natural generalisation of Choi-Schoen's compactness theorem. In this paper, we borrow the idea in [38] to prove the following smooth compactness theorem for the space of  $f$ -minimal hypersurfaces.

**Theorem 1.3.** *Let  $(M^{n+1}, g, e^{-f}d\mu)$  be a complete smooth metric measure space with  $2 \leq n \leq 6$  and  $Ric_f \geq \kappa > 0$  for some positive constant  $\kappa$ . Given an integer  $I$  and a constant  $0 < \Lambda < \infty$ , the space  $S_{\Lambda, I}$  of smooth complete embedded  $f$ -minimal hypersurfaces  $\Sigma \subset M^{n+1}$  with*

- $f$ -index at most  $I$ ,
- $\partial\Sigma = \emptyset$ ,
- $\int_{\Sigma} e^{-f}dv \leq \Lambda$  where  $dv$  is the volume form of  $\Sigma$  with respect to the metric induced from  $(M, g)$

*is compact in the smooth topology. Namely, any sequence of  $S_{\Lambda, I}$  has a subsequence that converges in  $C^k$  topology on compact subsets to a hypersurface in  $S_{\Lambda, I}$  for any  $k \geq 2$ .*

Our Theorem 1.3 generalises Theorem 1.2, as when  $f = \text{const}$ , an  $f$ -minimal hypersurface is just a minimal hypersurface with  $Ric_f = Ric$ . Note that an  $f$ -minimal hypersurface  $\Sigma$  in  $(M^{n+1}, g, e^{-f}d\mu)$  is also minimal in  $(M^{n+1}, \tilde{g})$ , where  $\tilde{g} = e^{-\frac{2}{n}f}g$  is a conformal metric of  $g$ , see §2.2. However, Theorem 1.3 cannot follow from Theorem 1.2 directly by a conformal transformation, because the condition  $Ric_f \geq \kappa > 0$  cannot guarantee the Ricci tensor of the conformal metric  $\tilde{g}$  has a positive lower bound  $\tilde{Ric} \geq c > 0$ . Moreover, the Riemannian manifold  $(M, g)$  with  $Ric \geq \kappa > 0$  must be compact by Myers' theorem, while this is not true for a smooth metric measure space  $(M^{n+1}, g, e^{-f}d\mu)$  with  $Ric_f \geq \kappa > 0$ . In Theorem 1.3, the  $f$ -index of an  $f$ -minimal hypersurface is defined as the dimension of the subspace such that the second variation of the weighted volume is negative, see §2.1 for details. The proof of Theorem 1.3 will be given in §3. Since a self-shrinker is an  $f$ -minimal hypersurface in the Gaussian soliton which has constant  $Ric_f = \frac{1}{2}$ , Theorem 1.3 has the following immediate corollary, which can be viewed as a generalisation of Theorem 1.1 to higher dimensions.

**Corollary 1.4.** *For  $2 \leq n \leq 6$ , given an integer  $I$  and a constant  $0 < \Lambda < \infty$ , the space of smooth complete embedded self-shrinkers  $\Sigma \subset \mathbb{R}^{n+1}$  with*

- $f$ -index at most  $I$ ,
- $\partial\Sigma = \emptyset$ ,
- $Vol(B_R(x_0) \cap \Sigma) \leq \Lambda R^n$  for all  $x_0 \in \mathbb{R}^{n+1}$  and all  $R > 0$

*is compact in smooth topology.*

*Remark 1.1.* The relationship between Corollary 1.4 and Theorem 1.1 is not well understood presently, in the sense that it is not clear which is stronger than the other. However, we expect there to be a more explicit relationship between the index, genus and number of ends for self-shrinkers (or more generally  $f$ -minimal surfaces). We mention some partial results below for closed surfaces which follow easily from the estimates available for minimal surfaces in Riemannian manifolds - the relationship in this case is comparatively well understood.

Note that the third condition in Corollary 1.4 implies that the weighted volume  $\int_{\Sigma} e^{-\frac{|x|^2}{4}} dv \leq \tilde{\Lambda}$  for some positive constant  $\tilde{\Lambda}$ . Then the conclusion of Corollary 1.4 follows directly from Theorem 1.3.

In particular for closed self shrinkers we have the following which follows easily from the estimates for minimal surfaces:

**Proposition 1.5.** *Let  $\Sigma^n \subset \mathbb{R}^{n+1}$  be a closed smooth embedded self-shrinker satisfying  $|x| \leq c_1$ , i.e.,  $\Sigma$  is contained in an Euclidean ball of radius  $c_1$ . Then the  $f$ -index of  $\Sigma$  satisfies*

$$\text{ind}_f(\Sigma) \leq \begin{cases} C(n, c_1) \int_{\Sigma} (|A|^2 + \frac{1}{2})^{\frac{n}{2}} e^{-\frac{|x|^2}{4}} dv & \text{if } n \geq 3 \\ C(c_1) (\int_{\Sigma} e^{-\frac{|x|^2}{4}} dv + \gamma - 1) & \text{if } n = 2, \end{cases} \quad (1.5)$$

where  $C(n, c_1)$  is a uniform constant depending on  $n, c_1$ .

*Remark 1.2.* Cao-Li [5] proved that any compact self-shrinker  $\Sigma^n \subset \mathbb{R}^{n+1}$  satisfying  $H^2 = \frac{1}{4}|x^\perp|^2 \leq n/2$  is the sphere  $\mathbb{S}^n(\sqrt{2n})$ , see also [6, 29, 37] for related results.

We moreover have the analogous result for general closed  $f$ -minimal surfaces in smooth metric measure spaces, from which Proposition 1.5 follows;

**Proposition 1.6.** *Let  $\Sigma^n$  be a smooth closed embedded  $f$ -minimal hypersurface in  $(M^{n+1}, g, e^{-f} d\mu)$ . Suppose that  $\Sigma \subset\subset K$  where  $K$  is compact with open interior, and  $\partial K$  is smooth. Then the  $f$ -index of  $\Sigma$  satisfies*

$$\text{ind}_f(\Sigma) \leq \begin{cases} C(n, K, M, f) \int_{\Sigma} \max\{1, |A|^2 + \text{Ric}_f(\nu, \nu)\}^{\frac{n}{2}} e^{-f} dv & \text{if } n \geq 3 \\ C(K, M, f) (\int_{\Sigma} e^{-f} dv + \gamma - 1) & \text{if } n = 2, \end{cases} \quad (1.6)$$

where  $C(n, K, M, f)$  is a uniform constant depending on  $n, K, f$  and  $M$ .

*Remark 1.3.* We note that since we are working in a compact region of  $M$ , the quantities on the right hand side of (1.5), (1.6) are equivalent (up to a constant depending only on  $K, c_1$  and  $f$ ) to the  $(M, g)$ -area  $\int_{\Sigma} 1 dv$ , plus the usual (un-weighted) total curvature  $\int_{\Sigma} |A|^n dv$ .

From Corollary 8 in [30], we know that for a closed embedded  $f$ -minimal surface  $\Sigma$  in a closed smooth metric measure space  $(M^3, g, e^{-f} d\mu)$  with  $\text{Ric}_f \geq \kappa > 0$ , the weighted area of  $\Sigma$  satisfies

$$\int_{\Sigma} e^{-f} dv \leq \frac{16\pi}{\kappa} (\gamma + 1) e^{-\min_{\Sigma} f}. \quad (1.7)$$

Combining (1.6), (1.7) and Theorem 1.3, we recover Li-Wei's [30] theorem.

