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MULTIPLICITY BOUNDS IN PRIME CHARACTERISTIC

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Dedicated to Gennady Lyubeznik on the occasion of his sixtieth birthday

Abstract. We extend a result by Huneke and Watanabe ([HW15]) bounding the multiplicity of F-pure local rings of prime characteristic in terms of their dimension and embedding dimensions to the case of F-injective, generalized Cohen-Macaulay rings. We then produce an upper bound for the multiplicity of any local Cohen-Macaulay ring of prime characteritic in terms of their dimensions, embedding dimensions and HSL numbers. Finally, we extend the upper bounds for the multiplicity of generalized Cohen-Macaulay rings in characteristic zero which have dense F-injective type.

1. Introduction

In [HW15], Huneke and Watanabe proved that, if R is a noetherian, F-pure local ring of dimension d and embedding dimension v, then $e(R) \leq \binom{v}{d}$ where $e(R)$ denotes the Hilbert-Samuel multiplicity of R. The following was left as an open question in [HW15, Remark 3.4]:

Question 1.1 (Huneke-Watanabe)**.** Let R be a noetherian F-injective local ring with dimension d and embedding dimension v. Is it true that $e(R) \leq \binom{v}{d}$?

In this note, we answer this question in the affirmative when R is generalized Cohen-Macaulay.

Theorem 1.1. *Let R be a d-dimensional noetherian F-injective generalized Cohen-Macaulay local ring of embedding dimension v. Then*

$$e(R) \leq \binom{v}{d}.$$

Using reduction mod p, one can prove an analogous result for generalized Cohen-Macaulay rings of dense F-injective type in characteristic 0, *cf.* Theorem 5.2.

We also generalize these result to Cohen-Macaulay, non-F-injective rings as follows.

Definition 1.2 (cf. section 4 in [Lyu97])**.** Let A be a commutative ring and let H be an A-module with Frobenius map $\theta : H \to H$ (i.e., an additive map such that $\theta(ah) = a^p\theta(h)$ for all $a \in A$ and $h \in H$). Write $\operatorname{Nil} H = \{h \in H \mid \theta^e h = 0 \text{ for some } e \geq 0\}$. The *Hartshorne-Speiser-Lyubeznik number* (henceforth *abbreviated HSL number*) is defined as

$$\inf\{e \geq 0 \mid \theta^e \operatorname{Nil} H = 0\}.$$

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The HSL number of a local, Cohen-Macaulay ring $(R,\mathfrak{m})$ is defined as the HSL number of the top local cohomology module $\mathrm{H}^{\dim R}_{\mathfrak{m}}(R)$ with its natural Frobenius map.

For artinian modules over a quotient of a regular ring, HSL numbers are finite. ([Lyu97, Proposition 4.4]).

Without the F-injectivity assumption, we have the following upper bound in the Cohen-Macaulay case which involves the HSL number of R.

Theorem 1.2 (Theorem 3.1)**.** *Assume that $(R,\mathfrak{m})$ is a reduced, Cohen-Macaulay noetherian local ring of dimension d and embedding dimension v. Let η be the HSL number of R and write $Q = p^\eta$. Then*

$$e(R) \leq Q^{v-d}\binom{v}{d}.$$

This bound is asymptotically sharp as shown in Remark 3.2.

2. Bounds on F-injective rings

For each commutative noetherian ring R, let R^o denote the set of elements of R that are not contained in any minimal prime ideal of R.

Remark 2.1. If R is a reduced noetherian ring, then each $c \in R^o$ is a non-zero-divisor.

