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The Sylvester equation and the elliptic Korteweg-de Vries system

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The elliptic potential Korteweg-de Vries lattice system is a multi-component extension of the lattice potential Korteweg-de Vries equation, whose soliton solutions are associated with an elliptic Cauchy kernel (i.e., a Cauchy kernel on the torus). In this paper we generalize the class of solutions by allowing the spectral parameter to be a full matrix obeying a matrix version of the equation of the elliptic curve, and for the Cauchy matrix to be a solution of a Sylvester type matrix equation subject to this matrix elliptic curve equation. The construction involves solving the matrix elliptic curve equation by using Toeplitz matrix techniques, and analysing the solution of the Sylvester equation in terms of Jordan normal forms. Furthermore, we consider the continuum limit system associated with the elliptic potential Korteweg-de Vries system, and analyse the dynamics of the soliton solutions, which reveals some new features of the elliptic system in comparison to the non-elliptic case. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4977477]

I. INTRODUCTION

The elliptic lattice potential Korteweg-de Vries KdV (elpKdV) system is a two-parameter extension of the lattice potential KdV equation which arises naturally by generalising the relevant Cauchy kernel, underlying the solutions structure, to a Cauchy kernel on the torus, i.e., the one where the spectral parameter takes values on an elliptic curve. This leads to the following multi-component lattice system:[1]

$$(a+b+u-\widehat{\widetilde{u}})(a-b+\widehat{u}-\widetilde{u})=a^2-b^2+g(\widetilde{s}-\widehat{s})(\widehat{\widetilde{s}}-s), \tag{1.1a}$$

$$(\widehat{\widetilde{s}}-s)(\widetilde{w}-\widehat{w})=[(a+u)\widetilde{s}-(b+u)\widehat{s}]\widehat{\widetilde{s}}-[(a-\widehat{\widetilde{u}})\widehat{s}-(b-\widehat{\widetilde{u}})\widetilde{s}]s, \tag{1.1b}$$

$$(\widehat{s}-\widetilde{s})(\widehat{\widetilde{w}}-w)=[(a-\widetilde{u})s+(b+\widetilde{u})\widehat{\widetilde{s}}]\widehat{s}-[(a+\widehat{u})\widehat{\widetilde{s}}+(b-\widehat{u})s]\widetilde{s}, \tag{1.1c}$$

$$(a+u-\frac{\widetilde{w}}{\widetilde{s}})(a-\widetilde{u}+\frac{w}{s})=a^2-R(s\widetilde{s}), \tag{1.1d}$$

$$(b+u-\frac{\widehat{w}}{\widehat{s}})(b-\widehat{u}+\frac{w}{s})=b^2-R(s\widehat{s}). \tag{1.1e}$$

This is a coupled set of partial difference equations for dependent variables $u=u_{n,m}, \widetilde{s}=s_{n,m}, \widetilde{w}=w_{n,m}$ for discrete variables, where the accents denote shifts, e.g. (1.4) and where a, b are parameters associated with those lattice shifts. A related continuous elliptic system is the elliptic potential Korteweg-de Vries KdV (epKdV) system

$$s_t=4s_{xxx}+6s_x[R(s^2)-A^2-\frac{2As_x}{s}-\frac{2s_{xx}}{s}], \tag{1.2a}$$

$$A_t=4A_{xxx}-6A^2A_x+6A_xR(s^2)-\frac{6s_x}{s}(R(s^2))_x, \tag{1.2b}$$

with $A=-u+\frac{w}{s}$ derived in the same paper. Here $R(x)$ is associated with the elliptic curve

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$$y^2 = R(x) = \frac{1}{x} + 3e_1 + gx, \tag{1.3}$$

where $e_1, g \in \mathbb{C}$ are moduli of the elliptic curve. In Equation (1.1) we use the conventional tilde-hat notations to express shifts with respect to discrete variables, e.g.,

$$u \doteq u_{n,m},\ \widetilde{u} \doteq u_{n+1,m},\ \widehat{u} \doteq u_{n,m+1},\ \widehat{\widetilde{u}} \doteq u_{n+1,m+1}. \tag{1.4}$$

The direct linearisation approach used in Ref. 1 leads to a description in terms of an infinite order matrix $\boldsymbol{U}$, from which closed form equations for the entries of $\boldsymbol{U}$ are derived yielding nonlinear lattice equations (see Refs. 1–4). A special class of soliton type solutions was presented in terms of elliptic Cauchy matrices.

In the present paper, we will derive a novel class of solutions of the above two elliptic systems using a generalization of the Cauchy matrix approach in terms of spectral parameters which are full matrices, and where the Cauchy kernel is a solution of a Sylvester type matrix equation. The scalar Cauchy matrix approach was successfully applied in Refs. 5 and 6 to derive integrable lattice equations and to analyse their underlying structures. Subsequently, the generalized Cauchy matrix approach, involving solutions of Sylvester type matrix equations, was used to generate a far more general class of solutions for those same systems.[7,8] The latter is the approach we adopt in the current paper in the case of elliptic systems (1.1) and (1.2).

The Cauchy matrix approach is purely an algebraic procedure which enables us to obtain various integrable equations, their explicit soliton solutions, and their Lax pairs. In the Cauchy matrix approach, the Sylvester equation

$$\boldsymbol{AX} - \boldsymbol{XB} = \boldsymbol{C} \tag{1.5}$$

can be viewed as a starting point.[7,8] The matrix $\boldsymbol{X}$ is a dressed Cauchy matrix (see the factorization (3.13a)) and is used to introduce τ-function.[5–7]

In the present paper we start from the following Sylvester equation:

$$\boldsymbol{kM} + \boldsymbol{Mk} = \boldsymbol{rc}^T - g\boldsymbol{K}^{-1}\boldsymbol{rc}^T\boldsymbol{K}^{-1}, \tag{1.6}$$

where $\boldsymbol{r} = (r_1, r_2, \ldots, r_N)^T$, $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)^T$, and $\boldsymbol{k},\ \boldsymbol{K} \in \mathbb{C}_{N\times N}$ obey the matrix relation

$$\boldsymbol{k}^2 = \boldsymbol{K} + 3e_1\boldsymbol{I} + g\boldsymbol{K}^{-1},\ \ \boldsymbol{kK} = \boldsymbol{Kk}, \tag{1.7}$$

in which $\boldsymbol{I}$ is the $N \times N$ unit matrix. Based on the above Sylvester equation, the dispersion relations for the elpKdV system are defined by

$$(a\boldsymbol{I} - \boldsymbol{k})\widetilde{\boldsymbol{r}} = (a\boldsymbol{I} + \boldsymbol{k})\boldsymbol{r},\ \ (b\boldsymbol{I} - \boldsymbol{k})\widehat{\boldsymbol{r}} = (b\boldsymbol{I} + \boldsymbol{k})\boldsymbol{r}, \tag{1.8}$$

and for the epKdV system by

$$\boldsymbol{r}_x = \boldsymbol{kr},\ \ \boldsymbol{r}_t = 4\boldsymbol{k}^3\boldsymbol{r},\ \ \boldsymbol{c}_x^T = \boldsymbol{c}^T\boldsymbol{k},\ \ \boldsymbol{c}_t^T = 4\boldsymbol{c}^T\boldsymbol{k}^3. \tag{1.9}$$

In Sec. II, we will focus on system (1.7) as a full matrix equation governed by an elliptic curve (1.3). In Sec. III, we will concentrate on the Sylvester equation (1.6) and the scalar functions $S^{(i,j)}$ defined in (3.23). Explicit solution $\boldsymbol{M}$ of (1.6) will be given, distinguishing between the cases where $\boldsymbol{k}$ is either diagonal or of Jordan block form, as well as on their combinations. Solutions of the elpKdV and epKdV systems are consequently obtained, in terms of a generic element $S^{(i,j)}$ composed in an infinite order matrix $\boldsymbol{S}$. The entries $S^{(i,j)}$ satisfy some recurrence relations which can be viewed as discrete equations of $S^{(i,j)}$ defined in $\mathbb{Z}\times\mathbb{Z}$ and which will play a crucial role in deriving the continuous epKdV system. In Secs. IV and V we then derive the elpKdV system and epKdV system together with their Lax pairs, respectively. Some analysis of the dynamics of the solutions is presented in Sec. VI, which illustrates the novel aspects of the solutions. In Sec. VII we discuss continuum limits of the elpKdV system, and draw some conclusions in Sec. VIII. In Appendix A we list properties of lower triangular Toeplitz matrices which play important roles in our paper.

II. POINTS ON THE ELLIPTIC CURVE: PARAMETRIZATION AND SELECTION

A. Scalar case

Consider the elliptic curve

$$k^2 = R\Big(\frac{1}{K}\Big) = K + 3e_1 + \frac{g}{K} \tag{2.1}$$

which is the elliptic curve (1.3). The discrete plane wave factor is defined as

$$\rho_i = \left(\frac{a+k_i}{a-k_i}\right)^n \left(\frac{b+k_i}{b-k_i}\right)^m \rho_i^0, \tag{2.2}$$

where ρ_i^0 is a phase factor and k_i together with K_i obeys the elliptic curve (2.1), i.e.,

$$k_i^2 = K_i + 3e_1 + \frac{g}{K_i}, \quad i = 1, 2, \ldots, N. \tag{2.3}$$

In the case of classical soliton solutions (see Refs. 6 and 9), $\{k_i\}$ play the role of wave numbers, which should be distinct so that they can represent different solitons. Since k_i and K_i are coupled through the elliptic curve (2.3) (say, both k_i and $-k_i$ correspond to the same K_i), we consequently require that they can identify each other, i.e.,

$$k_i \neq k_j \Leftrightarrow K_i \neq K_j. \tag{2.4}$$

Note that for the arbitrary two points (k_i,K_i) and (k_j,K_j) on the elliptic curve (2.1) we always have the relation

$$(k_i + k_j)(k_i - k_j) = (K_i - K_j)\,\frac{K_iK_j - g}{K_iK_j}. \tag{2.5}$$

This means that if we take

$$(k_i + k_j)(K_iK_j - g) \neq 0, \tag{2.6}$$

then (2.4) is guaranteed. Equation (2.6) is the criteria that we select points from the elliptic curve (2.1). Consequently, in (2.6) $k_i \neq 0$.

The elliptic curve (2.1) can be parameterized using Weierstrass's elliptic function $\wp(\kappa)$ as follows (cf. Ref. 1):

$$K = \wp(\kappa) - e_1, \quad k = \frac{\wp'(\kappa)}{2(\wp(\kappa) - e_1)}, \tag{2.7a}$$

$$e_1 = \wp(\omega), \quad g = (e_1 - e_2)(e_1 - e_3), \tag{2.7b}$$

where $e_2 = \wp(\omega + \omega')$, $e_3 = \wp(\omega')$, and ω and ω' are respectively the half periods of $\wp(\kappa)$.

Under the parametrization (2.7), for the points on the curve (2.1) we have

$$K_i = \wp(\kappa_i) - e_1, \quad k_i = \frac{\wp'(\kappa_i)}{2(\wp(\kappa_i) - e_1)}. \tag{2.8}$$

Then the criteria (2.6) can alternatively be described through the following requirements for κ_i:

$$\kappa_i \in \mathbb{D}' = \mathbb{D} \setminus \{0, \omega, \omega', \omega + \omega'\}, \tag{2.9a}$$

$$(\wp(\kappa_i) - e_1)(\wp(\kappa_j) - e_1) \neq g, \tag{2.9b}$$

$$\kappa_i + \kappa_j \neq 0, \tag{2.9c}$$

for $i, j = 1, 2, \ldots, N$, where $\mathbb{D}$ is a fundamental period parallelogram $ABCD$ as described in Fig. 1. We note that here and henceforth when we talk about $\kappa_i + \kappa_j$ we always mean that it is the remainder of the Euclidean division of $\kappa_i + \kappa_j$ by the periodic lattice, i.e., $\kappa_i + \kappa_j \bmod(2\omega, 2\omega')$. In fact, 0 is the pole of $\wp(\kappa)$ and thus we require $\kappa_i \neq 0$ to avoid singularities. Besides, k_i cannot be zero as a consequence of $k_i + k_j \neq 0$. In light of Liouville's theorems (cf. Ref. 10), since $\wp'(\kappa)$ is a third-order elliptic function in the period parallelogram $\mathbb{D}$ $\wp'(\kappa)$ only has 3 zeros which are ω, ω', and $\omega + \omega'$. To avoid breaking the one-to-one correspondence of k_i and K_i, we require $\kappa_i \notin \{\omega, \omega', \omega + \omega'\}$. Thus we have (2.9a). (2.9b) is from the requirement $K_iK_j \neq g$. For (2.9c), we can prove that under (2.9a) and (2.9b) the following holds:

$$\kappa_i + \kappa_j = 0 \ \Leftrightarrow k_i + k_j = 0. \tag{2.10}$$

In fact, since $\wp(\kappa)$ is even and $\wp'(\kappa)$ is odd, from the parametrization (2.8), we immediately find that if $\kappa_i + \kappa_j = 0$ then $k_i + k_j = 0$. On the other hand, under (2.9b), from the factorization (2.5), if $k_i + k_j = 0$ and (2.9b) holds, there must be $K_i = K_j$, which means $\kappa_i = \pm\kappa_j$ in $\mathbb{D}$ in light of Liouville's theorems (cf. Ref. 10). The case $\kappa_i = \kappa_j$ is impossible because this case yields $k_i = k_j = 0$ due to $k_i + k_j = 0$ but $k_i = 0$ requires $\wp'(\kappa_i) = 0$ which is impossible in $\mathbb{D}'$. Thus, $\kappa_i = -\kappa_j$ is the only choice, i.e., $\kappa_i + \kappa_j = 0$.

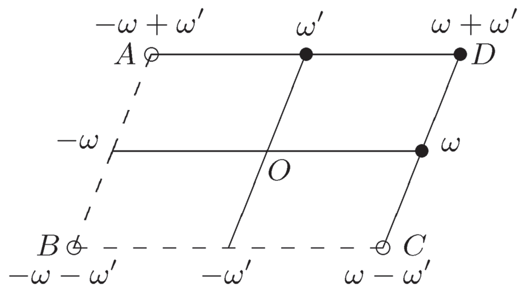


FIG. 1. Fundamental period parallelogram $\mathbb{D}$.

B. Matrix system (1.7)

Let us come to the matrix relation (1.7). Although at first glance (1.7) suggests an interpretation of this matrix relation as a matrix version of an elliptic curve, it just represents the coordination of a collection of points on the given elliptic curve (2.1).

To understand this, let us consider a similarity transformation

$$\boldsymbol{k}_1 = \boldsymbol{T}\boldsymbol{k}\boldsymbol{T}^{-1}, \ \ \boldsymbol{K}_1 = \boldsymbol{T}\boldsymbol{K}\boldsymbol{T}^{-1}, \tag{2.11}$$

where $\boldsymbol{T}$ serves as the transform matrix. Obviously, under the above similarity transformation, (1.7) is formally invariant

$$\boldsymbol{k}_1{}^2 = \boldsymbol{K}_1 + 3e_1\boldsymbol{I} + g\boldsymbol{K}_1{}^{-1}, \ \ \boldsymbol{k}_1\boldsymbol{K}_1 = \boldsymbol{K}_1\boldsymbol{k}_1. \tag{2.12}$$

Thus, in the following we only need to consider the relation

$$\boldsymbol{\Gamma}^2 = \boldsymbol{K} + 3e_1\boldsymbol{I} + g\boldsymbol{K}^{-1}, \ \ \boldsymbol{\Gamma}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{\Gamma}, \tag{2.13}$$

where $\boldsymbol{\Gamma}$ is the canonical form of $\boldsymbol{k}$.

