The analysis of nonlinear systems in the frequency domain using Nonlinear Output Frequency Response Functions

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Abstract

The Nonlinear Output Frequency Response Functions (NOFRFs) are a concept which provides a new extension of the well-known concept of the Frequency Response Function (FRF) of linear systems to the nonlinear case. The present study introduces a NOFRFs based approach for the analysis of nonlinear systems in the frequency domain. It is well known that a nonlinear system can, under rather general conditions, be represented by a polynomial type Nonlinear Auto Regressive with eXogenous input (NARX) model. From the NARX model of a nonlinear system under study, the NOFRFs based approach for the frequency analysis of nonlinear systems involves solving a set of linear difference equations known as the Associated Linear Equations (ALEs) to determine the system nonlinear output responses and then the NOFRFs of the system up to an arbitrary order of nonlinearity of interests. The results enable a representation of the frequency domain characteristics of nonlinear systems by means of a series of Bode diagram like plots that can be used for nonlinear system frequency analyses for various purposes including, for example, condition monitoring, fault diagnosis, and nonlinear modal analysis.

Key words: Volterra series, Associated Linear Equations, Nonlinear Output Frequency Response Functions, Frequency domain analysis

1. Introduction

The frequency domain approach of linear systems is the very basis of control, signal processing, and communication and has been applied in almost all science and engineering areas. The key concept associated with the linear system frequency analysis is the Frequency Response Function (FRF), which is the foundation of Bode diagrams, Nyquist stability criterion, modal analysis, filter designs, among other well-known and well-estabilished theories and methods.

The direct extension of the FRF concept to the nonlinear case is known as the Generalized Frequency Response Functions (GFRFs) (George, 1959), which were proposed under the assumption that the output of the nonlinear systems under study can be described by a convergent Volterra series (Boyd and Chua, 1985). The difficulties with the practical application of the GFRFs are that the GFRFs can only be graphically studied up to the second order (Yue et al., 2005). This implies that the well-established Bode or Nyquist diagram based frequency domain analysis cannot be generally extended to the nonlinear case. Therefore, although some specific applications can be found in literatures such as, e.g., in image processing (Ramponi, 1986), channel equalization (Karam and Sari, 1989) and fault detection (Tang et al., 2010), a systematic approach for the analysis of nonlinear systems in the frequency domain that can be generally applied in practice still does not exist.

It is worth mentioning that describing functions (Khalil, 2002) are a traditional frequency domain analysis approach to nonlinear systems which only involve a one dimensional function of frequency and have been used in practical nonlinear system control problems. However, describing functions are defined for specific nonlinear components and can only be applied in the context of simple control systems with an a priori given structure.

Nonlinear FRF and associated nonlinear Bode plots (Pavlov et al, 2007, Rijlaarsdam et al. 2017) were introduced based on the exact evaluation of the bound on the output response of nonlinear systems under a harmonic excitation. These are the concepts of the nature and properties similar to that of the describing functions.

In order to resolve these difficulties, researchers have made considerable efforts to develop new concepts that can capture the system essential features while keeping problem dimensionality low. Examples of such approaches are the best linear approximation (Schoukens et al., 2003), the High Order Sinusoidal Input Describing Functions (HOSIDF) (Nuij et al., 2006) and the Associated Frequency Response Functions (AFRFs) (Feijoo et al., 2004). These approaches have also been applied to solve many engineering problems (Rijlaarsdam et al., 2012, 2013; Feijoo et al., 2006). However, these approaches have many limitations. For instance, HOSIDF can only deal with sinusoidal inputs and require complex computations that must be repeated for each frequency of interest while AFRFs can be evaluated only when the differential equation model of the system under study is available. In addition, these approaches have only been studied for simple and particular cases. It is difficult to assess the efficiency of these approaches in situations where systems are described by more general nonlinear models.

