Article:

Chaudhuri, K orcid.org/0000-0002-7492-1369, Sen, R and Tan, Z (2018) Testing extreme dependence in financial time series. Economic Modelling, 73. pp. 378-394. ISSN 0264-9993

https://doi.org/10.1016/j.econmod.2018.04.016

Testing Extreme Dependence in Financial Time Series[1]

Abstract

Financial interdependence indicates a process through which transmission of shock originating in the financial market of one economy spreads to others. This paper provides a new idea of Residual and Recurrence Times of high or low values for bivariate time series to detect extreme dependence or contagion. In presence of financial extreme dependence, the distributions of residual and recurrence times are not the same. We examine the equality of two distributions using the permutation test. In comparison to other methods in multivariate extreme value theory, our proposed method does not need the i.i.d. assumption. Our method can handle the situation where the extremes for different components do not occur at the same time. We justify our methods in two ways: first using thorough simulation studies and then applying the proposed method to real data on weekly stock indices from sixteen markets.

KEYWORDS: Financial Dependence, Residual and Recurrence Times, GARCH.

JEL Classification: C14, C53, G12.

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[1] We are most grateful to Sushanta Mallick and three anonymous referees for their valuable comments and suggestions.

1 Introduction

In financial market research, interdependence mostly quoted as contagion is a widely researched term. Financial interdependence can be at both levels: at the domestic level, for example, the crash of Lehman Brothers and subsequent depressed United States financial markets, and, at the international level, for example, the Mexican crisis in 1994, the Asian crisis in 1997 or the most recent global, and Eurozone crisis. King and Wadhwani (1990) demonstrate that the correlations between the United States, the United Kingdom, and other developed markets increased significantly following the 1987 stock market crash whereas Lee and Kim (1993) extend this analysis to more countries including emerging markets to show presence of increased correlations during the 1987 crash. In financial markets, contagion refers to the transmission of a financial shock in one market to other interdependent markets. In this paper, we are proposing a test for extreme dependence, which is of interest as it indicates contagion.

Despite the wide use of the term contagion, several definitions exist in the literature (see Pericoli and Sbracia (2003)). Contagion could refer to i) significant rise in the probability of a crisis conditional on a crisis in another country, ii) spill overs of volatility from the crisis country to the financial markets of other countries, iii) significant increase in co-movements of across different markets following a crisis in one or group of markets, iv) disproportionate co-movements following a shock in one country and v) co-movements across markets that can not be explained by fundamentals. Therefore three characteristics are important: the presence of a crisis, the dynamics of the interdependencies and the way of measuring these interdependencies.

According to Forbes and Rigobon (2001), there have been four methods to test and measure contagion effects. The first and most straightforward one is based on cross-markets correlation coefficient in asset returns and examines whether inter-related financial markets exhibit anomalous patterns of correlation in returns during two different periods: stable period and the period following a shock. If the correlation coefficient increases significantly after the shock, this implies the presence of contagion. Forbes and Rigobon (2002) distinguish between interdependence and contagion.[4] Bekaert et

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[4] Shift contagion refers to a situation where the propagation of shocks during crisis periods increases systematically from that observed during normal times. A broader definition refers to contagion as the transmission of shocks through any channels that cause markets to co-vary. We acknowledge that our proposed test cannot distinguish between interdependence and (shift) contagion, given that there is no control for common shocks. We use the term contagion and interdependence interchangeably throughout the paper.

al. (2005) have defined contagion as excess correlation on top of the correlation in economic fundamentals. In this framework, test for contagion is being conducted by examining the correlation in the residuals obtained from employing a multi-factor model. In a recent paper, Blatt et al. (2015) detect changes in correlation matrix, hence extending the approach of the aforementioned papers to multiple dimensions. But correlation is a moment based statistic and therefore is affected by extreme values. However, correlation may be high even without extreme events moving together.

The second approach as in Hamao et al. (1990), the ARCH or GARCH models are used to test for the presence of significant volatility spillover from one market to another during or after the crisis. Leung et al. (2017) employed GARCH model to examine the volatility spillover between the exchange rates markets and the equity markets during the global financial crisis and the euro debt crisis and test whether the increased spillover is the resultant of fundamental contagion (see Dornbusch et al. (2000)) or pure contagion (see Lin (1994)). Shen et al. (2015) have used the Kalman filter to estimate the time-varying correlation coefficients of the stock market indices between the Eurozone and China to tests for pure contagion and interdependence. See also Ahmad et al. (2013), and Hemche et al. (2016) for dynamic conditional correlation using GARCH model for an application.

The third test, implemented by Longin and Solnik (1995), examines whether there is significant change in the cross-market correlations over time. The fourth procedure applies probit model to ascertain the probability of a crisis occurring in one economy conditional on a crisis that has already occurred in another economy (Eichengreen et al. (1996) and Kaminsky and Reinhart (2000)). We propose a new method based on test of dependence between point processes using Residual and Recurrence Times (RRT). Our proposed method is closest to the fourth procedure that applies probit model to ascertain the probability of a crisis occurring in one economy conditional on a crisis that has already occurred in another economy. However our method is nonparametric in nature. The all other procedures relying on correlation, volatility etc. concentrates on the center of the data and not on the extremes. On the other hand, we do not treat the periods between the even and the contagion period as independent. However, they could be still uncorrelated. Tests based on correlation would not pick up the dependence but our proposed RRT test can not only detect dependence effect between the two components but also can tell the direction of the effect. This is our first contribution in the existing literature.

Interdependence/contagion is observed when one or more entities are going through extreme high or low economic phases. Such phases are economically the most interesting and high-impact periods. Standard methods of multivariate time series analysis are not suitable in this setting, since they concentrate on the joint behavior during stable and stationary periods. Another approach is through tail dependence in multivariate extreme value theory. Sivapulle et al. (2016) have used a robust copula method to model tail dependence and test for contagion effects. See Sen and Tan (2012) and references therein for a survey. Our method contributes in the existing literature by overcoming the limitation of standard multivariate extreme value theory that extreme events in different series occur concurrently. Given the presence of finite time lag in propagation of financial shocks originated in one market to be transmitted from one market to another, the extremes do not necessarily occur in both series at the same point of time. The same assumption is made in other works on contagion like Bae et al. (2002). Although block maxima methods address this to some extent, the length of the block is ad hoc in nature.

To illustrate our point, we present the returns of stock indices of Korea and Thailand in October-November 2008 in Figure 1.

[Figure 1 about here]

The small return of Thailand in early October is followed by small value of Korea in late October. The high value of Korea in early November is followed by high value of Thailand. Thus, contagion effect is present. We observe that in Figure 1, the extreme events in the two series do not necessarily occur at the same time point. Thus, here bivariate extreme value theory may not be a good tool. Furthermore, we find that extremogram, see Davis and Mikosch (2009), for this kind of series is very small and hence, extremogram may not be appropriate to detect the extremal dependence largely due to an implicit assumption of fixed time lag between extreme events. Our proposed RRT method does not suffer from fixed time lag problem and removes the assumption of i.i.d. observations. Related studies in actuarial risk theory exist with renewal times. For instance Doss (1989) derives the asymptotic distribution of relevant estimators as the sample size goes to infinity. Our test is non-parametric and distribution-free and hence valid for any sample size. This is our third contribution in the existing literature.

We carry out a thorough simulation study to better emphasize that the test proposed outperforms various standard methods in the literature in terms of size and power. The comparison with the Longin and Solnik (1995) test is incomplete, since the latter has problems with computational feasibility. We consider independent iid series and GARCH type series to compute the size of the proposed test and that of the competing candidates. We consider correlated bivariate normal or bivariate dynamic conditional correlation GARCH DGP$^{\prime}$s to assess the power of the proposed test and compare it with that of the other competing tests. We also consider the case of a DGP characterized by extreme dependent bivariate series. We believe that this is our final contribution in the existing literature.

Our results using simulation studies show that the size and the power of the proposed test outperforms the other tests in general. We demonstrate that many small-sized countries (defined in terms of their market capitalization) have contagion effects between each other, whereas larger economies like USA, and China cannot be easily affected. Our results remain valid irrespective of whether we use return or volatility.

The rest of the paper is organized as follows: We introduce our proposed test and testing procedure in Section 2. The method validation for the RRT method under different scenarios is presented in Section 3. Section 4 provides the simulation study and section 5 the results from our proposed test to real data from financial markets. Section 6 contains concluding comments.

2 Testing Procedure

In this section, we describe the testing procedure for the RRT approach. In Section 2.1, we introduce the used notations and definitions whereas Section 2.2 outlines the procedure for testing contagion.

2.1 Some Notations and Definitions

Let X and Y be two time series, for example, return on assets, volatility, volume, etc.

Definition 2.1. Extreme Event: An extreme event for series X (Y) is an event defined as being beyond a chosen threshold, say an upper or lower percentile of the empirical distribution of series X (Y).

Definition 2.2. Recurrence Time: Recurrence time, denoted by U_i (V_j), for series X (Y) is the time lag between two consecutive occurrences of extreme events in series X (Y).

Definition 2.3. Raw Residual Time: Raw Residual time, denoted by Z_k, for series X given series Y is the time lag from an extreme event in series Y to the following occurrence of an extreme event in series X. Mathematically,

$$\begin{aligned} Z_k &= \sum_{i=1}^{N} U_i - \sum_{j=1}^{k} V_j + 1, \\ \text{where} \quad N &= \text{arg min}_n \{ \sum_{i=1}^{n} U_i - \sum_{j=1}^{k} V_j \geq 0 \}. \end{aligned} \tag{1}$$

Remark 1. The Raw Residual Time is a measure of time needed for the transmission an extreme shock from one series to another. In the above definition for Raw Residual Times $\{Z_k\}$, it has "+1" on the right hand side of the equation. This is just a matter of convention. The "+1" means that if two extreme events occur at the same time, we consider the transmission time to be 1.

Definition 2.4. Residual Time: The sequence of residual times, denoted by $\{W_k\}$, for series X given series Y is a subset of $\{Z_k\}$ by eliminating the overlapping raw residual times, and is equal to $\{Z_k\} \backslash \{Z_{k'} : Z_{k'} + V_{k'} = Z_{k'-1}\}$.

In Figure 2, we demonstrate the definitions that we have introduced so far. The successive events in each series are marked with dots. Let U denote the waiting times between successive events of return series for country 1 and V the waiting times between successive events of return series for country 2. $W's$ are the residual times according to our definition. The interpretation of W is, after an event is observed in return series 2, what is the waiting time for an event in return series 1. Our claim is that if there is no contagion from return series 2 to 1, then the distribution of U's will be same as that of W's.

[Figure 2 about here]

2.2 *Details of the test*

Given a bivariate time series (X, Y), we can choose a threshold (say 95th percentile or 5th percentile of the empirical series) and find the extreme events as the values beyond (above or below) the threshold. Then the recurrence times for X, $\{U_i\}$, and the residual times for X given Y, $\{W_k\}$ can also be obtained. The null hypothesis is that there is no contagion effect transmitted from Y to X, and the alternative hypothesis is that there exists contagion effect transmitted from Y to X. The testing rule is constructed as follows. If the distribution of recurrence times $\{U_i\}$ and the distribution of residual times $\{W_k\}$ are significantly different, one would reject the null hypothesis. There are well established non-parametric procedures for testing the equality of two distributions, such as Kolmogorov-Smirnov test (K-S test) and Permutation test. We use the Permutation test as this test is more appropriate for integer valued data with ties, while the K-S test is mainly suitable continuous distributions.[5]

Application of permutation test directly to residual times and recurrence times is problematic as it requires independence between two samples. But residual times and recurrence times are dependent. Simulation studies show that this problem leads to a very small size. To avoid this problem, we propose the following alternative. In order to obtain critical values, we permute the combined group $\{U_i\} \cup \{V_j\}$ since $\{U_i\}$ and $\{V_j\}$ are independent under null hypothesis. We use these critical values for the test statistic $\overline{U} - \overline{W}$. We summarize the testing procedure for existence of contagion effect from Y (for example Mexican return) to X (say USA return) as follows:.

(a) Input two series X and Y, and find the time points of extreme events.