**Theorem 1.7** ([30]). *Let  $(M^3, g, e^{-f} d\mu)$  be a closed (i.e. compact with empty boundary) smooth metric measure space with  $\text{Ric}_f \geq \kappa > 0$ . Then the space of closed embedded  $f$ -minimal surfaces of fixed topological type in  $(M^3, g, e^{-f} d\mu)$  is compact in the smooth topology.*

Further to the above we know that if  $\Sigma^2$  is a closed embedded  $f$ -minimal surface in a complete non-compact smooth metric measure space  $(M^3, g, e^{-f} d\mu)$  with  $\text{Ric}_f \geq k > 0$ , and  $\Sigma \subset D$  where  $D$  is some compact domain with  $g$ -convex boundary, then by [14, Proposition 8] we have that (1.7) is satisfied. Combining this with (1.6), we consider the compactness for  $f$ -minimal surfaces in a complete non-compact manifold and recover Cheng-Meija-Zhou's [14] theorem.

**Theorem 1.8** ([14]). *Let  $(M^3, g, e^{-f} d\mu)$  be a complete non-compact smooth metric measure space with  $\text{Ric}_f \geq k$ , where  $k$  is a positive constant. Assume that  $M^3$  admits an exhaustion by bounded domains with convex boundary. Then the space, denoted by  $S_{D, g}$ , of closed embedded  $f$ -minimal surfaces in  $M^3$  with genus at most  $g$  and diameter at most  $D$  is compact in the  $C^m$  topology, for any  $m \geq 2$ .*

*Remark 1.4.* We would like to point out that Theorem 1.3 can now be thought of as an extension of Theorems 1.7 and 1.8 to higher dimensions.

**Acknowledgements.** The first and second authors would like to thank André Neves for countless illuminating discussions on minimal surfaces. The first author would also like to thank Professor Neves and Imperial College London for their hospitality during his time in London, during which this work was carried out.

## 2. $f$ -MINIMAL HYPERSURFACES

In this section, we collect and prove some properties about  $f$ -minimal hypersurfaces. The readers can refer to [13–15, 19, 25, 28, 30] for more information about  $f$ -minimal hypersurfaces.

**2.1. Variational properties.** Let  $(M^{n+1}, g, e^{-f}d\mu)$  be a complete smooth metric measure space. Suppose that  $\Sigma$  is a smooth embedded hypersurface in  $M^{n+1}$ . For any normal variation  $\Sigma_s$  of  $\Sigma$  with compactly supported variation vector field  $X \in T^\perp\Sigma$ , the first variation formula for the weighted volume of  $\Sigma$

$$Vol_f(\Sigma) := \int_{\Sigma} e^{-f} dv$$

is given by

$$\left. \frac{d}{ds} \right|_{s=0} Vol_f(\Sigma_s) = - \int_{\Sigma} g(X, \mathbf{H}_f) e^{-f} dv \quad (2.1)$$

where  $dv$  is the volume form of  $\Sigma$  with respect to the metric induced from  $(M, g)$ . We say that  $\Sigma$  is  $f$ -minimal if  $\left. \frac{d}{ds} \right|_{s=0} Vol_f(\Sigma_s) = 0$  for any compactly supported normal variation  $\Sigma_s$ . Then from (2.1),  $\Sigma$  is  $f$ -minimal if and only if  $\mathbf{H}_f = 0$  on  $\Sigma$ .

We also have the second variation formula for  $Vol_f(\Sigma)$  (see e.g., [13, 28]):

$$\left. \frac{d^2}{ds^2} \right|_{s=0} Vol_f(\Sigma_s) = \int_{\Sigma} \left( |\nabla^\perp X|^2 - (|A|^2 |X|^2 + Ric_f(X, X)) \right) e^{-f} dv \quad (2.2)$$

where  $\nabla^\perp$  is the connection on the normal bundle  $T^\perp\Sigma$ ,  $|A|^2$  is the squared norm of the second fundamental form  $A$  of  $\Sigma$  in  $(M, g)$  and  $Ric_f$  is the Bakry-Émery Ricci tensor of  $(M, g, e^{-f}d\mu)$ . We associate the right hand side of (2.2) to a bilinear form on the space of compactly supported normal vector field on  $\Sigma$

$$Q_f^\Sigma(X, X) = \int_{\Sigma} \left( |\nabla^\perp X|^2 - (|A|^2 |X|^2 + Ric_f(X, X)) \right) e^{-f} dv. \quad (2.3)$$

An  $f$ -minimal hypersurface  $\Sigma$  is called  $f$ -stable if  $Q_f^\Sigma(X, X) \geq 0$  for any compactly supported  $X \in T^\perp\Sigma$ . The  $f$ -index of  $\Sigma$ , denoted by  $ind_f(\Sigma)$ , is defined to be the dimension of the subspace such that  $Q_f^\Sigma$  is strictly negative definite.

If  $\Sigma$  is two-sided, then there exists a globally defined unit normal  $\nu$  on  $\Sigma$ . Then  $f$ -mean curvature vector field of  $\Sigma$  in  $(M, g)$  satisfies  $\mathbf{H}_f = -H_f\nu = 0$ . Set  $X = \varphi\nu$  for some smooth compactly supported function  $\varphi$  on  $\Sigma$ . Then the quadratic form (2.3) is equivalent to

$$\begin{aligned} Q_f^\Sigma(\varphi, \varphi) &= \int_{\Sigma} \left( |\nabla\varphi|^2 - (|A|^2 + Ric_f(\nu, \nu))\varphi^2 \right) e^{-f} dv \\ &= - \int_{\Sigma} \varphi L_f \varphi e^{-f} dv, \end{aligned} \quad (2.4)$$

where  $\nabla$  is the Levi-Civita connection on  $\Sigma$  w.r.t the metric induced from  $(M, g)$  - thus in particular there are no two-sided  $f$ -minimal surfaces which have zero index (or are stable) if  $Ric_f > 0$ . The operator  $L_f$  is defined by

$$L_f = \Delta_f + |A|^2 + Ric_f(\nu, \nu), \quad (2.5)$$

where  $\Delta_f = \Delta - \nabla f \cdot \nabla$  is the  $f$ -Laplacian on  $\Sigma$ , which is self-adjoint in the weighted  $L^2$  sense, i.e., for any two  $u, v \in C_c^\infty(\Sigma)$ ,

$$\int_{\Sigma} u \Delta_f v e^{-f} dv = - \int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-f} dv. \quad (2.6)$$

Then  $L_f$  is also self-adjoint in the weighted  $L^2$  sense. For any compact domain  $\Omega \subset \Sigma$ , consider the following Dirichlet eigenvalue problem

$$-L_f u = \lambda u, \quad u|_{\partial\Omega} = 0, \quad u \in C^\infty(\Omega). \quad (2.7)$$

The standard spectrum theory implies that the set of eigenvalues of the problem (2.7) is a discrete increasing sequence  $\lambda_1(\Omega) < \lambda_2(\Omega) \leq \dots \rightarrow \infty$ . There is some number  $k$ , counted with multiplicity, such that  $\lambda_1, \dots, \lambda_k < 0 \leq \lambda_{k+1}$ . The Morse index of  $L_f$  on  $\Omega$  is defined to be the number  $k$  so that  $\lambda_{k+1}$  is the first nonnegative eigenvalue of  $L_f$  on  $\Omega$ , which we denote by  $ind_f(\Omega)$ . By the minimax characterisation of eigenvalues we have

$$ind_f(\Omega) \leq ind_f(\Omega'), \quad \text{if } \Omega \subset \Omega' \subset \Sigma.$$

As in the complete two-sided minimal hypersurface case (see [21]), the  $f$ -index of complete two-sided  $f$ -minimal hypersurface  $\Sigma$  is equal to the supremum over compact domains of  $\Sigma$  of the  $f$ -index of compact domain  $\Omega \subset \Sigma$ , i.e.,  $ind_f(\Sigma) = \sup_{\Omega \subset \Sigma} ind_f(\Omega)$ .