Given any local ring $(R,\mathfrak{m})$, we can pass to $S = R[x]_{\mathfrak{m}\,R[x]}$ which admits an infinite residue field: this does not affect the multiplicity, dimension, embedding dimension and Cohen-Macaulyness (cf. [HS06, Lemma 8.4.2]). In addition, since S is a faithfully flat extension of R, $\mathrm{H}^i_{\mathfrak{m}\,S}(S) = \mathrm{H}^i_{\mathfrak{m}}(R) \otimes_R S$ and, if $\phi_i : \mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(R)$ is the natural Frobenius map induced by the Frobenius map $r \mapsto r^p$ on R, then the natural Frobenius map on $\mathrm{H}^i_{\mathfrak{m}\,S}(S)$ takes an element $a \otimes x^\alpha$ to $\phi_i(a) \otimes x^{\alpha p}$. Therefore, passing to S preserves HSL numbers (and hence also F-injectivity). Therefore, for the purpose of seeking an upper bound of multiplicity, we may assume that that all mentioned local rings $(R,\mathfrak{m})$ have infinite residue fields; consequently, $\mathfrak{m}$ admits a minimal reduction generated by $\dim R$ elements (cf. [HS06, Proposition 8.3.7]).

We begin with a Skoda-type theorem for F-injective rings which may be viewed as a generalization of [HW15, Theorem 3.2].

Theorem 2.2. *Let $(R,\mathfrak{m})$ be a commutative noetherian ring of characteristic p and let $\mathfrak{a}$ be an ideal that can be generated by ℓ elements. Assume that each $c \in R^o$ is a non-zero-divisor. Then*

$$\overline{\mathfrak{a}^{\ell+1}} \subseteq \mathfrak{a}^F,$$

where $\overline{\mathfrak{a}^{\ell+1}}$ is the integral closure of $\mathfrak{a}^{\ell+1}$ and $\mathfrak{a}^F$ the Frobenius closure of $\mathfrak{a}$.

Proof. For each $x \in \overline{\mathfrak{a}^{\ell+1}}$ pick $c \in R^o$ such that for $N \gg 1$, $cx^N \in \mathfrak{a}^{(\ell+1)N}$ ([HS06, Corollary 6.8.12]). Note that c is a non-zero-divisor by our assumptions. We have $cx^N \in c(\mathfrak{a}^{(\ell+1)N} : c) \subseteq cR \cap \mathfrak{a}^{(\ell+1)N}$. An application of the Artin-Rees Lemma gives a $k \geq 1$ such that $cx^N \in c\,\mathfrak{a}^{(\ell+1)N-k}$ for all large N, and so $x^N \in \mathfrak{a}^{(\ell+1)N-k}$ for all large N. For any large enough $N = p^e$ we have $x^{p^e} \in \mathfrak{a}^{[p^e]}$, i.e., x, and hence $\overline{\mathfrak{a}^{d+1}}$ is in the Frobenius closure of $\mathfrak{a}$. $\square$

Corollary 2.3. *Let $(R,\mathfrak{m})$ be a d-dimensional noetherian local ring of characteristic p. Assume that $\mathfrak{m}$ admits a minimal reduction J. Then*

(a) $\mathfrak{m}^{d+1} \subseteq \overline{\mathfrak{m}^{d+1}} = \overline{J^{d+1}} \subseteq J^F$, *and*
(b) $e(R) \leq \binom{v}{d} + \ell(J^F/J)$.

Proof. Since $\mathfrak{m}^{d+1} \subseteq \overline{\mathfrak{m}^{d+1}} = \overline{J^{d+1}}$, (a) follows from Theorem 2.2.

For part (b), since $\overline{J^{d+1}} \subseteq J^F$ and J is generated by d elements, we have $\ell(R/J^F) \leq \binom{v}{d}$ (as in the proof of [HW15, Theorem 3.1]). Then

$$e(R) \leq \ell(R/J) = \ell(R/J^F) + \ell(J^F/J) \leq \binom{v}{d} + \ell(J^F/J).$$