(b) Compute the corresponding recurrence times, $\{U_i\}$ and $\{V_j\}$; denote the sample sizes as n_u and n_v for $\{U_i\}$ and $\{V_j\}$, respectively.

(c) Find residual times $\{W_k\}$ of X based on Y, and calculate $\overline{U} - \overline{W}$, denoted as Δ_0.

(d) Combine $\{U_i\}$ and $\{V_j\}$ as one group, then permute and divide it into two subgroups, denoted as $\{\widetilde{U}_i\}$ and $\{\widetilde{V}_j\}$, with one sample size equal to n_u, and the other sample size equal to n_v. Find the corresponding residual times $\{\widetilde{W}_k\}$ and calculate $\overline{\widetilde{U}} - \overline{\widetilde{W}}$, denoted as $\widetilde{\Delta}_i$.

(e) Repeat step (d) for all possible permutations (or permute the combined group randomly for many times), then we have the empirical distribution of $\{\widetilde{\Delta}_i\}$, which will be considered to be the null distribution of the test statistic.

(f) Finally, the p-value is the proportion of the times when the absolute value of $\widetilde{\Delta}_i$ is larger than or equal to Δ_0.

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[5] Moreover, permutation test is an exact test which can deal with small sample size situations and we sometimes have small sample sizes (less than 30) for the sequences of residual times.

3 Method Validation

This section gives the detailed theoretical justification for the RRT method under different scenarios. Section 3.1 verifies the RRT method under the i.i.d. series scenario. Section 3.2 gives the limit theorems for extremal events in ARCH and GARCH converging to those of Compound Poisson Processes. Based on Compound Poisson Processes, one can find the validation of the RRT method in section 3.3.

3.1 Residual times for two independent i.i.d series

The idea of RRT method comes from a simple scenario for i.i.d. series. Since the goal is to test the existence of contagion effect or extreme dependence among series, we would consider extreme events over or below a specified threshold. For two i.i.d. series, called X and Y, we are interested in the recurrence times (denoted as $\{U_i\}$ for X, and $\{V_j\}$ for Y) over (below) a high (low) threshold. When the threshold is a fixed number (for example, the theoretical pth percentile of the distribution) then the recurrence times follow Geometric distribution. If the two recurrence times are independent (or the two series X and Y are independent over high thresholds), by using “Memoryless" property for Geometric distribution, the residual times of X given on Y have the same distribution as the recurrence times of X. We summarize this in the following theorem.

Theorem 3.1. *Let X and Y be two i.i.d. series with cumulative distribution function $F(x)$ and $G(y)$, respectively. Define $\{U_i\}$ ($\{V_j\}$) to be the recurrence times for series X (Y) above the $100p_1th$ ($100p_2th$) percentile of the distribution $F(x)$ ($G(y)$), and $\{W_k\}$ be the residual times of X given Y. Then,*

(a) $\{U_i\}$ is i.i.d. Geometrically distributed with success probability $(1-p_1)$, and $\{V_j\}$ is i.i.d. Geometrically distributed with success probability $(1-p_2)$.

(b) If X and Y are independent, $\{W_k\}$ is i.i.d. Geometrically distributed with success probability $(1-p_1)$, that is, $\{U_i\}$ and $\{W_k\}$ have the same distribution.

The following theorem proves the feasibility of the method under the i.i.d. series scenario when the threshold is a pth sample percentile of the empirical distribution. Detailed proof can be found in Appendix 7.1.

Theorem 3.2. *Let* $\{X_i\}_{i=1}^m$ *and* $\{Y_j\}_{j=1}^m$ *be two i.i.d. series with length m. Their empirical distributions are* $\hat{F}(x)$ *and* $\hat{G}(y)$*, respectively. Define* $\{U_i\}_{i=1}^M$ $(\{V_j\}_{i=1}^{M'})$ *to be the recurrence times for series* $\{X_i\}_{i=1}^m$ $(\{Y_j\}_{j=1}^m)$ *above the* p_1*th* $(p_2$*th*$)$ *sample percentile, and* $\{W_k\}_{k=1}^K$ *be the residual times of* $\{X_i\}_{i=1}^m$ *given* $\{Y_j\}_{j=1}^m$*. Then,*

(a) $(U_1,\dots,U_M) \overset{D}{=} (R_1,R_2,...,R_M | R_1+\cdots+R_M \leq m, R_1+\cdots+R_{M+1} > m)$*, where* $\{R_i\}_{i=1}^{M+1} \overset{i.i.d.}{\sim}$ *Geometric*$(1-p_1)$*, and a similar result also holds for* $(V_1,\dots,V_{M'})$*.*

(b) For any finite integer k*,* $\{U_i\}_{i=1}^k$ *asymptotically* $\overset{i.i.d.}{\sim}$ *Geometric*$(1-p_1)$ *as* $m \to \infty, \frac{M}{m} \to (1-p_1)$*, and* $\{V_j\}_{i=1}^k$ *asymptotically* $\overset{i.i.d.}{\sim}$ *Geometric*$(1-p_2)$ *as* $m \to \infty, \frac{M'}{m} \to (1-p_2)$*.*

(c) If $\{X_i\}_{i=1}^m$ *and* $\{Y_j\}_{j=1}^m$ *are independent, for any finite integer* k'*,* $\{W_k\}_{k=1}^{k'}$ *asymptotically* $\overset{i.i.d.}{\sim}$ *Geometric*$(1-p_1)$ *as* $m \to \infty, \frac{M}{m} \to (1-p_1)$*, that is,* $\{U_i\}$ *and* $\{W_k\}$ *have the same asymptotic distribution, as* $m \to \infty, \frac{M}{m} \to (1-p_1)$*.*

Remark 2. According to above theorem part (c), an asymptotic hypothesis testing procedure for contagion

$$H_0 : \text{No contagion vs } H_1 : \text{Exists Contagion}$$

can be constructed as

$$H_0 : \{U_i\} \text{ and } \{W_k\} \text{ follow the same distribution}$$

$$\text{vs} \quad H_1 : \{U_i\} \text{ and } \{W_k\} \text{ follow different distributions.}$$

3.2 Limit Theorems for Exceedances of ARCH/GARCH models

Let $\{X_t\}$ be a stationary ARCH(1) process with tail index κ. For basics in ARCH (GARCH), see Embrechts et al. (1997) and Jacod and Shiryaev (2003).

For $x > 0$, the point process of exceedances of the threshold $xn^{1/(2\kappa)}$ by $X_1,\dots,X_n$, is given by

$$N_n(\cdot) = \sum_{i=1}^n \varepsilon_{n^{-1}i}(\cdot) I_{\{X_i > xn^{1/(2\kappa)}\}}$$

where ε_x denotes *Dirac measure* at x. de Haan et al. (1989) obtain the result that

$$N_n \xrightarrow{d} N, \quad n \to \infty, \tag{2}$$

where N is a compound Poisson process with intensity $c\theta^{-2\kappa}$ and cluster probabilities given explicitly. The weak convergence is in $\mathscr{M}_p((0,1])$, the collection of Radon point (or counting) measures on $(0,1]$ equipped with the Borel sigma algebra. Convergence in distribution of a sequence of point processes $\{N_n\}$ toward a point process N, $N_n \xrightarrow{d} N$, is well explained in Kallenberg (1983), Daley and Vere-Jones (1998), Resnick (1987).

Theorem 3.1 of Mikosch and Starica (2000) gives a similar result for GARCH(1,1) processes.

We need to extend this result to a pair of processes. Denote a metric space by S, and let $\mathscr{S}$ be the Borel σ-field, the one generated by the open sets. Let P be a probability measure on $\mathscr{S}$. Assume that the product $T = S' \times S''$ is separable, which implies that S' and S'' are separable and that the three Borel σ-fields are related by $\mathscr{T} = \mathscr{S}' \times \mathscr{S}''$. Denote the marginal distribution of a probability measure P on $\mathscr{T}$ by P' and P'': $P'(A') = P(A' \times S'')$ and $P''(A'') = P(S' \times A'')$.

Theorem 3.3.
If $T = S' \times S''$ is separable, then $P_n' \times P_n'' \xrightarrow{w} P' \times P''$ if and only if $P_n' \xrightarrow{w} P'$ and $P_n'' \xrightarrow{w} P''$

Based on the above Theorem 3.3 and extremal results about ARCH and GARCH in section 3.2, point processes N_n^X and N_n^Y for the extreme exceedances of two independent ARCH (GARCH) processes have convergence property as follows:

$$(N_n^X, N_n^Y) \xrightarrow{d} (N^X, N^Y) \tag{3}$$

where N^X and N^Y are two independent compound Poisson processes.

3.3 *Residual times for two independent compound Poisson processes*

The following theorem derives distribution of residual and recurrence times for compound Poisson processes. Detailed proof can be found in Appendix 7.2.

Theorem 3.4. *Assume that there are two compound poisson processes N^X and N^Y on $[0,1]$ with intensity λ_1 and λ_2, respectively. Let U_i and V_j be the recurrence times for N^X and N^Y, respectively. Also, let $\{W_k\}$ be the residual times. Then,*

(a) $U_i \overset{i.i.d.}{\sim} \exp(\lambda_1)$ *and* $V_j \overset{i.i.d.}{\sim} \exp(\lambda_2)$.

(b) If N^X and N^Y are independent, $\{W_k\} \overset{i.i.d.}{\sim} \exp(\lambda_1)$.

Let U_i^n and V_j^n be the recurrence time for the point processes N_n^X and N_n^Y, and $\{W_k^n\}$ be the residual time, which are shown in Figure 2.

From Equation (2) and Theorem 3.4 part (a), we have

$$\begin{aligned} U_i^n &\xrightarrow{d} U_i,\ \ i=1,2,\ldots, \\ \text{that is,}\quad \{U_i^n\}\text{ asymptotically } &\overset{i.i.d.}{\sim} \exp(\lambda_1). \end{aligned} \tag{4}$$

From Equation (3) and Theorem 3.4, residual times $\{W_k^n\}$ have convergence property as

$$\begin{aligned} W_k^n &\xrightarrow{d} W_k,\ \ k=1,2,\ldots \\ \text{that is,}\quad \{W_k^n\}\text{ asymptotically } &\overset{i.i.d.}{\sim} \exp(\lambda_1). \end{aligned} \tag{5}$$

Therefore, based on the above Equations (4) and (5), one can construct a hypothesis for contagion as in remark 2.

4 Simulation Study

In this section, simulations under different scenarios are given. We compare the performance of our method with those of Censored Likelihood Method (CLM) of Ledford and Tawn (1996), Extremogram of Davis and Mikosch (2009) and some other methods in testing tail independence following Falk and Michel (2006), namely, Neyman-Pearson (NP) test, Fisher's κ (Fish) test, Kolmogorov-Smirnov (KS) test and Chi-square goodness-of-fit (ChiSq) test. Further details about these methods are in the Appendix. Section 4.1 shows simulations from an artificial time series where two independent series are superimposed with dependent extremes. In this case the correlation and extremogram cannot capture the extreme dependence, but RRT method can, as seen from the power. Section 4.2 and Section 4.3 obtain the power and size of RRT method by using simulated data under different models: the i.i.d. normal distribution model and the GARCH model (using indices time series to estimate parameters).

All results are reported with threshold 0.9 and significance level 0.05. Simulations for other values give similar results and are available from the authors on request.

4.1 Series with Dependent Extremes but Independent Non-extremes

This section describes a procedure for generating a bivariate sequence with dependent extremes but independent non-extremes, where the dependent extremes account for a small proportion (say, 10%) and the independent non-extremes take up a large proportion (say, 90%). For such a bivariate sequence, the cross-correlation between the two components are very small. Thus, cross-correlation does not indicate extreme dependence.

The steps of series generating procedure is shown as below.

(a) First generate two independent i.i.d. standard normal series, called series x and series y (the cross-autocorrelation between x and y is close to zero, since they are independent).

(b) Find the time points (denoted as $\{t_j\}_{j=1}^{J}$) of series y, where extreme events occur (above 90th percentile of y), and make these values more extreme by adding 1 to each of them, then call the modified series, Y.