The concept of Nonlinear Output Frequency Response Functions (Lang and Billings, 2005) - NOFRFs - is a new extension of the FRF to the nonlinear case. One of its most attractive feature is its one-dimensional nature, which has many advantages, as has been demonstrated by a wide range of studies (Peng et al., 2007; Lang and Peng, 2008). However, current applications of the NOFRFs use a Least Squares (LS) based method to evaluate the NOFRFs (Lang and Billings, 2005). This requires an appropriate selection of the maximum order of the system nonlinearity, which is sometimes difficult and may suffer from numerical issues. In addition, the method requires the system response data from several simulation or experimental tests, which may not be convenient for implementation.

The present study is motivated by the need of addressing these problems. A systematic NOFRF-based approach for the nonlinear system frequency analysis is developed based on a polynomial type Nonlinear Auto Regressive with eXogenous input (NARX) model, which can either be obtained by discretizing the system's nonlinear differential equation model or determined by a data driven system identification method (Billings, 2013). The work involves the derivation of an algorithm which solves a set of linear difference equations to determine the nonlinear output responses and then the NOFRFs of a nonlinear system up to an arbitrary order of interest.

The results enable a representation of the frequency domain characteristics of nonlinear systems by means of a series of Bode diagram like plots that can be used for nonlinear system frequency analyses for various purposes including, for example, condition monitoring, fault diagnosis, and nonlinear modal analysis (Zhang et al, 2016; Xia et al, 2017). The application of the proposed new analysis to the detection and quantification of cracks in a beam structure is finally demonstrated in a case study.

2. The NOFRFs based approach for the analysis of nonlinear systems in the frequency domain

2.1. Nonlinear Output Frequency Response Functions (NOFRFs)

Let $y(k)$ and $u(k)$ respectively denote the output and input of a discrete time fading memory system (Boyd and Chua, 1985) with a zero equilibrium, and k represent the discrete time. The system output response around the origin can be described by the Volterra series:

$$y(k)=\sum_{n=1}^{+\infty} y_n(k)=\sum_{n=1}^{+\infty}\sum_{\mathbb{Z}^n} h_n(\boldsymbol{\tau}_n)\prod_{i=1}^{n} u(k-\tau_i) \qquad (1)$$

where $y_n(k)$ denotes a degree-n polynomial functional of $u(k)$, $h_n(\boldsymbol{\tau}_n)=h_n(\tau_1,\ldots,\tau_n)$ is the degree-n kernel.

Functionals can be described in the frequency domain using integral transforms such as the the Z transform or the normalised Discrete-Time Fourier Transform (DTFT). For example, the normalised DTFT of $y_n(k)$ can be described as (Lang and Billings, 1996)

$$Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int H_n(j\boldsymbol{\omega}_n)\prod_{i=1}^{n} U(j\omega_i)\,d\sigma_{n,\omega} \qquad (2)$$

where the integration is carried out over the hyperplane $\omega_1+\cdots+\omega_n=\omega$ with $-\pi\le\omega/f_s\le\pi$, where f_s is the sampling frequency.

The function $H_n(j\boldsymbol{\omega}_n)=H_n(\omega_1,\ldots,\omega_n)$ is the n-th order GFRF defined as the n-th order normalised DTFT of $h_n(\boldsymbol{\tau}_n)$

$$H_n(j\boldsymbol{\omega}_n)=\Delta t\sum_{\mathbb{Z}^n} h_n(\boldsymbol{\tau}_n)\prod_{i=1}^{n} e^{-j\omega_i\tau_i\Delta t} \qquad (3)$$

and $U(j\omega)$ is the DTFT of $u(k)$.

Definition 1. Let $u(k)$ be the sequence of a finite energy signal. In the discrete time domain, the n-th order generalized spectrum of $u(k)$ is defined as (Lang and Billings, 2005):

$$U_n(j\omega)=DF\left\{u^n(k)\right\}\Delta t=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int\prod_{i=1}^{n} U(j\omega_i)\,d\sigma_{n,\omega} \qquad (4)$$

where DF denotes the DTFT.

Definition 2. The n-th order NOFRF is defined as (Lang and Billings, 2005):

$$G_n(j\omega)=\frac{Y_n(j\omega)}{U_n(j\omega)};\ \omega\in\Omega\subseteq[-\pi f_s,\pi f_s] \qquad (5)$$

where Ω is the frequency support of $|U_n(j\omega)|$, which can be determined using the results about the output frequencies of nonlinear systems (Lang and Billings, 1996).