□

Proof of Theorem 1.1. Let $\hat{R}$ denote the completion of R. Then R is F-injective and generalized Cohen-Macaulay if and only if $\hat{R}$ is so, and $e(R) = e(\hat{R})$. Hence we may assume that R is complete. Since R is F-injective, it is reduced ([SZ13, Remark 2.6]) and hence each $c \in R^o$ is a non-zero-divisor by Remark 2.1. It is proved in [Ma15, Theorem 1.1] that a generalized Cohen-Macaulay local ring is F-injective if and only if every parameter ideal is Frobenius closed. Let J denote a minimal reduction of $\mathfrak{m}$, then $J^F = J$. Our theorem follows immediately from Corollary 2.3. □

3. Bounds on multiplicity using HSL numbers

Theorem 3.1. *Assume that $(R,\mathfrak{m})$ is a reduced, Cohen-Macaulay noetherian local ring of dimension d and embedding dimension v. Let η be the HSL number of R and write $Q = p^\eta$. Then $e(R) \leq Q^{v-d}\binom{v}{d}$.*

Proof. We may assume that R is complete since $e(R) = e(\hat{R})$. Hence $\mathfrak{m}$ admits a minimal reduction J (generated by d elements). We have $e(R) = \ell(R/J)$, and Theorem 2.2 shows that $\mathfrak{m}^{d+1} \subseteq J^F$. Now $\left(J^F\right)^{[Q]} = J^{[Q]}$ for $Q = p^\eta$ hence $\left(\mathfrak{m}^{d+1}\right)^{[Q]} \subseteq J^{[Q]}$.

Extend a set of minimal generators $x_1, \dots, x_d$ of J to a minimal set of generators $x_1, \dots, x_d, y_1, \dots y_{v-d}$ of $\mathfrak{m}$. Now $R/J^{[Q]}$ is spanned by monomials

$$x_1^{\gamma_1} \dots x_d^{\gamma_d} y_1^{\alpha_1 Q + \beta_1} \dots y_{v-d}^{\alpha_{v-d} Q + \beta_{v-d}}$$

where $0 \leq \gamma_1, \dots \gamma_d, \beta_1, \dots, \beta_{v-d} < Q$ and $0 \leq \alpha_1 + \dots + \alpha_{v-d} < d+1$. The number of such monomials is $Q^v\binom{v}{d}$ and so $\ell(R/J^{[Q]}) \leq Q^v\binom{v}{d}$.

Note that as J is generated by a regular sequence, $\ell(R/J^{[Q]}) = Q^d \ell(R/J)$ and we conclude that

$$\ell(R/J) = \ell(R/J^{[Q]})/Q^d \leq Q^{v-d}\binom{v}{d}.$$

□

Remark 3.2. The next family of examples shows that the bound in Theorem 3.1 is asymptotically sharp.

Let $\mathbb{F}$ be a field of prime characteristic p, let $n \geq 2$, and let S be $\mathbb{F}[x_1, \dots, x_n]$. Let $\mathfrak{m} = (x_1, \dots, x_n)S$, and let E denote the injective hull of the residue field of $S_\mathfrak{m}$.

Define $f=\sum_{i=1}^n x_1^p \dots x_{i-1}^p x_i x_{i+1}^p \dots x_n^p$ and $h=x_1\dots x_{n-1}$. We claim that f is square-free: if this is not the case write $f=r^\alpha s$ where r is irreducible of positive degree, and $\alpha\geq 2$. Let ∂ denote the partial derivative with respect to x_n. Note that $\partial f = h^p$ and so

$$h^p=\alpha r^{\alpha-1}(\partial r)s + r^\alpha(\partial s) = r^{\alpha-1}\left(\alpha(\partial r)s + r(\partial s)\right).$$

We deduce that r divides h, but this would imply that x_i^2 divides all terms of f for some $1\leq i\leq n-1$, which is false. We conclude that S/fS is reduced.

Let R be the localization of S/fS at $\mathfrak{m}$. We compute next the HSL number η of R using the method described in sections 4 and 5 in [Kat08]. It is not hard to show that $\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\cong \operatorname{ann}_E f$ where $E=\mathrm{H}^n_{\mathfrak{m}}(S)$, and that, after identifying these, the natural Frobenius action on $\operatorname{ann}_E f$ is given by $f^{p-1}T$ where T is the natural Frobenius action on E.