(c) Add 4 to the value in series x at each time point $t_j + k_j$, where k_j is an independent random variable taking value $\{0,1,2\}$ with probability $\{1/6,1/3,1/2\}$ and $1 \leq j \leq J$. Call the modified series, X. This step makes the modified series X and Y extreme dependent, since an extreme event in series Y will trigger another extreme event in series X in a few, say 0-2, time points later.

By using the above procedure, one can generate 1000 bivariate series X and Y with length 1000, then apply the RRT method to test contagion effects and obtain the power of the test. Since the extreme events only account for 10% of the data and the independent part accounts for 90%, thus the cross-autocorrelation between X and Y are still close to zero (no cross-autocorrelation). Additionally, extreme events in the two series do not necessarily occur simultaneously and time lags between two extreme events in the two series are not fixed. We calculate the extremogram (with $A = B = (1,\infty) \times (1,\infty)$) for each generated series as above, with different thresholds (90%, 95%, 99%) and lags from 1 to 100. The values in the extremogram plot are all very small (nearly all are less than 0.05). Thus, using the extremogram cannot detect any extremal dependence in the simulated series. Using the same simulated series, we obtain the power for the other methods. The results in Table 3 show that the CLM test outperforms the other four tests, but is still far worse than RRT.

4.2 *i.i.d. Normal Simulation*

In this section, we generate i.i.d. univariate and bivariate normal series to obtain the size and power of our RRT test. The simulation study is as follows.

(a) Size: simulate two independent series of i.i.d. normal samples with $\sigma_1 = 1, \sigma_2 = 10$, and length equal to 1000. Apply our algorithm to test independence of the two simulated series with significant level 0.05. We call this model $\mathfrak{M}_0$. Repeat the above procedure 1000 times, then we can obtain the size of our algorithm.

(b) Power: simulate i.i.d. bivariate normal mean zero random vector series from 2 models, $\mathfrak{M}_1$ and $\mathfrak{M}_2$. The former has positive correlated covariance matrix $\begin{pmatrix} 10 & 2 \\ 2 & 3 \end{pmatrix}$, the latter has negative correlated covariance matrix $\begin{pmatrix} 10 & -2 \\ -2 & 3 \end{pmatrix}$. The length for both is 1000. Apply our algorithm to test the independence of the two components in each of two models (with significant level 0.05). Repeat the above procedure for 1000 times, then we can obtain the powers of our algorithm.

There are four possible directions of contagion and corresponding tests, Upper vs Upper, Upper vs Lower, Lower vs Upper and Lower vs Lower. We shall denote them by A, B, C and D respectively. We report the results from all the tests in Table 4.

The first row of Table 4 report sizes of our test for the simulated series for case A. Using the same simulated series, one can also obtain similar size for the other 3 cases. In the second row of Table 4, we report the powers of our test for $\mathfrak{M}_1$ for case A. Using the same simulated series, one can also obtain a similar table of rejection rates for case D. The power for case B and C under $\mathfrak{M}_2$ also similar. In the last row of Table 4, we have rejection rates of our test for $\mathfrak{M}_1$ in case of B. We refrain from calling these power as the underlying model has positive correlation and the test is set up to reject for contagion in the opposite direction. Ideally one would like these values to be small. Using the same simulated series, one can also obtain a similar table of rejection rates for case C. The rejection rates case A and D under $\mathfrak{M}_2$ are also similar.

Considering the first column of Table 4, one can find that for the positively corrected series, the rejection rate of RRT for case A (and D) reported in row 2 is much larger than that in case B (and C) reported in row 3. It shows that the RRT test can not only detect contagion effect between the two components but also can tell in which quadrant contagion effect exists. None of the other tests except NP have this property .

It should be noted that the CLM test is based on maximum likelihood method and often encounters bad results, for example, warning messages, errors and NaN. We present the percentage of such bad results in brackets in the tables. We present this percentage in brackets. The powers for the Fisher's κ test, and the ChiSq test are quite low.

4.3 GARCH Simulation

In this section, we use GARCH model to fit real indices series and use fitted model to simulate time series in order to find the size and power of our method. An introduction to DCC-GARCH model can be found in Nakatani and Tersvirta (2008). The real data being used is weekly indices for Mexico and USA from May 2003 to May 2007 (about 4 years weekly data), since this time period data shows contagion effects from USA to Mexico with p-value very close to 0 for case D and the p-value in case B is 0.685, which indicates no contagion transmission in Upper vs Lower quadrant.

(a) Size: use above data to fit two univariate GARCH models (GARCH(1,1)), then use the fitted models to simulate two series. Apply our algorithm to test independence of the simulated series. Repeat the above process for 1000 times, then we can obtain the size of our algorithm.

(b) Power: use the same data to fit a bivariate GARCH model (DCC-GARCH), then use the fitted model to simulate log returns. Apply our algorithm to test independence of the simulated series. Repeat the above process for 1000 times, then we can obtain the power of our algorithm. It is desirable that the power be high for case D and rejection rate be low for case B, since that is true in the underlying model.

Size, power and rejection rate of all the tests for the simulated series are reported in Table 5. The Fisher's κ test shows low power. Although the size and power for the other tests are good, the rejection rate for case B is always high except for RRT. So none of the other tests can distinguish between the directions of contagion. As before, CLM runs into computational problems too often.

5 Empirical Study for Stock Indices Data

Our data set consists of stock indices time series for 16 economies, namely Argentina, Brazil, Chile, Colombia, Mexico, Peru, China, India, Indonesia, Korea, Malaysia, Philippines, Taiwan, Thailand, USA and Japan. These are MSCI indices obtained from Datastream in US dollars and the data is weekly. The study is based on weekly data from January 1993 to December 2011, giving 992 observations for each of the 16 time series. We use weekly data to reduce the potential problem of nonsynchronous data partially.[6]

We define the return as $r_{it} = log(P_{it}/P_{i,t-1}); i = 1,...17, t = 1,...,T$. In Table 1 and 2, we report basic descriptive statistics of our return series. Except China and Japan, all countries show positive returns. Highest average return was observed for Peru, followed by Brazil, Colombia, Chile and Mexico. Our volatility measure is calculated using a moving average over a rolling window of four weeks of squared returns. We observe highest volatility in case of Indonesia followed by Brazil. The US market shows the lowest volatility. Among the emerging market economies, volatility in Chile and in Taiwan market is lower compared to others. Most return series show negative skewness except for Taiwan. All the return series displays excess kurtosis above zero and the $Jarque - Bera(JB)$ test rejects the null hypothesis of normality for each of the 16 stock markets. The correlations are all positive and the highest value is 0.64.

Figure 3 plots the rolling average return using a window of 52 weeks for some of these pairs, namely, Argentina-Brazil, Korea-Thailand, Mexico-USA and Korea-USA. The Asian crisis is visible from Korea-Thailand case but not so for other countries whereas the impact of sub-prime crisis is clear in all pairs. We also observe that the Mexican “Tequila crisis" in 1994 did not have much impact in USA but in Argentina and Brazil. Similar inferences can be drawn about the Asian Crisis. Our aim is to find an objective testing criterion for such statements. We get back to these specific examples in section 5.

Following RRT methods, we examine the contagion effects in the return as well as in the volatility. Subsection 5.1 gives a table (Table 6) of pairwise p-values resulting from applying RRT test to the return data where we have used the data from 2006 to the end of 2011 to test for directed contagion effects. The result using the volatility data is presented in Table 7. Using return data, we in Subsec-

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[6] Nonsynchronous data may arise due to markets closure in one country and open in another.

tion 5.2, focus on four specific pairs: Argentina given Brazil, Korea given Thailand, and Mexico given USA using data from 1993 to 2011.

5.1 Directed contagion effect

We use RRT method to test for directed contagion effect using the return and the change in volatility data set from 2006 to the end of 2011.[7] We consider the change in volatility for two reasons: first the volatility series could be non-stationary and second we are more interested in the volatility transmission: whether a big increase in volatility in one market is followed by a big increase in the other. Pairwise p-values for return are presented for directed contagion effect in Table 6, using 0.1 vs 0.1 threshold. This implies that extreme events are defined to be below 10th percentile of the corresponding series data. For change in volatility, we only consider upper quantiles (using 0.85 vs 0.85 threshold, results in Table 7). The choice of threshold is data dependent.

The evidence favouring the contagion is evident in those cases where the obtained p-value is less than 0.05 (in bold) or less than (0.10) (in italics). Let us take the case for Argentina: the contagion effect is significant either at 5% or at 10% level from Argentina to Brazil, Colombia, Mexico, Peru and Philippines. From Brazil, the same set of countries appear with the addition of Argentina, Taiwan and Thailand. However from Chili, there appears to be significant contagion to Argentina, Peru and Thailand. Looking at the East-Asian countries, we observe that for Korea, significant contagion exists in case of Peru, Philippines and Taiwan. For Thailand, we observe significant contagion effects towards Brazil, Colombia, Mexico and Peru. Note none of these are East-Asian countries. However, significant contagion exits from China, Indonesia, Philippines, Taiwan and to some extent from Malaysia and Japan towards Thai Market. Chinese stock market gets significant contagion effect from Brazil, Peru, Philippines, Taiwan and Thailand. Note, Brazil is one of the BRICS countries. Similarly for India; Brazil, Colombia and Peru exerts significant contagion effects, but none of the other East-Asian countries. Many small-sized countries (defined in terms of their market capitalization) have contagion effects between each other. On the other hand, developed markets like Japan and USA almost get no significant contagion effects.

Our results in terms of directed contagion effects in the higher volatility (as presented in Table 7)

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[7] This reflects the time period of sub-prime crisis and events thereafter.

shows that the contagion effect is significant either at 5% or at 10% level from Argentina, China, Colombia, India, Peru and Philippines to Brazil. For Argentina, presence of contagion effect in volatility is only from Brazil. Looking at Korea, we observe that there exists no significant contagion in higher volatility from other countries and vice versa. The Chinese market on the other hand shows significant directed contagion effect towards Brazil, Peru, India, Malaysia and Philippines. We observe no significant contagion effect from other countries to USA or Japan. Also, except for Philippines, the East-Asian and the Latin-American groups seem to be well-segregated.

In order to see the long term evolution of contagion in return series, we present four-year snapshots directed contagion effect for all countries. The nodes of the graph are the countries positioned at the latitude and longitude of their capital city. The arrows are for directed contagion effect detected at significance level 0.001 with thresholds at 1%. The color denotes the nature of contagion: Red for lower to lower, Green for Lower to Upper, Blue for Upper to Lower and Black for Upper to Upper. We focus on 1997−2000 (the East Asian Crisis) and 2007−2010 (the Global Financial Crisis).[8] During the East-Asian Crisis (Figure 4), we observe that for upper−upper return (color Black), directed contagion effect is present from Philippines to Thailand and China, Thailand to Indonesia and vice versa, Thailand to Malaysia, China to Malaysia and Taiwan to Brazil. For the same period, for lower−lower return (color Red), we observe presence of more directed contagion effects: from USA and India to Japan, Mexico to Brazil, from Taiwan to India, China to Korea, from China, Philippines and Malaysia to Argentina, and Chile to Philippines. The only directed contagion between Lower to Upper return exists from Thailand to China.

For the global financial crisis period (Figure 5), we observe that amongst the countries, there is absence of significant contagion effect from any countries in our sample to India and Japan. On the other hand, the crisis seems to be global in nature as there is contagion effect across countries irrespective of their geographical locations. Fr example, we observe the presence of significant contagion from China to Mexico, Thailand and USA for lower to upper return (color green), to Colombia, Peru, and Taiwan for the lower−lower return (color red) and to Philippines for the upper−lower return (color blue). Similarly, significant contagion is present from Chile to Argentina, Brazil, Colombia, Mexico, Peru, Indonesia, Thailand, Taiwan and USA for lower to lower return (color red); to Korea for the

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[8] All other plots are available from the authors on request.

upper−lower return (color blue).

Given the above set of results, we also try to examine two other events. First, we use only the Asian countries (China, Indonesia, Korea, Malaysia,Philippines, Taiwan, Thailand and Japan) and the return data from 1998-2000 for the East Asian Crisis.[9] The obtained results show that significant directed contagion effect from Korea exists for China, Indonesia, Taiwan, and Thailand. On the other hand, Korean market gets significantly influenced by China and Japan. The Japanese market does not get significantly affected by any other markets.

