Stochastic user equilibrium with a bounded choice model

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ABSTRACT

Stochastic User Equilibrium (SUE) models allow the representation of the perceptual and preferential differences that exist when drivers compare alternative routes through a transportation network. However, as an effect of the used choice models, conventional applications of SUE are based on the assumption that all available routes have a positive probability of being chosen, however unattractive. In this paper, a novel choice model, the Bounded Choice Model (BCM), is presented along with network conditions for a corresponding Bounded SUE. The model integrates an exogenously-defined *bound* on the random utility of the set of paths that are used at equilibrium, within a Random Utility Theory (RUT) framework. The model predicts which routes are used and unused (the choice sets are equilibrated), while still ensuring that the distribution of flows on used routes accords to a Discrete Choice Model. Importantly, conditions to guarantee existence and uniqueness of the Bounded SUE are shown. Also, a corresponding solution algorithm is proposed and numerical results are reported by applying this to the Sioux Falls network.

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1. Introduction and motivation

For many decades, the two dominant approaches for modelling transport network equilibrium have been Deterministic User Equilibrium (DUE; Wardrop, 1952) and Stochastic User Equilibrium (SUE; Daganzo & Sheffi, 1977), including variants of the basic models to handle issues such as route correlation, time-dependent congestion, multiple classes of traveller/vehicle, risk-related phenomena and non-additive travel costs. In spite of these many advances, the basic assumptions that remain provide a choice between two extreme cases, namely i) that only routes with minimum cost are used in DUE, and ii) that all possible routes are used in SUE regardless of their costs.

In a recent paper (Watling et al., 2015), we illustrated the implausibility of these extreme assumptions by considering the *unused routes* in a highly-converged DUE solution of the Sioux Falls network (LeBlanc et al., 1975). For each distinct unused route, we calculated the relative travel cost with respect to the DUE travel cost on that route's OD movement. The frequency distribution of the relative travel costs for unused routes, across all OD movements, is presented in Fig. 1.

We particularly highlight two features from Fig. 1 that motivate our study:

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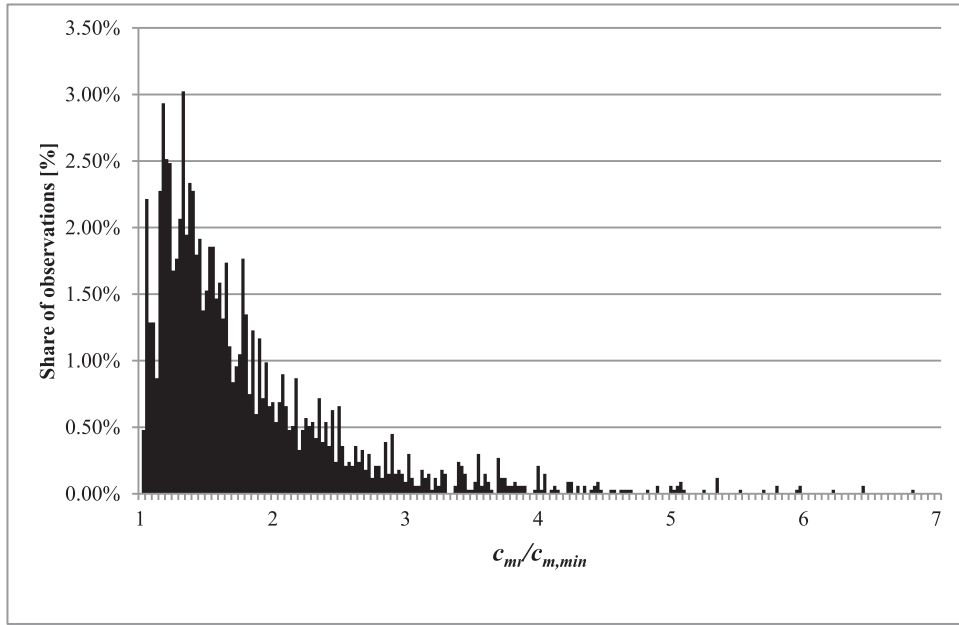*E-mail address:* d.p.watling@its.leeds.ac.uk (D.P. Watling).