To find the HSL number η of $\mathrm{H}^{n-1}_{\mathfrak{m}}(R)$ we readily compute $I_1(f)$ (cf. [Kat08, Proposition 5.4]) to be the ideal generated by $\{x_1\dots x_{i-1}x_{i+1}\dots x_n \,|\, 1\leq i\leq n\}$ and

$$\begin{aligned} I_2(f^{p+1}) &= I_1\left(f I_1(f)\right)\\ &= \sum_{i=1}^n I_1\Big(\sum_{j=1}^{i-1} x_1^{p+1}\dots x_{j-1}^{p+1}x_j^2x_{j+1}^{p+1}\dots x_{i-1}^{p+1}x_i^p x_{i+1}^{p+1}\dots x_n^{p+1}\\ &+ x_1^{p+1}\dots x_{i-1}^{p+1}x_i x_{i+1}^{p+1}\dots x_n^{p+1}\\ &+ \sum_{j=i+1}^{n} x_1^{p+1}\dots x_{i-1}^{p+1}x_i^p x_{i+1}^{p+1}\dots x_{j-1}^{p+1}x_j^2x_{j+1}^{p+1}\dots x_n^{p+1}\Big)\\ &= I_1(f)\end{aligned}$$

and we deduce that $\eta=1$.

We now compute

$$\Gamma_{n,p} := \frac{\deg f}{\binom{n}{n-1}p^\eta} = \frac{(n-1)p+1}{np}.$$

We have $\lim_{n\to\infty}\Gamma_{n,p}=1$ and $\lim_{p\to\infty}\Gamma_{n,p}=(n-1)/n$, so we can find values of $\Gamma_{n,p}$ arbitrarily close to 1.

4. Examples

The injectivity of the natural Frobenius action on the top local cohomology $H^d_{\mathfrak{m}}(R)$ does *not* imply $e(R)\leq\binom{v}{d}$ as shown by the following example.

Example 4.1. Let $S=\mathbb{Z}/2\mathbb{Z}[x,y,u,v]$, let $\mathfrak{m}$ be its ideal generated by the variables, define $I=(v,x)\cap(u,x)\cap(v,y)\cap(u,y)\cap(y,x)\cap(v,u)\cap(y-u,x-v) = (xv(y-u), yu(x-v), yuv(y-u), xuv(x-v))$, and let $R=S/I$: this is a reduced 2-dimensional ring.

We compute the following graded S-free resolution of I

$$0\longrightarrow S(-6)\xrightarrow{B} S^4(-5)\xrightarrow{A} S^2(-3)\oplus S^2(-4)\longrightarrow I\longrightarrow 0$$

where

$$A = \begin{bmatrix} u(x-v) & yu & 0 & 0 \\ 0 & 0 & xv & v(y-u) \\ 0 & -x & 0 & v-x \\ u-y & 0 & -y & 0 \end{bmatrix}, \quad B = \begin{bmatrix} y \\ v-x \\ u-y \\ x \end{bmatrix}$$

and note that R has projective dimension 3, hence depth 1 and so it is not Cohen-Macaulay. Also, we can read the Hilbert series of R from its graded resolution and we obtain

$$\frac{1-2t^3-2t^4+4t^5-t^6}{(1-t)^4} = \frac{1+2t+3t^2+2t^3-t^4}{(1-t^2)}$$

and so the multiplicity of R is $1+2+3+2-1=7$ exceeding $\binom{4}{2}=6$ (cf. [HH11, §6.1.1].)

Note that R is not F-injective, but the natural Frobenius action on the top local cohomology module is injective.

From the proof of Theorem 1.1, we can see that if a minimal reduction of the maximal ideal in an F-injective local ring R is Frobenius closed then the bound $e(R) \leq \binom{v}{d}$ will hold. Hence we may ask whether minimal reductions would be Frobenius-closed in such rings (cf. Theorem 6.5 and Problem 3 in [QS17]). However, the following example shows this not to be the case.