**2.2. An  $f$ -minimal surface as a minimal surface in a conformally changed metric.** The  $f$ -minimal hypersurface can be viewed as a minimal hypersurface under a conformal metric. Define a conformal metric  $\tilde{g} = e^{-\frac{2}{n}f}g$  on  $M$ . Then the volume of  $\Sigma$  with respect to the induced metric from  $\tilde{g}$  is

$$\widetilde{Vol}(\Sigma) = \int_{\Sigma} d\tilde{v} = \int_{\Sigma} e^{-f} dv = Vol_f(\Sigma). \quad (2.8)$$

The first variation formula for  $\widetilde{Vol}$  is given by

$$\left. \frac{d}{ds} \right|_{s=0} \widetilde{Vol}(\Sigma_s) = - \int_{\Sigma} \tilde{g}(X, \tilde{\mathbf{H}}) d\tilde{v} = - \int_{\Sigma} e^{-\frac{2}{n}f} g(X, \tilde{\mathbf{H}}) e^{-f} dv, \quad (2.9)$$

where  $\tilde{\mathbf{H}}$  is the mean curvature vector field of  $\Sigma$  in  $(M, \tilde{g})$ . Comparing (2.1) and (2.9), and noting (2.8), we conclude that  $\tilde{\mathbf{H}} = e^{\frac{2}{n}f} \mathbf{H}_f$  and  $\Sigma$  is  $f$ -minimal in  $(M, g)$  if and only if  $\Sigma$  is minimal in  $(M, \tilde{g})$ .

When  $\Sigma$  is two-sided, there exists a globally defined unit normal  $\tilde{\nu}$  on  $\Sigma$ . Then  $\tilde{\nu} = e^{\frac{f}{n}} \nu$ , where  $\nu$  is the unit normal of  $\Sigma$  in  $(M, g)$ . The mean curvature vector field of  $\Sigma$  in  $(M, \tilde{g})$  satisfies  $\tilde{\mathbf{H}} = -\tilde{H}\tilde{\nu} = -\tilde{H}e^{\frac{f}{n}}\nu$ . Thus, we have

$$\tilde{H} = e^{\frac{f}{n}} H_f. \quad (2.10)$$

The variation vector field  $X$  can be rewritten as  $X = \varphi\nu = e^{-\frac{f}{n}}\varphi\tilde{\nu}$ . The second variation formula for  $\widetilde{Vol}(\Sigma)$  is given by

$$\left. \frac{d^2}{ds^2} \right|_{s=0} \widetilde{Vol}(\Sigma_s) = - \int_{\Sigma} \tilde{g}(X, \tilde{L}(X)) d\tilde{v} = - \int_{\Sigma} e^{-\frac{f}{n}} \varphi \tilde{L}(e^{-\frac{f}{n}} \varphi) e^{-f} dv, \quad (2.11)$$

where  $\tilde{L}$  is the Jacobi operator of  $\Sigma \subset (M, \tilde{g})$ :

$$\tilde{L} = \Delta_{\tilde{g}} + |\tilde{A}|_{\tilde{g}}^2 + \widetilde{Ric}(\tilde{\nu}, \tilde{\nu}). \quad (2.12)$$

Since  $\widetilde{Vol}(\Sigma) = Vol_f(\Sigma)$  and the normal direction of  $\Sigma$  in  $M$  is unchanged under the conformal change of the ambient metric, we know that the second variation

$$\left. \frac{d^2}{ds^2} \right|_{s=0} Vol_f(\Sigma_s) = \left. \frac{d^2}{ds^2} \right|_{s=0} \widetilde{Vol}(\Sigma_s)$$

for any normal variation  $\Sigma_s$  of  $\Sigma$ . Thus  $\Sigma$  is  $f$ -stable in  $(M, g)$  if and only if  $\Sigma$  is stable in  $(M, \tilde{g})$  in the usual sense, and the  $f$ -index  $ind_f$  of  $\Sigma \subset (M, g)$  is equal to the index of  $\Sigma \subset (M, \tilde{g})$  denoted by  $\widetilde{ind}$ .

**2.3. The relationship of the geometric quantities from the different viewpoints.** We can relate the second fundamental form  $\tilde{A}$  of  $\Sigma$  in  $(M, \tilde{g})$  to the second fundamental form  $A$  of  $\Sigma$  in  $(M, g)$  as follows: Choose an orthonormal basis  $e_1, \dots, e_n$  for  $T\Sigma$  with respect to the metric induced from  $(M, g)$ . Then  $A(e_i, e_j) = -g(\bar{\nabla}_{e_i} e_j, \nu)$ . Under the conformal metric  $\tilde{g}$ ,  $\{\tilde{e}_1 = e^{\frac{f}{n}} e_1, \dots, \tilde{e}_n = e^{\frac{f}{n}} e_n\}$  and  $\tilde{\nu} = e^{\frac{f}{n}} \nu$  are the orthonormal basis for  $T\Sigma$  and unit normal vector field of  $\Sigma$  in  $(M, \tilde{g})$ . Then a direct calculation gives that

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y - \frac{1}{n} df(X)Y - \frac{1}{n} df(Y)X + \frac{1}{n} g(X, Y) \bar{\nabla} f \quad (2.13)$$

for any tangent vector fields  $X, Y$  on  $M$ , where  $\tilde{\nabla}$  is the Levi-Civita connection on  $(M, \tilde{g})$ . Thus the second fundamental form  $\tilde{A}$  satisfies

$$\begin{aligned} \tilde{A}(\tilde{e}_i, \tilde{e}_j) &= -\tilde{g}(\tilde{\nabla}_{\tilde{e}_i} \tilde{e}_j, \tilde{\nu}) \\ &= -\tilde{g}(\bar{\nabla}_{\tilde{e}_i} \tilde{e}_j - \frac{1}{n} df(\tilde{e}_i) \tilde{e}_j - \frac{1}{n} df(\tilde{e}_j) \tilde{e}_i + \frac{1}{n} g(\tilde{e}_i, \tilde{e}_j) \bar{\nabla} f, \tilde{\nu}) \\ &= -\tilde{g}(\bar{\nabla}_{\tilde{e}_i} \tilde{e}_j + \frac{1}{n} g(\tilde{e}_i, \tilde{e}_j) \bar{\nabla} f, \tilde{\nu}) \\ &= -e^{-\frac{2}{n} f} g(e^{\frac{2}{n} f} \bar{\nabla}_{e_i} e_j + \frac{1}{n} e^{\frac{2}{n} f} g(e_i, e_j) \bar{\nabla} f, e^{\frac{f}{n}} \nu) \\ &= e^{\frac{f}{n}} \left( A(e_i, e_j) - \frac{1}{n} g(e_i, e_j) g(\bar{\nabla} f, \nu) \right). \end{aligned} \quad (2.14)$$

Letting  $i = j$  and summing from 1 to  $n$  in (2.14) can also give the relation (2.10). From (2.14), we can also obtain

$$\begin{aligned} |\tilde{A}|_{\tilde{g}}^2 &= \sum_{i,j=1}^n (\tilde{A}(\tilde{e}_i, \tilde{e}_j))^2 \\ &= e^{\frac{2f}{n}} \left( \sum_{i,j=1}^n (A(e_i, e_j))^2 + \frac{1}{n} g(\bar{\nabla} f, \nu)^2 - \frac{2}{n} H g(\bar{\nabla} f, \nu) \right) \\ &= e^{\frac{2f}{n}} \left( |A|_g^2 - \frac{1}{n} H^2 \right), \end{aligned} \quad (2.15)$$

where in the last equality we used the fact  $H = g(\bar{\nabla} f, \nu)$ , since  $\Sigma$  is  $f$ -minimal in  $(M, g)$ .