When

$$\boldsymbol{\Gamma} = \mathrm{Diag}(k_1, k_2, \ldots, k_N), \tag{2.14a}$$

$\boldsymbol{K}$ is taken as

$$\boldsymbol{K} = \mathrm{Diag}(K_1, K_2, \ldots, K_N), \tag{2.14b}$$

where (k_i,K_i) are the points on (2.1), i.e., satisfying (2.3). k_i is the eigenvalue set of $\boldsymbol{k}$. Here we require that each $k_i \neq 0$ and $k_i^2 \neq k_j^2$ for $i \neq j$. Under such a requirement one can see that the criteria (2.6) are satisfied in light of the factorization (2.5).

It is interesting to consider the case that $\boldsymbol{\Gamma}$ is a Nth order Jordan block

$$\boldsymbol{\Gamma} = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & k_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & k_1 \end{pmatrix}, \ \ k_1 \neq 0. \tag{2.15}$$

In this case, k_1 is the only eigenvalue of $\boldsymbol{k}$ with algebraic multiplicity N (and geometric multiplicity 1). In terms of the parametrization (2.7) we need κ_1 to satisfy

$$\kappa_1 \in \mathbb{D} \setminus \{0, \omega, \omega', \omega+\omega'\}, \ \ \wp(\kappa_1) \neq \sqrt{g} + e_1. \tag{2.16}$$

To find $\boldsymbol{K}$ that corresponds to the Jordan block (2.15), we will make use of properties of lower triangular Toeplitz matrices (LTT). For more details about LTT matrices please see Appendix A. According to Proposition A.1, for the Jordan block (2.15), when $\boldsymbol{\Gamma}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{\Gamma}$ there must be $\boldsymbol{K} \in \mathfrak{T}^{[N]}$ where $\mathfrak{T}^{[N]}$, which is commutative, denotes the set composed of all Nth order LTT matrices. By taking derivatives with respect to k of the elliptic curve (2.1) at the point (k_1,K_1) where $K_1 = K(k_1)$ and $K(k)$ is viewed as an implicit function of k determined by the curve (2.1), we find (for $i \geq j$)

$$\frac{1}{j!}\partial_k^j k^2 = \frac{1}{j!}\partial_k^j K(k) + 3e_1\delta_{j,0} + \frac{g}{j!}\partial_k^j \frac{1}{K(k)}, \ \ (\text{at } k = k_1). \tag{2.17}$$

The l.h.s., the first term and third term on the r.h.s., respectively, correspond to the elements of the LTT matrices generated (see Definition 1 in Appendix A) by $f(k) = k^2$, $K(k)$ and $1/K(k)$ at $k = k_1$. This means that for the Jordan block case (2.15) if we take $\boldsymbol{K} = \boldsymbol{T}^{[N]}[K(k_1)]$ (for this notation see Definition 1 in Appendix A) then the relation (2.13) holds.

III. THE SYLVESTER EQUATION AND INFINITE MATRIX STRUCTURE

In this section we will first investigate solutions of the Sylvester equation (1.6) and derive an explicit expression of the solution $\boldsymbol{M}$. Then, with the help of some special matrices we will investigate recurrence relations of scalar function $S^{(i,j)}$ (defined in (3.23)) and properties of the infinite matrix $\boldsymbol{S}$ composed of $S^{(i,j)}$.

A. Solvability of (1.6)

For the solution of the Sylvester equation (1.5), there is the following well known result.[11]

Proposition 3.1. Denote the eigenvalue sets of $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\mathcal{E}(\boldsymbol{A})$ and $\mathcal{E}(\boldsymbol{B})$, respectively. For the known $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the Sylvester equation (1.5) has a unique solution $\boldsymbol{M}$ if and only if $\mathcal{E}(\boldsymbol{A}) \cap \mathcal{E}(\boldsymbol{B}) = \varnothing$.

Based on this proposition, we find the following.

Proposition 3.2. Consider the Sylvester equation (1.6) where the matrices $\boldsymbol{k}$ and $\boldsymbol{K}$ satisfy

$$\mathcal{E}(\boldsymbol{k}) \cap \mathcal{E}(-\boldsymbol{k}) = \varnothing, \tag{3.1a}$$

$$\mathcal{E}(g\boldsymbol{K}^{-1}) \cap \mathcal{E}(\boldsymbol{K}) = \varnothing, \tag{3.1b}$$

and the matrix relation (1.7). Then, the "dual" matrix equation

$$\boldsymbol{KM} - \boldsymbol{MK} = \boldsymbol{k}\,\boldsymbol{rc}^T - \boldsymbol{rc}^T\boldsymbol{k} \tag{3.2}$$

holds.

Proof. Here we note that condition (3.1a) is necessary to guarantee the solvability of Equation (1.6) in light of Proposition 3.1. Then, left multiplying $\boldsymbol{k}$ and (1.6) yields

$$\boldsymbol{k}^2\boldsymbol{M} + \boldsymbol{kMk} = \boldsymbol{k}(\boldsymbol{rc}^T - g\boldsymbol{K}^{-1}\boldsymbol{rc}^T\boldsymbol{K}^{-1}). \tag{3.3a}$$

By simple algebraic substitution, Equation (3.3a) can be written as

$$\boldsymbol{k}^2\boldsymbol{M} - \boldsymbol{Mk}^2 = -\boldsymbol{rc}^T\boldsymbol{k} + \boldsymbol{krc}^T + g\boldsymbol{K}^{-1}\boldsymbol{rc}^T\boldsymbol{K}^{-1}\boldsymbol{k} - g\boldsymbol{kK}^{-1}\boldsymbol{rc}^T\boldsymbol{K}^{-1}. \tag{3.3b}$$

Applying (1.7) in (3.3b), we get

$$g\boldsymbol{K}^{-1}(\boldsymbol{KM} - \boldsymbol{MK} - \boldsymbol{krc}^T + \boldsymbol{rc}^T\boldsymbol{k})\boldsymbol{K}^{-1} = \boldsymbol{KM} - \boldsymbol{MK} - \boldsymbol{krc}^T + \boldsymbol{rc}^T\boldsymbol{k}, \tag{3.4}$$

which can be rewritten as a Sylvester equation

$$g\boldsymbol{K}^{-1}\boldsymbol{W} - \boldsymbol{WK} = 0, \quad \boldsymbol{W} = \boldsymbol{KM} - \boldsymbol{MK} - \boldsymbol{krc}^T + \boldsymbol{rc}^T\boldsymbol{k}. \tag{3.5}$$

Based on Proposition 3.1 and noting that $\mathcal{E}(g\boldsymbol{K}^{-1}) \cap \mathcal{E}(\boldsymbol{K}) = \varnothing$ in (3.1b), Equation (3.5) has a unique solution $\boldsymbol{W} = 0$, which means (3.2) holds. □

Here, we note that condition (3.1) is obvious because it is actually the criteria (2.6) for selecting points from the elliptic curve (2.1). We also note that we cannot derive (1.6) from (3.2). In fact, we start from (3.2), replace $\boldsymbol{K}$ using (1.7), and we get

$$\boldsymbol{k}^2\boldsymbol{M} - \boldsymbol{Mk}^2 - g(\boldsymbol{K}^{-1}\boldsymbol{M} - \boldsymbol{MK}^{-1}) = \boldsymbol{krc}^T - \boldsymbol{rc}^T\boldsymbol{k}.$$

In the meantime, from (3.2) we also have

$$\boldsymbol{K}^{-1}\boldsymbol{M} - \boldsymbol{MK}^{-1} = -\boldsymbol{K}^{-1}\boldsymbol{krc}^T\boldsymbol{K}^{-1} + \boldsymbol{K}^{-1}\boldsymbol{rc}^T\boldsymbol{kK}^{-1}.$$

Manipulating these two equations leads to

$$\boldsymbol{kY} - \boldsymbol{Yk} = 0, \quad \boldsymbol{Y} = \boldsymbol{kM} + \boldsymbol{Mk} - \boldsymbol{rc}^T + g\boldsymbol{K}^{-1}\boldsymbol{rc}^T\boldsymbol{K}^{-1}.$$

Obviously, $\boldsymbol{Y}=0$ is a solution to the above equation but it is not unique. This means (1.6) and (3.2) are not equivalent. (1.6) is more general while (3.2) is a by-product of the former. In the following discussion it is sufficient that we only consider (1.6).

B. Solution to the Sylvester equation (1.6)

1. Canonical form of (1.6)

Using the similarity transformation (2.11) and denoting

$$\boldsymbol{M}_1=\boldsymbol{T}\boldsymbol{M}\boldsymbol{T}^{-1},\quad \boldsymbol{r}_1=\boldsymbol{T}\boldsymbol{r},\quad \boldsymbol{c}_1^T=\boldsymbol{c}^T\boldsymbol{T}^{-1}, \tag{3.6}$$

it follows from (1.6) that

$$\boldsymbol{M}_1\boldsymbol{k}_1+\boldsymbol{k}_1\boldsymbol{M}_1=\boldsymbol{r}_1\boldsymbol{c}_1^T-g\boldsymbol{K}_1^{-1}\boldsymbol{r}_1\boldsymbol{c}_1^T\boldsymbol{K}_1^{-1},$$

which takes the same form as (1.6). This means that when we solve the Sylvester equation (1.6) we only need to consider the following canonical form:

$$\boldsymbol{M}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}\boldsymbol{M}=\boldsymbol{r}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1}, \tag{3.7a}$$

together with (2.13), where

$$\boldsymbol{r}=(r_1,r_2,\ldots,r_N)^T,\quad \boldsymbol{c}=(c_1,c_2,\ldots,c_N)^T, \tag{3.7b}$$

and we suppose that $\boldsymbol{\Gamma}$ is the canonical form of $\boldsymbol{k}$.

2. Solutions to the matrix system (2.13)

Here we list the solutions using the notations given in Appendix B.

Proposition 3.3. The matrix system (2.13) satisfies the following three cases of solutions:

(1) *Diagonal case:*

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_D^{[N]}(\{k_j\}_1^N),\quad \boldsymbol{K}=\boldsymbol{\Gamma}_D^{[N]}(\{K_j\}_1^N), \tag{3.8}$$

where

$$k_j^2=K_j+3e_1+gK_j^{-1},\quad j=1,2,\ldots,N. \tag{3.9}$$

(2) *Jordan block case:*

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_J^{[N]}(k_1),\quad \boldsymbol{K}=\boldsymbol{T}^{[N]}[K(k_1)]. \tag{3.10}$$

(3) *Generic case:*

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_G^{[N]}, \tag{3.11a}$$

$$\boldsymbol{K}=\mathrm{Diag}\big(\boldsymbol{\Gamma}_D^{[N_1]}(\{K_j\}_1^{N_1}),\boldsymbol{T}^{[N_2]}[K(k_{N_1+1})],\ldots,\boldsymbol{T}^{[N_s]}[K(k_{N_1+(s-1)})]\big). \tag{3.11b}$$

3. Solutions to (3.7a)

Now let us come to the solutions to the Sylvester equation (3.7a).

Case 1. $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_D^{[N]}(\{k_j\}_1^N)$.
Solution to (3.7a) is given by

$$\boldsymbol{M}=\boldsymbol{F}\boldsymbol{G}_D^{[N]}(\{k_j\}_1^N)\boldsymbol{H}=\Big(\frac{1-g/(K_iK_j)}{k_i+k_j}\,r_ic_j\Big)_{N\times N}, \tag{3.12a}$$

where

$$\boldsymbol{F}=\mathrm{Diag}(r_1,r_2,\ldots,r_N),\quad \boldsymbol{H}=\mathrm{Diag}(c_1,c_2,\ldots,c_N). \tag{3.12b}$$

Case 2. $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_J^{[N]}(k_1)$.
This is also referred to as the Jordan block case. In this case, $\boldsymbol{\Gamma}$ and $\boldsymbol{K}$ take the form of (3.10). To find solution $\boldsymbol{M}$ of equation (3.7a), we factorize

$$\boldsymbol{M}=\boldsymbol{F}\boldsymbol{G}\boldsymbol{H}, \tag{3.13a}$$

where

$$\boldsymbol{F}=\boldsymbol{T}^{[N]}(\{r_j\}_1^N),\quad \boldsymbol{H}=\boldsymbol{H}^{[N]}(\{c_j\}_1^N),\tag{3.13b}$$

and $\boldsymbol{G}$ is a $N\times N$ unknown matrix. Note that $\boldsymbol{r}$ and $\boldsymbol{c}$ can be expressed through $\boldsymbol{F}$ and $\boldsymbol{H}$ as

$$\boldsymbol{r}=\boldsymbol{F}\,\mathbf{e}_1,\quad \boldsymbol{c}=\boldsymbol{H}\,\mathbf{e}_1,\tag{3.14}$$

where $\mathbf{e}_1=\mathbf{e}_1{}^{[N]}$ is defined in (B2b). Then, the Equation (3.7a) is rewritten as

$$\boldsymbol{F}\boldsymbol{G}\boldsymbol{H}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}\boldsymbol{F}\boldsymbol{G}\boldsymbol{H}=\boldsymbol{F}\,\mathbf{e}_1\,\mathbf{e}_1^T\,\boldsymbol{H}-g\boldsymbol{K}^{-1}\boldsymbol{F}\,\mathbf{e}_1\,\mathbf{e}_1^T\,\boldsymbol{H}\boldsymbol{K}^{-1}.\tag{3.15}$$

Further, from proposition A.4, one has

$$\boldsymbol{F}\boldsymbol{G}\boldsymbol{\Gamma}^T\boldsymbol{H}+\boldsymbol{F}\boldsymbol{\Gamma}\boldsymbol{G}\boldsymbol{H}=\boldsymbol{F}\,\mathbf{e}_1\,\mathbf{e}_1^T\,\boldsymbol{H}-g\boldsymbol{F}\boldsymbol{K}^{-1}\,\mathbf{e}_1\,\mathbf{e}_1^T\,\boldsymbol{K}^{-1^T}\boldsymbol{H},\tag{3.16}$$

and then

$$\boldsymbol{G}\boldsymbol{\Gamma}^T+\boldsymbol{\Gamma}\boldsymbol{G}=\mathbf{e}_1\,\mathbf{e}_1^T-g\boldsymbol{K}^{-1}\,\mathbf{e}_1\,\mathbf{e}_1^T\,\boldsymbol{K}^{-1^T}.\tag{3.17}$$

To solve (3.17), we set

$$\boldsymbol{G}=(G_1,G_2,\ldots,G_N)\tag{3.18}$$

with column vectors $\{G_j\}$. (3.17) is expanded to the following equation set:

$$(k_1\boldsymbol{I}+\boldsymbol{\Gamma})G_1=\mathbf{e}_1-\frac{g}{K_1}A_1,\tag{3.19a}$$

$$(k_1\boldsymbol{I}+\boldsymbol{\Gamma})G_{j+1}+G_j=-\frac{g}{j!}(\partial_{k_1}^j\frac{1}{K_1})A_1,\quad (j=1,2,\ldots,N-1),\tag{3.19b}$$

where $K_1=K(k_1)$ and

$$A_1=(a_1,a_2,\ldots,a_N)^T,\quad a_j=\frac{1}{(j-1)!}\partial_{k_1}^{j-1}\frac{1}{K_1},\quad (j=1,2,\ldots,N).\tag{3.19c}$$

The above equation set is solved by

$$G_j=\frac{\partial_a^{j-1}g^{[N]}(a)|_{a=2k_1}}{(j-1)!}+g\sum_{i=1}^{j}\frac{(-1)^i}{(j-i)!}(\partial_{k_1}^{j-i}\frac{1}{K_1})\boldsymbol{\Gamma}_J^{[N]}(2k_1)^{-i}A_1,\quad (j=1,2,\cdots,N).\tag{3.20}$$

The matrix $\boldsymbol{G}$ is symmetric.

Case 3. $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_G^{[N]}$.