Example 4.2. Let $S = \mathbb{Z}/2\mathbb{Z}[x,y,u,v,w]$, let $\mathfrak{m}$ be its ideal generated by the variables and let $I_1 = (x,y) \cap (x+y, u+w, v+w)$, $I_2 = (u,v,w) \cap (x,u,v) \cap (y,u,v) = (u,v,xyw)$, and $I = I_1 \cap I_2$. Fedder's Criterion [Fed83, Proposition 1.7] shows that S/I_1, S/I_2 and $S/(I_1+I_2)$ are F-pure, and [QS17, Theorem 5.6] implies that S/I is F-injective. Also, S/I is almost Cohen-Macaulay: it is 3-dimensional and its localization at $\mathfrak{m}$ has depth 2.

Its not hard to check that the ideal J generated by the images in S/I of $w, y+v, x+u$ is a minimal reduction. However $J^F \neq J$: while $v^2 \notin J$, we have

$$v^4 = xyw^2 + v^2(y+v)^2 + yvw(x+y) + (v+w)(y^2v+xyw),$$

hence $v^2 \in J^F \setminus J$.

5. Bounds in Characteristic zero

Throughout this section K will denote a field of characteristic zero, $T = K[x_1, \dots, x_n]$, R will denote the finitely generated K-algebra $R = T/I$ for some ideal $I \subseteq T$, and $\mathfrak{m} = (x_1, \dots, x_n)R$; d and v will denote the dimension and embedding dimension, respectively, of $R_\mathfrak{m}$. We also choose $\mathbf{y} = y_1, \dots, y_d \in \mathfrak{m}$ whose images in $R_\mathfrak{m}$ form a minimal reduction of $\mathfrak{m}R_\mathfrak{m}$.

We may, and do assume that the only maximal ideal containing $\mathbf{y}$ is $\mathfrak{m}$. Otherwise, if $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ are all the maximal ideals distinct from $\mathfrak{m}$ which contain $\mathbf{y}$, we can pick $f \in (\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_t) \setminus \mathfrak{m}$, and now the only maximal ideal containing $\mathbf{y}$ in R_f is $\mathfrak{m} R_f$. We may now replace R with $R' = K[x_1, \dots, x_n, x_{n+1}]/I + \langle x_{n+1}f - 1 \rangle \cong R_f$ and since $R_\mathfrak{m} = (R_f)_\mathfrak{m}$ we are not affecting any local issues.

The main tool used in this section descent techniques described in [HH06]. We start by introducing a flavour of it useful for our purposes.

Definition 5.1. By *descent objects* we mean

(1) a finitely generated K-algebra R as above,
(2) a finite set of finitely generated T-modules,

(3) a finite set of T linear maps between T-modules in (2),
(4) a finite set of finite complexes involving maps in (3),

By *descent data* for these descent objects we mean

(a) A finitely generated $\mathbb{Z}$-subalgebra A of K, $T_A = A[x_1, \ldots, x_n]$, $I_A \subseteq T_A$ such that with $R_A = T_A/I_A$
  - $R_A \subseteq R$ induces an isomorphism $R_A \otimes_A K \cong R \otimes_A K = R$, and
  - R_A is A-free.

(b) For each M in (2), a finitely generated free A-submodule $M_A \subseteq M$ such that this inclusion induces an isomorphism $M_A \otimes_A K \cong M \otimes_A K = M$.
(c) For every $\phi : M \to N$ in (3) an A linear map $\phi_A : M_A \to N_A$ such that
  - $\phi_A \otimes 1 : M_A \otimes_A K \to N_A \otimes_A K$ is the map ϕ, and
  - $\operatorname{Im} \phi$, $\operatorname{Ker} \phi$ and $\operatorname{Coker} \phi$ are A-free.