Fig. 1. Frequency distribution of relative travel costs of unused DUE paths for Sioux Falls network (route cost on route r for OD-relation m, c_{mr}, relative to cost on minimum cost route for OD-relation m ($c_{m,\min}$)).

i) there are many unused routes with a travel cost only slightly higher than the cost of the used routes, i.e. with a relative travel cost near to one;

ii) there are many unused routes with a travel cost more than twice as high as the cost of the used routes, i.e. with a relative travel cost greater than two.

Point i) exemplifies the implausibility of the assumption in DUE that no such path would actually be used. This seems unreasonable given imperfections in drivers' knowledge (even if obtained through contemporary information systems), in the modeller's knowledge of the factors that motivate driver preferences, and given the natural variations in real-life systems. Conceptual understanding of the underlying behavioural processes supports this criticism: travellers have limitations in the consideration of alternative routes prior to choosing their preferred route (Bovy and Stern, 1990; Bovy, 2009) and, most relevantly, they consider spatio-temporal constraints for limiting the consideration set (Papinski et al., 2009; Kaplan and Prato, 2012). Moreover, empirical evidence supports this criticism: only a fraction of commuters were observed to select the shortest path (by distance or travel time) in Copenhagen (Nielsen, 1996, 2004), Lexington (Jan et al., 2000), Nagoya (Morikawa et al., 2005), Boston (Bekhor et al., 2006), Turin (Prato and Bekhor, 2006), and Minneapolis (Zhu, 2011).

There have been several approaches proposed in the literature for addressing deficiency i); however, they all (in our view) have undesirable consequences. With the aim of clarifying what these consequences are, and at the same time motivating our current study, we need first to analyse each of these approaches in detail. We group the approaches into four categories:

(I) *Conventional SUE models:* In the first class of approaches, deficiency i) is addressed by adopting an SUE model, based on conventional Random Utility Theory (RUT) distributions with unbounded error terms (such as that underlying the logit and probit families of models, for example). This, however, then raises point ii), since all routes will then attract some flow, regardless of how implausible they are. This is at odds with our understanding of the behavioural processes drivers are capable of adopting, as travellers have spatiotemporal constraints that limit the consideration of alternative routes (Papinski et al., 2009; Kaplan and Prato, 2012) as well as limitations to their cognitive capacity (Bovy and Stern, 1990; Bovy, 2009; Gao et al., 2011). It is also at odds with empirical evidence: commuters have a limit in the excess distance or travel time with respect to the minimum cost path, with heterogeneous preferences going from large variations in Copenhagen, Lexington and Nagoya (Jan et al., 2000; Nielsen, 2004; Morikawa et al., 2005) to small variations in Minneapolis and Turin (Zhu, 2011; Kaplan and Prato, 2012). This is also exemplified in Fig. 2, which is based on 16,618 GPS observations collected among car travellers in the Greater Copenhagen Area over an extended period of time (Rasmussen et al., 2017). The figure illustrates the cumulative share of observations as a function of the ratio between the cost on the observed path (path obtained from GPS data) and the cost on the minimum cost path between the corresponding locations. As can be seen, only 1.8% of the trips are made using a path that is more than 50% more costly than the corresponding shortest. Also note that approximately 50% of the trips use the shortest path. It should also be remarked that while it is true that many numerical *solution algorithms* for solving SUE will, after a finite number of iterations, only identify a subset of the available routes, this is not
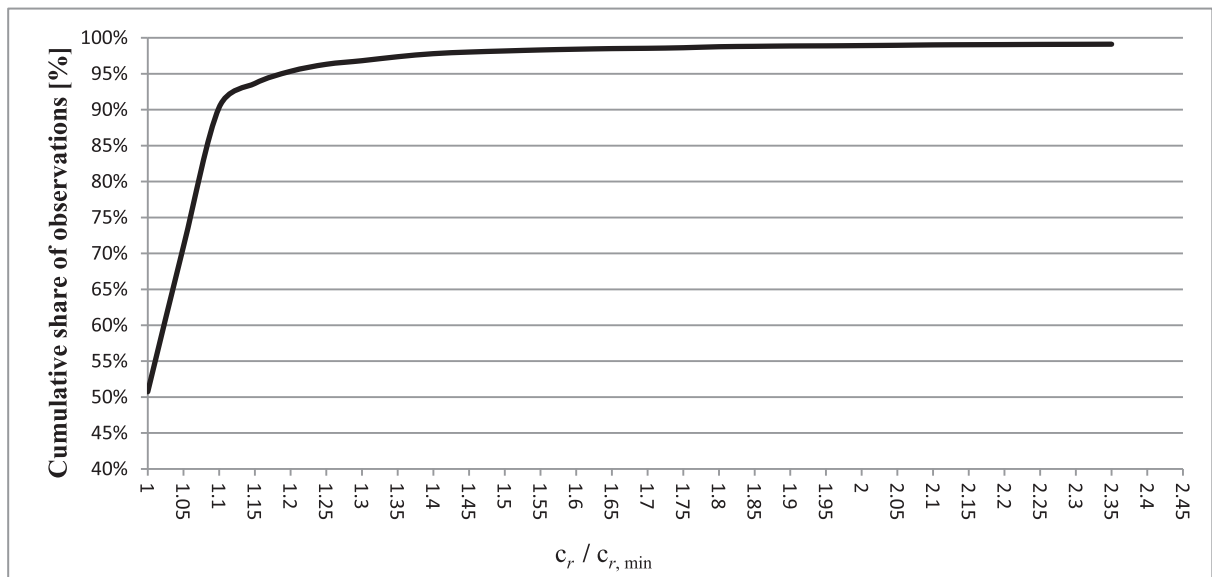
Fig. 2. Cumulative share of observations as a function of the ratio between the cost on the observed path r, c_r, and the cost on the corresponding minimum cost path $c_{r,\min}$. The figure is based on 16,618 observed path choices obtained from a large GPS dataset collected among car drivers in the Greater Copenhagen Area during 2011. Path costs of the observed/corresponding shortest paths are based on equilibrated network travel costs obtained from the Danish National Transport Model.