We next relate the  $\widetilde{Ric}(\tilde{\nu}, \tilde{\nu})$  and  $Ric_f(\nu, \nu)$ . Under the conformal change of the metric  $\tilde{g} = e^{-\frac{2}{n} f} g$  on  $M$ , we know that the Ricci curvature changes by the following formula

$$\widetilde{Ric} = (n-1) \left( \frac{\bar{\nabla}^2 f}{n} + \frac{1}{n^2} df \otimes df - \frac{1}{n^2} |\bar{\nabla} f|^2 g \right) + \frac{1}{n} \bar{\Delta} f g + Ric, \quad (2.16)$$



where  $\bar{\nabla}, \bar{\Delta}$  are Levi-Civita connection and Laplacian operator on  $(M, g)$ . Then

$$\begin{aligned} \widetilde{Ric}(\tilde{\nu}, \tilde{\nu}) &= e^{\frac{2f}{n}} \left( \frac{n-1}{n^2} (n\bar{\nabla}^2 f(\nu, \nu) + g(\bar{\nabla}f, \nu)^2 - |\bar{\nabla}f|^2) + \frac{1}{n} \bar{\Delta}f + Ric(\nu, \nu) \right) \\ &= e^{\frac{2f}{n}} \left( Ric_f(\nu, \nu) - \frac{1}{n} \bar{\nabla}^2 f(\nu, \nu) + \frac{n-1}{n^2} (H^2 - |\bar{\nabla}f|^2) + \frac{1}{n} \bar{\Delta}f \right) \\ &= e^{\frac{2f}{n}} \left( Ric_f(\nu, \nu) + \frac{1}{n} (\Delta f + H^2) - \frac{n-1}{n^2} |\nabla f|^2 \right), \end{aligned}$$

where in the last equality we used the elementary fact

$$\Delta f = \bar{\Delta}f - \bar{\nabla}^2 f(\nu, \nu) - Hg(\bar{\nabla}f, \nu),$$

and  $f$ -minimal equation  $H = g(\bar{\nabla}f, \nu)$ . Then we have

$$|\tilde{A}|_{\tilde{g}}^2 + \widetilde{Ric}(\tilde{\nu}, \tilde{\nu}) = e^{\frac{2f}{n}} \left( |A|_g^2 + Ric_f(\nu, \nu) + \frac{1}{n} \Delta f - \frac{n-1}{n^2} |\nabla f|^2 \right). \quad (2.17)$$

Taking the trace of (2.16), we have that the scalar curvature changes after the conformal change of metric by

$$\tilde{R} = e^{\frac{2f}{n}} \left( R + 2\bar{\Delta}f - \frac{n-1}{n} |\bar{\nabla}f|^2 \right). \quad (2.18)$$

Recall that the natural scalar curvature in smooth metric measure space is the Perelman's scalar curvature  $R_f^P$  (see [36]) defined by

$$R_f^P = R + 2\bar{\Delta}f - |\bar{\nabla}f|^2, \quad (2.19)$$

which is different from  $tr(Ric_f) = R + \bar{\Delta}f$ . Then combining (2.18) and (2.19), we have

$$\tilde{R} = e^{\frac{2f}{n}} \left( R_f^P + \frac{1}{n} |\bar{\nabla}f|^2 \right). \quad (2.20)$$

Thus if the Perelman's scalar curvature  $R_f^P$  is nonnegative, the the scalar curvature  $\tilde{R}$  of the conformal metric  $\tilde{g}$  is also nonnegative.

In particular, for self-shrinkers  $\Sigma^n$  in  $\mathbb{R}^{n+1}$ , (which are  $f$ -minimal with  $f = |x|^2/4$ ), we have

$$|\tilde{A}|_{\tilde{g}}^2 + \widetilde{Ric}(\tilde{\nu}, \tilde{\nu}) = e^{\frac{|x|^2}{2n}} \left( |A|_g^2 + 1 - \frac{1}{4n} |x|^2 + \frac{1}{4n^2} |x^\top|^2 \right), \quad (2.21)$$

where  $x^\top$  denotes the tangential part of the position vector  $x$  of  $\Sigma$ . From (2.18) we also have that the scalar curvature of  $(\mathbb{R}^{n+1}, \tilde{g})$  satisfies

$$\tilde{R} = e^{\frac{|x|^2}{2n}} \left( n + 1 - \frac{n-1}{4n} |x|^2 \right), \quad (2.22)$$

which appeared in [8]. It is easy to see from (2.22) that  $\tilde{R}$  tends to negative infinity as  $|x| \rightarrow \infty$ .

### 3. PROOF OF MAIN THEOREM

We first have the following local singular compactness theorem for the space of embedded minimal hypersurfaces with bounded volume and index.

**Proposition 3.1.** *Given a point  $p$  in  $M^{n+1}$  with  $2 \leq n \leq 6$ . There exists  $R > 0$  such that the following is true: Suppose  $\Sigma_j$  is a sequence of embedded minimal hypersurfaces in  $B_{2R} = B_{2R}(p) \subset M$  with  $\partial\Sigma_j \subset \partial B_{2R}$ . If each  $\Sigma_j$  has volume at most  $\Lambda$  and index at most  $I$  for some fixed  $\Lambda > 0, I \in \mathbb{N}$ , then there exists a smooth embedded minimal hypersurface  $\Sigma \subset B_R$  with  $\partial\Sigma \subset \partial B_R$ ,*

and at most  $I$  points  $\{x_k\} \subset \Sigma$ , and a subsequence of  $\Sigma_j$  that converges in  $B_R$  with finite multiplicity to  $\Sigma$  away from  $\{x_k\}$ .

*Proof.* This proposition is a local version of the second author's [38] compactness theorem, and the proof is essentially the same. We refer the readers to [38, §4] for detail of the proof. But for convenience of reader, we would like to recall the key part of the proof to show how the index bound is used. Basically, the index bound is used to show the limit hypersurface  $\Sigma$  is smooth, and the convergence is smooth away from at most  $I$  points.

Since each  $\Sigma_j$  is minimal and has bounded volume, the Allard's compactness theorem tells us that a subsequence, still denoted by  $\Sigma_j$ , converges to  $\Sigma$  in the varifold sense with  $\Sigma$  stationary and integral. Without loss of generality, we can assume that each  $\Sigma_j$  has index  $I$ . Firstly, we can show that  $\Sigma \cap B_{2R}$  has at most  $I$  singular points. Suppose for contradiction that there has at least  $I + 1$  singular points  $\{x_k\}_{k=1}^{I+1} \subset \text{sing}(\Sigma \cap B_{2R})$ . Let  $\epsilon_0 = \frac{1}{2} \min\{\min_{k \neq l} d_g(x_k, x_l), \text{inj}(M)\}$ , where  $\text{inj}(M)$  denote the injectivity radius of  $M$ . The convergence is not smooth and graphical over  $\Sigma_i \cap B_{\epsilon_0}(x_k)$  for each  $k$ . Then Schoen-Simon's regularity theorem [39, Corollary 1] implies that there is a subsequence  $\Sigma_i$  and sufficiently large  $i_0$  such that for all  $i \geq i_0$ , the index of  $\Sigma_i \cap B_{\epsilon_0}(x_k)$  is at least one for all  $1 \leq k \leq I + 1$ . Since the  $I + 1$  subsets  $\Sigma_i \cap B_{\epsilon_0}(x_k)$  are mutually disjoint. From Lemma 3.1 in [38], we have that for each  $\Sigma_i$ , at least one  $B_{\epsilon_0}(x_k) \cap \Sigma_i$  has index zero, which is a contradiction.