(d) For every homological complex

$$\mathcal{C}_\bullet = \ldots \xrightarrow{\partial_{i+2}} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \ldots$$

in (4), an homological complex

$$\mathcal{C}_{A\bullet} = \ldots \xrightarrow{(\partial_{i+2})_A} (C_{i+1})_A \xrightarrow{(\partial_{i+1})_A} (C_i)_A \xrightarrow{(\partial_i)} \ldots$$

such that $\mathrm{H}_i(\mathcal{C}_A \otimes_A K) = \mathrm{H}_i(\mathcal{C}_A) \otimes_A K$. For every cohomological complex in (4), a similar corresponding contruction.

Descent data exist: see [HH06, Chapter 2].

Notice that for any maximal ideal $\mathfrak{p} \subset A$, the fiber $\kappa(\mathfrak{p}) = A/\mathfrak{p}$ is a finite field. Given any property $\mathcal{P}$ of rings of prime characteristic, we say that R as in the definition above as *dense* $\mathcal{P}$ *type* if there exists descent data (A, R_A) and such that for all maximal ideals $\mathfrak{p} \subset A$ the fiber $R_A \otimes_A \kappa(\mathfrak{p})$ has property $\mathcal{P}$.

Notice also that for any complex $\mathcal{C}$ of free A modules where the kernels and cokernels of all maps are A-free (as in Definition 5.1(c) and (d)), $\mathrm{H}_i(\mathcal{C} \otimes_A \kappa(\mathfrak{p})) = \mathrm{H}_i(\mathcal{C}) \otimes_A \kappa(\mathfrak{p})$.

The main result in this section is the following theorem.

Theorem 5.2. *If $R_\mathfrak{m}$ is Cohen-Macaulay on the punctured spectrum and has dense F-injective type, then $e(R_\mathfrak{m}) \leq \binom{v}{d}$.*

Lemma 5.3. *There exists descent data (A, R_A) for R with the following properties.*

(a) $y_1, \ldots, y_d \in R_A$,
(b) *for all maximal ideals $\mathfrak{p} \subset A$ the images of $y_1, \ldots, y_d$ in $R_{\kappa(\mathfrak{p})}$ are a minimal reduction of $\mathfrak{m} R_{\kappa(\mathfrak{p})}$,*
(c) *if $R_\mathfrak{m}$ is Cohen-Macaulay on its punctured spectrum, so is $R_{\kappa(\mathfrak{p})}$ for all maximal ideals $\mathfrak{p} \subset A$.*
(d) *if $R_\mathfrak{m}$ is unmixed, so is $R_\mathfrak{p}$ for all maximal ideals $\mathfrak{p} \subset A$.*

Proof. Start with some descent data (A, R_A) where A contains all K-coefficients among a set of generators $g_1, \ldots, g_\mu$ of I, I_A is the ideal of $A[x_1, \ldots, x_n]$ generated by $g_1, \ldots, g_\mu$ and $R_A = A[x_1, \ldots, x_n]/I_A$. Let $\mathbf{x}$ denote $(x_1, \ldots, x_n)$. For (a) write $y_i = Q_i(x_1, \ldots, x_n) + I$ for all $1 \leq i \leq d$ and extend A to include all the K-coefficients in $Q_1, \ldots, Q_d$.

Assume that $\mathfrak{m}^{s+1} \subseteq \mathbf{y}\mathfrak{m}^s$ for some s. Write each monomial of degree $s+1$ in the form $r_1(\mathbf{x})Q_1(\mathbf{x}) + \cdots + r_d(\mathbf{x})Q_d(\mathbf{x}) + a(\mathbf{x})$ where $r_1, \ldots, r_d$ are polynomials of degrees at least s and $a(\mathbf{x}) \in I$; enlarge A to include all the K-coefficients of $r_1, \ldots, r_d, a$.