Second, we turn our focus to Latin-American countries (Argentina, Brazil, Chile, Colombia, Mexico and Peru). We also include USA here. Our sample runs from 1998 to 2000 broadly covering the period of Brazilian Crisis.[10] We observe significant directed contagion effect is present from Brazil and Mexico. The Mexican market gets significant contagion from Argentina and USA but not from the Brazilian market. Countries like Colombia, Chile and Peru does not receive any significant contagion from others and also do not exert any significant contagion to others. The USA market remains unaffected by any other countries in the sample.

5.2 Moving Window Plots

In this section, we examine contagion effects for specific pair of countries. We concentrate on three specific pairs: Korea given Thailand (for the “Asian Flu" in 1997), Argentina given Brazil (for the crisis in Brazil in 1998/1999 and Argentina being the largest trading partner of Brazil), and Mexico given USA (given the Tequila crisis and the Trade agreement between Mexico and USA). Plots of p-value against initial time for a period of 4 weeks are shown in Figure 6, Figure 7, and in Figure 8, with a moving window of 3 years (about 156 data points) and 4 years (about 208 data points), and step of one month (about 4 data points). p-values below the horizontal line indicate significant contagion at 5% level for the corresponding time period. Here, we vary the quantile (the top and bottom panels of Figures 4-7) and observe that the result of the test is quite robust to the choice of quantile.

The “Asian Flu" impact is clear from Figure 6. The crisis that has generated in 1997 following the devaluation of Thai Baht exerts its impact on Korea and the contagion effect is stronger until 2004. This is in contrast with the results that we have obtained from directed contagion effect test when we use

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[9] The results are not reported but available on request.

[10] The results are not reported but available on request.

sample from 2006-2011. Figure 7 demonstrates that for Argentina and Brazil, the impact was strong during 1998-1999 and then again from late 2004. Note our directed contagion effect tests also have detected the presence of significant contagion from Argentina to Brazil and vice-versa. For Mexico and USA, the contagion effect is stronger around year 2005 and it also shows large contagion effects especially around the sub-prime crisis and the events thereafter.

6 Concluding Comments

Most of the development in recurrence time has been so far with univariate time series. This paper provides a new idea of RRT method of high or low values for bivariate time series to detect contagion. We document that our proposed method does not need the i.i.d. assumption and can handle the situation where the extremes for different components do not occur at the same time.

Our primary intent is to reveal that there is a good source of information on contagion contained in the recurrence and residual time distribution of a certain characteristic event, if properly chosen. We have chosen the characteristic event as the returns as well as change in volatility hitting a threshold. The choice of threshold is data dependent, in that, it is a particular quantile of the data. The definition of some given percentile for the tail of the distribution is current practice in finance for eg. the value at risk. Most studies of contagion use fixed threshold, for example Bae et al. (2002). In the real example, we vary this quantile (the top and bottom panels of fig 4-7 are for different alpha) and observe that the result of the test is quite robust to the choice of quantile.

Our results show that RRT test helps to detect in which quadrant contagion effect exists. The simulation study show that the size and the power of the proposed test outperforms the other tests in general. Third, we demonstrate that many small-sized countries (defined in terms of their market capitalization) have contagion effects between each other, whereas larger economies like USA and China cannot be easily affected. Finally the East-Asian and the Latin-American groups seem to be well-segregated irrespective of our use of return or change in volatility.

Several extensions are possible: first, in order to choose optimal threshold parameters, one can follow adaptive model selection criterion of Fushing et al. (2012). Other ways to select the threshold value are suggested in Longin and Solnik (2001) and Danielsson et al. (2001). Another possibility is to fit bivariate or multivariate VAR model and then apply the RRT method with the estimated residuals.

Recently Dias et al. (2015) using an extended Hidden Markov Model, show the presence of three regimes: the bull, the bear and a stable regime with the stable regime occurring most frequently. One can use this method to detect regime changes. Changes from stable to bull will count as hitting upper threshold and changes from stable to bear will count as hitting lower threshold. The full development in our paper can then be carried out by combining our method with that of Dias et al. (2015). It will detect contagion in the sense of entering bull or bear states in one market drives the same behavior in the other.

Although we apply the RRT method to financial series, this method can be a valid tool in many areas, for example, one can apply this to the credit-rating literature to examine the spatial patterns. The same could be applied to housing price bubbles and the transmission mechanism form one country to another or from a regional perspective using data from single country. Finally one can examine lead-lag relationships using high frequency return data (for example see Huth and Abergel (2014)) where the Residual and Recurrence time method can be used. In all, we are looking forward to a deeper development of this method and more applications in the future.

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7 Appendix

7.1 *Proof of Theorem 3.2*

Proof. (a) Using the definition of $\{U_i\}$, it follows $P(U_1 = u_1, U_2 = u_2, ..., U_M = u_M) = \frac{1}{\binom{m}{M}}$.

Let $R := (R_1, R_2, ..., R_{M+1})$ be $(M+1)$ i.i.d. Geometric(q_1) random variables, where $q_1 = 1 - p_1$. Then,

$$\begin{aligned} & P(R_1 = r_1, R_2 = r_2, ..., R_M = r_M | R_1 + \cdots + R_M \le m, R_1 + \cdots + R_{M+1} > m) \\ = & \frac{P(R_1 = r_1, R_2 = r_2, ..., R_M = r_M, R_1 + \cdots + R_M \le m, R_1 + \cdots + R_{M+1} > m)}{P(R_1 + \cdots + R_M \le m, R_1 + \cdots + R_{M+1} > m)} \\ = & \frac{q_1^M p_1^{m-M}}{q_1^M p_1^{m-M} \binom{m}{M}} = \frac{1}{\binom{m}{M}}. \end{aligned}$$

As a result, $(U_1, \ldots, U_M) \overset{D}{=} (R_1, R_2, ..., R_M | R_1 + \cdots + R_M \le m, R_1 + \cdots + R_{M+1} > m)$.

(b) For any integer $k \le M$,

$$\begin{aligned} & P(U_i = u_i, i = 1, \ldots, k) \\ = \quad & \frac{\binom{m-u_1}{M-1} / \binom{m}{M}}{1} \times \frac{\binom{m-u_1-u_2}{M-2} / \binom{m}{M}}{\binom{m-u_1}{M-1} / \binom{m}{M}} \times \cdots \frac{\binom{m-u_1-\ldots-u_k}{M-k} / \binom{m}{M}}{\binom{m-u_1-\cdots-u_{M-1}}{M-(k-1)} / \binom{m}{M}} \\ \xrightarrow{M \approx (1-p_1)m} \quad & (1-p_1) p_1^{u_1-1} \times (1-p_1) p_1^{u_2-1} \times \cdots \times (1-p_1) p_1^{u_k-1}. \end{aligned}$$

As a result, for any finite k, $\{U_i\}_{i=1}^k$ is asymptotically $\overset{\text{i.i.d.}}{\sim}$ Geometric$(1-p_1)$ as $m \to \infty$, $\frac{M}{m} \to (1-p_1)$.

In the same way, $\{V_j\}_{j=1}^k$ is asymptotically $\overset{\text{i.i.d.}}{\sim}$ Geometric$(1-p_2)$ as $m \to \infty$, $\frac{M'}{m} \to (1-p_2)$.

(c) In order to find the asymptotic joint distribution for the residual times $\{W_k\}$, one would first find the asymptotic joint distribution for raw residual times $\{Z_k\}$. Let us define S_k and T_k as the partial sums of the U_i and V_i series. Also, we shall denote $\underset{\sim}{V} = (V_1, \cdots, V_M)$.

Note that $P(U_n = s - t | S_{n-1} = t) = \frac{\binom{m-s}{M-n}}{\binom{m-t}{M-n+1}}$ and $P(U_n \ge \nu - t | S_{n-1} = t) = \frac{\binom{m-\nu+1}{M-n+1}}{\binom{m-t}{M-n+1}}$. Using this, for any $t < \nu$, it is straightforward to derive that $P(S_n = s | S_{n-1} = t, S_n \ge \nu) = \frac{\binom{m-s}{M-n}}{\binom{m-\nu+1}{M-n+1}}$, which is not depending on t.

This in turn leads to the result, $P(S_n = s | S_{n-1} < \nu, S_n \ge \nu) = \frac{\binom{m-s}{M-n}}{\binom{m-\nu+1}{M-n+1}}$.

Using the definition of N in equation 1, we get $P(N = n | T_k = \nu) = \frac{\binom{\nu-1}{n-1}\binom{m-\nu+1}{M-n+1}}{\binom{m}{M}}$. Thus,

$$P(S_N = s | T_k = \nu) = \frac{\binom{m-s+\nu-1}{M-1}}{\binom{m}{M}}.$$

Since $Z_k = S_N - \sum_{j=1}^{k} V_j + 1$, for any $k < M$,

$$\begin{array}{rcl} P(Z_k = z_k|\underset{\sim}{V}) & = & P(S_N = z_k + \nu - 1|T_k = \nu) \\ & = & \frac{\binom{m-z_k-\nu+1+\nu-1}{M-1}}{\binom{m}{M}} \\ & = & \frac{\binom{m-z_k}{M-1}}{\binom{m}{M}} \\ & \overset{M\approx(1-p_1)m}{\longrightarrow} & (1-p_1)p_1^{z_k-1} \text{as} \quad m \to \infty. \end{array}$$

Thus, $(Z_k|\underset{\sim}{V}) \overset{\text{appr.}}{\sim}$ Geometric$(1-p_1)$, which does not depend on k or $\underset{\sim}{V}$.

Now we proceed to find the asymptotic joint distribution for $(W_1,\dots,W_k')$, where k' is the number of non-overlapping Z's as in definition 3.4.

For $k' = 1, W_1 = Z_1$. So we already have the result above. For $k' = 2$, one needs to find the conditional distribution of W_2 given W_1 and $\underset{\sim}{V}$. Note that,

$$W_2 = \begin{cases} Z_2 & \text{if} \quad w_1 \le v_2 \\ Z_3 & \text{if} \quad v_2 < w_1 \le v2 + v3 \\ \vdots & \end{cases}$$

In each case, $P(W_2 = w_2|W_1 = w1, \underset{\sim}{V}) = \frac{\binom{m-w_1-w_2}{M-2}}{\binom{m-w_1}{M-1}} \to (1-p_1)p_1^{w_2-1}$. Therefore,

$$\begin{array}{rcl} P(W_1 = w_1, W_2 = w_2|\underset{\sim}{V}) & = & P(W_1 = w_1|\underset{\sim}{V})P(W_2 = w_2|W_1 = w_1, \underset{\sim}{V}) \\ & = & \left(\frac{\binom{m-w_1}{M-1}}{\binom{m}{M}}\right) \cdot \left(\frac{\binom{m-w_1-w_2}{M-2}}{\binom{m-w_1}{M-1}}\right) \\ & \to & (1-p_1)p_1^{w_1-1} \cdot ((1-p_1)p_1^{w_2-1}). \end{array}$$

By applying the same technique $P(W_{i+1} = w_{i+1}|W_1 = w_1, \cdots, W_i = w_i, \underset{\sim}{V}) \to ((1-p_1)p_1^{w_{i+1}-1})$ where $w = \sum_{j=1}^{i} w_j$. In conclusion, $P(W_1 = w_1, \cdots, W_{k'} = w_{k'}|\underset{\sim}{V}) \to ((1-p_1)p_1^{w_1-1}) \cdot ((1-p_1)p_1^{w_2-1}) \cdots ((1-p_1)p_1^{w_{k'}-1})$. Thus, $\{W_k\}$ are asymptotically i.i.d. Geometric$(1-p_1)$.

$\square$

7.2 *Proof of Theorem 3.4*

proof. (a) This is obvious by using the “memoryless" property of exponential distribution.