Next, we can use the index bound to show that for each  $x_k \in \text{sing}(\Sigma \cap B_{2R})$ , there exists some  $\epsilon_k$  such that  $(B_{\epsilon_k}(x_k) \setminus \{x_k\}) \cap \Sigma$  is stable (see claim 2 in [38, §4]). Then

$$\mathcal{H}^{n-2}(\text{sing}(\Sigma \cap B_{2R})) \begin{cases} = 0, & \text{if } n \geq 3, \\ < \infty, & \text{if } n = 2, \end{cases}$$

which by the regularity results of Schoen-Simon [39, Corollary 1] (and [38, Remark 2.2]) implies that  $\Sigma \cap B_R$  is smooth.  $\square$

*Remark 3.1.* In the case of  $n = 2$ , the above proposition recovers Proposition 2.1 in Colding-Minicozzi's paper [8]. This follows since the local genus and area bound implies the index bound, thus the conclusion in [8, Proposition 2.1] follows from the above proposition.

Using Proposition 3.1 and a covering argument as in [8, 13], we can prove a global singular compactness theorem for complete  $f$ -minimal hypersurfaces.

**Proposition 3.2.** *Let  $(M^{n+1}, g, e^{-f} d\mu)$  be a complete smooth metric measure space with  $2 \leq n \leq 6$ . Given an integer  $I$  and a constant  $0 < \Lambda < \infty$ . Suppose that  $\Sigma_j \subset M$  is a sequence of smooth complete embedded  $f$ -minimal hypersurfaces with  $f$ -index at most  $I$ , without boundary and with weighted volume  $\int_{\Sigma_j} e^{-f} dv$  at most  $\Lambda$ . Then there exists a smooth embedded minimal hypersurface  $\Sigma \subset M$ , a subsequence of  $\Sigma_j$  and a locally finite collection of points  $\mathcal{S} \subset \Sigma$  such that  $\Sigma_j$  converges smoothly (possibly with finite multiplicity) to  $\Sigma$  away from  $\mathcal{S}$ .*

*Proof.* Recall that an  $f$ -minimal hypersurface is a minimal hypersurface under the conformal change of the ambient metric, see §2.2. Then each  $\Sigma_j$  is a minimal hypersurface in  $(M^{n+1}, \tilde{g})$ , where  $\tilde{g} = e^{-\frac{2}{n}f} g$  is the conformal metric on  $M$ . Let  $\tilde{B}_{2R}(p) \subset M$  denote the geodesic ball in  $(M, \tilde{g})$  of radius  $2R$  centred at some point  $p \in M$ . Since on each geodesic ball  $\tilde{B}_{2R}$ , the area of  $\Sigma_j \cap \tilde{B}_{2R}$  is at most  $\int_{\Sigma_j} d\tilde{v} = \int_{\Sigma_j} e^{-f} dv \leq \Lambda$ , and the index of  $\Sigma_j \cap \tilde{B}_{2R}$  is at most  $I$ , then Proposition 3.2 follows by covering  $M^{n+1}$  with a countable collection of small geodesic balls  $\tilde{B}_{2R}$ , applying Proposition 3.1 and passing to a diagonal subsequence. Moreover, Proposition 3.1 tells us the convergence is smooth away from a locally finite set  $\mathcal{S}$ , in fact on each  $\tilde{B}_R$ , the number of points  $\mathcal{S} \cap \tilde{B}_R$  is at most  $I$ .  $\square$

Now we can prove our main theorem.

*Proof of Theorem 1.3.* By the assumption  $Ric_f \geq \kappa > 0$  on  $M$ , we have that the fundamental group of  $M$  is finite, see [32, 40]. After passing to a covering argument, without loss of generality, we can assume that  $M$  is simply connected. Since each  $\Sigma_j$  has finite weighted volume, then each  $\Sigma_j$  is two-sided, see [13, Proposition 8]. From Proposition 3.2; thus to prove our theorem it remains to show the convergence is smooth everywhere. By Allard's regularity theorem, it suffices to check that the convergence has multiplicity one.

If the multiplicity of the convergence is greater than one, by following the same argument as in the proof of Proposition 3.2 in [8] (see also [38]), there exists a smooth positive function  $u$  on  $\Sigma$  satisfying  $L_f u = 0$ , which implies (see e.g., [8, p 471], [7, Lemma 1.36]) that  $\Sigma$  is a two-sided  $f$ -stable  $f$ -minimal hypersurface. However, there exists no complete two-sided  $f$ -stable  $f$ -minimal hypersurface without boundary and with finite weighted volume in complete smooth metric measure space  $(M^{n+1}, g, e^{-f} d\mu)$  satisfying  $Ric_f \geq \kappa > 0$ , see [13, Theorem 3]. This is a contradiction, so the multiplicity of the convergence must be one and the convergence is smooth everywhere.

Finally, we show that the  $f$ -index of the limit hypersurface  $\Sigma$  is at most  $I$  by adapting the argument in [38]. Suppose the  $f$ -index of  $\Sigma$  is at least  $I + 1$ , then there exists a sufficiently large  $R_1$  such that the  $f$ -index of  $B_\rho(p)$  is at least  $I + 1$  for all  $\rho \geq R_1$ , where  $B_\rho(p)$  is a geodesic ball on  $\Sigma$  of radius  $\rho$  centred at  $p \in \Sigma$ . There exist  $I + 1$  functions  $\{u_k\}_{k=1}^{I+1}$  compactly supported in  $B_{R_1}$  which are weighted  $L^2$  orthogonal, i.e.,

$$\int_{\Sigma} u_k u_l e^{-f} dv = \delta_{kl}, \quad k, l = 1, \dots, I + 1,$$

such that

$$Q_f^{\Sigma}(u_k, u_k) = \lambda_k, \quad \lambda_k < 0, \quad k = 1, \dots, I + 1.$$

Let  $\phi_k$  be an arbitrary  $C^1$  extension of  $u_k \nu$  to the rest of  $M$  and  $u_k^i = g(\phi_k, \nu_i)$ , where  $\nu_i$  is the unit normal vector of  $\Sigma_i$ . Then there exists sufficiently large  $i_0$  such that for all  $i \geq i_0$ , we have  $Q_f^{\Sigma_i}(u_k^i, u_k^i) \leq \frac{3}{4} \lambda_{I+1} < 0$  for each  $k$ , where we have ordered the eigenvalues such that  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_{I+1} < 0$ . Choose  $p_i \in \Sigma_i$  such that  $p_i \rightarrow p$  as  $i \rightarrow \infty$ . Let  $\xi(\rho) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a smooth cut-off function satisfying

$$\xi(\rho) = \begin{cases} 1, & \text{for } \rho \in [0, R_2], \\ 0 & \text{for } \rho \in [2R_2, \infty), \end{cases} \quad 0 \leq \xi \leq 1, \quad |\xi'| \leq C/R_2,$$