In this case, we still suppose the factorization (3.13a), where

$$\boldsymbol{F}=\mathrm{Diag}(\boldsymbol{\Gamma}_D^{[N_1]}(\{r_j\}_1^{N_1}),\boldsymbol{T}^{[N_2]}(\{r_j\}_{N_1+1}^{N_1+N_2}),\ldots,\boldsymbol{T}^{[N_s]}(\{r_j\}_{N_1+N_2+\ldots+N_{s-1}+1}^{N_1+N_2+\ldots+N_s})),\tag{3.21a}$$

$$\boldsymbol{H}=\mathrm{Diag}(\boldsymbol{\Gamma}_D^{[N_1]}(\{c_j\}_1^{N_1}),\boldsymbol{H}^{[N_2]}(\{c_j\}_{N_1+1}^{N_1+N_2}),\ldots,\boldsymbol{H}^{[N_s]}(\{c_j\}_{N_1+N_2+\ldots+N_{s-1}+1}^{N_1+N_2+\ldots+N_s})),\tag{3.21b}$$

$\boldsymbol{G}$ is a symmetric matrix with block structure

$$\boldsymbol{G}=\boldsymbol{G}^T=(\boldsymbol{G}_{i,j})_{s\times s}\tag{3.21c}$$

and each $\boldsymbol{G}_{i,j}$ is a $N_i\times N_j$ matrix. Clearly,

$$\boldsymbol{G}_{1,1}=\boldsymbol{G}_D^{[N]}(\{k_j\}_1^N),\tag{3.22a}$$

$$\boldsymbol{G}_{1,j}=(G_{11},G_{12},\ldots,G_{1N_j}),\quad (1<j\le s),\tag{3.22b}$$

$$\boldsymbol{G}_{i,j}=(G_{i1},G_{i2},\ldots,G_{iN_j}),\quad (1<i\le j\le s),\tag{3.22c}$$

with

$$G_{1l}=(-1)^{l-1}\Big[\boldsymbol{I}-g\sum_{i=1}^{l}\frac{(-1)^{l-i}}{(l-i)!}\Big(\partial_k^{l-i}\frac{1}{K(k)}\Big)\Big|_{k=k_{N_1+1}}(\boldsymbol{\Gamma}_D^{[N_1]}(\{K_j\}_1^{N_1}))^{-1}\Big]\boldsymbol{\Gamma}_D^{[N_1]}(\{\alpha_j\})\mathbf{e}^{[N_1]},$$

$$G_{il}=\frac{\partial_{\beta_{ij}}^{l-1}g^{[N_i]}(\beta_{ij})}{(l-1)!}$$

$$+\,g\sum_{m=1}^{l}\frac{(-1)^m}{(l-m)!}\Big(\partial_k^{l-m}\frac{1}{K(k)}\Big)\Big|_{k=k_{N_1+(j-1)}}(\boldsymbol{\Gamma}_J^{[N_i]}(\beta_{ij}))^{-m}\boldsymbol{T}^{[N_i]}(\frac{1}{K(k)})\Big|_{k=k_{N_1+(i-1)}}\mathbf{e}_1^{[N_i]},$$

for $1 < i \leq j \leq s$, where $\alpha_j = \frac{1}{k_j + k_{N_1+(j-1)}}$, $\beta_{ij} = k_{N_1+(i-1)} + k_{N_1+(j-1)}$, and k and $K(k)$ satisfy the elliptic curve (2.1), i.e., $k^2 = K^2(k) + 3e_1 + g/K(k)$.

C. Infinite matrix *S*

In the Cauchy matrix approach a generic element $S^{(i,j)}$ plays a crucial role. Such entries $S^{(i,j)}$ compose an infinite order matrix $\boldsymbol{S}$ and they satisfy some recurrence relations which are used in deriving nonlinear equations as well as expressing solutions of the obtained equations.

Referring to the Sylvester equation (1.6) and the matrix relation (1.7) and using the elements $\{\boldsymbol{M}, \boldsymbol{k}, \boldsymbol{K}, \boldsymbol{r}, \boldsymbol{c}\}$ in (1.6) and (1.7), we introduce an $\infty \times \infty$ matrix $\boldsymbol{S} = (S^{(i,j)})_{\infty\times\infty}$, $i, j \in \mathbb{Z}$, where the elements $S^{(i,j)}$ are defined as (cf. Ref. 1)

$$S^{(2i,2j)} = \boldsymbol{c}^T\, \boldsymbol{K}^j(\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{3.23a}$$

$$S^{(2i+1,2j)} = \boldsymbol{c}^T\, \boldsymbol{K}^j(\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{k}\boldsymbol{K}^i\boldsymbol{r}, \tag{3.23b}$$

$$S^{(2i,2j+1)} = \boldsymbol{c}^T\, \boldsymbol{K}^j\boldsymbol{k}(\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{3.23c}$$

$$S^{(2i+1,2j+1)} = \boldsymbol{c}^T\, \boldsymbol{K}^j\boldsymbol{k}(\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{k}\boldsymbol{K}^i\boldsymbol{r}. \tag{3.23d}$$

1. Recurrence relations of $S^{(i,j)}$

For these elements (3.23) we present the following relations.

Proposition 3.4. For the scalar functions $S^{(i,j)}$ defined in (3.23) with $\{\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{k}, \boldsymbol{r}, \boldsymbol{c}\}$ satisfying the Sylvester equation (1.6) and the matrix relation (1.7), we have the following relations:

$$S^{(i,j+2s)} = S^{(i+2s,j)} - \sum_{l=0}^{s-1}(S^{(2s-2l-1,j)}S^{(i,2l)} - S^{(2s-2l-2,j)}S^{(i,2l+1)}), \tag{3.24a}$$

$$S^{(i,j-2s)} = S^{(i-2s,j)} + \sum_{l=1}^{s}(S^{(-2s+2l-1,j)}S^{(i,-2l)} - S^{(2l-2s-2,j)}S^{(i,-2l+1)}), \tag{3.24b}$$

where $s = 1, 2, \cdots$. In particular, when s = 1, one has

$$S^{(i,j+2)} = S^{(i+2,j)} - S^{(1,j)}S^{(i,0)} + S^{(0,j)}S^{(i,1)}, \tag{3.25a}$$

$$S^{(i,j-2)} = S^{(i-2,j)} + S^{(-1,j)}S^{(i,-2)} - S^{(-2,j)}S^{(i,-1)}. \tag{3.25b}$$

Proof. First, from the Sylvester equation (1.6) we have the following relation:

$$\boldsymbol{k}^s\boldsymbol{M} - (-1)^s\boldsymbol{M}\boldsymbol{k}^s = \sum_{j=0}^{s-1}(-1)^j\boldsymbol{k}^{s-j-1}(\boldsymbol{r}\boldsymbol{c}^T - g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1})\boldsymbol{k}^j, \quad (s = 1, 2, \ldots). \tag{3.26}$$

In fact, the Sylvester equation (1.6) itself is the case when $s = 1$ of (3.26), while (3.3b) is the case when $s = 2$. Making use of mathematical inductive approach we can reach (3.26). Similarly, from (3.2) one has a parallel result

$$\boldsymbol{K}^s\boldsymbol{M} - \boldsymbol{M}\boldsymbol{K}^s = \sum_{j=0}^{s-1}\boldsymbol{K}^{s-j-1}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T - \boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^j, \tag{3.27a}$$

or

$$\boldsymbol{M}\boldsymbol{K}^{-s} - \boldsymbol{K}^{-s}\boldsymbol{M} = \sum_{j=1}^{s}\boldsymbol{K}^{-(s-j+1)}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T - \boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^{-j}, \quad (s = 1, 2, \ldots). \tag{3.27b}$$

Now let us prove the relation (3.24a). We introduce the auxiliary vectors

$$\boldsymbol{u}^{(2i)} = (\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \quad \boldsymbol{u}^{(2i+1)} = (\boldsymbol{I} + \boldsymbol{M})^{-1}\boldsymbol{k}\boldsymbol{K}^i\boldsymbol{r}, \quad i \in \mathbb{Z}. \tag{3.28}$$

From this we immediately have

$$\boldsymbol{K}^s\boldsymbol{u}^{(2i)} + \boldsymbol{K}^s\boldsymbol{M}\boldsymbol{u}^{(2i)} = \boldsymbol{K}^{s+i}\boldsymbol{r}, \tag{3.29a}$$

$$\boldsymbol{K}^s\boldsymbol{u}^{(2i+1)} + \boldsymbol{K}^s\boldsymbol{M}\boldsymbol{u}^{(2i+1)} = \boldsymbol{k}\boldsymbol{K}^{s+i}\boldsymbol{r}. \tag{3.29b}$$

Replacing $\boldsymbol{K}^s\boldsymbol{M}$ using the relation (3.27a), one finds

$$(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{K}^s\boldsymbol{u}^{(2i)}=\boldsymbol{K}^{s+i}\boldsymbol{r}-\sum_{l=0}^{s-1}\boldsymbol{K}^{s-l-1}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T-\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^l\boldsymbol{u}^{(2i)}, \tag{3.30a}$$

$$(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{K}^s\boldsymbol{u}^{(2i+1)}=\boldsymbol{k}\boldsymbol{K}^{s+i}\boldsymbol{r}-\sum_{l=0}^{s-1}\boldsymbol{K}^{s-l-1}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T-\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^l\boldsymbol{u}^{(2i+1)}. \tag{3.30b}$$

These relations, left-multiplied by $\boldsymbol{c}^T\boldsymbol{K}^j(\boldsymbol{I}+\boldsymbol{M})^{-1}$, yield

$$S^{(2i,2j+2s)}=S^{(2i+2s,2j)}-\sum_{l=0}^{s-1}(S^{(2s-2l-1,2j)}S^{(2i,2l)}-S^{(2s-2l-2,2j)}S^{(2i,2l+1)}),$$

$$S^{(2i+1,2j+2s)}=S^{(2i+2s+1,2j)}-\sum_{l=0}^{s-1}(S^{(2s-2l-1,2j)}S^{(2i+1,2l)}-S^{(2s-2l-2,2j)}S^{(2i+1,2l+1)}),$$

and left-multiplied by $\boldsymbol{c}^T\boldsymbol{K}^j\boldsymbol{k}(\boldsymbol{I}+\boldsymbol{M})^{-1}$, yield

$$S^{(2i,2j+2s+1)}=S^{(2i+2s,2j+1)}-\sum_{l=0}^{s-1}(S^{(2s-2l-1,2j+1)}S^{(2i,2l)}-S^{(2s-2l-2,2j+1)}S^{(2i,2l+1)}),$$

$$S^{(2i+1,2j+2s+1)}=S^{(2i+2s+1,2j+1)}-\sum_{l=0}^{s-1}(S^{(2s-2l-1,2j+1)}S^{(2i+1,2l)}-S^{(2s-2l-2,2j+1)}S^{(2i+1,2l+1)}).$$

The above four equations are merged into the relation (3.24a).

The relation (3.24b) can be proved in a similar procedure, in which we use the following counterpart of (3.30):

$$(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{K}^{-s}\boldsymbol{u}^{(2i)}=\boldsymbol{K}^{-s+i}\boldsymbol{r}+\sum_{l=1}^{s}\boldsymbol{K}^{-(s-l+1)}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T-\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^{-l}\boldsymbol{u}^{(2i)}, \tag{3.31a}$$

$$(\boldsymbol{I}+\boldsymbol{M})\boldsymbol{K}^{-s}\boldsymbol{u}^{(2i+1)}=\boldsymbol{k}\boldsymbol{K}^{-s+i}\boldsymbol{r}+\sum_{l=1}^{s}\boldsymbol{K}^{-(s-l+1)}(\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T-\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k})\boldsymbol{K}^{-l}\boldsymbol{u}^{(2i+1)}, \tag{3.31b}$$

with $s=1,2,\ldots$. We note that (3.25) are corresponding to the algebraic relations (2.13) in Ref. 1. □

2. *Invariance and symmetry property of $S^{(i,j)}$*

In Secs. II B and III B A we have shown that the matrix relation (1.7) and the Sylvester equation (1.6) preserve invariance formally in terms of the similarity transformation (2.11) and notations (3.6). In following we will see that $S^{(i,j)}$ defined in (3.23) are the same as those defined with $\{\boldsymbol{c}_1,\boldsymbol{r}_1,\boldsymbol{k}_1,\boldsymbol{K}_1,\boldsymbol{M}_1\}$. Besides, $S^{(i,j)}$ satisfy symmetry property $S^{(i,j)}=S^{(j,i)}$.

Proposition 3.5. The matrix **S** *(or the element $S^{(i,j)}$) preserves invariance under the similarity transformation (2.11) and notations (3.6).*

Proof. Using (2.11) and (3.6) one can rewrite (3.23) and find

$$S^{(2i,2j)}=\boldsymbol{c}_1^T\boldsymbol{K}_1^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{3.32a}$$

$$S^{(2i+1,2j)}=\boldsymbol{c}_1^T\boldsymbol{K}_1^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{k}_1\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{3.32b}$$

$$S^{(2i,2j+1)}=\boldsymbol{c}_1^T\boldsymbol{K}_1^j\boldsymbol{k}_1(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{3.32c}$$

$$S^{(2i+1,2j+1)}=\boldsymbol{c}_1^T\boldsymbol{K}_1^j\boldsymbol{k}_1(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}\boldsymbol{k}_1\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{3.32d}$$

which means $S^{(i,j)}$ preserve invariance formally. □

In addition, we have the following symmetry property.

Proposition 3.6. Suppose that $\boldsymbol{M},\boldsymbol{K},\boldsymbol{k},\boldsymbol{r},\boldsymbol{c}$ satisfy the Sylvester equation (1.6) together with the matrix system (1.7) in which $\mathcal{E}(\boldsymbol{k})\cap\mathcal{E}(-\boldsymbol{k})=\varnothing$ and $\mathcal{E}(g\boldsymbol{K}^{-1})\cap\mathcal{E}(\boldsymbol{K})=\varnothing$. Then the scalar elements $S^{(i,j)}$ defined by (3.23) satisfy the symmetry property

$$S^{(i,j)}=S^{(j,i)}, \tag{3.33}$$

i.e., the infinite matrix $\boldsymbol{S}$ is symmetric.