(b) Define $\{Z_k\}$ to be raw residual times as in section 2.1 and as shown in Figure 2

$$
\begin{aligned}
& P(Z_1 = z_1, \ldots, Z_n = z_n | V_j, j = 1, \ldots, n) \\
= & \begin{cases}
\text{if } z_{n-1} < V_n : \\
P(Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1} | V_j, j = 1, \ldots, n-1) \times \\
\times P(Z_n = z_n | V_j, j = 1, \ldots, n), \\
\text{if } z_{n-1} \geq V_n : \\
P(Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1} | V_j, j = 1, \ldots, n-1) \ (note : z_{n-1} + V_n = z_n)
\end{cases} \\
= & P(Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1} | V_j, j = 1, \ldots, n) \times \\
& \times P(Z_n = z_n | V_j, j = 1, \ldots, n)^{I_{\{z_{n-1} < V_n\}}} \\
= & P(Z_1 = z_1 | V_1) \cdot P(Z_2 = z_2 | V_1, V_2)^{I_{\{z_1 < V_2\}}} \times \cdots \\
& \cdots \times P(Z_n = z_n | V_j, j = 1, \ldots, n)^{I_{\{z_{n-1} < V_n\}}}
\end{aligned}
$$

By using “memoryless” property for exponential random variable, one would have

$$
P(Z_k = z_k | V_j, j = 1, \ldots, k) = \lambda_1 e^{-\lambda_1 z_k}
$$

Therefore,

$$
\begin{aligned}
& P(Z_1 = z_1, \ldots, Z_n = z_n | V_j, j = 1, \ldots, n) \\
= & (\lambda_1 e^{-\lambda_1 z_1})(\lambda_1 e^{-\lambda_1 z_2})^{I_{\{z_1 < V_2\}}} \cdots (\lambda_1 e^{-\lambda_1 z_n})^{I_{\{z_{n-1} < V_n\}}}
\end{aligned}
$$

Then, the distribution for residual times $\{W_k\}$ is

$$
\{W_k\} \overset{i.i.d.}{\sim} \exp(\lambda_1).
$$

□

Table 1: Descriptive Statistics

Country	Mean	Median	SD	Skewness	Kurtosis	J-B Test Statistics
Argentina	0.075	0.329	5.287	-0.732	5.975	1562.590
Brazil	0.242	0.595	5.777	-0.578	3.037	435.987
Chile	0.149	0.293	3.456	-1.275	12.288	6503.458
China	-0.064	0.122	4.822	-0.190	2.360	235.976
Colombia	0.236	0.224	4.056	-0.726	5.088	1156.163
India	0.126	0.393	4.067	-0.353	1.998	185.495
Indonesia	0.100	0.259	6.570	-0.926	17.279	12469.401
Japan	-0.006	-0.028	2.961	-0.026	1.599	105.708
Korea	0.114	0.330	5.498	-0.933	12.071	6160.741
Malaysia	0.078	0.246	4.194	-0.982	22.510	21082.735
Mexico	0.144	0.515	4.547	-0.781	6.655	1929.592
Peru	0.265	0.219	4.328	-0.211	4.242	750.403
Philippines	0.012	0.133	4.135	-0.618	4.750	994.692
Taiwan	0.048	0.245	3.944	0.052	1.846	141.158
Thailand	0.002	0.061	5.188	-0.068	3.907	631.049
USA	0.109	0.217	2.491	-0.758	6.549	1865.783

Note: SD refers to standard deviation. The $J-B$ test statistic is distributed as a χ^2 and the critical value is 5.99 at 5% level.

Table 2: Correlation matrix

	Argentina	Brazil	Chile	Colombia	Mexico	Peru	China	India	Indonesia	Korea	Malaysia	Philippines	Taiwan	Thailand	USA	Japan
Arg	1.000	0.559	0.518	0.302	0.585	0.486	0.304	0.296	0.288	0.340	0.215	0.326	0.288	0.334	0.455	0.251
Bra	0.559	1.000	0.601	0.361	0.636	0.548	0.341	0.361	0.309	0.395	0.206	0.360	0.334	0.329	0.544	0.308
Chile	0.518	0.601	1.000	0.404	0.564	0.528	0.398	0.371	0.336	0.366	0.258	0.390	0.359	0.381	0.526	0.291
Colombia	0.302	0.361	0.404	1.000	0.353	0.380	0.209	0.299	0.263	0.226	0.125	0.283	0.214	0.226	0.350	0.166
Mexico	0.585	0.636	0.564	0.353	1.000	0.520	0.380	0.359	0.293	0.410	0.288	0.395	0.341	0.347	0.637	0.319
Peru	0.486	0.548	0.528	0.380	0.520	1.000	0.342	0.336	0.285	0.337	0.199	0.328	0.321	0.313	0.440	0.289
China	0.304	0.341	0.398	0.209	0.380	0.342	1.000	0.368	0.358	0.410	0.399	0.432	0.440	0.418	0.368	0.361
India	0.296	0.361	0.371	0.299	0.359	0.336	0.368	1.000	0.279	0.397	0.258	0.279	0.343	0.298	0.370	0.257
Indonesia	0.288	0.309	0.336	0.263	0.293	0.285	0.358	0.279	1.000	0.417	0.477	0.532	0.278	0.522	0.257	0.267
Korea	0.340	0.395	0.366	0.226	0.410	0.337	0.410	0.397	0.417	1.000	0.314	0.371	0.444	0.471	0.419	0.415
Malaysia	0.215	0.206	0.258	0.125	0.288	0.199	0.399	0.258	0.477	0.314	1.000	0.456	0.340	0.445	0.240	0.279
Phili	0.326	0.360	0.390	0.283	0.395	0.328	0.432	0.279	0.532	0.371	0.456	1.000	0.359	0.502	0.326	0.299
Taiwan	0.288	0.334	0.359	0.214	0.341	0.321	0.440	0.343	0.278	0.444	0.340	0.359	1.000	0.363	0.339	0.353
Thailand	0.334	0.329	0.381	0.226	0.347	0.313	0.418	0.298	0.522	0.471	0.445	0.502	0.363	1.000	0.290	0.363
USA	0.455	0.544	0.526	0.350	0.637	0.440	0.368	0.370	0.257	0.419	0.240	0.326	0.339	0.290	1.000	0.346
Japan	0.251	0.308	0.291	0.166	0.319	0.289	0.361	0.257	0.267	0.415	0.279	0.299	0.353	0.363	0.346	1.000

Table 3: Power, Dependent extremes with Independent non-extremes

Method	RRT	CLM	NP	Fish	KS	ChiSq
Power	1	0.358	0.223	0.092	0.130	0.082

Table 4: Size, Power and Rejection Rate. Normal Positively correlated.

Method	RRT	CLM	NP	Fish	KS	ChiSq
Size	0.042	0.051(0.176)	0.068	0.048	0.035	0.032
Power	0.954	1.000(0.185)	0.870	0.144	0.467	0.329
Rejection Rate	0.161	1.000(0.165)	0.000	0.379	0.667	0.474

Note: Numbers in brackets denote proportion of times the CLM algorithm gave error messages as the likelihood could not be maximized.

Table 5: Size, Power and Rejection Rate. GARCH Mex | USA

Method	RRT	CLM	NP	Fish	KS	ChiSq
Size	0.038	0.066(0.119)	0.067	0.050	0.060	0.055
Power	0.911	1.000(0.153)	1.000	0.169	0.876	0.785
Rejection Rate	0.195	0.823(0.156)	0.924	0.111	0.533	0.419

Note: Numbers in brackets denote proportion of times the CLM algorithm gave error messages as the likelihood could not be maximized.

Table 6: Directed Contagion Effect in Return: p-values with 0.1 vs 0.1 threshold

	Argentina	Brazil	Chile	Colombia	Mexico	Peru	China	India	Indonesia	Korea	Malaysia	Philippines	Taiwan	Thailand	USA	Japan
Arg	NA	*0.06*	0.32	**0.04**	*0.10*	**0.01**	0.30	0.18	0.11	0.25	0.56	*0.05*	0.13	0.11	0.99	0.31
Bra	*0.06*	NA	0.38	**0.02**	**0.03**	**0.00**	0.22	0.10	0.14	0.14	0.34	**0.05**	*0.06*	**0.01**	0.50	0.26
Chile	*0.07*	0.13	NA	0.13	0.15	**0.01**	0.44	0.22	0.32	0.52	0.65	0.10	0.11	*0.07*	0.84	0.22
Colombia	0.18	**0.01**	0.84	NA	*0.08*	**0.00**	0.42	0.36	0.19	0.53	0.63	0.10	0.15	**0.03**	0.91	0.99
Mexico	*0.09*	*0.07*	0.66	*0.08*	NA	**0.00**	0.93	0.34	0.37	0.55	0.98	0.39	0.35	0.10	0.28	0.24
Peru	**0.02**	**0.00**	0.23	**0.02**	0.02	NA	0.39	0.17	0.15	0.08	0.92	0.02	0.18	0.01	0.84	0.25
China	0.13	**0.03**	0.42	0.24	0.23	**0.01**	NA	0.24	0.13	0.19	0.39	**0.02**	*0.07*	**0.03**	0.80	0.30
India	0.18	*0.06*	0.71	*0.05*	0.24	**0.04**	0.30	NA	0.21	0.38	0.39	0.17	0.27	0.12	0.56	0.62
Indonesia	0.30	**0.05**	0.70	0.12	0.20	**0.03**	*0.10*	0.17	NA	0.51	0.41	*0.06*	*0.06*	**0.01**	0.90	0.42
Korea	0.17	0.21	0.24	0.38	0.20	**0.05**	0.43	0.45	0.37	NA	0.43	*0.10*	*0.07*	0.16	0.69	0.22
Malaysia	0.13	0.29	0.39	0.23	0.23	*0.06*	0.17	0.25	0.30	0.35	NA	*0.09*	0.12	0.10	0.93	0.20
Phili	0.20	**0.05**	0.38	**0.03**	0.13	**0.01**	0.33	0.24	*0.07*	0.29	0.80	NA	*0.05*	**0.03**	0.92	0.31
Taiwan	0.19	*0.07*	0.46	0.13	0.21	**0.03**	0.17	0.22	0.12	0.21	0.43	**0.05**	NA	**0.02**	0.78	*0.09*
Thailand	0.40	**0.01**	0.98	**0.02**	**0.04**	**0.00**	0.69	0.41	0.22	0.20	0.87	0.42	0.18	NA	0.91	0.19
USA	0.12	0.30	0.66	0.29	0.34	0.13	0.53	0.54	0.47	0.61	0.71	0.28	0.55	0.55	NA	0.55
Japan	0.17	0.17	0.70	0.15	0.25	*0.08*	0.83	0.43	0.28	0.99	0.63	0.73	0.19	0.10	0.80	NA

Table 7: Directed Contagion Effect in Volatility: p-values with 0.85 vs 0.85 threshold