With this enlarged A we have $(\mathbf{x}R_A)^{s+1} \subseteq (\mathbf{y}R_A)(\mathbf{x}R_A)^s$ and tensoring with any $\kappa(\mathfrak{p})$ gives $(\mathbf{x}R_{\kappa(\mathfrak{p})})^{s+1} \subseteq (\mathbf{y}R_{\kappa(\mathfrak{p})})(\mathbf{x}R_{\kappa}(\mathfrak{p}))^s$.

If $R_\mathfrak{m}$ is Cohen-Macaulay on its punctured spectrum, then we can find a localization of R at one element whose only point at which it can fail to be non-Cohen-Macaulay is $\mathfrak{m}$. After adding a new variable to R as at the beginning of this section, we may assume that the non-Cohen-Macaulay locus of R is contained in $\{\mathfrak{m}\}$. The hypothesis in (c) is now equivalent to the existence of a $k \geq 1$ such that $\mathfrak{m}^k \operatorname{Ext}^i_T(R,T) = 0$ for all $\operatorname{ht} I < i \leq n$. Let $\mathcal{F}$ be a free T-resolution of R. Include $\mathfrak{m}$, $\mathcal{F}$ and $\mathcal{C} = \operatorname{Hom}(\mathcal{F}, T)$ in the descent objects. Now, with the corresponding descent data, $\mathcal{F}_A$ is a T_A-free resolution of R_A. Localize A at one element, if necessary, so that $\mathfrak{m}_A^k \operatorname{Ext}^i_{T_A}(R_A, T_A)$ is A-free for all $\operatorname{ht} I < i \leq n$. Fix any $\operatorname{ht} I < i \leq n$; we have

$$\operatorname{Ext}^i_{T_A}(R_A, T_A) \otimes_A K = \mathrm{H}^i(\operatorname{Hom}(\mathcal{F}_A, T_A)) \otimes_A K = \mathrm{H}^i(\mathcal{C}_A) \otimes_A K = \mathrm{H}^i(\mathcal{C})$$

and hence $\mathfrak{m}_A^k \operatorname{Ext}^i_{T_A}(R_A, T_A) \otimes_A K = 0$ so $\mathfrak{m}_A^k \operatorname{Ext}^i_{T_A}(R_A, T_A) = 0$. Now for any maximal ideal $\mathfrak{p} \subset A$, $\mathfrak{m}^k_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{T_{\kappa(\mathfrak{p})}}(R_{\kappa(\mathfrak{p})}, T_{\kappa(\mathfrak{p})}) = 0$, and hence $R_{\kappa(\mathfrak{p})}$ is Cohen-Macaulay on its punctured spectrum.

The last statement is [HH06, Theorem 2.3.9].

□

Proof of Theorem 5.2. Using [BH93, Theorem 4.6.4] we write $e(R_\mathfrak{m}) = \chi(\mathbf{y}; R_\mathfrak{m})$, and using the fact that R was constructed so that $\mathfrak{m}$ is the only maximal ideal containing $\mathbf{y}$, we deduce that $e(R_\mathfrak{m}) = \chi(\mathbf{y}; R) = \sum_{i=0}^d (-1)^i \ell_R \mathrm{H}_i(\mathbf{y}, R)$. We add to the descent objects in Lemma 5.3 the Koszul complex $\mathcal{K}_\bullet(\mathbf{y}; R)$ and extend the descent data in Lemma 5.3 to cater for these.

For all $0 \leq i \leq d$ we have $\mathrm{H}_i(\mathbf{y}; R) \cong \mathrm{H}_i(\mathbf{y}; R_A) \otimes_A K$ and $\ell(\mathrm{H}_i(\mathbf{y}; R)) = \operatorname{rank} \mathrm{H}_i(\mathbf{y}; R_A)$.

Pick any maximal ideal $\mathfrak{p} \subset A$. We have $\mathrm{H}_i(\mathbf{y}; R_A) \otimes_A \kappa(\mathfrak{p}) \cong \mathrm{H}_i(\mathbf{y}; R_{\kappa(\mathfrak{p})})$.