Proof. Based on the invariance of $S^{(i,j)}$ presented in proposition 3.5, we only need to consider the proof when $\boldsymbol{k}$ is in its canonical form, i.e., $\boldsymbol{k}=\boldsymbol{\Gamma}_D^{[N]}(\{k_j\}_1^N)$, $\boldsymbol{k}=\boldsymbol{\Gamma}_J^{[N]}(k_1)$, and $\boldsymbol{k}=\boldsymbol{\Gamma}_G^{[N]}$. Corresponding to these three cases, $\boldsymbol{M}$ is given respectively in the three cases in Sec. III B 2. As an example let us consider the case $\boldsymbol{k}=\boldsymbol{\Gamma}_J^{[N]}(k_1)\in\mathcal{T}^{[N]}$. We note that from Sec. II B $\boldsymbol{K}\in\mathcal{T}^{[N]}$ as well. Let us look at the scalar function $S^{(2i,2j+1)}$ defined in (3.23c). We have

$$\begin{aligned}S^{(2i,2j+1)}&=(S^{(2i,2j+1)})^T\\&=\boldsymbol{r}^T\,(\boldsymbol{K}^i)^T(\boldsymbol{I}+\boldsymbol{M}^T)^{-1}(\boldsymbol{K}^j\boldsymbol{k})^T\boldsymbol{c}.\end{aligned} \tag{3.34}$$

Making use of the fact that $\boldsymbol{M}=\boldsymbol{F}\boldsymbol{G}\boldsymbol{H}$, $\boldsymbol{r}=\boldsymbol{F}\mathbf{e}_1^{[N]}$, $\boldsymbol{c}=\boldsymbol{H}\mathbf{e}_1^{[N]}$, $\boldsymbol{G}=\boldsymbol{G}^T$, $\boldsymbol{F}\in\mathcal{T}^{[N]}$, $\boldsymbol{H}\in\bar{\mathcal{T}}^{[N]}$, and Proposition A.3, we find from (3.34) that

$$\begin{aligned}S^{(2i,2j+1)}&=(\mathbf{e}_1^{[N]})^T\,\boldsymbol{F}^T(\boldsymbol{K}^i)^T(\boldsymbol{I}+\boldsymbol{H}\boldsymbol{G}\boldsymbol{F}^T)^{-1}(\boldsymbol{K}^j\boldsymbol{k})^T\boldsymbol{H}\mathbf{e}_1^{[N]}\\&=(\mathbf{e}_1^{[N]})^T\,(\boldsymbol{K}^i)^T\boldsymbol{F}^T(\boldsymbol{I}+\boldsymbol{H}\boldsymbol{G}\boldsymbol{F}^T)^{-1}\boldsymbol{H}\boldsymbol{K}^j\boldsymbol{k}\mathbf{e}_1^{[N]}\\&=(\mathbf{e}_1^{[N]})^T\,(\boldsymbol{K}^i)^T[\boldsymbol{H}^{-1}(\boldsymbol{F}^T)^{-1}+\boldsymbol{G}]^{-1}\boldsymbol{K}^j\boldsymbol{k}\mathbf{e}_1^{[N]}\\&=(\mathbf{e}_1^{[N]})^T\,(\boldsymbol{K}^i)^T(\boldsymbol{F}^{-1}\boldsymbol{H}^{-1}+\boldsymbol{G})^{-1}\boldsymbol{K}^j\boldsymbol{k}\mathbf{e}_1^{[N]}\\&=(\mathbf{e}_1^{[N]})^T\,(\boldsymbol{K}^i)^T\boldsymbol{H}(\boldsymbol{I}+\boldsymbol{F}\boldsymbol{G}\boldsymbol{H})^{-1}\boldsymbol{F}\boldsymbol{K}^j\boldsymbol{k}\mathbf{e}_1^{[N]}\\&=(\mathbf{e}_1^{[N]})^T\,\boldsymbol{H}\boldsymbol{K}^i(\boldsymbol{I}+\boldsymbol{F}\boldsymbol{G}\boldsymbol{H})^{-1}\boldsymbol{K}^j\boldsymbol{k}\boldsymbol{F}\mathbf{e}_1^{[N]}\\&=\boldsymbol{c}^T\,\boldsymbol{K}^i(\boldsymbol{I}+\boldsymbol{F}\boldsymbol{G}\boldsymbol{H})^{-1}\boldsymbol{K}^j\boldsymbol{k}\boldsymbol{r}\\&=S^{(2j+1,2i)}.\end{aligned}$$

In a similar way we can prove $S^{(i,j)}=S^{(j,i)}$ for arbitrary i,j. For the cases of $\boldsymbol{k}=\boldsymbol{\Gamma}_D^{[N]}(\{k_j\}_1^N)$ and $\boldsymbol{k}=\boldsymbol{\Gamma}_G^{[N]}$, we can also prove the symmetric property and for the later case we need to use Proposition A.4. □

Hereafter we always require that $\{\boldsymbol{M},\boldsymbol{K},\boldsymbol{k},\boldsymbol{r},\boldsymbol{c}\}$ satisfy the assumption of Proposition 3.6, under which we proceed with further discussions.

IV. THE ELLIPTIC LATTICE POTENTIAL KdV SYSTEM

In this section we derive the elpKdV system together with its Lax pair using the Cauchy matrix approach. In this approach $S^{(i,j)}$ play elementary roles. Since the elpKdV system can be viewed as an elliptic extension of the lattice potential KdV equation, we use the same dispersion relation and with the help of the Sylvester equation we can first derive a set of recurrence relations of $S^{(i,j)}$. Then the elpKdV system can be derived as closed forms.

A. Discrete dispersion relation and recurrence relations

Now let us impose discrete dispersion relation on $\boldsymbol{r}$ as follows:

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{r}}=(a\boldsymbol{I}+\boldsymbol{k})\boldsymbol{r},\quad(b\boldsymbol{I}-\boldsymbol{k})\widehat{\boldsymbol{r}}=(b\boldsymbol{I}+\boldsymbol{k})\boldsymbol{r},\quad a,b\notin\mathcal{E}(\pm\boldsymbol{k}), \tag{4.1}$$

while we take $\boldsymbol{c}$ to be a constant vector.

By applying a similar procedure as done in Ref. 8 to the Sylvester equation (1.6), matrix system (1.7), and the dispersion relation (4.1), one can derive the following shift relations of $\boldsymbol{M}$:

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{M}}=(a\boldsymbol{I}+\boldsymbol{k})\boldsymbol{M}, \tag{4.2a}$$

$$(b\boldsymbol{I}-\boldsymbol{k})\widehat{\boldsymbol{M}}=(b\boldsymbol{I}+\boldsymbol{k})\boldsymbol{M}, \tag{4.2b}$$

and

$$\widetilde{\boldsymbol{M}}(a\boldsymbol{I}+\boldsymbol{k})-(a\boldsymbol{I}+\boldsymbol{k})\boldsymbol{M}=\widetilde{\boldsymbol{r}}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\widetilde{\boldsymbol{r}}\boldsymbol{c}^T\boldsymbol{K}^{-1}, \tag{4.3a}$$

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{M}}-\boldsymbol{M}(a\boldsymbol{I}-\boldsymbol{k})=\boldsymbol{r}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1}, \tag{4.3b}$$

$$\widehat{\boldsymbol{M}}(b\boldsymbol{I}+\boldsymbol{k})-(b\boldsymbol{I}+\boldsymbol{k})\boldsymbol{M}=\widehat{\boldsymbol{r}}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\widehat{\boldsymbol{r}}\boldsymbol{c}^T\boldsymbol{K}^{-1}, \tag{4.3c}$$

$$(b\boldsymbol{I}-\boldsymbol{k})\widehat{\boldsymbol{M}}-\boldsymbol{M}(b\boldsymbol{I}-\boldsymbol{k})=\boldsymbol{r}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1}. \tag{4.3d}$$

These relations lead to the following results.

Proposition 4.1. Under the assumption of Proposition 3.6 and the dispersion relation (4.1), the scalar functions $S^{(i,j)}$ defined by (3.23) satisfy the following recurrence relations:

$$a\widetilde{S}^{(2i,2j)}-\widetilde{S}^{(2i,2j+1)}=aS^{(2i,2j)}+S^{(2i+1,2j)}-\widetilde{S}^{(2i,0)}S^{(0,2j)}+g\widetilde{S}^{(2i,-2)}S^{(-2,2j)}, \tag{4.4a}$$

$$aS^{(2i,2j)}+S^{(2i,2j+1)}=a\widetilde{S}^{(2i,2j)}-\widetilde{S}^{(2i+1,2j)}+S^{(2i,0)}\widetilde{S}^{(0,2j)}-gS^{(2i,-2)}\widetilde{S}^{(-2,2j)}, \tag{4.4b}$$

$$b\widehat{S}^{(2i,2j)}-\widehat{S}^{(2i,2j+1)}=bS^{(2i,2j)}+S^{(2i+1,2j)}-\widehat{S}^{(2i,0)}S^{(0,2j)}+g\widehat{S}^{(2i,-2)}S^{(-2,2j)}, \tag{4.4c}$$

$$bS^{(2i,2j)}+S^{(2i,2j+1)}=b\widehat{S}^{(2i,2j)}-\widehat{S}^{(2i+1,2j)}+S^{(2i,0)}\widehat{S}^{(0,2j)}-gS^{(2i,-2)}\widehat{S}^{(-2,2j)}, \tag{4.4d}$$

$$\begin{aligned}a\widetilde{S}^{(2i+1,2j)}-\widetilde{S}^{(2i+1,2j+1)}=&\,aS^{(2i+1,2j)}+S^{(2i+2,2j)}-\widetilde{S}^{(2i+1,0)}S^{(0,2j)}\\&+gS^{(2i-2,2j)}+g\widetilde{S}^{(2i+1,-2)}S^{(-2,2j)}+3e_1S^{(2i,2j)},\end{aligned} \tag{4.4e}$$

$$\begin{aligned}aS^{(2i+1,2j)}+S^{(2i+1,2j+1)}=&\,a\widetilde{S}^{(2i+1,2j)}-\widetilde{S}^{(2i+2,2j)}+S^{(2i+1,0)}\widetilde{S}^{(0,2j)}\\&-g\widetilde{S}^{(2i-2,2j)}-g\widetilde{S}^{(2i+1,-2)}S^{(-2,2j)}-3e_1\widetilde{S}^{(2i,2j)},\end{aligned} \tag{4.4f}$$

$$\begin{aligned}b\widehat{S}^{(2i+1,2j)}-\widehat{S}^{(2i+1,2j+1)}=&\,bS^{(2i+1,2j)}+S^{(2i+2,2j)}-\widehat{S}^{(2i+1,0)}S^{(0,2j)}\\&+gS^{(2i-2,2j)}+g\widehat{S}^{(2i+1,-2)}S^{(-2,2j)}+3e_1S^{(2i,2j)},\end{aligned} \tag{4.4g}$$

$$\begin{aligned}bS^{(2i+1,2j)}+S^{(2i+1,2j+1)}=&\,b\widehat{S}^{(2i+1,2j)}-\widehat{S}^{(2i+2,2j)}+S^{(2i+1,0)}\widehat{S}^{(0,2j)}\\&-g\widehat{S}^{(2i-2,2j)}-g\widehat{S}^{(2i+1,-2)}S^{(-2,2j)}-3e_1\widehat{S}^{(2i,2j)}.\end{aligned} \tag{4.4h}$$

The proof of this proposition is similar to the one for theorem 2 in Ref. 8. Here we skip the details. We also note that these relations correspond to the discrete matrix Riccati type of relations (2.12) in Ref. 1.

B. Elliptic lattice equations

To obtain elliptic lattice equations, we introduce scalar functions (cf. Ref. 1)

$$u=S^{(0,0)},\quad s=S^{(-2,0)},\quad h=S^{(-2,-2)},\quad v=1-S^{(-1,0)},\quad w=1+S^{(-2,1)}. \tag{4.5}$$

It then follows from (4.4) that

$$a(u-\widetilde{u})=-(\widetilde{S}^{(0,1)}+S^{(0,1)})-gs\widetilde{s}+u\widetilde{u}, \tag{4.6a}$$

$$a(h-\widetilde{h})=-(\widetilde{S}^{(-1,-2)}+S^{(-1,-2)})-gh\widetilde{h}+s\widetilde{s}, \tag{4.6b}$$

$$a(s-\widetilde{s})=g\widetilde{h}s+\widetilde{w}-v-u\widetilde{s}, \tag{4.6c}$$

$$a(s-\widetilde{s})=gh\widetilde{s}+w-\widetilde{v}-\widetilde{u}s, \tag{4.6d}$$

$$a(v-\widetilde{v})=\widetilde{S}^{(-1,1)}+3e_1s+u\widetilde{v}+gvh+gs(S^{(-2,-1)}+\widetilde{S}^{(-2,-1)}), \tag{4.6e}$$

$$a(v-\widetilde{v})=S^{(-1,1)}+3e_1\widetilde{s}+\widetilde{u}v+g\widetilde{v}\widetilde{h}+g\widetilde{s}(S^{(-2,-1)}+\widetilde{S}^{(-2,-1)}), \tag{4.6f}$$

$$a(w-\widetilde{w})=\widetilde{S}^{(-1,1)}+3e_1s+uw+gh\widetilde{w}-s(S^{(0,1)}+\widetilde{S}^{(0,1)}), \tag{4.6g}$$

$$a(w-\widetilde{w})=S^{(-1,1)}+3e_1\widetilde{s}+\widetilde{u}\widetilde{w}+g\widetilde{h}w-\widetilde{s}(S^{(0,1)}+S^{(0,1)}), \tag{4.6h}$$

where we have made use of the symmetric property $S^{(i,j)} = S^{(j,i)}$ and the relation (3.25b) with $(i,j)=(0,-2)$, i.e.,

$$S^{(0,-4)} = S^{(-2,-2)} + S^{(-1,-2)}S^{(0,-2)} - S^{(-2,-2)}S^{(0,-1)} = hv + sS^{(-2,-1)}.$$

One more relation obtainable from (3.25b) is

$$sS^{(-1,1)} = 1 - vw, \tag{4.6i}$$

by taking $(i,j) = (0,1)$. The shift relations with respect to $\{b,\widehat{\cdot}\}$ can be obtained by interchanging the relations $\{a,\widetilde{\cdot}\}$ with $\{b,\widehat{\cdot}\}$ in (4.6).

With these relations in hand, we can combine them, eliminate $S^{(0,1)}$, $S^{(-1,-2)}$, h and v in order to obtain the closed form of elliptic lattice equations of the variables u, s, and w, (cf. Ref. 1)

$$(a+b+u-\widehat{\widetilde{u}})(a-b+\widehat{u}-\widetilde{u}) = a^2-b^2+g(\widetilde{s}-\widehat{s})(\widehat{\widetilde{s}}-s), \tag{4.7a}$$

$$(\widehat{\widetilde{s}}-s)(\widetilde{w}-\widehat{w}) = [(a+u)\widetilde{s}-(b+u)\widehat{s}]\widehat{\widetilde{s}}-[(a-\widehat{\widetilde{u}})\widehat{s}-(b-\widehat{\widetilde{u}})\widetilde{s}]s, \tag{4.7b}$$

$$(\widehat{s}-\widetilde{s})(\widehat{\widetilde{w}}-w) = [(a-\widetilde{u})s+(b+\widetilde{u})\widehat{\widetilde{s}}]\widehat{s}-[(a+\widehat{u})\widehat{\widetilde{s}}+(b-\widehat{u})s]\widetilde{s}, \tag{4.7c}$$

$$(a+u-\frac{\widetilde{w}}{\widetilde{s}})(a-\widetilde{u}+\frac{w}{s}) = a^2-R(s\widetilde{s}), \tag{4.7d}$$

$$(b+u-\frac{\widehat{w}}{\widehat{s}})(b-\widehat{u}+\frac{w}{s}) = b^2-R(s\widehat{s}), \tag{4.7e}$$

where

$$y^2 = R(x) = \frac{1}{x} + 3e_1 + gx. \tag{4.8}$$

Clearly, the system (4.7) is an elliptic generalization of the lpKdV equation. If $g=0$, (4.7a) decouples and becomes the standard lattice potential KdV equation.

Furthermore, solutions to the elpKdV system (4.7) can be expressed in the explicit structure

$$u = S^{(0,0)} = \boldsymbol{c}^T\,(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{r}, \tag{4.9a}$$

$$s = S^{(-2,0)} = \boldsymbol{c}^T\,(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \tag{4.9b}$$

$$h = S^{(-2,-2)} = \boldsymbol{c}^T\,\boldsymbol{K}^{-1}(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}, \tag{4.9c}$$

$$v = 1 - S^{(-1,0)} = 1 - \boldsymbol{c}^T\,(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{k}\boldsymbol{K}^{-1}\boldsymbol{r}, \tag{4.9d}$$

$$w = 1 + S^{(-2,1)} = 1 + \boldsymbol{c}^T\,\boldsymbol{k}(\boldsymbol{I}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r}. \tag{4.9e}$$

We only need to solve for $\boldsymbol{r}$ in (4.1) with $\boldsymbol{k}$ taking the three cases in (B1). Explicit forms of $\boldsymbol{r}$ are given in Appendix C.