The NOFRFs as defined in (5) have the following attractive properties.

Property 1. (Lang and Billings, 2005) Let K be a non-zero constant and $G_n(j\omega)$ the n-th order NOFRF computed for $U(j\omega)$. Then, the NOFRF computed for $KU(j\omega)$ are also $G_n(j\omega)$.

Property 2. (Lang and Billings, 2005) The frequency support of $G_n(j\omega)$, $Y_n(j\omega)$ and $U_n(j\omega)$, i.e., the frequency range where these functions of frequency are well defined, are the same.

2.2. The NOFRFs based approach for the analysis of nonlinear systems in the frequency domain

It is obvious that the NOFRFs are an extension of the FRF to the nonlinear case, as when $n=1$, $G_n(j\omega)$ $=G_1(j\omega)$ reduces to the FRF of a linear system.

The NOFRFs of higher orders are generally dependent on the system input (Lang and Billings, 2005). However, different systems have different NOFRFs when probed by the same input. Consequently, the NOFRFs evaluated under the same input can be exploited to reveal the differences between systems in the frequency domain. This is the fundamental idea of the NOFRFs based system frequency analysis.

Based on these ideas, a general approach for the analysis of nonlinear systems in the frequency domain using the NOFRFs can be proposed as follows.

(i) Find an NARX model of the nonlinear system.

(ii) Determine the NOFRFs of the system from the NARX model under a probing input dependent on the specific application.

(iii) Analyze the system in the frequency domain from the determined NOFRFs for the specific application related objective.

In this approach, the nonlinear system identification approach in (Billings, 2013) can be applied to complete step (i) if the system differential equation model is not available. This approach is known as the NARMAX method that includes an integration of model structure determination, parameter estimation, and model validation and can produce a reliable NARX model as has been demonstrated in many real applications (Billings, 2013). The NOFRFs based system analysis in step (iii) is generally application dependent. The focus of the present study is therefore to investigate how to determine the NOFRF up to an arbitrary order from an NARX model for a given probing input in step (ii).

In order to resolve this problem, a new algorithm will be developed to enable an accurate evaluation of the NOFRFs up to an arbitrary order of interest in the following.

3. Determining the NOFRFs of the NARX model up to an arbitrary order of interest

Consider a general polynomial NARX model

$$Ay(k) = Bu(k) + \sum_{m=1}^{M} c_m F_m(k) \quad (6)$$

where

$$F_m(k) = \prod_{l=1}^{L} y(k-l)^{p(m,l)} u(k-l)^{q(m,l)} \quad (7)$$

$p(m,l)$ and $q(m,l)$ represent the non-negative integers such that $q(m,l) + p(m,l) > 1$, and A and B denote linear time shifting operators such that:

$$Ay(k) = y(k) + \sum_{l=1}^{L} a_l y(k-l) \quad (8)$$

$$Bu(k) = \prod_{l=1}^{L} b_l u(k-l) \quad (9)$$

It is worth noting that the Wiener and Hammerstein models of nonlinear systems (Wills et al., 2013) are the special cases of the NARX model (6), (7) (Billings, 2013).

The basic idea of the new algorithm for the evaluation of the NOFRFs comes from the observation that:

- The n-th order NOFRF can directly be obtained from the ratio of the normalised DTFT of the n-th order system output $Y_n(j\omega)$ and the n-th order generalized input spectrum $U_n(j\omega)$, and
- The n-th order system output $y_n(k)$ can be determined by solving a set of difference equations known as the Associated Linear Equations (ALEs) of the system.