We can choose  $R_2 \gg R_1$  sufficiently large such that  $Q_f^{\Sigma_i}(\xi u_k^i, \xi u_k^i) \leq \frac{1}{2} \lambda_{I+1} < 0$  for each  $k$  and all  $i \geq i_0$ , where we view  $\xi(\rho)$  as a composite function of  $\xi$  and the distance function  $\rho$  on  $\Sigma_i$  from a fixed point  $p_i$ . Note that each  $\xi u_k^i$  is compactly supported in  $B_{2R_2}(p_i) \subset \Sigma_i$  and the  $f$ -index of  $\Sigma_i$  is at most  $I$ ,  $\{\xi u_k^i\}_{k=1}^{I+1}$  must be linearly dependent for all  $i \geq i_0$ . After passing to a subsequence and re-ordering, without loss of generality, we can assume that

$$\xi u_{I+1}^i = a_1^i \xi u_1^i + \dots + a_I^i \xi u_I^i, \quad |a_k^i| \leq 1, \quad 1 \leq k \leq I, \quad i \geq i_0. \quad (3.1)$$

Note that

$$\int_{\Sigma_i} |\phi_k^{\top i}|^2 e^{-f} dv_{\Sigma_i} \rightarrow 0, \quad \text{as } i \rightarrow \infty,$$

since we know that  $\Sigma_i$  converges to  $\Sigma$  smoothly everywhere, where  $\phi_k^\top$  means the tangential part of the vector field  $\phi_k$  on each  $\Sigma_i$ . Then

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\Sigma_i} \xi^2 u_k^i u_l^i e^{-f} dv_{\Sigma_i} &= \lim_{i \rightarrow \infty} \int_{\Sigma_i} \xi^2 (g(\phi_k, \phi_l) - g(\phi_k^\top, \phi_l^\top)) e^{-f} dv_{\Sigma_i} \\ &= \int_{\Sigma} u_k u_l e^{-f} dv = \delta_{kl}, \quad k, l = 1, \dots, I+1, \end{aligned} \quad (3.2)$$

where in the second equality we used the fact that  $u_k$  are compactly supported in  $B_{R_1}(p)$  and  $\xi \equiv 1$  in  $B_{R_2}(p)$ . Using (3.1) and (3.2), we have for all  $k = 1, 2, \dots, I$ ,

$$0 = \lim_{i \rightarrow \infty} \int_{\Sigma_i} \xi^2 u_k^i u_{I+1}^i e^{-f} dv_{\Sigma_i} = \lim_{i \rightarrow \infty} a_k^i. \quad (3.3)$$

Combining (3.1), (3.2) and (3.3), we obtain

$$\lim_{i \rightarrow \infty} \int_{\Sigma_i} (\xi u_{I+1}^i)^2 e^{-f} dv_{\Sigma_i} = \lim_{i \rightarrow \infty} \sum_{k=1}^I (a_k^i)^2 \int_{\Sigma_i} (\xi u_k^i)^2 e^{-f} dv_{\Sigma_i} = 0,$$

which contradicts (3.2). Thus the  $f$ -index of  $\Sigma$  is at most  $I$  and we have completed the proof of Theorem 1.3.  $\square$

#### 4. ESTIMATE ON THE UPPER BOUND OF THE $f$ -INDEX

Here we prove some upper bounds on compact  $f$ -minimal surfaces. We will use results of Ejiri-Micallef (when  $n = 2$ ) and Cheng-Tysk (when  $n \geq 3$ ) which together give the following Theorem.

**Theorem 4.1** ([12] [20]). *Let  $\Sigma^n \subset \tilde{M}^m$  be a closed minimally immersed manifold in a closed Riemannian manifold  $\tilde{M}$ . Then the index of  $\Sigma$  satisfies*

$$\text{ind}(\Sigma) \leq \begin{cases} C(n, \tilde{M})(m-n) \int_{\Sigma} \max\{1, |A|^2 + \text{Ric}_{\tilde{M}}(\nu, \nu)\}^{\frac{n}{2}} dv & \text{if } n \geq 3 \\ C(\tilde{M})m(\int_{\Sigma} dv + \gamma - 1) & \text{if } n = 2, \end{cases} \quad (4.1)$$

where  $C(\tilde{M})$  is a constant depending on an isometric embedding of  $\tilde{M}$  into some Euclidean space.

*Remark 4.1.* We point out here that if  $\tilde{M}$  is a compact manifold with boundary and  $\Sigma^n \subset \tilde{M}$  is closed, then the conclusion of the above theorem is still true. This follows by exactly the same proof as the above authors', except we have to isometrically embed our manifold with boundary into Euclidean space so that the second fundamental form is uniformly bounded in  $L^\infty$  - but this follows easily since the Nash embedding theorem holds for manifolds with boundary and  $\tilde{M}$  is compact.

*Proof of Proposition 1.6.* Under the assumptions of Proposition 1.6 we can let  $(\tilde{M}, \tilde{g}) = (K, e^{-\frac{2f}{n}} g)$  and apply Theorem 4.1 (and the preceding remark) to give

$$\widetilde{\text{ind}}(\Sigma) \leq \begin{cases} C(n, \tilde{M}) \int_{\Sigma} \max\{1, |\tilde{A}|_{\tilde{g}}^2 + \widetilde{\text{Ric}}_{\tilde{M}}(\tilde{\nu}, \tilde{\nu})\}^{\frac{n}{2}} d\tilde{v} & \text{if } n \geq 3 \\ C(\tilde{M})(\int_{\Sigma} d\tilde{v} + \gamma - 1) & \text{if } n = 2. \end{cases} \quad (4.2)$$

It is an easy exercise using (2.17) to finish the proof.  $\square$

We will now prove an estimate on the  $f$ -index of an  $f$ -minimal hypersurface which is not necessarily compact with  $n \geq 3$ . We do this by following the methods of Cheng-Tysk [12]. Firstly, we recall a Sobolev inequality in the weighted setting proved by Impera-Rimoldi [25].

**Theorem 4.2** ([25]). *Let  $\Sigma^n$  be an immersed hypersurface in  $(M^{n+1}, g, e^{-f}d\mu)$ . Suppose that the sectional curvature  $\text{sect}(M) \leq 0$  and there exists a constant  $c_n > 0$  such that*

$$\limsup_{\rho \rightarrow 0^+} \frac{\text{Vol}_f(S_\rho(\xi))}{\rho^n} \geq c_n, \quad (4.3)$$

for all  $\xi \in \Sigma$ , where  $S_\rho(\xi)$  denotes

$$S_\rho(\xi) = \{x \in \Sigma \mid \text{dist}_M(x, \xi) \leq \rho\}.$$

Then there exists a constant  $C_n$  such that for any nonnegative compactly supported  $C^1$  function  $\varphi$  on  $\Sigma$ , we have

$$\left( \int_\Sigma \varphi^{\frac{n}{n-1}} e^{-f} dv \right)^{\frac{n-1}{n}} \leq C_n \int_\Sigma (|\nabla \varphi| + \varphi(|H_f| + |\overline{\nabla} f|)) e^{-f} dv. \quad (4.4)$$

The condition 4.3 is satisfied if we assume that  $\sup_\Sigma f < \infty$ , since

$$\limsup_{\rho \rightarrow 0^+} \frac{\text{Vol}(S_\rho(\xi))}{\rho^n} \geq \omega_n,$$

for all  $\xi \in \Sigma$ , where  $\omega_n$  denotes the volume of  $n$ -dimensional unit ball in  $\mathbb{R}^n$ . We remark that Batista-Mirandola [3] also obtained a similar Sobolev inequality in the weighted setting independently.

The proof of Theorem 4.1 in [12] used the heat kernel and Michael-Simon Sobolev inequality, inspired by a similar argument in Li-Yau's paper [31]. In the following, we will adapt the idea in [12] to obtain an upper bound on the  $f$ -index of  $f$ -minimal hypersurface.