C. Lax pair for the elpKdV system (4.9)

Rewriting (3.28) results in

$$\boldsymbol{K}^i\boldsymbol{r} = (\boldsymbol{I}+\boldsymbol{M})\boldsymbol{u}^{(2i)}, \tag{4.10a}$$

$$\boldsymbol{k}\boldsymbol{K}^i\boldsymbol{r} = (\boldsymbol{I}+\boldsymbol{M})\boldsymbol{u}^{(2i+1)}, \quad i\in\mathbb{Z}, \tag{4.10b}$$

then we perform a tilde shift to (4.10a) and multiply the result by $(a\boldsymbol{I}-\boldsymbol{k})$ to give

$$\boldsymbol{K}^i(a\boldsymbol{I}+\boldsymbol{k})\boldsymbol{r} = (\boldsymbol{I}+\boldsymbol{M})(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{u}}^{(2i)} + (\boldsymbol{r}\boldsymbol{c}^T - g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1})\widetilde{\boldsymbol{u}}^{(2i)}, \tag{4.11}$$

where we also make use of the relation $\boldsymbol{k}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{k}$, the dispersion relation (4.1), the shift relation (4.2a), and the Sylvester equation (1.6). (4.11) is further transformed to

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{u}}^{(2i)} = a\boldsymbol{u}^{(2i)} + \boldsymbol{u}^{(2i+1)} - \widetilde{S}^{(2i,0)}\boldsymbol{u}^{(0)} + g\widetilde{S}^{(2i,-2)}\boldsymbol{u}^{(-2)}. \tag{4.12}$$

Taking $i=0$ leads to

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{u}}^{(0)} = (a-\widetilde{u})\boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)} + g\widetilde{s}\boldsymbol{K}^{-1}(w\boldsymbol{u}^{(0)} - s\boldsymbol{u}^{(1)}), \tag{4.13}$$

which is obtained by replacing $\boldsymbol{u}^{(-2)}$ with

$$\boldsymbol{u}^{(-2)} = \boldsymbol{K}^{-1}(w\boldsymbol{u}^{(0)} - s\boldsymbol{u}^{(1)}). \tag{4.14}$$

In a similar way, from (4.10b) we have

$$(a\boldsymbol{I}-\boldsymbol{k})\widetilde{\boldsymbol{u}}^{(1)}=(\boldsymbol{K}+g\widetilde{w}w\boldsymbol{K}^{-1})\boldsymbol{u}^{(0)}+(3e_1+g\widetilde{s}s+a(u-\widetilde{u})-u\widetilde{u})\boldsymbol{u}^{(0)}+(a+u-g\widetilde{w}s\boldsymbol{K}^{-1})\boldsymbol{u}^{(1)}, \tag{4.15}$$

where

$$\boldsymbol{u}^{(2)}=\boldsymbol{K}\boldsymbol{u}^{(0)}-\boldsymbol{S}^{(0,1)}\boldsymbol{u}^{(0)}+u\boldsymbol{u}^{(1)}. \tag{4.16}$$

Clearly, under the similarity transformation (2.11), Equations (4.13) and (4.15) are formally invariant if we define $\boldsymbol{u}_1^{(0)}=\boldsymbol{T}\boldsymbol{u}^{(0)}$, $\boldsymbol{u}_1^{1}=\boldsymbol{T}\boldsymbol{u}^{1}$. This means that we can directly consider that $\boldsymbol{k}$ is in its canonical form and $\boldsymbol{K}$ is consequently defined by (1.7). Thus, the first row of $\boldsymbol{k}$ will be $(k,0,0,\ldots,0)$, while the first rows of $\boldsymbol{K}$ and $\boldsymbol{K}^{-1}$ have to be $(K,0,0,\ldots,0)$ and $(1/K,0,0,\ldots,0)$ respectively, where (k,K) obeys the elliptic curve (2.1).

Let us denote the first element of $\boldsymbol{u}^{(0)}$ by $(\boldsymbol{u}^{(0)})_1$ and the first element of $\boldsymbol{u}^1$ by $(\boldsymbol{u}^1)_1$, and introduce the vector

$$\phi=\begin{pmatrix}(\boldsymbol{u}^{(0)})_1\\(\boldsymbol{u}^{(1)})_1\end{pmatrix}. \tag{4.17}$$

Then, from (4.13) and (4.15) and with a replaced by b and $\widetilde{\ }$ by $\widehat{\ }$ we obtain the following discrete linear system:

$$(a-k)\widetilde{\phi}=L(K)\phi, \tag{4.18a}$$

$$(b-k)\widehat{\phi}=M(K)\phi, \tag{4.18b}$$

where

$$L(K)=\begin{pmatrix}a-\widetilde{u}+\dfrac{g}{K}\widetilde{s}w & 1-\dfrac{g}{K}\widetilde{s}s\\ K+3e+g\widetilde{s}s+a(u-\widetilde{u})-\widetilde{u}u+\dfrac{g}{K}\widetilde{w}w & a+u-\dfrac{g}{K}\widetilde{w}s\end{pmatrix} \tag{4.19}$$

which is the same as in Ref. 1, and $M(K)$ is the $(b,\widehat{\ })$ counterpart of $L(K)$. The point (k,K) obeys the elliptic curve (2.1) and here they play the roles of spectral parameters. The compatibility condition

$$\widehat{L}M=\widetilde{M}L \tag{4.20}$$

yields the whole elpKdV system (4.7) with the exception of the Equation (4.7c).

V. THE ELLIPTIC POTENTIAL KdV SYSTEM

In this section we derive a continuous elliptic potential KdV system. The procedure is similar to the one for the KdV system in Ref. 7. This is a continuous version of the Cauchy matrix approach, where the recurrence relations of $S^{(i,j)}$ (see (2.3) in Ref. 7 and (3.25) in this paper) play key roles.

A. Evolution of *M*

We assume that $\boldsymbol{M},\boldsymbol{r},\boldsymbol{c}$ are functions of (x,t) while $\boldsymbol{k}$ is still a non-trivial constant matrix. The dispersion relation is now defined through the evolution of $\boldsymbol{r}$ and $\boldsymbol{c}$ as follows:

$$\boldsymbol{r}_x=\boldsymbol{k}\boldsymbol{r},\quad \boldsymbol{c}_x=\boldsymbol{k}^T\boldsymbol{c}, \tag{5.1a}$$

$$\boldsymbol{r}_t=4\boldsymbol{k}^3\boldsymbol{r},\quad \boldsymbol{c}_t=4(\boldsymbol{k}^T)^3\boldsymbol{c}. \tag{5.1b}$$

Taking the derivative of the Sylvester equation (1.6) with respect to x and making use of (5.1a) we have

$$\begin{aligned}\boldsymbol{k}\boldsymbol{M}_x+\boldsymbol{M}_x\boldsymbol{k}&=\boldsymbol{r}_x\boldsymbol{c}^T+\boldsymbol{r}\boldsymbol{c}_x^T-g\boldsymbol{K}^{-1}\boldsymbol{r}_x\boldsymbol{c}^T\boldsymbol{K}^{-1}-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}_x^T\boldsymbol{K}^{-1}\\&=\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T+\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k}-g\boldsymbol{K}^{-1}\boldsymbol{k}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1}-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{k}\boldsymbol{K}^{-1},\end{aligned}$$

i.e.,

$$\boldsymbol{k}(\boldsymbol{M}_x-\boldsymbol{r}\boldsymbol{c}^T+g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1})+(\boldsymbol{M}_x-\boldsymbol{r}\boldsymbol{c}^T+g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1})\boldsymbol{k}=0,$$

where we have made use of the relation $\boldsymbol{k}\boldsymbol{K}=\boldsymbol{K}\boldsymbol{k}$. Using Proposition 3.1, this yields

$$\boldsymbol{M}_x=\boldsymbol{r}\boldsymbol{c}^T-g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1}, \tag{5.2}$$

i.e.,

$$\boldsymbol{M}_x = \boldsymbol{k}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{k}, \tag{5.3}$$

if we use the Sylvester equation (1.6). In a similar way, for the time evolution of $\boldsymbol{M}$, we have

$$\boldsymbol{M}_t = 4(\boldsymbol{k}^3\boldsymbol{M} + \boldsymbol{M}\boldsymbol{k}^3). \tag{5.4}$$

B. Evolution of $S^{(i,j)}$

With the evolution formulas (5.1)–(5.4), we can derive the evolution of $S^{(i,j)}$. We make use of the auxiliary vectors $\boldsymbol{u}^{(i)}$ defined in (3.28), i.e.,

$$(\boldsymbol{I} + \boldsymbol{M})\boldsymbol{u}^{(2i)} = \boldsymbol{K}^i\boldsymbol{r}, \tag{5.5a}$$

$$(\boldsymbol{I} + \boldsymbol{M})\boldsymbol{u}^{(2i+1)} = \boldsymbol{k}\boldsymbol{K}^i\boldsymbol{r}. \tag{5.5b}$$

By using the auxiliary vectors $\boldsymbol{u}^{(i)}$, $S^{(i,j)}$ are expressed as

$$S^{(2i,2j)} = \boldsymbol{c}^T\,\boldsymbol{K}^j\boldsymbol{u}^{(2i)}, \tag{5.6a}$$

$$S^{(2i+1,2j)} = \boldsymbol{c}^T\,\boldsymbol{K}^j\boldsymbol{u}^{(2i+1)}, \tag{5.6b}$$

$$S^{(2i,2j+1)} = \boldsymbol{c}^T\,\boldsymbol{K}^j\boldsymbol{k}\boldsymbol{u}^{(2i)}, \tag{5.6c}$$

$$S^{(2i+1,2j+1)} = \boldsymbol{c}^T\,\boldsymbol{K}^j\boldsymbol{k}\boldsymbol{u}^{(2i+1)}. \tag{5.6d}$$

Taking x-derivative on (5.5a) we have

$$\boldsymbol{M}_x\boldsymbol{u}^{(2i)} + (\boldsymbol{I} + \boldsymbol{M})\boldsymbol{u}_x^{(2i)} = \boldsymbol{K}^i\boldsymbol{r}_x = \boldsymbol{K}^i\boldsymbol{k}\boldsymbol{r}, \tag{5.7}$$

and further, by substitution of (5.2), we have

$$(\boldsymbol{I} + \boldsymbol{M})\boldsymbol{u}_x^{(2i)} = \boldsymbol{K}^i\boldsymbol{k}\boldsymbol{r} - (\boldsymbol{r}\boldsymbol{c}^T - g\boldsymbol{K}^{-1}\boldsymbol{r}\boldsymbol{c}^T\boldsymbol{K}^{-1})\boldsymbol{u}^{(2i)}, \tag{5.8}$$

which indicates the evolution of $\boldsymbol{u}^{(2i)}$ in x-direction

$$\boldsymbol{u}_x^{(2i)} = \boldsymbol{u}^{(2i+1)} - S^{(2i,0)}\boldsymbol{u}^{(0)} + gS^{(2i,-2)}\boldsymbol{u}^{(-2)}. \tag{5.9a}$$

Similarly we can derive the evolution of $\boldsymbol{u}^{(2i+1)}$ in x-direction,

$$\boldsymbol{u}_x^{(2i+1)} = \boldsymbol{u}^{(2i+2)} - S^{(2i+1,0)}\boldsymbol{u}^{(0)} + gS^{(2i+1,-2)}\boldsymbol{u}^{(-2)} + 3e_1\boldsymbol{u}^{(2i)} + g\boldsymbol{u}^{(2i-2)}, \tag{5.9b}$$

and consequently the evolution of $\boldsymbol{u}^{(i)}$ in t-direction

$$\begin{aligned}\boldsymbol{u}_t^{(2i)} = 4[&g\boldsymbol{u}^{(-2)}(S^{(2i,0)} + 3e_1S^{(2i,-2)} + gS^{(2i,-4)}) + gS^{(2i,-2)}(\boldsymbol{u}^{(0)} + 3e_1\boldsymbol{u}^{(-2)} + g\boldsymbol{u}^{(-4)})\\ &- \boldsymbol{u}^{(0)}(S^{(2i,2)} + 3e_1S^{(2i,0)} + gS^{(2i,-2)}) + S^{(2i,1)}\boldsymbol{u}^{(1)} - g\boldsymbol{u}^{(-1)}S^{(2i,-1)} + \boldsymbol{u}^{(2i+3)}\\ &+ 3e_1\boldsymbol{u}^{(2i+1)} + g\boldsymbol{u}^{(2i-1)} - S^{(2i,0)}(\boldsymbol{u}^{(2)} + 3e_1\boldsymbol{u}^{(0)} + g\boldsymbol{u}^{(-2)})],\end{aligned} \tag{5.10a}$$

$$\begin{aligned}\boldsymbol{u}_t^{(2i+1)} = 4[&g\boldsymbol{u}^{(-2)}(S^{(2i+1,0)} + 3e_1S^{(2i+1,-2)} + gS^{(2i+1,-4)}) - g\boldsymbol{u}^{(-1)}S^{(2i+1,-1)} + \boldsymbol{u}^{(2i+4)}\\ &- \boldsymbol{u}^{(0)}(S^{(2i+1,2)} + 3e_1S^{(2i+1,0)} + gS^{(2i+1,-2)}) + (9e_1^2 + 2g)\boldsymbol{u}^{(2i)} + g^2\boldsymbol{u}^{(2i-4)}\\ &+ 6e_1g\boldsymbol{u}^{(2i-2)} + 6e_1\boldsymbol{u}^{(2i+2)} - S^{(2i+1,0)}(\boldsymbol{u}^{(2)} + 3e_1\boldsymbol{u}^{(0)} + g\boldsymbol{u}^{(-2)})\\ &+ S^{(2i+1,1)}\boldsymbol{u}^{(1)} + gS^{(2i+1,-2)}(\boldsymbol{u}^{(0)} + 3e_1\boldsymbol{u}^{(-2)} + g\boldsymbol{u}^{(-4)})].\end{aligned} \tag{5.10b}$$

These evolutions of $\boldsymbol{u}^{(i)}$ can be transformed into the evolution of $S^{(i,j)}$,

$$S_x^{(2i,2j)} = S^{(2i+1,2j)} + S^{(2i,2j+1)} - S^{(2i,0)}S^{(0,2j)} + gS^{(2i,-2)}S^{(-2,2j)}, \tag{5.11a}$$

$$\begin{aligned}S_x^{(2i,2j+1)} = {}&S^{(2i,2j+2)} + 3e_1S^{(2i,2j)} + gS^{(2i,2j-2)} - S^{(2i,0)}S^{(0,2j+1)} + S^{(2i+1,2j+1)}\\ &+ gS^{(2i,-2)}S^{(-2,2j+1)},\end{aligned} \tag{5.11b}$$

$$\begin{aligned}S_x^{(2i+1,2j+1)} = {}&S^{(2i+1,2j+2)} + 3e_1S^{(2i+1,2j)} + gS^{(2i+1,2j-2)} - S^{(2i+1,0)}S^{(0,2j+1)} + S^{(2i+2,2j+1)}\\ &+ gS^{(2i+1,-2)}S^{(-2,2j+1)} + 3e_1S^{(2i,2j+1)} + gS^{(2i-2,2j+1)},\end{aligned} \tag{5.11c}$$

and

$$
\begin{aligned}
S_t^{(2i,2j)} = &\, 4[gS^{(-2,2j)}(S^{(2i,0)} + 3e_1S^{(2i,-2)} + gS^{(2i,-4)}) + gS^{(2i,-2)}(S^{(0,2j)} + 3e_1S^{(-2,2j)} \\
& + gS^{(-4,2j)}) - S^{(0,2j)}(S^{(2i,2)} + 3e_1S^{(2i,0)} + gS^{(2i,-2)}) + S^{(2i,1)}S^{(1,2j)} \\
& - gS^{(-1,2j)}S^{(2i,-1)} + S^{(2i+3,2j)} + 3e_1S^{(2i+1,2j)} + gS^{(2i-1,2j)} - S^{(2i,0)}(S^{(2,2j)} \\
& + 3e_1S^{(0,2j)} + gS^{(-2,2j)}) + S^{(2i,2j+3)} + 3e_1S^{(2i,2j+1)} + gS^{(2i,2j-1)}], \qquad (5.12a)\\
S_t^{(2i,2j+1)} = &\, 4[gS^{(-2,2j+1)}(S^{(2i,0)} + 3e_1S^{(2i,-2)} + gS^{(2i,-4)}) + gS^{(2i,-2)}(S^{(0,2j+1)} \\
& + 3e_1S^{(-2,2j+1)} + gS^{(-4,2j+1)}) - S^{(0,2j+1)}(S^{(2i,2)} + 3e_1S^{(2i,0)} + gS^{(2i,-2)}) \\
& + S^{(2i,1)}S^{(1,2j+1)} - gS^{(-1,2j+1)}S^{(2i,-1)} + S^{(2i+3,2j+1)} + 3e_1S^{(2i+1,2j+1)} \\
& + gS^{(2i-1,2j+1)} - S^{(2i,0)}(S^{(2,2j+1)} + 3e_1S^{(0,2j+1)} + gS^{(-2,2j+1)}) + S^{(2i,2j+4)} \\
& + 6e_1S^{(2i,2j+2)} + (9e_1^2 + 2g)S^{(2i,2j)} + 6e_1gS^{(2i,2j-2)} + g^2S^{(2i,2j-4)}], \qquad (5.12b)\\
S_t^{(2i+1,2j+1)} = &\, 4[gS^{(-2,2j+1)}(S^{(2i+1,0)} + 3e_1S^{(2i+1,-2)} + gS^{(2i+1,-4)}) + gS^{(2i+1,-2)}(S^{(0,2j+1)} \\
& + 3e_1S^{(-2,2j+1)} + gS^{(-4,2j+1)}) - S^{(0,2j+1)}(S^{(2i+1,2)} + 3e_1S^{(2i+1,0)} + gS^{(2i+1,-2)}) \\
& + S^{(2i+1,1)}S^{(1,2j+1)} - gS^{(-1,2j+1)}S^{(2i+1,-1)} + S^{(2i+4,2j+1)} + 6e_1S^{(2i+2,2j+1)} \\
& + 6e_1S^{(2i+1,2j+2)} + (9e_1^2 + 2g)S^{(2i+1,2j)} + 6e_1gS^{(2i+1,2j-2)} + g^2S^{(2i+1,2j-4)} \\
& - S^{(2i+1,0)}(S^{(2,2j+1)} + 3e_1S^{(0,2j+1)} + gS^{(-2,2j+1)}) + S^{(2i+1,2j+4)} \\
& + (9e_1^2 + 2g)S^{(2i,2j+1)} + 6e_1gS^{(2i-2,2j+1)} + g^2S^{(2i-4,2j+1)}]. \qquad (5.12c)
\end{aligned}
$$