The concept of the ALEs was proposed in Feijoo et al (2005, 2006) but the available results about the construction of ALEs can only be used for a special differential equation model known as the Duffing model. In order to apply the above idea of the NOFRFs evaluation to a much wider class of nonlinear systems, an important theoretical result about the ALEs of the NARX model (6) and (7) is first established in Proposition 1 below:

Proposition 1. The ALEs of the NARX model (6)(7) are a series of linear difference equations described by:

$$Ay_1(k) = Bu(k) \quad (10)$$

$$Ay_n(k) = \upsilon_n(k);\ n \geq 2 \quad (11)$$

where

$$\upsilon_n(k) = \sum_{m=1}^{M} c_m \psi_m(k) \sum_{S_m} \rho_m \phi_m(k) \quad (12)$$

$$\rho_m = \frac{\prod_{l=1}^{L} p(m,l)!}{\prod_{l=1}^{L} \prod_{j=1}^{J_m} r(m,l,j)!} \quad (13)$$

$$\psi_m(k) = \prod_{l=1}^{L} u(k-l)^{q(m,l)} \quad (14)$$

$$\phi_m(k) = \prod_{l=1}^{L} \prod_{j=1}^{J_m} y_k(k-l)^{r(m,l,j)} \quad (15)$$

$$J_m = n - \sum_{l=1}^{L} [q(m,l) + p(m,l)] + 1 \quad (16)$$

and S_m is the set of all non-negative integer solutions of the linear Diophantine system

$$\sum_{j=1}^{J_m} r(m,l,j) = p(m,l) \quad \forall l \quad (17)$$

$$\sum_{l=1}^{L} \sum_{j=2}^{J_m} (j-1) r(m,l,j) = J_m - 1 \quad (18)$$

Proof. See Appendix A.

Proposition 1 implies that given the order n of the system nonlinearity, the ALE of the NARX model (6)(7) can be obtained by first solving the Diophantine equations (17)-(18) to find S_m and then building the right-hand side of equation (11) from S_m and equations (12)-(16).

Having established the ALEs (10)(11), the n th order NOFRF can readily be evaluated as follows.

New Algorithm for the Computation of NOFRFs

1. Write down the n-th order ALEs, using Proposition 1, for $n = \{1, \ldots, N\}$. N is the maximum order of system nonlinearity of interest.
2. For $n = \{1, \ldots, N\}$
   - 2.1. Solve the n-th order ALE for $y_n(k)$
   - 2.2. Compute $Y_n(j\omega) = DF\{y_n(k)\}\Delta t$
   - 2.3. Compute $U_n(j\omega) = DF\{u_n(k)\}\Delta t$
   - 2.4. Compute $G_n(j\omega) = Y_n(j\omega)/U_n(j\omega)$

This new method can obviously determine the NOFRFs up to an arbitrary high order N and for any probing input.

It is worth pointing out that, the NOFRFs evaluated under a sinusoidal probing input are input independent and can directly reflect the systems' intrinsic behaviors (Peng et al., 2007).

In Section 4, a case study will be discussed to demonstrate the application of the new NOFRFs based frequency analysis to the detection and quantification of cracks in cantilever beams, where the NOFRFs under a sinusoidal probing input will be used.

4. A case study

4.1. Fault detection problem with cracked structures

Beam like structures are widely applied in engineering practice and the fault detection of such structures is widely concerned by researchers (Peng et al., 2007; Ma et al., 2016). A simple cantilever beam with crack is illustrated in Fig.1.

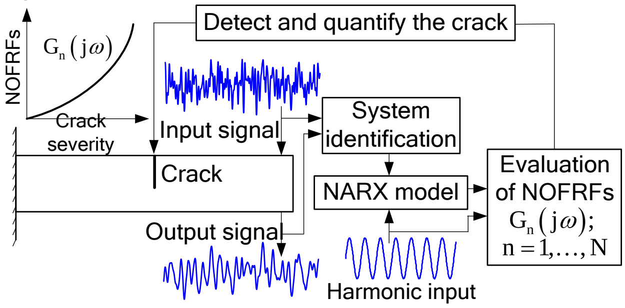

Fig.1 The NOFRFs based fault detection of cracked beams

Cracks in beam like structures can often be detected by analyzing the output spectra under a harmonic excitation (Ma et al., 2016), and the higher order super-harmonic output spectrum is expected to be monotonously increase/ decrease along the increase of the severity of a crack. However, there are many cracks that can generate more complex output responses, making the output spectrum analysis based detection of cracks not applicable in these situations (Zeng et al., 2017; Zhang et al., 2017).