**Theorem 4.3.** *Suppose  $n \geq 3$ , and let  $\Sigma^n$  be a smooth complete  $f$ -minimal hypersurface in  $(M^{n+1}, g, e^{-f}d\mu)$ , where  $M$  satisfies  $\text{sect}(M) \leq 0$  and (4.3) as in Theorem 4.2, and  $|\overline{\nabla} f| \leq c_1$  on  $\Sigma$ . Then the  $f$ -index of  $\Sigma$  satisfies*

$$\text{ind}_f(\Sigma) \leq C(n, M, c_1) \int_\Sigma (\max\{\frac{1}{2}, |A|^2 + \text{Ric}_f(\nu, \nu)\})^{\frac{n}{2}} e^{-f} dv, \quad (4.5)$$

where  $C(n, M, c_1)$  is a uniform constant depending on  $n, M, c_1$ .

*Proof.* The proof is similar as that in [12, Theorem 1] (see also [31]), so we just sketch it. Assume that  $\Sigma$  is orientable, otherwise we pass to the orientable double cover i.e. replace  $\Sigma$  by the double cover  $\tilde{\Sigma}$ , thus inducing an immersion of an orientable hypersurface. Now we can pull back the second variation operator onto  $\tilde{\Sigma}$  for which we estimate the  $f$ -index via the below method. Finally notice that the  $f$ -index of  $\Sigma$  is bounded from above by that of  $\tilde{\Sigma}$ .

Recall that the  $L_f$  operator is

$$L_f = \Delta_f + |A|^2 + \text{Ric}_f(\nu, \nu).$$

Consider the weighted manifold  $(\Sigma^n, g, e^{-f}dv)$ . Denote  $p(x) = \max\{\frac{1}{2}, |A|^2 + \text{Ric}_f(\nu, \nu)\}$  and  $D \subset \Sigma$  any compact domain. We know that (see [27]) there exists a unique heat kernel  $H(x, y, t)$ , a  $C^\infty$  function on  $\Sigma \times \Sigma \times \mathbb{R}^+$ , of the operator

$$\frac{\partial}{\partial t} - \frac{1}{p(x)} \Delta_f$$

on  $D$ , which satisfies Dirichlet boundary condition if  $\partial D \neq \emptyset$ . Let  $\{\lambda_i\}_{i=0}^\infty$  be the Dirichlet eigenvalues of  $\frac{1}{p(x)} \Delta_f$  on  $D$ . Define

$$h(t) = \sum_{i=0}^{\infty} e^{-2\lambda_i t}.$$

Then

$$h(t) = \int_D \int_D H^2(x, y, t) p(x) p(y) e^{-f(x)} dv(x) e^{-f(y)} dv(y).$$

Differentiation in  $t$  and integration by parts yield

$$\frac{dh}{dt} = -2 \int_D \int_D |\nabla_y H(x, y, t)|^2 p(x) e^{-f(x)} dv(x) e^{-f(y)} dv(y).$$

Using the Hölder inequality and the property of heat kernel, by arguing as the proof of Theorem 1 in [12], we have

$$\begin{aligned} h^{\frac{n+2}{n}}(t) & \left( \int_D p(x)^{\frac{n}{2}} e^{-f(x)} dv(x) \right)^{-\frac{2}{n}} \\ & \leq \int_D p(x) \left( \int_D H^{\frac{2n}{n-2}}(x, y, t) e^{-f(y)} dv(y) \right)^{\frac{n-2}{n}} e^{-f(x)} dv(x). \end{aligned} \quad (4.6)$$

Replacing  $\varphi = H^{\frac{2(n-1)}{n-2}}$  in the weighted Sobolev inequality (4.4), and applying the Hölder-inequality, we obtain

$$\left( \int_{\Sigma} H^{\frac{2n}{n-2}} e^{-f} dv \right)^{\frac{n-2}{n}} \leq C \int_{\Sigma} (|\nabla H|^2 + c_1^2 H^2) e^{-f} dv, \quad (4.7)$$

where we used the fact  $H_f = 0$  since  $\Sigma$  is  $f$ -minimal; and  $|\bar{\nabla} f| \leq c_1$  on  $M$ . Applying (4.7) to (4.6), we obtain

$$h^{\frac{n+2}{n}}(t) \left( \int_D p(x)^{\frac{n}{2}} e^{-f(x)} dv(x) \right)^{-\frac{2}{n}} \leq -\frac{1}{2} C \frac{dh}{dt} + 2C c_1^2 h(t), \quad (4.8)$$

where we used the fact  $p(y) \geq 1/2$  for all  $y \in \Sigma$ . From the differential inequality (4.8), using the fact that  $h(t) \rightarrow \infty$  as  $t \rightarrow 0^+$ , we have

$$h(t) \leq \frac{(2C)^{\frac{n}{2}} c_1^n}{(1 - e^{-\frac{8t}{n} c_1^2})^{\frac{n}{2}}} \int_{\Sigma} p(x)^{\frac{n}{2}} e^{-f(x)} dv(x).$$

As in [12, 31], we have

$$ind_f(D) \leq \inf_{t>0} e^{2t} h(t) \leq C \int_{\Sigma} p(x)^{\frac{n}{2}} e^{-f(x)} dv(x),$$

where  $C$  is a constant independent of  $D$ . Since  $ind_f(\Sigma) = \sup_{D \subset \Sigma} ind_f(D)$ , this finishes the proof of Theorem 4.3.  $\square$

*Remark 4.2.* From the formula (2.17), we can see that although  $f$ -minimal hypersurface can be viewed as a minimal hypersurface after the conformal change of the ambient metric, the conclusion in Theorem 4.3 can not follow directly from Cheng-Tsyk's result [12] after the conformal transformation. The conditions on  $M$  and  $f$  assumed in Theorem 4.3 are only used to guarantee that the Sobolev inequality (4.7) holds. Thus the index upper bound (4.5) also holds for  $f$ -minimal hypersurfaces  $\Sigma$  in  $(M^{n+1}, g, e^{-f} d\mu)$  if (4.7) holds on  $\Sigma$ .

Theorem 4.3 and Proposition 1.6 do not cover the general complete non-compact  $f$ -minimal hypersurfaces and self-shrinkers. We know that if a self-shrinker  $\Sigma^n \subset \mathbb{R}^{n+1}$  arises from the Type-I singularity of mean curvature flow, then the second fundamental form of  $\Sigma$  must be bounded and  $\Sigma$  has at most Euclidean volume growth. Then  $\Sigma$  has finite weighted volume, and the right-hand side of (1.5) is bounded above. Thus, if the conclusion in Proposition 1.5 also holds for general complete non-compact self-shrinker, then we can obtain an upper bound on the  $f$ -index of the self-shrinker arising from Type-I singularity, and Corollary 1.4 would imply that the space of Type-I singularities of the mean curvature flow is compact in smooth topology with multiplicity one. About the lower

bound of the  $f$ -index, there is one result by Hussey [24] for self-shrinkers. In his thesis, Hussey proved that a flat hyperplane  $\mathbb{R}^n$  through the origin, as a self-shrinker, has index 1; the cylinder  $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ ,  $1 \leq k \leq n$  has index  $n + 2$ ; any other smooth complete embedded self-shrinker without boundary and with polynomial volume growth has index at least  $n + 3$  - see [24] for details.