It is unnecessary to write $S_x^{(2i+1,2j)}$ and $S_t^{(2i+1,2j)}$ due to the symmetry property $S^{(i,j)} = S^{(j,i)}$. One can repeatedly use (5.11) and easily get higher-order x-derivatives of $S^{(i,j)}$ by means of Mathematica, amongst other computer applications.

These derivatives of $S^{(i,j)}$ bring the following epKdV system:[1]

$$u_t = u_{xxx} + 6u_x^2 - 6gs_x^2, \qquad (5.13a)$$

$$s_t = s_{xxx} + 6u_xs_x - 6gs_xh_x, \qquad (5.13b)$$

$$h_t = h_{xxx} + 6s_x^2 - 6gh_x^2, \qquad (5.13c)$$

$$v_t = v_{xxx} + 6v_xu_x + 6gs_xS_x^{(-1,-2)}, \qquad (5.13d)$$

$$w_t = w_{xxx} + 6s_xS_x^{(0,1)} - 6gw_xh_x, \qquad (5.13e)$$

where u, v, s, w, h are as defined in (4.5) or (4.9), and from (5.11a) we have

$$S^{(-1,-2)} = \frac{1}{2}(h_x + s^2 - gh^2), \qquad (5.14a)$$

$$S^{(0,1)} = \frac{1}{2}(u_x + u^2 - gs^2). \qquad (5.14b)$$

To achieve the derivation of (5.13), one needs to perform long and tedious iterations in which the recurrence relations (3.25) are successively used. Taking the first equation (5.13a) as an example, we substitute the expressions of u_t, u_{xxx}, u_x, s_x and obtain

$$
\begin{aligned}
&\frac{1}{6}(u_t - u_{xxx} - 6u_x^2 + 6gs_x^2)\\
&= g^2S^{(0,-2)}[S^{(0,-4)} - S^{(-1,-2)}S^{(0,-2)} + S^{(-2,-2)}(-1 + S^{(0,-1)})]\\
&\quad + g[-S^{(0,-2)}(-S^{(0,0)}(1 + S^{(1,-2)}) + S^{(2,-2)} + S^{(0,-2)}S^{(1,0)})\\
&\quad - S^{(1,-2)} + S^{(0,-1)}(1 + S^{(1,-2)}) - S^{(1,-1)}S^{(0,-2)}]\\
&\quad - {S^{(1,0)}}^2 + S^{(0,0)}S^{(1,1)} - S^{(2,1)} + S^{(3,0)}.
\end{aligned}
$$

This equation vanishes in light of the recurrence relations (3.25) with $(i, j) = (0, -2)$ and $(0, 1)$. Equations (5.13b)–(5.13e) can rigorously be derived in a similar manner.

In Ref. 1, the relation

$$A = -u + \frac{w}{s} \tag{5.15}$$

is introduced, through which the epKdV system (5.13) yielded the following coupled equation, (i.e. (1.2)):

$$s_t = 4s_{xxx} + 6s_x[R(s^2) - A^2 - \frac{2As_x}{s} - \frac{2s_{xx}}{s}], \tag{5.16a}$$

$$A_t = 4A_{xxx} - 6A^2A_x + 6A_xR(s^2) - \frac{6s_x}{s}(R(s^2))_x, \tag{5.16b}$$

with the elliptic curve $R(x)$ given in (1.3). Since this coupled system admits a continuous Lax pair[1] (also see Sec. V C) we conclude that it is integrable.

C. Lax pair

Let us consider (5.9) with $i = 0$,

$$\boldsymbol{u}_x^{(0)} = \boldsymbol{u}^{(1)} - u\boldsymbol{u}^{(0)} + gs\boldsymbol{u}^{(-2)}, \tag{5.17a}$$

$$\boldsymbol{u}_x^{(1)} = \boldsymbol{u}^{(2)} - S^{(1,0)}\boldsymbol{u}^{(0)} + gw\boldsymbol{u}^{(-2)} + 3e_1\boldsymbol{u}^{(0)}. \tag{5.17b}$$

After replacing $\boldsymbol{u}^{(-2)}$, $\boldsymbol{u}^2$, and $S^{1,0}$ with (4.14), (4.16), and (5.14b) respectively, we have

$$\boldsymbol{u}_x^{(0)} = \boldsymbol{u}^{(1)} - u\boldsymbol{u}^{(0)} + gs\boldsymbol{K}^{-1}(w\boldsymbol{u}^{(0)} - s\boldsymbol{u}^{(1)}), \tag{5.18a}$$

$$\boldsymbol{u}_x^{(1)} = (\boldsymbol{K} + gw^2\boldsymbol{K}^{-1})\boldsymbol{u}^{(0)} + (3e_1 - u_x - u^2 + gs^2)\boldsymbol{u}^{(0)} + (u\boldsymbol{I} - gsw\boldsymbol{K}^{-1})\boldsymbol{u}^{(1)}. \tag{5.18b}$$

Similarly to the discrete case, we consider $\boldsymbol{k}$ to be in its canonical form and then from the first rows of (5.18a) and (5.18b) we find the linear form

$$\phi_x = \begin{pmatrix} -u + \frac{g}{K}sw & 1 - \frac{g}{K}s^2 \\ K + 3e + gs^2 - u^2 - u_x + \frac{g}{K}w^2 & u - \frac{g}{K}ws \end{pmatrix}\phi, \tag{5.19a}$$

where ϕ is defined in (4.17). In a similar way from (5.10) we can find the time evolution of ϕ, which is formulated as

$$\phi_t = -\begin{pmatrix} S_x^{(0,1)} & u_x \\ S_x^{(1,1)} & S_x^{(0,1)} \end{pmatrix}\phi - \frac{g}{K}\begin{pmatrix} (1-vw)\frac{s_x}{s} + v_xw & vs_x - sv_x \\ (1-vw)\frac{w_x}{s} - w(\frac{1-vw}{s})_x & -(1-vw)\frac{s_x}{s} - v_xw \end{pmatrix}\phi, \tag{5.19b}$$

where

$$S_x^{(1,1)} = \frac{1}{2}S_{xx}^{(0,1)} + uS_x^{(0,1)} - gsw_x, \tag{5.20}$$

and $S^{(0,1)}$ is given by (5.14b).

(5.19) can be viewed as a Lax pair of the system (5.13), which can also be derived from the direct linearization approach.[1] The compatibility property gives equations (5.13a), (5.13b), and (5.13e).

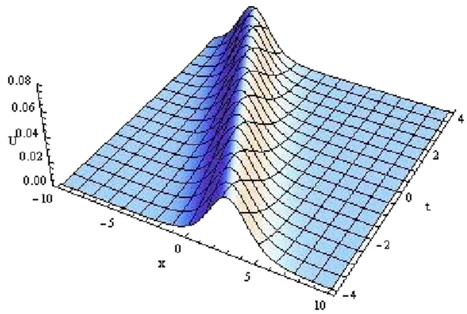
VI. DYNAMICS OF SOLUTIONS

In the section, we investigate dynamics of two-soliton solutions for epKdV system (5.13) by taking the solution u as an example. One will find that what we illustrate in the following is the derivative of u with respect to x since the system (5.13) is a potential system.

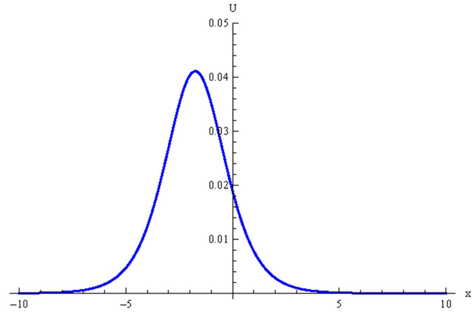
Soliton solutions of u are given by (4.9) with (3.12), (C1), and (C5). For one-soliton solution, u is written as

$$u_{1ss} = \frac{2k_1e^{2(k_1x+4k_1{}^3t)}}{2k_1 + (1 - g/K_1{}^2)e^{2(k_1x+4k_1{}^3t)}}. \tag{6.1}$$

Here we note that it is possible to get real solitons if we have real-valued $\wp(\kappa_1)$. This can be done by taking real invariants g_2 and g_3 and real or pure imaginary κ_1 (cf. Ref. 10). Hereafter, we set $U = u_x$ and the expression for U is



(a)



(b)

FIG. 2. The one-soliton solution given by (6.2) for $e_1 = 1, e_2 = 1.5$ (which consequently determine the invariants $g_2 = 19$, $g_3 = -15$ and then the Weierstrass elliptic $\wp$-function) and $\kappa_1 = -1.1$ (which indicates the values of k_1,K_1,e_1,g by (2.7)). (a) Shape and motion. (b) Soliton wave at $t = 0$.

$$U_{1ss} = \frac{8{k_1}^3{K_1}^4 e^{2(k_1x+4{k_1}^3t)}}{[2k_1{K_1}^2 + (-g + {K_1}^2)e^{2(k_1x+4{k_1}^3t)}]^2}, \tag{6.2}$$

which is depicted in Fig. 2. From Fig. 2, we can see that this soliton is identified by the amplitude and the top trace (i.e., the straight line on (x,t)-plane on which U_{1ss} takes maximum value) clearly, and after some calculations we get the amplitude

$$\mathrm{Amp} = \frac{{k_1}^2{K_1}^2}{-g + {K_1}^2}, \tag{6.3}$$

and the top trace

$$x(t) = -4{k_1}^2 t + \frac{1}{2k_1}\ln\left|\frac{2k_1{K_1}^2}{-g + {K_1}^2}\right|. \tag{6.4}$$

Obviously, the soliton of U is a single-direction wave with the velocity $-4{k_1}^2$ and the amplitude can be negative when $-g + {K_1}^2 < 0$.

Next, let us look at the 2-soliton solution. The 2-soliton solution for U is written as

$$U = \left(\frac{f}{g}\right)_x, \tag{6.5a}$$

where

$$\begin{aligned} f &= 2(k_1 + k_2)[2k_1k_2{K_1}^2{K_2}^2(k_1 + k_2)(e^{2\xi_1} + e^{2\xi_2}) + ({K_1}^2{K_2}^2(k_1 - k_2)^2 - g({k_1}^2{K_1}^2 \\ &\quad + {k_2}^2{K_2}^2) - k_1k_2({K_1}^2 - 4K_1K_2 + {K_2}^2))e^{2(\xi_1+\xi_2)}], \end{aligned} \tag{6.5b}$$

$$\begin{aligned} g &= -2k_2{K_2}^2(k_1 + k_2)^2(g - {K_1}^2)e^{2\xi_1} - 2k_1{K_1}^2(k_1 + k_2)^2(g - {K_2}^2)e^{2\xi_2} \\ &\quad + 4k_1k_2{K_1}^2{K_2}^2(k_1 + k_2)^2 + [(g^2 + {K_1}^2)(k_1 - k_2)^2 - g((k_1 + k_2)^2(K_1 + K_2)^2 \\ &\quad + 2k_1k_2({K_1}^2 - 4K_1K_2 + {K_2}^2))]e^{2(\xi_1+\xi_2)}, \quad \xi_i = k_ix + 4k_i^3t + \xi_i^{(0)}, \ i = 1, 2, \end{aligned} \tag{6.5c}$$

and U is depicted in Fig. 3. One can see that the amplitudes of the two solitons are changed after interaction. This can be demonstrated by analyzing asymptotic behaviors of the two-soliton solution by a similar procedure as in Ref. 12. For convenience we call the two solitons k_1-soliton and k_2-soliton, respectively. Then we rewrite the two-soliton solution (6.5) in terms of the following coordinates:

$$(X_1 = x + 4{k_1}^2t, t), \tag{6.6}$$

which gives

$$U = \left(\frac{F_1}{G_1}\right)_x, \tag{6.7a}$$

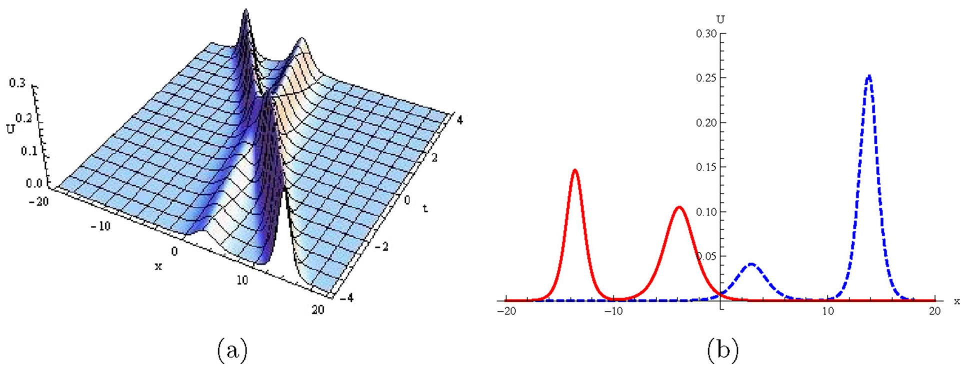


FIG. 3. Two-soliton interactions given by (6.5) for $e_1 = 1, e_2 = 1.5, \kappa_1 = -1.1, \kappa_2 = -1$ and $\xi_1^{(0)} = \xi_2^{(0)} = 0$. (a) Shape and motion. (b) Amplitude-change interaction of two solitons, in which the shape at $t = -4$ is denoted by blue dashed line and the shape at $t = 4$ is denoted by the red solid line.

where

$$\begin{aligned} F_1 = {} & 2(k_1 + k_2)[2k_1k_2{K_1}^2{K_2}^2(k_1 + k_2)(e^{2k_1X_1} + e^{2k_2(X_1-4{k_1}^2t+4{k_2}^2t)}) + ({K_1}^2{K_2}^2(k_1 - k_2)^2 \\ & - g((k_1K_1 - k_2K_2)^2 + k_1k_2(K_1 - K_2)^2))e^{2(k_1+k_2)X_1-8k_2({k_1}^2-{k_2}^2)t}], \end{aligned} \tag{6.7b}$$

$$\begin{aligned} G_1 = {} & -2k_2{K_2}^2(g - {K_1}^2)(k_1 + k_2)^2e^{2k_1X_1} - 2k_1K_1^2(k_1 + k_2)^2(g - K_2^2)e^{2k_2(-4k_1^2t+4k_2^2t+X_1)} \\ & + 4k_1k_2{K_1}^2{K_2}^2(k_1 + k_2)^2 + [(g^2 + K_1^2{K_2}^2)(k_1 - k_2)^2 \\ & - g(k_1 + k_2)^2({K_1}^2 + {K_2}^2) - 8gk_1k_2K_1K_2]e^{-8{k_1}^2k_2t+8{k_2}^3t+2k_1X_1+2k_2X_1}, \end{aligned} \tag{6.7c}$$

where we have taken $\xi_1^{(0)} = \xi_2^{(0)} = 0$ without loss of generality. Noting that if we suppose $|k_1| > |k_2|$ and keep X_1 to be constant together with t going to infinity, we find that there is only k_1-soliton left along the line $X_1 = constant$ and also find how the k_1-soliton is asymptotically identified by its top trace and amplitude, for both $t \to \pm\infty$.