This issue will now be addressed by using the NOFRFs which are more sensitive to variations in nonlinear characteristics in structural systems (Peng et al., 2007). The basic idea of the NOFRFs based fault detection follows the three steps in Section 2.2. The procedure is illustrated in Fig.1 and the details will be explained in Sections 4.2-4.4.

4.2. The NARX model of cracked cantilever beams

In practice, the dynamic properties of a cantilever beam with cracks can often be investigated by using a nonlinear differential equation model with second and forth order nonlinearities such as (Zhang et al., 2017; Zeng et al., 2017)

$$\ddot{y}(t)+c\dot{y}(t)+ky(t)+k_2y^2(t)-k_4y^4(t)=u(t) \quad (19)$$

In this case study, $c=20\ \mathrm{N/ms^{-1}}$, $k=1\times10^3\ \mathrm{N/m}$, k_2 and k_4 are the model nonlinear parameters determined by crack characteristics.

According to the NOFRFs based approach for nonlinear system analysis proposed above, the NARX models of the cracked cantilever beam, under different values of nonlinear parameter k_4, are identified using the input and the output data generated using model (19) and the nonlinear system identification method in (Billings, 2013).

For the specific case of

$k_2=1\times10^5\ \mathrm{N/m^2}$, $k_4=[0,2,3.5,5,7]\times10^{10}\ \mathrm{N/m^4}$,

and the sampling frequency of $f_s=1024\ \mathrm{Hz}$ the identified NARX model are

$$Ay(k)=Bu(k)+c_1y^2(k-1)+c_2y^4(k-1) \quad (20)$$

where

$$\begin{cases} Ay(k)=y(k)+a_1y(k-1)+a_2y(k-2) \\ Bu(k)=b_1u(k-1) \end{cases} \quad (21)$$

with the model coefficients shown in Tab.1.

Tab.1 NARX model coefficients under $k_2=1\times10^5\ \mathrm{N/m^2}$

$k_4\times10^{10}$ $\mathrm{N/m^4}$	$a_1\times10^{-6}$	a_2	b_1	c_1	$c_2\times10^4$
0	0.9436	1.9797	-0.9807	-0.0938	-
2.0	0.9438	1.9797	-0.9807	-0.0935	1.7690
3.5	0.9437	1.9797	-0.9807	-0.0935	3.1558
5.0	0.9436	1.9797	-0.9807	-0.0930	4.3261
7.0	0.9435	1.9797	-0.9807	-0.0940	6.4804

In the next, the newly proposed NOFRFs based analysis will be applied to the NARX models (20) to demonstrate how the novel analysis can reveal the changes of the system nonlinear parameter k_4 so as to enable the detection and quantification of cracks in cantilever beams.

4.3. Evaluation of the NOFRFs

(1) Determination of the ALEs

Given the NARX model (20), the ALEs of the system up to 4th order are obtained as follows. For $n=1$:

$$Ay_1(k)=b_1u(k-1) \quad (22)$$

For $n=2$, $J_1=2-2+1=1$ and $J_2=2-4+1=-1$, yielding the Diophantine system:

$$\begin{cases} r(1,1,1)=2 \\ 0=0 \end{cases} \text{and} \begin{cases} 0=4 \\ 0=-2 \end{cases} \quad (23)$$

The first Diophantine system has only one solution which is $r(1,1,1)=2$, and the second Diophantine system is inconsistent so that can be ignored. Consequently, the second order ALE can be obtained as

$$Ay_2(k)=c_1y_1^2(k-1) \quad (24)$$

For $n=3,4$, similar procedures can be followed to produce the 3rd and 4th order ALEs as

$$Ay_3(k)=2c_1y_1(k-1)y_2(k-1) \quad (25)$$

and

$$Ay_4(k)=c_1y_2^2(k-1)+2c_1y_1(k-1)\times y_3(k-1)+c_2y_1^4(k-1) \quad (26)$$

respectively.