	Argentina	Brazil	Chile	Colombia	Mexico	Peru	China	India	Indonesia	Korea	Malaysia	Philippines	Taiwan	Thailand	USA	Japan
Arg	NA	**0.02**	0.68	0.44	0.55	0.30	0.1	0.11	0.84	0.94	0.98	0.10	0.47	0.17	0.73	0.31
Bra	*0.06*	NA	0.55	**0.04**	0.50	**0.04**	**0.03**	**0.01**	0.77	0.96	0.79	**0.02**	0.58	0.11	0.64	0.3
Chile	**0.03**	**0.03**	NA	0.32	0.54	0.14	0.11	*0.09*	0.84	0.86	0.36	*0.05*	0.51	0.23	0.88	0.18
Colombia	0.2	0.14	0.79	NA	0.46	0.31	*0.07*	0.10	0.74	0.73	0.76	**0.04**	0.28	0.20	0.99	0.67
Mexico	0.32	*0.08*	0.58	0.24	NA	0.20	0.16	0.26	0.69	0.85	0.65	0.65	0.28	0.40	0.47	0.4
Peru	*0.060*	**0.03**	0.66	0.12	0.31	NA	**0.04**	0.46	0.91	0.48	0.89	*0.07*	0.50	0.41	0.82	0.38
China	0.28	*0.08*	0.52	0.75	0.46	**0.04**	NA	*0.06*	0.75	0.46	0.81	*0.05*	0.29	0.23	0.66	0.3
India	0.91	0.10	0.59	0.21	0.99	0.21	**0.04**	NA	0.62	0.84	0.43	**0.03**	0.36	0.28	0.69	0.9
Indonesia	0.69	0.41	0.88	0.39	0.66	0.51	0.43	0.42	NA	0.87	0.69	0.29	0.25	0.27	0.89	0.59
Korea	0.52	0.40	0.38	0.43	1.00	0.52	0.20	0.18	0.85	NA	0.85	0.14	0.34	0.33	0.89	0.29
Malaysia	0.48	0.15	0.82	0.94	0.65	0.14	**0.00**	0.25	0.97	0.56	NA	0.22	0.30	0.26	0.93	0.46
Phili	0.24	*0.06*	0.71	0.22	0.52	*0.06*	**0.00**	0.20	0.71	0.88	0.31	NA	0.72	0.10	0.65	0.3
Taiwan	0.48	0.11	0.42	0.18	0.77	0.28	0.16	0.40	0.85	0.61	0.64	0.13	NA	*0.05*	0.56	*0.05*
Thailand	0.81	0.11	0.63	0.44	0.53	*0.08*	0.11	0.16	0.50	0.74	0.45	0.34	0.21	NA	0.78	0.15
USA	0.27	0.21	0.52	0.42	0.67	0.39	0.23	0.40	0.49	0.76	0.53	0.28	0.54	0.88	NA	0.87
Japan	0.47	0.38	0.44	0.21	0.76	0.39	0.30	0.53	0.95	0.86	0.48	0.21	0.93	0.11	0.58	NA

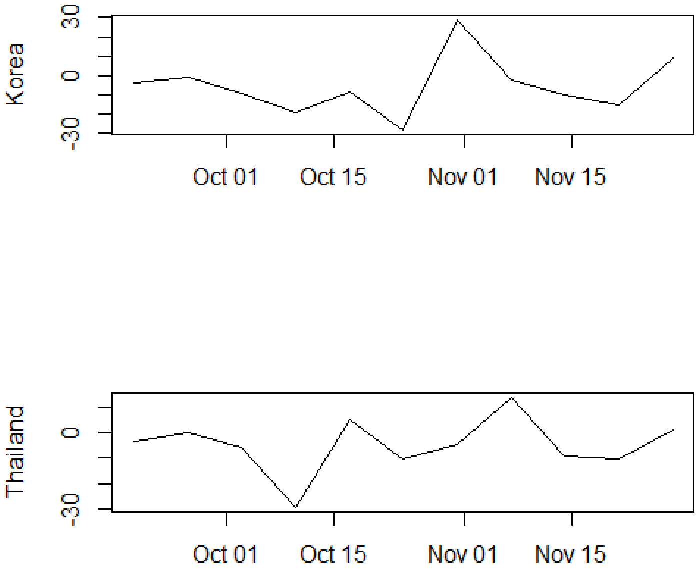


Figure 1: Percentage return series of Korea and Thailand in Oct-Nov 2008

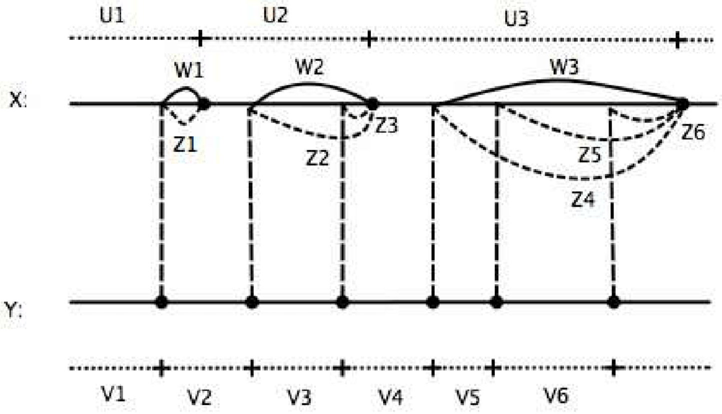
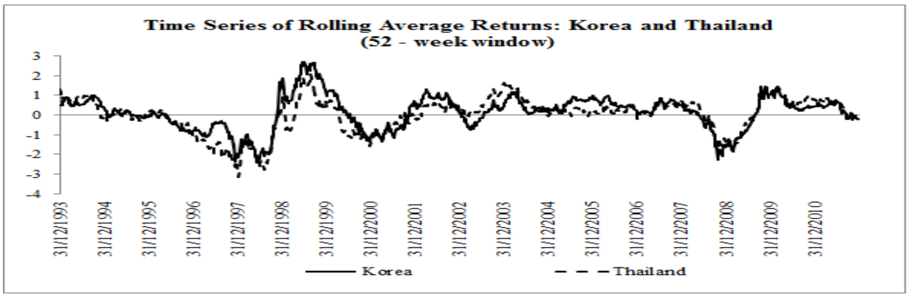
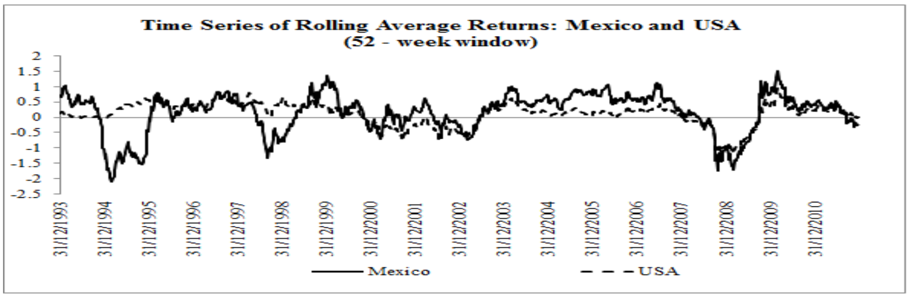


Figure 2: Recurrence, Raw Residual and Residual Times





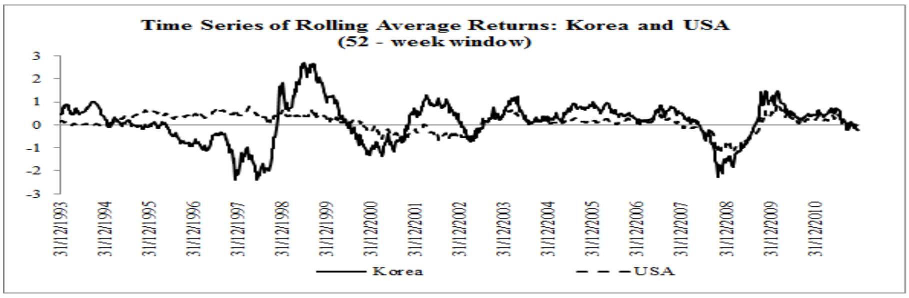


Figure 3: Rolling Average Returns for Different Country Pairs: 52 week Window. The pairs considered from top to bottom are: (1) Argentina-Brazil (2) Korea-Thailand (3) Mexico-USA and (4) Korea-USA.