## REFERENCES

- [1] S. Angenent, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states (Gregynog 1989), Progr. Nonlinear Differential Equations Appl. vol. 7, Birkhäuser, Boston, 1992.
- [2] Simon Brendle, *Embedded self-similar shrinkers of genus 0*, Annals of Mathematics, **183**, no 2 (2016): 715–728.
- [3] Marcio Batista and Heudson Mirandola, *Sobolev and isoperimetric inequalities for submanifolds in weighted ambient spaces*, Annali di Matematica, online first, doi:10.1007/s10231-014-0449-8.
- [4] Huai-Dong Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, 1-38, Adv. Lect. Math., **11**, Int. Press, Somerville, MA, 2010.
- [5] Huai-Dong Cao and Haizhong Li, *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*, Calc. Var. PDE, **46**, no.3–4 (2013): 879–889.
- [6] Marcos P. Cavalcante and Jose M. Espinar, *Halfspace type Theorems for self-shrinkers*, Bull. London Math. Soc., **48** (2016), 242–250.
- [7] Tobias H. Colding and William P. Minicozzi II, *A course in minimal surfaces*, Graduate studies in Math., vol **121**, Amer. Math. Soc., Providence, Rhode Island, 2011.
- [8] Tobias H. Colding and William P. Minicozzi II, *Smooth compactness of self-shrinkers*, Comm. Math. Helv., **87**(2012), no. 2, 463–475.
- [9] Tobias H. Colding and William P. Minicozzi II, *Generic mean curvature flow I; generic singularities*, Annals of Math., **175**(2012), no. 2, 755–833.
- [10] Hyeon In Choi and Richard Schoen, *The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature*, Invent. Math., **81** (1985), 387–394.
- [11] H. I. Choi and A. N. Wang, *A first eigenvalue estimate for minimal hypersurfaces*, J. Differential Geom. **18**(3) (1983), 559–562.
- [12] Shiu-Yuen Cheng and Johan Tysk, *Schrödinger operators and index bounds for minimal submanifolds*, Rocky Mountain J. of Mathematics, **24**, no. 3 (1994), 977–996.
- [13] Xu Cheng, Tito Mejia and Detang Zhou, *Stability and compactness for complete  $f$ -minimal surfaces*, Trans. of Amer. Math. Soc., **367** (2015), 4041–4059.
- [14] Xu Cheng, Tito Mejia and Detang Zhou, *Eigenvalue estimate and compactness for closed  $f$ -minimal surfaces*, Pacific J. Math., **271** (2014), no. 2, 347–367.
- [15] Xu Cheng and Detang Zhou, *Stability properties and gap theorem for complete  $f$ -minimal hypersurfaces*, Bulletin of the Brazilian Mathematical Society, **46**(2) (2015), 251–274.
- [16] Qi Ding and Yuanlong Xin, *Volume growth, eigenvalue and compactness for self-shrinkers*, Asian J. Math., **17** (2013), no. 3, 443–456.
- [17] G. Drugan, *An immersed  $S^2$  self-shrinker*, Trans. Amer. Math. Soc., **367**(5) (2015), 3139–3159.
- [18] G. Drugan and S.J. Kleene, *Immersed self-shrinkers*, [arXiv:1306.2383](https://arxiv.org/abs/1306.2383).
- [19] Jose M. Espinar, *Manifolds with density, applications and gradient Schrödinger operators*, [arXiv:1209.6162](https://arxiv.org/abs/1209.6162).
- [20] N. Ejiri and M. Micallef, *Comparison between second variation of area and second variation of energy of a minimal surface*, Adv. Calc. Var. **1**(3) (2008), 223–239.
- [21] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three manifolds*, Invent. Math. **82** (1985), no. 1, 121–132.
- [22] D. Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature*, Communications on Pure and Applied Mathematics, **33**, no.2 (1980): 199–211.
- [23] P. T. Ho, *The structure of  $\phi$ -stable minimal hypersurfaces in manifolds of nonnegative  $P$ -scalar curvature*, Math. Ann. **392**, (2010): 319–332.
- [24] Caleb Hussey, *Classification and Analysis of Low Index Mean Curvature Flow Self-Shrinkers*. Doctoral dissertation, Johns Hopkins University, 2012.
- [25] Debora Impera, Michele Rimoldi, *Stability properties and topology at infinity of  $f$ -minimal hypersurfaces*, Geom. Dedicata, **178**, no.1 (2015): 21–47.
- [26] N. Kapouleas, S.J. Kleene and N.M. Møller, *Mean curvature self-shrinkers of high genus: non-compact examples*, to appear in J. Reine Angew. Math., [arXiv:1106.5454](https://arxiv.org/abs/1106.5454).
- [27] Alexander Grigoryan, *Heat kernels on weighted manifolds and applications*, Cont. Math **398** (2006): 93–191.

- [28] Gang Liu, *Stable weighted minimal surfaces in manifolds with nonnegative Bakry-Emery tensor*, *Comm. Anal. Geom.*, **21** (2013), no.5, 1061–1079.
- [29] Haizhong Li and Yong Wei, *Classification and rigidity of self-shrinkers in the mean curvature flow*, *J. Math. Soc. Japan*, 2014, 66(3):709–734.
- [30] Haizhong Li and Yong Wei,  *$f$ -minimal surface and manifold with positive  $m$ -Bakry-Émery Ricci curvature*, *J. Geom. Anal.*, Vol 25, issue 1 (2015), 421–435.
- [31] Peter Li and Shing-Tung Yau, *On the Schrödinger equation and the eigenvalue problem*. *Comm. Math. Phys.*, **88**, no.3 (1983): 309–318.
- [32] Frank Morgan. *Manifolds with density*. *Notices of the AMS* (2005): 853–858.
- [33] N. M. Møller, *Closed self-shrinking surfaces in  $\mathbb{R}^3$  via the torus*. [arXiv:1111.7318](https://arxiv.org/abs/1111.7318)
- [34] Ovidiu Munteanu and Jiaping Wang, *Smooth metric measure spaces with non-negative curvature*, *Comm. Anal. Geom.*, **19** (2011), no.5, 451–486.
- [35] X. H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. Part III*, *Duke Math. J.*, **163**, no. 11, 2014: 2023–2056.
- [36] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, [arXiv math/0211159](https://arxiv.org/abs/math/0211159).
- [37] Stefano Pigola and Michele Rimoldi, *Complete self-shrinkers confined into some regions of the space*, *Annals of Global Analysis and Geometry*, **45** no.1 (2014): 47–65.
- [38] Ben Sharp, *Compactness of minimal hypersurfaces with bounded index*, [arXiv: 1501.02703](https://arxiv.org/abs/1501.02703), accepted *J. Diff. Geom.*
- [39] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, *Comm. Pure Appl. Math.* **34** (1981), 741–797.
- [40] Guofang Wei and Will Wylie, *Comparison geometry for the Bakry-Émery Ricci tensor*, *J. Diff. Geom.* **83**(2009), 377–405.
- [41] Lu Wang, *A Bernstein type theorem for self-similar shrinkers*, *Geom. Dedicata*, **151** (2011), 297–303.

UNIVERSIDADE FEDERAL DE MINAS GERAIS (UFMG), DEPARTAMENTO DE MATEMÁTICA, CAIXA POSTAL 702, 30123-970, BELO HORIZONTE, MG, BRAZIL  
*E-mail address:* ezequiel@mat.ufmg.br

UNIVERSITY OF WARWICK, DEPARTMENT OF MATHEMATICS, COVENTRY, CV4 7AL, UK  
*E-mail address:* b.sharp@warwick.ac.uk

MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, JOHN DEDMAN BUILDING 27, UNION LANE, CANBERRA ACT 2601, AUSTRALIA  
*E-mail address:* yong.wei@anu.edu.au