Note that that $\mathrm{H}_i(\mathbf{y}; R_{\kappa(\mathfrak{p})})$ is only supported at $\mathfrak{m} R_{\kappa(\mathfrak{p})}$. Otherwise, we can find an $x \in \mathfrak{m} R_{\kappa(\mathfrak{p})}$ such that $0 \neq \mathrm{H}_i(\mathbf{y}; R_{\kappa(\mathfrak{p})})_x \cong \mathrm{H}_i(\mathbf{y}; R_A)_x \otimes_A \kappa(\mathfrak{p})$, hence $\mathrm{H}_i(\mathbf{y}; R_A)_x \neq 0$ and $(\mathrm{H}_i(\mathbf{y}; R_A) \otimes_A K)_x \cong \mathrm{H}_i(\mathbf{y}; R)_x = 0$, contradicting the fact that $\operatorname{Supp} \mathrm{H}_i(\mathbf{y}; R) \subseteq \{\mathfrak{m}\}$.

Now

$$\begin{aligned} e((R_{\kappa(\mathfrak{p})})_\mathfrak{m}) = \chi(\mathbf{y}; (R_{\kappa(\mathfrak{p})})_\mathfrak{m}) &= \chi(\mathbf{y}; R_{\kappa(\mathfrak{p})}) \\ &= \sum_{i=0}^d (-1)^i \ell_R \mathrm{H}_i(\mathbf{y}, R_{\kappa(\mathfrak{p})}) \\ &= \sum_{i=0}^d (-1)^i \operatorname{rank} \mathrm{H}_i(\mathbf{y}, R_A) \end{aligned}$$

and so Theorem 1.1 implies that $e(R_\mathfrak{m}) = e((R_{\kappa(\mathfrak{p})})_\mathfrak{m}) \leq \binom{v}{d}$.

□

Remark 5.4. In [Sch09] it is conjectured that being a K-algebra with dense F-injective type is equivalent to being a Du Bois singularity. Recently, the multiplicity of Cohen-Macaulay Du Bois singularities has been bounded by $\binom{v}{d}$ (see [Shi17]) and hence the results of this section provide further evidence for the conjecture above.

References

[BH93] W. Bruns and J. Herzog: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)

[Fed83] R. Fedder: *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 461–480. MR701505 (84h:13031)

[HH11] J. Herzog and T. Hibi: *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London, Ltd., London, 2011.

[HH06] M. Hochster and C. Huneke: *Tight closure in equal characteristic zero*, A preprint of a manuscript, 2006.

[HS06] C. Huneke and I. Swanson: *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR2266432 (2008m:13013)

[HW15] C. Huneke and K.-i. Watanabe: *Upper bound of multiplicity of F-pure rings*, Proc. Amer. Math. Soc. **143** (2015).

[Kat08] M. Katzman: *Parameter-test-ideals of Cohen-Macaulay rings*, Compos. Math. **144** (2008), no. 4, 933–948. MR2441251 (2009d:13030)

[Lyu97] G. Lyubeznik: *F-modules: applications to local cohomology and D-modules in characteristic $p > 0$*, J. Reine Angew. Math. **491** (1997), 65–130. MR1476089 (99c:13005)

[Ma15] L. Ma: *F-injectivity and Buchsbaum singularities*, Math. Ann. **362** (2015), no. 1-2, 25–42. 3343868

[QS17] P. H. Quy and K. Shimomoto: *F-injectivity and Frobenius closure of ideals in Noetherian rings of characteristic $p > 0$*, Adv. Math. **313** (2017), 127–166. 3649223

[Sch09] K. Schwede: *F-injective singularities are Du Bois*, Amer. J. Math. **131** (2009), no. 2, 445–473. MR2503989

[SZ13] K. Schwede and W. Zhang: *Bertini theorems for F-singularities*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 4, 851–874. 3108833

[Shi17] K. Shibata: *Upper bound of the multiplicity of a Du Bois singularity*, Proc. Amer. Math. Soc. **145** (2017), no. 3, 1053–1059.

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