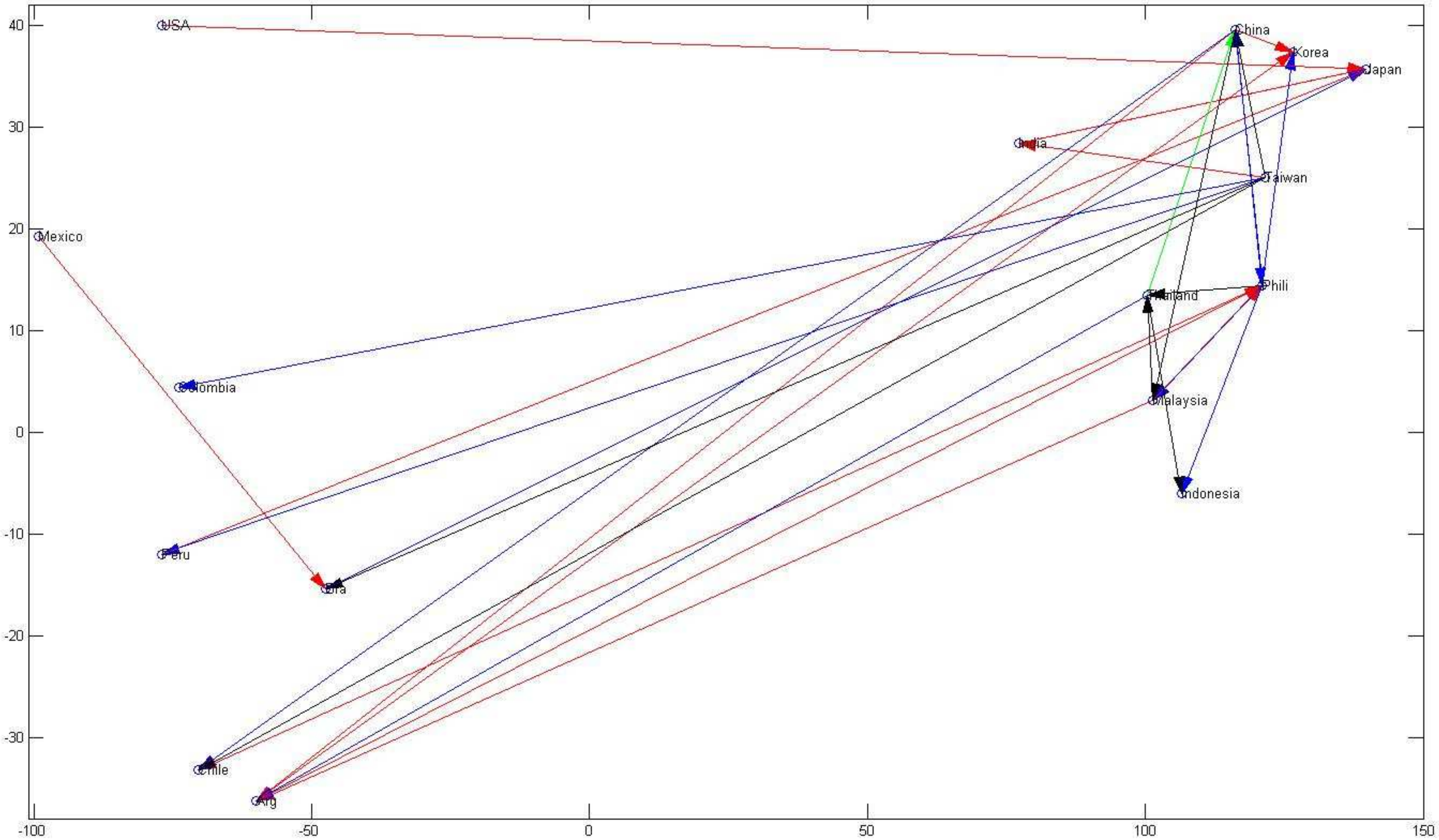


Figure 4: Directed Contagion Effects: Asian Crisis

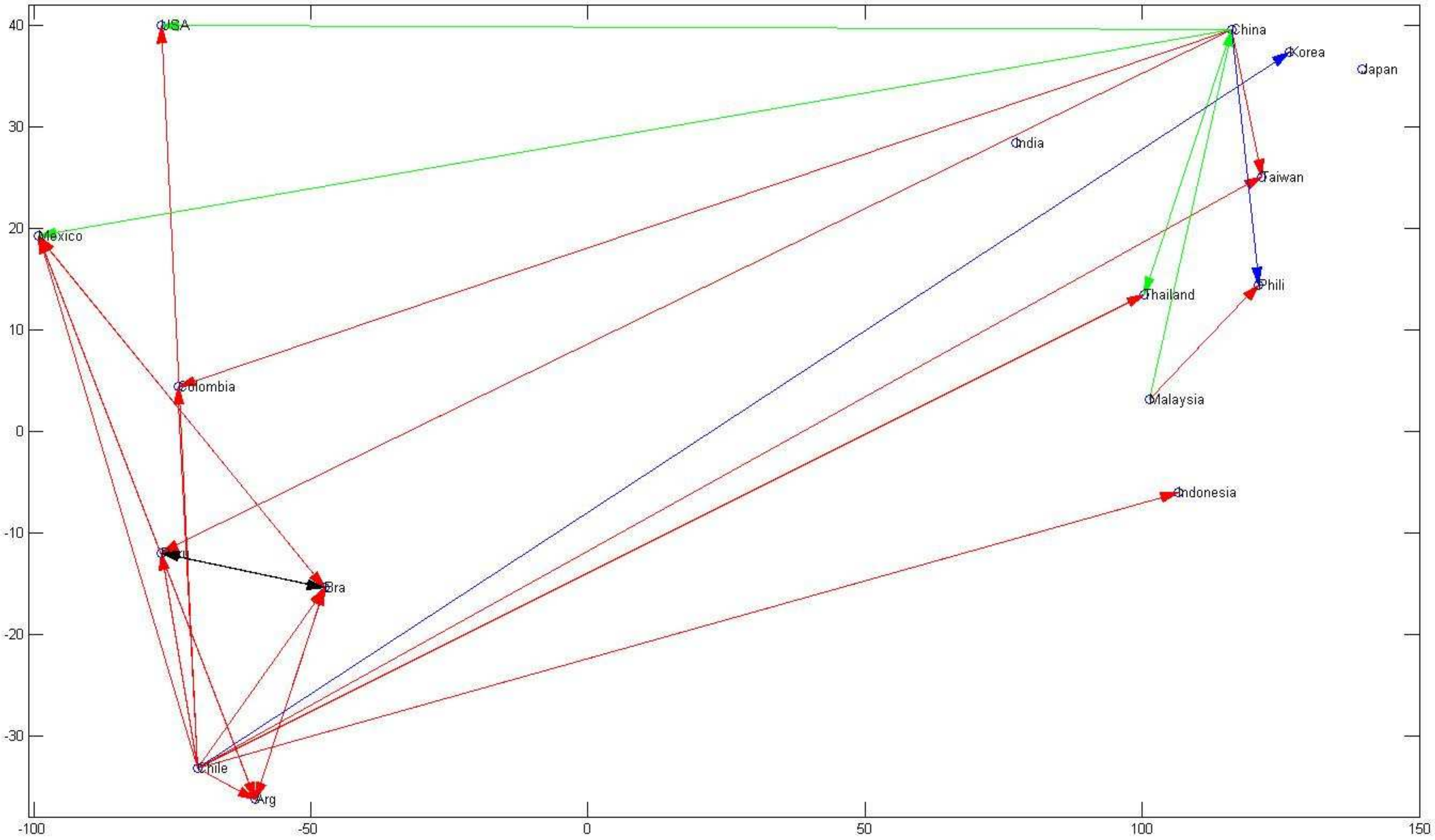


Figure 5: Directed Contagion Effects: Global Financial Crisis

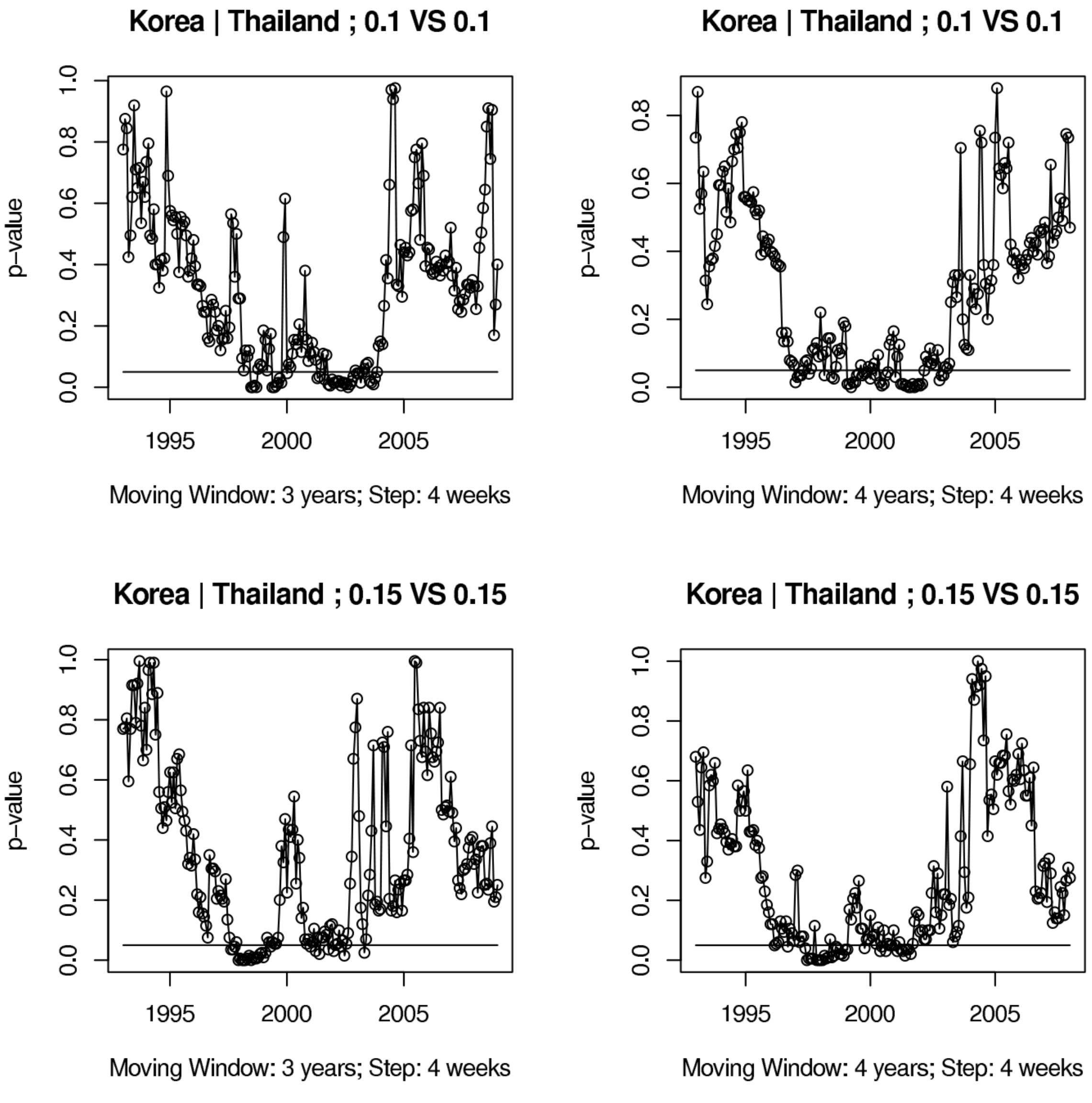


Figure 6: Moving Window Plot for Korea | Thailand

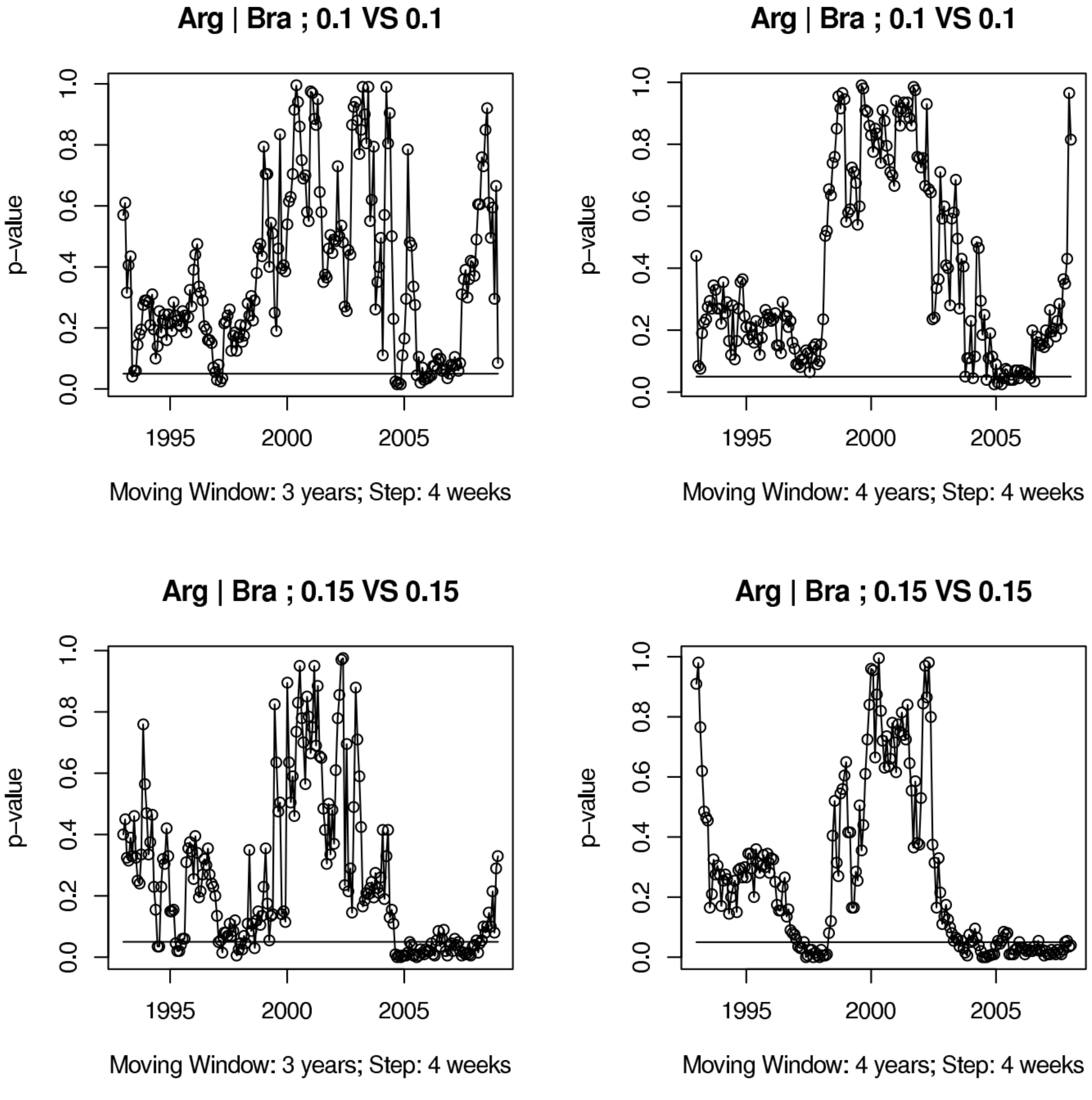


Figure 7: Moving Window Plot for Argentina | Brazil

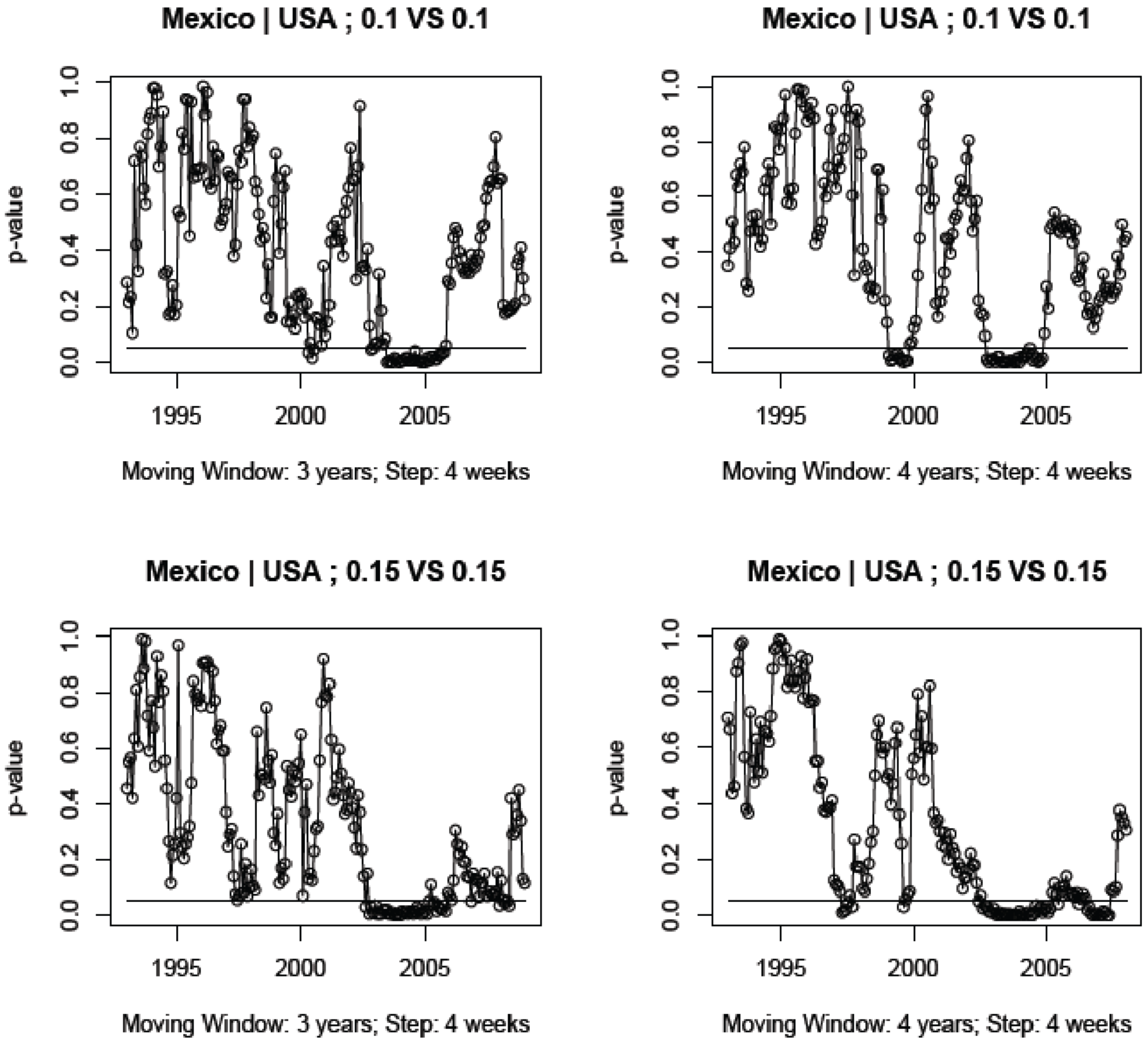


Figure 8: Moving Window Plot for Mexico | USA.

A Censored Likelihood Method

As suggested in Tawn (1990), the joint distribution above a threshold is assumed to be of the form,

$$F(x_1,\dots,x_D) = \exp\{-V(-1/\log(1-\lambda_1 t_1(x_1)),\dots,-1/\log(1-\lambda_D t_D(x_D)))\},$$

for $x_j > u_j,\ j=1,\dots,D.$ where $t_j(x_j) = \{1+\xi_j(x_j-u_j)/\sigma_j\}_+^{-1/\xi_j}$, and λ_j is some small probability such that the thresholds u_j are taken to be the $1-\lambda_j$ quantiles of the marginal distributions.

A particular dependence structure for V, the multivariate logstic structure is defined as

$$V(z_1,\dots,z_D) = (z_1^{-1/\alpha}+\dots+z_D^{-1/\alpha})^{\alpha},$$

where α is the dependence parameter ($0<\alpha\le 1$).

Remark A.1. *(a) When $\alpha=1$, $V(z_1,\dots,z_D)=z_1+\dots+z_D$, the marginal variables are independent. (b) When $\alpha\to 0$, the marginal variables are totally dependent. (c) The dependence weakens as α increases from 0 to 1.*

The parameters $\Theta = (\{\lambda_j,\xi_j,\sigma_j\}_{j=1}^D)$ can be estimated by maximum likelihood estimation. However, notice that the above model is only specified on the region $[\mathbf{u},\infty)$. The likelihood cannot be obtained directly. Instead, the observations such that $x_j < u_j$ need to be censored.

Let $J=\{J_1,\dots,J_m\}\subset\{1,\dots,D\}$, then

$$L(\mathbf{x}) \propto \mathbf{P}(\mathbf{X_j}\in\mathbf{dx_j},\ \mathbf{j}\in\mathbf{J};\mathbf{X_i}\le\mathbf{u_i},\ \mathbf{i}\notin\mathbf{J}) \propto \frac{\partial^{\mathbf{m}}\mathbf{F}}{\partial\mathbf{x_{j_1}}\cdots\partial\mathbf{x_{j_m}}(\mathbf{x}\vee\mathbf{u})}.$$

Define the total score,

$$U_{(n)} = \sum_{i=1}^n \frac{\partial}{\partial\alpha}\log L_i(\tilde{\Theta},\alpha)|_{\alpha=1},$$

where $\tilde{\Theta}$ is the maximizer of $L(\mathbf{x_i}_{i=1}^n;\Theta,1)$.

As shown by the Proposition 1 in Ledford and Tawn (1996),

Theorem A.1. *If the marginal variables are independent, then $U_{(n)}/c_n \to N(0,1)$ in distribution as $n\to\infty$, where $c_n = ((n\ \log n)/2)^{1/2}$.*

The above theorem is used to test independence in simulation study.

B Extremogram

In the paper of Davis and Mikosch (2009), the authors define the extremogram, which depends only on the extreme events in the series, as an analog of autocorrelation function. The details are as follows.

Let $(\mathbf{X_t})$ be a strictly stationary, regularly varying sequence of a d dimension random vector (with index $\alpha > 0$), i.e., for any $h \geq 1$, the vector $\mathbf{Y_h} = \mathrm{vec}(\mathbf{X_1}, \ldots, \mathbf{X_h})$ satisfies

$$\frac{P(x^{-1}\mathbf{Y_h} \in \cdot)}{P(|\mathbf{Y_h}| > x)} \xrightarrow{v} \mu_h(\cdot) \tag{6}$$

where $\mu_h(tS) = t^{-\alpha}\mu_h(S), t > 0$, for any Borel set $S \subset \bar{\mathbb{R}}^{hd} \backslash \{\mathbf{0}\} = \bar{\mathbb{R}}_0^{hd}$, and $\xrightarrow{v}$ means vague convergence.

If $P(|\mathbf{X}| > a_n) \sim n^{-1}$ for a sequence $a_n \uparrow \infty$, then formula (6) holds if and only if there exist constants $b_h > 0$ such that $nP(a_n^{-1}\mathbf{Y_h} \in \cdot) \xrightarrow{v} b_h\mu_h(\cdot) = \upsilon_h(\cdot)$, then,

$$nP(a_n^{-1}\mathbf{X_0} \in A, a_n^{-1}\mathbf{X_h} \in B) \to \upsilon_{h+1}(A \times \bar{\mathbb{R}}_0^{d(h-1)} \times B) = \gamma_{AB}(h).$$

Let A, B be two Borel sets, bounded away from zero, then

$$n\,\mathrm{cov}(I_{\{a_n^{-1}\mathbf{X_0} \in A\}}, I_{\{a_n^{-1}\mathbf{X_h} \in B\}}) \sim \gamma_{AB}(h).$$

Define the extremogram at lag h to be