(2) Evaluation of the NOFRFs

Consider the case where system (20) is subject to the sinusoidal input $u(t)=A\sin(\omega_h t)$. From the ALEs of system (20) determined above, the nonlinear output responses $y_1(k),\ldots,y_N(k)$ of the system are obtained. Then, the system output spectra contributed by up to the 4th order system nonlinearity, namely, $Y_1(j\omega_h)$, $Y_2(j2\omega_h)$, $Y_3(j\omega_h)$, $Y_3(j3\omega_h)$, $Y_4(j2\omega_h)$, $Y_4(j4\omega_h)$ are obtained by evaluating the normalised DTFT of $y_1(k),\ldots,y_4(k)$. Consequently, the NOFRFs $G_1(j\omega_h)$, $G_2(j2\omega_h)$, $G_3(j\omega_h)$ and $G_4(j2\omega_h)$ are evaluated as

$$G_1(j\omega_h)=\frac{Y_1(j\omega_h)}{U(j\omega_h)};G_3(j\omega_h)=\frac{Y_3(j\omega_h)}{U_3(j\omega_h)};$$
$$G_2(j2\omega_h)=\frac{Y_2(j2\omega_h)}{U_2(j2\omega_h)};G_4(j2\omega_h)=\frac{Y_4(j2\omega_h)}{U_4(j2\omega_h)} \tag{27}$$

where $U(j\omega_h)$, $U_2(j2\omega_h)$, $U_3(j\omega_h)$, $U_4(j2\omega_h)$ are obtained from evaluating the normalised DTFT of $u(k),u^2(k),u^3(k)$ and $u^4(k)$, respectively.

Moreover, for a specific ω_h, the NOFRFs in (27) are evaluated, which is expected to produce an effective index whose value increases/decreases monotonically with the severity of cracks so as to be able to be used to detect and quantify the cracks in beam structures.

4.4. The NOFRF based crack detection

The sinusoidal input with the magnitude of $A=1\text{ N}$ and the frequency of $\omega_h=30\text{ rad/s}$, which is close to the resonant frequency of the system, is applied to the NARX model of (20) to evaluate $G_2(j2\omega_h)$, $G_3(j\omega_h)$ and $G_4(j2\omega_h)$ under five different values of $k_4=[0, 2, 3.5, 5, 7]\times 10^{10}\text{ N/m}^4$.

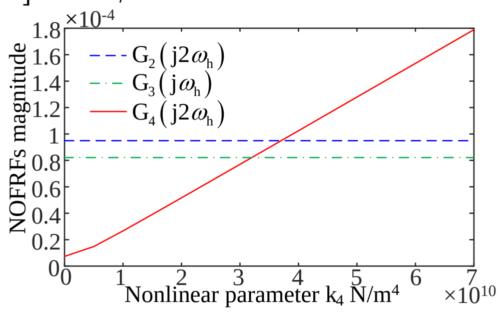

Fig.2 The change of the NOFRFs with respect to k_4 at $\omega_h=30\text{ rad/s}$

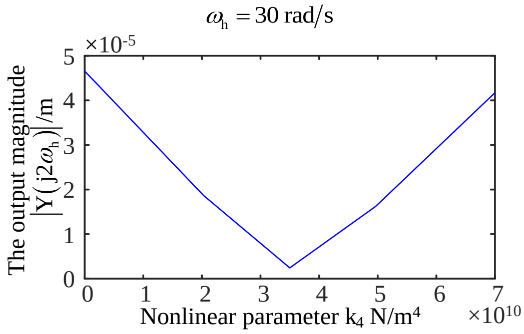

Fig.3 The change of $|Y(j2\omega_h)|$ with respect to k_4 at $\omega_h=30\text{ rad/s}$

The results are given in Fig.2, showing that $G_4(j2\omega_h)$ monotonously increases with the increase of k_4, while $G_2(j2\omega_h)$ and $G_3(j\omega_h)$ have no change with the increase of k_4, indicating that the severity of cracks in the beam can be detected and quantified using the NOFRF $G_4(j2\omega_h)$.