intended as a behavioural mechanism inherent in the model; which routes are identified will vary according to the implementation of the algorithm used, initial conditions, order of link- and OD-cell coding, and number of iterations, rather than depending on anything fundamental in the model (see, e.g., Bekhor et al., 2008).

(II) *Choice set pre-generation methods:* In order to address the problem that conventional SUE models identify too large a subset of used routes, a second class of approaches has been to apply the SUE model to only a subset of the available routes. This subset is identified in advance of running an SUE solution algorithm, through the use of an explicit choice set generation method (e.g., Friedrich et al., 2001; Prato and Bekhor, 2006; Bovy, 2009; Frejinger et al., 2009). While it addresses deficiencies i) and ii) above, it does so at a significant price, leading to new disadvantages, including: *inconsistency*, namely the level-of-service values (e.g., travel times) needed as input to the path generation will be different from those that arise after running the equilibrium algorithm and the behavioural assumptions of the choice set generation might differ from those underlying the choice model; and *policy insensitivity*, namely policy measures tested with the equilibrium model may make attractive some new options that were unattractive (and so not in the pre-defined route set) without the policy measure.

(III) *Boundedly Rational models:* In the field of route choice, the theory of *bounded rationality* has been interpreted as establishing the existence of 'indifference bands' on the excess cost of a route relative to the minimum cost route (Mahmassani and Chang, 1987; Hu and Mahmassani, 1997; Jayakrishnan et al., 1994; Mahmassani and Liu, 1999; Srinivasan and Mahmassani, 1999). A Boundedly Rational User Equilibrium (BRUE) has been proposed as a *space* of flow solutions representing drivers' inertia to route-switching, within which drivers are indifferent to route cost differences within their indifference bands (Mahmassani and Chang, 1987; Guo and Liu, 2011; Lou et al., 2010; Di et al., 2013; Di and Liu, 2016). The resulting BRUE solution set will typically contain flow solutions in which demand is divided between used and unused routes (thus addressing issue ii) above). In a sense the BRUE approach also addresses issue i), since it admits flow solutions in which sub-optimal routes *could* appear, while not ruling out DUE solutions. Thus, BRUE can be seen as a kind of weakening of the strict requirements of DUE. The price paid for this weakening, however, is that we no longer obtain point estimates of equilibrium, but what might be called *interval estimates*, though (unlike with statistical interval estimates) the model gives no higher weight to any possibility within the interval. This deliberate non-uniqueness results in what we believe to be significant difficulties in using this model for traffic forecasting, policy testing and cost-benefit analysis.