$$\rho_{AB}(h) = \frac{\gamma_{AB}(h)}{\upsilon_1(A)} = \lim_{n\to\infty} \frac{P(a_n^{-1}\mathbf{X_0} \in A, a_n^{-1}\mathbf{X_h} \in B)}{P(a_n^{-1}\mathbf{X} \in A)}. \tag{7}$$

A natural estimator of $\rho_{AB}(h)$ is the empirical extremogram

$$\hat{\rho}_{AB}(h) = \frac{\sum_{t=1}^{n-h} I_{\{a_m^{-1}\mathbf{X_t} \in A,\ a_m^{-1}\mathbf{X_{t+h}} \in B\}}}{\sum_{t=1}^{n} I_{\{a_m^{-1}\mathbf{X_t} \in A\}}}, \quad h = 0, 1, \ldots.$$

We can have several values for a_m, say 0.90, 0.95, 0.99 empirical quantile of the absolute values of the series, and A, B can chosen to be $(1, \infty), (-\infty, -1)$ or $(-\infty, -1) \cup (1, \infty)$.

C Some Other Methods in Testing Tail Independence

In Falk and Michel (2006), the authors introduce four methods in testing tail independence in the extreme value model setting. The basic idea is as follows.

Let (X,Y) be a random vector in $(-\infty,0]^2$, whose upper tail follows a bivariate extreme value distribution G with reverse exponential margins. They show that conditional distribution function of $X+Y$, given that $X+Y>c$, converges to $F(t)=t^2, t\in[0,1]$, as $c\uparrow 0$ if and only if X,Y are tail independent (or asymptotically independent), i.e. $\lim_{c\uparrow 0}P(Y>c|X>c)=0$. Otherwise, the limit is $F(t)=t$. Utilizing this property, they gives us four kinds of tests, Neyman-Pearson test, Fisher's κ test, Kolmogorov-Smirnov test, and Chi-square goodness-of-fit test.

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be n iid samples of (X,Y). For any fixed $c<0$, we only consider samples X_i+Y_i satisfying $X_i+Y_i>c$, and denote these by $Z_1,Z_2,\ldots,Z_{K(n)}$ in the order of their outcomes. Note that

$$F_c(t):=P(X+Y>tc|X+Y>c)=t^2(1+O(c)), 0\le t\le 1.$$

Then, Z_i/c are iid $\sim F_c$, and for c close to 0, F_c is independent of $K(n)$.

For the Neyman-Pearson test, we want to test whether $Z_i/c, i=1,2,\ldots$ is distributed as null hypothesis $F(t)=t^2$ or alternative $F(t)=t, 0\le t\le 1$. Assume that $K(n)=m>0$, the test statistics is based on log likelihood ratio,

$$T(Z_1,\ldots,Z_m):=\log\left(\Pi_{i=1}^m\frac{1}{2Z_i/c}\right).$$

and null hypothesis is rejected if T gets too large. Note that $-2\log(Z_i/c)$ has a distribution function $1-\exp(-x), x\ge 0$, under null hypothesis, thus T has distribution function $1-\exp(-x)\sum_{0\le i\le m-1}x^i/i!$, under null hypothesis. The corresponding p-value is

$$p_{NP}\simeq=\Phi\left(\frac{2\sum_{i=1}^m\log(Z_i/c)+m}{m^{1/2}}\right).$$

For Fisher's κ test, assuming $K(n)=m$, let $U_i:=F_c(Z_i/c), 1\le i\le m$. The corresponding order statistics are denoted by $U_{i:m}$. Then, define the Fisher's κ statistic to be

$$\kappa_m := (m+1) \max_{j \leq m+1} S_j \quad \text{where} \quad S_j = U_{j:m} - U_{j-1:m}, 1 \leq j \leq m+1.$$

Note that given $K(n) = m > 0$, κ_m has distribution function

$$P(\kappa_m \leq x) = G_{m+1}(x/(m+1)),$$

$$where \quad G_{m+1}(x) = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (max(0, 1-jx))^m, x > 0.$$

Then, the null hypothesis of tail independence is rejected for small values of the p-value given by

$$p_\kappa := 1 - G_{m+1}\left(\frac{\kappa_m}{m+1}\right).$$

For Kolmogorov-Smirnov test, assuming $K(n) = m > 0$, the Kolmogorov-Smirnov statistic is given by

$$\Delta_m := m^{1/2} \sup_{t \in [0,1]} |\hat{F}_m(t) - t|,$$

where $\hat{F}_m(t) = m^{-1} \sum_{i=1}^{m} 1_{[0,t](U_i)}$. Then, the null hypothesis is rejected if the approximate p-value

$$p_{KS} = 1 - KD(\Delta_m)$$

is small, where KD is the Kolomogorov distribution.

For Chi-square goodness-of-fit test, assuming $K(n) = m > 0$ and dividing the interval $[0,1]$ into k consecutive and disjoint intervals $I_1, \ldots, I_k$, the test statistic is

$$\chi^2_{m,k} := \sum_{i=1}^{k} \frac{(m_i - mp_i)^2}{mp_i},$$

where m_i is the number of observations among $U_1, \ldots, U_m$ that fall into the interval I_i and p_i is the length of I_i. If m is large and $mp_i > 5, 1 \leq i \leq k$, the null hypothesis is rejected for small values of the approximate p-value

$$p_{\chi^2} = 1 - \chi^2_{k-1}(\chi^2_{m,k}).$$