For a comparison, the traditional frequency response method introduced in Zhang et al. (2017) is also applied to quantify the increase of parameter k_4. The results are illustrated in Fig.3, indicating the second order super-harmonic magnitudes $|Y(j2\omega_h)|$ of the system varies non-monotonically with the increase of the value of k_4 and is therefore not suitable for use to detect cracks in this case.

5. Conclusions

In the present study, a new NOFRFs based approach for the analysis of nonlinear systems in the frequency domain has been proposed. The NOFRFs are a series of one-dimensional representations for the frequency properties of nonlinear systems, which are a new extension of the well known FRF to the nonlinear case and have been successfully used by many researchers to study nonlinear properties of engineering systems and structures. The new NOFRFs based nonlinear system analysis involves the determination of the NARX model of a system and evaluation of the NOFRFs of the NARX model for the purpose of the system analysis. The core technique is a novel algorithm derived in the present study that can accurately determines the NOFRFs of nonlinear systems up to an arbitrary order of interest, which has never been achieved before. The approach can be used for nonlinear system frequency analyses for various purposes including, for example, condition monitoring, fault diagnosis, and nonlinear modal analysis. A case study has been conducted, the results have demonstrated the potential application of the new NOFRFs based analysis to the detection and quantification of cracks in cantilever beam structures.

Appendix A. Proof of Proposition 1

Substituting the Volterra Series model (1) into (6) and (7) yields:

$$\sum_{j=1}^{\infty} Ay_j(k)=Bu(k)+\sum_{m=1}^{M} c_m F_m(k) \tag{A.1}$$

where

$$F_m(k)=\psi_m(k)\alpha_m(k) \tag{A.2}$$

$$\psi_m(k)=\prod_{l=1}^{L} u(k-l)^{q(m,l)} \tag{A.3}$$

$$\alpha_m(k)=\prod_{l=1}^{L}\left(\sum_{j=1}^{\infty} y_j(k-l)\right)^{p(m,l)} \tag{A.4}$$

In order to determine the n-th order ALE, (A.2) is expanded to identify all n-th order terms and equate them to those of the same order on the left-hand side of (A.1).

The products in (A.2) produce an expansion in terms of each $y_j(k)$ as

$$\alpha_m(k)=\sum\prod_{l=1}^{L}\beta(m,l)\prod_{j=1}^{\infty}y_j(k-l)^{r(m,l,j)} \quad (A.5)$$

where $r(m,l,j)$ are nonnegative integers

$$\beta(m,l)=\frac{p(m,l)!}{\prod_{j=1}^{\infty}r(m,l,j)!} \quad (A.6)$$

$$\sum_{j=1}^{\infty}r(m,l,j)=p(m,l);1\le l\le L \quad (A.7)$$

$$\sum_{l=1}^{L}\sum_{j=1}^{\infty}jr(m,l,j)=n-\sum_{l=1}^{L}q(m,l) \quad (A.8)$$

(A.7) and (A.8) are known as a Diophantine system as all unknowns are integers and can be further simplified, for every possible l, to yield

$$\sum_{l=1}^{L}\sum_{j=2}^{\infty}(j-1)r(m,l,j)=J_m-1 \quad (A.9)$$

$$J_m=1+n-\sum_{l=1}^{L}\left[q(m,l)+p(m,l)\right] \quad (A.10)$$

Notice that, since $r(m,l,j)\ge 0$, we must have $j\le J_m$, so that J_m can be used as upper limit to all summations and products in j, allowing system (A.7) and (A.8) to be rewritten as (17) and (18).

Let S_m denote the set of all nonnegative solutions of (17) and (18). By taking only the n-th order terms from the expansion of (A.2), the n-th order ALE can be written as:

$$Ay_n(k)=\sum_{m=1}^{M}c_m\psi_m(k)\alpha_m(k);n>1 \quad (A.11)$$

$$\alpha_m(k)=\sum_{S_m}\prod_{l=1}^{L}\beta(m,l)\prod_{j=1}^{J_m}y_j(k-l)^{r(m,l,j)} \quad (A.12)$$

Finally, by splitting the products in $\alpha_m(k)$ with respect to l, and defining ρ_m and $\phi_m(k)$ as (13) and (15), respectively, we obtain the n-th order ALE (11).

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