Let us present the detailed results. When $k_2 > 0,\ t \to +\infty$ or $k_2 < 0,\ t \to -\infty$, i.e., $\mathrm{sgn}[k_2]\cdot t \to +\infty$, the solution (6.7) becomes

$$\begin{aligned} U &= \left(\frac{2k_1e^{2(k_1X_1+4{k_1}^3t)}}{2k_1 + e^{2(k_1X_1+4{k_1}^3t)}(1 - g/{K_1}^2)}\right)_{X_1} \\ &= \frac{8{k_1}^3{K_1}^4e^{8{k_1}^3t+2k_1X_1}}{[2k_1{K_1}^2 + (-g + {K_1}^2)e^{2(k_1X_1+4{k_1}^3t)}]^2}, \end{aligned} \tag{6.8a}$$

and when $\mathrm{sgn}[k_2] \cdot t \to -\infty$ (6.7) becomes

$$U = \frac{F_2}{G_2}, \tag{6.9a}$$

$$F_2 = 8{k_1}^3{K_1}^2(k_1 + k_2)^2[g(k_1K_1 + k_2(K_1 - 2K_2)) + K_1(-k_1 + k_2){K_2}^2]^2e^{2k_1X1}, \tag{6.9b}$$

$$\begin{aligned} G_2 = {} & [-2K_1{K_1}^2(K_1 + K_2)^2(g - {K_2}^2) + [(g^2 + {K_1}^2{K_2}^2)(K_1 - K_2)^2 \\ & - g(({K_1}^2 + {K_2}^2)({K_1}^2 + {K_2}^2) + 2K_1K_2({K_1}^2 - 4K_1K_2 + {K_2}^2))]e^{2K_1X_1}]^2. \end{aligned} \tag{6.9c}$$

One can also rewrite the two-soliton solution (6.5) in terms of the coordinates

$$(X_2 = x + 4{k_1}^2t, t), \tag{6.10}$$

and do a similar asymptotic analysis for the k_2-soliton. We summarise the above analysis and reach the following theorem about how the two-soliton waves interact with each other.

Theorem 1. *The asymptotic behaviour is described as follows. Suppose that* $|k_1| > |k_2|$ *in (6.7). Then, when* $\mathrm{sgn}[k_2] \cdot t \to +\infty$*, the* k_1*-soliton asymptotically follows*

$$\text{top trace}: \ x(t) = -4{k_1}^2 t + \frac{1}{2k_1}\ln\Big|\frac{2k_1{K_1}^2}{-g+{K_1}^2}\Big|, \tag{6.11a}$$

$$\text{amplitude}: \ \mathrm{Amp} = \frac{k_1^2{K_1}^2}{-g+{K_1}^2}, \tag{6.11b}$$

and when $\mathrm{sgn}[k_2] \cdot t \to -\infty$*, it asymptotically follows*

$$\text{top trace}: \ x(t) = -4{k_1}^2 t + \frac{1}{2k_1}\ln\Big|\frac{2k_1{K_1}^2(k_1+k_2)^2(g-{K_2}^2)}{Q+2k_1k_2({K_1}^2+{K_2}^2)}\Big|, \tag{6.12a}$$

$$\text{amplitude}: \ \mathrm{Amp} = -\frac{{k_1}^2[gK_1(k_1+k_2)-2gk_2K_2+K_1K_2^2(k_2-k_1)]^2}{(g-K_2^2)Q}. \tag{6.12b}$$

When $\mathrm{sgn}[k_1] \cdot t \to -\infty$*, the* k_2*-soliton asymptotically follows*

$$\text{top trace}: \ x(t) = -4{k_2}^2 t + \frac{1}{2k_2}\ln\Big|\frac{2k_2{K_2}^2}{-g+{K_2}^2}\Big|, \tag{6.13a}$$

$$\text{amplitude}: \ \mathrm{Amp} = \frac{{k_2}^2{K_2}^2}{-g+{K_2}^2}, \tag{6.13b}$$

and when $\mathrm{sgn}[k_1] \cdot t \to +\infty$*, it asymptotically follows*

$$\text{top trace}: \ x(t) = -4{k_1}^2 t + \frac{1}{2k_1}\ln\Big|\frac{2k_2{K_1}^2(k_1+k_2)^2(g-{K_1}^2)}{Q+2k_1k_2({K_1}^2+{K_2}^2)}\Big|, \tag{6.14a}$$

$$\text{amplitude}: \ \mathrm{Amp} = -\frac{{k_2}^2[gK_2(k_1+k_2)-2gk_1K_1+{K_1}^2K_2(k_2-k_1)]^2}{(g-K_1^2)Q}, \tag{6.14b}$$

with $Q = g^2(k_1-k_2)^2 - g[({K_1}^2+{K_2}^2)(k_1+k_2)^2 - 8k_1k_2K_1K_2] + {K_1}^2{K_2}^2(k_1-k_2)^2$*. The phase shift for the* $k_j(j = 1,2)$*-soliton after interaction is* $-\frac{1}{2k_j}\ln\big|\frac{(k_1+k_2)^2(g-{K_1}^2)(g-{K_2}^2)}{Q+2k_1k_2({K_1}^2+{K_2}^2)}\big|$.

VII. STRAIGHT CONTINUUM LIMITS

The skew continuum limit of the elpKdV system (4.7) was considered in Ref. 1 where the authors studied initial value problems of the system. Such a limit is performed by introducing the skew-change of variables $(n,m) \mapsto (\mathcal{N} = n+m, m)$.

Let us consider the straight continuum limit, where we first take

$$m \to \infty, \quad b \to \infty, \quad \text{while } \frac{m}{b} = \tau - \tau_0 \sim O(1), \tag{7.1}$$

with τ_0 being a constant. We define

$$u = u_{n,m} =: u_n(\tau), \quad s = s_{n,m} =: s_n(\tau), \quad w = w_{n,m} =: w_n(\tau). \tag{7.2}$$

Then, applying the Taylor expansions into (4.7) at τ, the leading term (in terms of $1/b$) of each equation yields the following semi-discrete equations:

$$\partial_\tau(\widetilde{u}+u) = 2a(\widetilde{u}-u) - (\widetilde{u}-u)^2 + g(\widetilde{s}-s)^2, \tag{7.3a}$$

$$\partial_\tau(s\widetilde{s}) = (\widetilde{s}-s)(a\widetilde{s}+as-\widetilde{w}+w) + u\widetilde{s}^2 + \widetilde{u}s^2 - s\widetilde{s}(u+\widetilde{u}), \tag{7.3b}$$

$$(a+u-\frac{\widetilde{w}}{\widetilde{s}})(a-\widetilde{u}+\frac{w}{s}) = a^2 - R(s\widetilde{s}), \tag{7.3c}$$

$$\partial_\tau(u+\frac{w}{s}) + (u-\frac{w}{s})^2 = R(s^2), \tag{7.3d}$$

in which both (4.7b) and (4.7c) yield (7.3b) in the continuum limit. Here we note that this semi-discrete system can be viewed as an elliptic Bäcklund transformation of the epKdV system (5.13).

One natural question is whether the elpKdV system (1.1) can be rederived from this elliptic Bäcklund transformation as a superposition formula. This will be considered elsewhere. Here we also point out that the non-elliptic limit (7.3a) can perhaps be viewed as an elliptic version of the dressing chain given by Veselov and Shabat in Ref. 13.

For the full limit of (7.3), first we take

$$n \to \infty, \quad a \to \infty, \quad \text{while } \frac{n}{a} = \xi \sim O(a^2) \tag{7.4}$$

and then introduce continuous variables x and t,

$$x = \tau + \xi, \quad t = \frac{\xi}{12a^2}, \tag{7.5}$$

with ξ as an auxiliary variable. Then, in the coordinates (x,t) both (7.3c) and (7.3d) yield

$$\left(u + \frac{w}{s}\right)_x + \left(u - \frac{w}{s}\right)^2 = R(s^2), \tag{7.6}$$

(7.3a) yields (5.13a) and (7.3b) gives (5.13b). For this to be possible, we need to make use of the relation $s_{xx} = 2(gsh_x - us_x + w_x)$, which is obtained as a continuous limit of the summation of $(\widetilde{4.6c})$ and (4.6d).

If we employ the transformation $A = -u + \frac{w}{s}$ in the Equations (7.3b) and (7.3c), it turns out that

$$\partial_\tau(s\widetilde{s}) = (\widetilde{s} - s)(a\widetilde{s} + as - \widetilde{A}\widetilde{s} + As) + u\widetilde{s}^2 + \widetilde{u}s^2 - s\widetilde{s}(u + \widetilde{u}), \tag{7.7a}$$

$$(a + u - \widetilde{A} - \widetilde{u})(a - \widetilde{u} + A + u) = a^2 - R(s\widetilde{s}). \tag{7.7b}$$

The continuum limit of (7.7) gives the coupled system (5.16), where we use the relation $u_x = \frac{1}{2}(R(s^2) - A^2 - A_x)$, which is simply (7.6) written in terms of A and u, derived from (7.3d).

VIII. CONCLUSIONS

In this paper a new class of solutions of the elliptic KdV systems (both the discrete (1.1) and the continuous (1.2)) has been uncovered. We made use of Sylvester-type equation with elliptic ingredient. Solutions can be classified by the canonical form of $\boldsymbol{k}$, which are much richer than pure solitons. A typical feature of two soliton solutions is the amplitude change after interaction.

A Cauchy matrix dressed by dispersion relations usually satisfies a Sylvester equation. Starting from the Sylvester equation and dispersion relations, we find that not only integrable equations can be derived but also their solutions and Lax pairs can be constructed. The Cauchy matrix approach is particularly powerful in the study of discrete integrable systems (see Refs. 6 and 8), as well as continuous systems.[7] Dressed Cauchy matrices also play key roles in the so-called operator method,[14–19] trace method,[20] etc.

There are two ways elliptic curves can play a role in integrable systems: either as elliptic type solutions (i.e., solutions expressible in terms of elliptic functions) or as elliptic deformation of the equations themselves. In either way, the study of the elliptic case is often richer than the rational and trigonometric/hyperbolic case, and reveals many new features of the models in question, thus leading to new insights into the true nature of those integrable systems.[21–28] In this paper, we have shown that the Cauchy matrix approach works for the study of some elliptic integrable systems, i.e., some equations in these systems are formulated with an elliptic curve. Starting with the Sylvester equation (1.6), we derived the discrete as well as continuous elliptic KdV systems. Apart from finding and illustrating the solutions, we also obtained Lax representations. With regard to the solutions, the discrete plane wave factor (C2) and continuous one (C5b) are defined with the wave number k_i which together with K_i obeys the elliptic curve (1.3). For the Lax pairs (4.18) and (5.19), (k,K) plays the role of spectral parameters which also obeys the elliptic curve.

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APPENDIX A: LOWER TRIANGULAR TOEPLITZ MATRICES

Here we collect some properties of Lower triangular Toeplitz (LTT) matrices.

A Nth order LTT matrix is a matrix of the following form:

$$\boldsymbol{T}^{[N]}(\{a_j\}_1^N) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ a_N & a_{N-1} & a_{N-2} & \cdots & a_2 & a_1 \end{pmatrix}_{N\times N}. \tag{A1}$$

Let

$$\mathfrak{T}^{[N]} = \{\boldsymbol{T}^{[N]}(\{a_j\}_1^N)\}, \tag{A2}$$

then we have $\boldsymbol{AB} = \boldsymbol{BA}$, $\forall \boldsymbol{A}, \boldsymbol{B} \in \mathfrak{T}^{[N]}$, i.e., $\mathfrak{T}^{[N]}$ is a commutative set with respect to matrix multiplication. Particularly, the subset

$$\mathfrak{T}_1^{[N]} = \{\boldsymbol{F} \in \mathfrak{T}^{[N]} \mid \det(\boldsymbol{F}) \neq 0\} \tag{A3}$$

is an abelian group.

Obviously, the Jordan block matrix

$$\boldsymbol{\Gamma}_J^{[N]}(a) = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 1 & a & 0 & \cdots & 0 & 0 \\ 0 & 1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a \end{pmatrix} \tag{A4}$$

is a LTT matrix, and one can verify the following.

Proposition A.1. If $\boldsymbol{A} \in \mathbb{C}_{N\times N}$ and $\boldsymbol{\Gamma}_J^{[N]}(a)\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{\Gamma}_J^{[N]}(a)$, then there must be $\boldsymbol{A} \in \mathfrak{T}^{[N]}$.

If $a_j \in \mathbb{C}$, then the LTT matrix (A1) can be generated by certain functions. Suppose that $f(k)$ is an analytic function. Using Taylor coefficients

$$a_j = \frac{1}{(j-1)!}\partial_k^{j-1} f(k)|_{k=k_0}, \quad j = 1, 2, \ldots, N \tag{A5}$$

we can generate a LTT matrix.

Definition 1. The matrix (A1) with (A5) is called a LTT matrix generated by $f(k)$ at $k = k_0$, denoted by $\boldsymbol{T}^{[N]}[f(k_0)]$, and $f(k)$ is called the generating function.

In light of this definition, the Jordan block (A4) is generated by $f(k) = k$ at $k = a$, and the unit matrix $\boldsymbol{I}$ is generated by $f(k) \equiv 1$. On the other hand, for any LTT matrix (A1) with $a_j \in \mathbb{C}$, it can be generated by the polynomial

$$\alpha(k) = \sum_{j=1}^{N} a_j (k - k_0)^{j-1} \tag{A6}$$

with $\{a_j\}$ as coefficients. Next, by $[f(k)]_{k_0}{}^{[N]}$ we denote a set of functions (equivalence class) in which all the functions have the same $(N-1)$th order Taylor polynomial at $k = k_0$ as $f(k)$ has. Say, $f(k) \sim g(k)$ if they have the same $(N-1)$th order Taylor polynomial at $k = k_0$. Thus, the LTT matrix (A1) can be generated by any $f(k) \in [\alpha(k)]_{k_0}^{[N]}$. With such correspondence, we have the following.

Proposition A.2. If $\boldsymbol{A} = \boldsymbol{T}^{[N]}[f(k_0)]$ and $\boldsymbol{B} = \boldsymbol{T}^{[N]}[g(k_0)]$, then

$$\boldsymbol{C} = \boldsymbol{AB} = \boldsymbol{T}^{[N]}[f(k_0)g(k_0)], \tag{A7}$$

i.e., $\boldsymbol{AB}$ is a LTT matrix generated by $f(k)g(k)$ at k_0. As a result, we have

$$\prod_{j=1}^{s} \boldsymbol{T}^{[N]}[f_j(k_0)] = \boldsymbol{T}^{[N]}[\prod_{j=1}^{s} f_j(k_0)]$$

and

$$(T^{[N]}[f(k_0)])^{-1} = \boldsymbol{T}^{[N]}[1/f(k_0)]$$

iff $f(k_0) \neq 0$.

Proof. We only need to prove (A7). Suppose

$$\boldsymbol{A} = (a_{ij})_{N\times N}, \quad \boldsymbol{B} = (b_{ij})_{N\times N}, \quad \boldsymbol{C} = (c_{ij})_{N\times N}.$$

Then we have (with $k = k_0$)

$$a_{ij} = \begin{cases} \frac{1}{(i-j)!}\partial_k^{i-j} f(k), & i \geq j \\ 0, & i < j \end{cases}, \quad b_{ij} = \begin{cases} \frac{1}{(i-j)!}\partial_k^{i-j} g(k), & i \geq j \\ 0, & i < j \end{cases},$$

and (with $k = k_0$)

$$\begin{aligned} c_{ij} &= \sum_{s=1}^{N} a_{is} b_{sj} \\ &= \sum_{s=1}^{N} \frac{1}{(i-s)!}\partial_k^{i-s} f(k) \cdot \frac{1}{(s-j)!}\partial_k^{s-j} g(k) \\ &= \sum_{l=i-j}^{N} \frac{1}{(i-j-l)!\, l!}\partial_k^{i-j-l} f(k) \cdot \partial_k^{l} g(k) \quad (i \geq j) \\ &= \frac{1}{(i-j)!} \sum_{l=i-j}^{N} \partial_k^{i-j} (f(k) g(k)), \quad (i \geq j), \end{aligned}$$

and $c_{ij} = 0$ when $i < j$. Thus, (A7) is proved. □

In addition to the LTT matrices, we define the following skew triangular Hankel matrix:

$$\boldsymbol{H}^{[N]}(\{b_j\}_1^N) = \begin{pmatrix} b_1 & \cdots & b_{N-2} & b_{N-1} & b_N \\ b_2 & \cdots & b_{N-1} & b_N & 0 \\ b_3 & \cdots & b_N & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_N & \cdots & 0 & 0 & 0 \end{pmatrix}_{N\times N}. \tag{A8}$$

The following property holds.[7]

Proposition A.3. Let

$$\bar{\mathfrak{T}}^{[N]} = \{\boldsymbol{H}^{[N]}(\{b_j\}_1^N)\}. \tag{A9}$$

Then we have

(1) $\boldsymbol{H} = \boldsymbol{H}^T, \ \forall \boldsymbol{H} \in \bar{\mathfrak{T}}^{[N]}$.
(2) $\boldsymbol{H}\boldsymbol{A} = (\boldsymbol{H}\boldsymbol{A})^T = \boldsymbol{A}^T\boldsymbol{H}, \ \forall \boldsymbol{A} \in \mathfrak{T}^{[N]}, \ \forall \boldsymbol{H} \in \bar{\mathfrak{T}}^{[N]}$.

It can be extended to the following generic case.

Proposition A.4. Let

$$\mathfrak{G}^{[N]} = \{\mathrm{Diag}(\boldsymbol{\Gamma}_D^{[N]}(\{a_{1,j}\}_1^{N_1}), \boldsymbol{T}^{[N]}(\{a_{2,j}\}_1^{N_2}), \boldsymbol{T}^{[N]}(\{a_{3,j}\}_1^{N_3}), \cdots, \boldsymbol{T}^{[N]}(\{a_{s,j}\}_1^{N_s}))\}, \tag{A10a}$$

$$\bar{\mathfrak{G}}^{[N]} = \{\mathrm{Diag}(\boldsymbol{\Gamma}_D^{[N]}(\{b_{1,j}\}_1^{N_1}), \boldsymbol{H}^{[N]}(\{b_{2,j}\}_1^{N_2}), \boldsymbol{H}^{[N]}(\{b_{3,j}\}_1^{N_3}), \cdots, \boldsymbol{H}^{[N]}(\{b_{s,j}\}_1^{N_s}))\}, \tag{A10b}$$

where $0 \leq N_j \leq N$ *for* $j = 0, 1, \ldots, N$ *and* $\sum_{j=1}^{s} N_j = N$. *Then we have*

(1) $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}, \ \forall \boldsymbol{A}, \boldsymbol{B} \in \mathfrak{G}^{[N]}$.
(2) $\boldsymbol{H} = \boldsymbol{H}^T, \ \forall \boldsymbol{H} \in \bar{\mathfrak{G}}^{[N]}$.
(3) $\boldsymbol{H}\boldsymbol{A} = (\boldsymbol{H}\boldsymbol{A})^T = \boldsymbol{A}^T\boldsymbol{H}, \ \forall \boldsymbol{A} \in \mathfrak{G}^{[N]}, \ \forall \boldsymbol{H} \in \bar{\mathfrak{G}}^{[N]}$.

APPENDIX B: LIST OF NOTATIONS

Here we list some notations used in the paper.

$$\mathbf{\Gamma}_D^{[N]}(\{k_j\}_1^N) = \mathrm{Diag}(k_1, k_2, \cdots, k_N), \quad (k_i^2 \neq k_j^2,\ k_i \neq 0), \tag{B1a}$$

$$\mathbf{\Gamma}_J^{[N]}(k_1) = \begin{pmatrix} k_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & k_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & k_1 \end{pmatrix}, \tag{B1b}$$

$$\mathbf{\Gamma}_G^{[N]} = \mathrm{Diag}\big(\mathbf{\Gamma}_D^{[N_1]}(\{k_j\}_1^{N_1}), \mathbf{\Gamma}_J^{[N_2]}(k_{N_1+1}), \mathbf{\Gamma}_J^{[N_3]}(k_{N_1+2}), \ldots, \mathbf{\Gamma}_J^{[N_s]}(k_{N_1+(s-1)})\big), \tag{B1c}$$

where $\sum_{j=1}^{s} N_j = N$. The subscripts $_D$, $_J$, and $_G$ correspond to the cases of $\mathbf{\Gamma}$ being diagonal, being of Jordan block and generic canonical form, respectively. Besides,

$$N\text{-th order vector}: \ \mathbf{e}^{[N]} = (1, 1, 1, \ldots, 1)^T, \tag{B2a}$$

$$N\text{-th order vector}: \ \mathbf{e}_1^{[N]} = (1, 0, 0, \ldots, 0)^T, \tag{B2b}$$

$$N\text{-th order vector}: \ g^{[N]}(a) = \Big(\frac{1}{a}, \frac{-1}{a^2}, \frac{1}{a^3}, \ldots, \frac{(-1)^{N-1}}{a^N}\Big)^T, \tag{B2c}$$

$$N \times N \text{ matrix}: \ \boldsymbol{G}_D^{[N]}(\{k_j\}_1^N) = (G_{i,j})_{N\times N}, \quad G_{i,j} = \frac{1 - g/(K_iK_j)}{k_i + k_j}. \tag{B2d}$$

APPENDIX C: EXPLICIT FORMS OF r AND c

Here we list out the explicit forms of $\boldsymbol{r}$ and $\boldsymbol{c}$ satisfying (4.1) and (5.1), respectively.

1. Solution to (4.1)

(1) When $\mathbf{\Gamma} = \mathbf{\Gamma}_D^{[N]}(\{k_j\}_1^N)$, we have

$$\boldsymbol{r} = \boldsymbol{r}_D^{[N]}(\{k_j\}_1^N) = (r_1, r_2, \ldots, r_N)^T, \quad \text{with } r_i = \rho_i, \tag{C1}$$

where

$$\rho_i = \Big(\frac{a + k_i}{a - k_i}\Big)^n \Big(\frac{b + k_i}{b - k_i}\Big)^m \rho_i^0, \tag{C2}$$

and ρ_i^0 is a constant.

(2) When $\mathbf{\Gamma} = \mathbf{\Gamma}_J^{[N]}(k_1)$, we have

$$\boldsymbol{r} = \boldsymbol{r}_J^{[N]}(k_1) = (r_1, r_2, \ldots, r_N)^T, \quad \text{with } r_i = \frac{\partial_{k_1}^{i-1}\rho_1}{(i-1)!}, \tag{C3}$$

where ρ_1 is defined in (C2).

(3) When $\mathbf{\Gamma} = \mathbf{\Gamma}_G^{[N]}$, we have

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_D^{[N_1]}(\{k_j\}_1^{N_1}) \\ \boldsymbol{r}_J^{[N_2]}(k_{N_1+1}) \\ \boldsymbol{r}_J^{[N_3]}(k_{N_1+2}) \\ \vdots \\ \boldsymbol{r}_J^{[N_s]}(k_{N_1+(s-1)}) \end{pmatrix}, \tag{C4}$$

where $\boldsymbol{r}_D^{[N_1]}(\{k_j\}_1^{N_1})$ and $\boldsymbol{r}_J^{[N_i]}(k_j)$ are defined as in (C1) and (C3), respectively.

2. Solution to (5.1)

(1) When $\mathbf{\Gamma} = \mathbf{\Gamma}_D^{[N]}(\{k_j\}_1^N)$ we have $\boldsymbol{r} = \boldsymbol{r}_D^{[N]}(\{k_j\}_1^N)$, which is given in the form of (C1) and

$$\boldsymbol{c} = \boldsymbol{c}_D^{[N]}(\{k_j\}_1^N) = (c_1, c_2, \ldots, c_N)^T, \quad \text{with } c_i = r_i, \tag{C5a}$$

but here

$$\rho_i = e^{\xi_i}, \quad \xi_i = k_i x + 4k_i^3 t + \xi_i^{(0)}, \text{ with constant } \xi_i^{(0)}. \tag{C5b}$$

(2) When $\mathbf{\Gamma} = \mathbf{\Gamma}_J^{[N]}(k_1)$, we have $\boldsymbol{r} = \boldsymbol{r}_J^{[N]}(k_1)$, which is given in the form of (C3) and

$$\boldsymbol{c} = \boldsymbol{c}_J^{[N]}(k_1) = (c_1, c_2, \ldots, c_N)^T, \quad \text{with } c_i = r_{N-i+1}, \tag{C6}$$

where ρ_1 is defined in (C5b).

(3) When $\mathbf{\Gamma} = \mathbf{\Gamma}_G^{[N]}$ we have $\boldsymbol{r}$, which is given in the form of (C4) and

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_D^{[N_1]}(\{k_j\}_1^{N_1}) \\ \boldsymbol{c}_J^{[N_2]}(k_{N_1+1}) \\ \boldsymbol{c}_J^{[N_3]}(k_{N_1+2}) \\ \vdots \\ \boldsymbol{c}_J^{[N_s]}(k_{N_1+(s-1)}) \end{pmatrix}, \tag{C7}$$

where $\boldsymbol{c}_D^{[N_1]}(\{k_j\}_1^{N_1})$ and $\boldsymbol{c}_J^{[N_i]}(k_j)$ are defined as in (C5a) and (C6), respectively.

[1] F. W. Nijhoff and S. E. Puttock, "On a two-parameter extension of the lattice KdV system associated with an elliptic curve," J. Nonlinear Math. Phys. **10**(Suppl. 1), 107–123 (2003).
[2] F. W. Nijhoff, G. R. W. Quispel, and H. W. Capel, "Direct linearization of non-linear difference difference-equations," Phys. Lett. A **97**, 125–128 (1983).
[3] F. W. Nijhoff, "A higher-rank version of the Q3 equation," e-print arXiv:1104.1166 (2011).
[4] D. J. Zhang, S. L. Zhao, and F. W. Nijhoff, "Direct linearization of an extended lattice BSQ system," Stud. Appl. Math. **129**, 220–248 (2012).
[5] F. W. Nijhoff, *Discrete Systems and Integrability*, MATH4490 Lecture Notes (University of Leeds, 2004).
[6] F. W. Nijhoff, J. Atkinson, and J. Hietarinta, "Soliton solutions for ABS lattice equations: I. Cauchy matrix approach," J. Phys. A: Math. Theor. **42**, 404005 (2009).
[7] D. D. Xu, D. J. Zhang, and S. L. Zhao, "The Sylvester equation and integrable equations: I. The Korteweg-de Vries system and sine-Gordon equation," J. Nonlinear Math. Phys. **21**, 382–406 (2014).
[8] D. J. Zhang and S. L. Zhao, "Solutions to the ABS lattice equations via generalized Cauchy matrix approach," Stud. Appl. Math. **131**, 72–103 (2013).
[9] J. Hietarinta and D. J. Zhang, "Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization," J. Phys. A: Math. Theor. **42**, 404006 (2009).
[10] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Translations of Mathematical Monographs, edition by H. H. McFaden (AMS, Providence, RI, 1990), Vol. 79 (Translated from the 2nd Russian).
[11] J. Sylvester, "Sur l'equation en matrices $px = xq$," C. R. Acad. Sci. Paris **99**, 67–71 (1884).
[12] D. J. Zhang, J. Ji, and S. L. Zhao, "Soliton scattering with amplitude changes of a negative order AKNS equation," Physica D **238**(23-24), 2361–2367 (2009).
[13] A. P. Veselov and A. B. Shabat, "Dressing chains and the spectral theory of the Schrödinger operator," Funct. Anal. Appl. **27**(2), 81–96 (1993).
[14] V. A. Marchenko, *Nonlinear Equations and Operator Algebras* (D. Reidel, Dordrecht, 1988).
[15] H. Aden and B. Carl, "On realizations of solutions of the KdV equation by determinants on operator ideals," J. Math. Phys. **37**, 1833–1857 (1996).
[16] C. Schiebold, "An operator-theoretic approach to the Toda lattice equation," Physica D **122**, 37–61 (1998).
[17] B. Carl and C. Schiebold, "Nonlinear equations in soliton physics and operator ideals," Nonlinearity **12**, 333–364 (1999).
[18] B. Carl and C. Schiebold, "A direct approach to the study of soliton equations," J. Deutsch. Math.-Verein **102**(3), 102–148 (2000), English version is avilable on http://apachepersonal.miun.se/~corsch/.
[19] C. Schiebold, "Cauchy-type determinants and integrable systems," Linear Algebra Appl. **433**, 447–475 (2010).
[20] H. Blohm, "Solution of nonlinear equations by trace methods," Nonlinearity **13**, 1925–1964 (2000).
[21] F. W. Nijhoff and J. Atkinson, "Elliptic N-soliton solutions of ABS lattice equations," Int. Math. Res. Notices **2010**(20), 3837–3895.
[22] J. Atkinson and F. W. Nijhoff, "A constructive approach to the soliton solutions of integrable quadrilateral lattice equations," Commun. Math. Phys. **299**, 283–304 (2010).
[23] C. W. Cao and X. X. Xu, "A finite genus solution of the H1 model," J. Phys. A: Math. Theor. **45**, 055213 (2012).
[24] C. W. Cao and G. Y. Zhang, "A finite genus solution of the Hirota equation via integrable symplectic maps," J. Phys. A: Math. Theor. **45**, 095203 (2012).

[25] F. W. Nijhoff, *Elliptic Integrable Systems on the Lattice and Associated Continuous Systems, Talk in the Workshop on Elliptic Integrable Systems and Hypergeometric Functions* (Lorentz Center, Leiden, 2013).
[26] S. Yoo-Kong and F. W. Nijhoff, "Elliptic (N,N′)-soliton solutions of the lattice Kadomtsev-Petviashvili equation," J. Math. Phys. **54**, 043511 (2013).
[27] P. Jennings and F. W. Nijhoff, "On an elliptic extension of the Kadomtsev-Petviashvili equation," J. Phys. A: Math. Theor. **47**, 055205 (2014).
[28] N. Delice, F. W. Nijhoff, and S. Yoo-Kong, "On elliptic Lax systems on the lattice and a compound theorem for hyperdeterminants," J. Phys. A: Math. Theor. **48**, 035206 (2015).