(IV) *Endogenous choice set restriction methods:* Returning to SUE-type models, one recent stream of research has considered adding additional restrictions and/or constraints to a conventional SUE formulation, in order to arrive at a model in which it is possible that not all routes are used. In our own earlier research, we proposed a form of 'restricted' SUE in which the choice set of used routes is determined by some explicit constraint that is dependent on the equilibrium solution (Watling et al., 2015; Rasmussen et al., 2015). We recently further extended this approach to include a second restriction, determined partly by an exogenously defined cost threshold (Rasmussen et al., 2017). In a parallel piece of research in a similar spirit, Pel and Chaniotakis (2017) recently proposed a modified form of SUE in which all used

routes must have a flow above a user-defined flow-threshold. The disadvantage with all these approaches is that we are unable to guarantee that equilibrium solutions even *exist* in all cases, as indeed it is possible to develop examples where they do not. Even in cases where we have existence, we have no guarantee of the uniqueness of the solution. Again, this lack of theoretical guarantee of existence and uniqueness is a major price to pay when one considers the typical use of such models in policy testing, meaning that we cannot guarantee to attribute a unique forecasted benefit to any tested measure.

Taking together all of the perspectives discussed above, we set out the following *requirements* for the equilibrium modelling approach that we present in this piece of research:

- It should neither exclude all sub-optimal routes (as in DUE) nor include all available alternatives (as in the conventional SUE models).
- It should provide a *point estimate* of equilibrium, akin to DUE and SUE, unlike the interval estimate approach of BRUE.[1]
- It should take the attractive feature of DUE in its ability to address choice set equilibration simultaneously with the flow equilibrium, and consistently with the determination of potential choice sets. That is to say, it should provide an equilibrium prediction of the used/unused routes that is consistent with the travel costs.
- Like SUE, it should draw on RUT to represent travellers' misperceptions in travel costs and the significant empirical evidence of drivers' responsiveness to factors such as travel time that typically motivate route choice.
- Its properties should allow the establishment of theoretical conditions to guarantee both *existence* and *uniqueness* of the resulting equilibrium solution.

Accordingly, we first introduce a novel *closed-form* choice model that incorporates *bounds* within a discrete choice model, we then define *equilibrium conditions* corresponding to this choice model, and last we prove *existence* and *uniqueness* of the solution. The equilibrium conditions lead us to a solution algorithm for the consistent equilibration of both the flow and the choice sets.

The structure of the paper follows the presentation of our main contributions. We introduce in Section 2 the fundamental component of our formulation, namely we review existing literature concerning thresholds in Random Utility Models (RUM) before describing the details of our novel choice model upon which our work is based. In Section 3 we develop network equilibrium conditions for the newly introduced choice model, and go on to prove the existence and uniqueness properties of the model. We present in Section 4 numerical results of applying a solution algorithm for this model to a small network, as well as the Sioux Falls network. With these experiments we highlight some characteristics of the new model proposed as well as gain insights into the sensitivity of the model to input parameters. Finally, we present conclusions and future research directions in Section 5.

2. Theoretical approach to choice modelling with thresholds or bounds

In the present section we introduce a fundamental element of our study, namely how the notion of a threshold or bound might be incorporated within a discrete choice model. Throughout this paper the focus is on closed-form models. In Section 2.1, we begin with the simplest case of a problem with two alternatives, and present and discuss the relative merits of alternative formulations of logit-inspired models incorporating thresholds or bounds. From this discussion, we identify the most promising approach to take forward, and in Section 2.2 we generalise this approach to the case of any number of choice alternatives. Notably, Section 2 also serves an important role of allowing us to position our work in existing literature, and to distinguish our novel model from previously proposed ones.

2.1. Problems with two alternatives

2.1.1. Threshold indifference model

One of the most relevant existing studies, which motivated some of our own work below, is that reported by Krishnan (1977), who introduced the concept of a minimum perceivable difference within a binary logit formulation, inspired by ideas from bounded rationality. Krishnan referred to this as *Incorporating Thresholds of Indifference in Probabilistic Choice Models*, and hence for brevity we name this the Threshold indifference model. Particular features of this approach are that:

- the threshold is applied to the difference in random utilities;
- it is assumed that individuals display different kinds of behaviour depending on the threshold: namely they are indifferent to choices that have a random utility difference within the "indifference band" defined by some given threshold tolerance, whereas they behave according to a logit choice model when the random utility difference is outside this indifference band; and
- an additional parameter is introduced to represent the conditional probability of choosing one of the two alternatives, given that the choices have a (random) utility difference within the indifference band.

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[1] We recognise that interval estimates are also valuable. For models of the type we shall describe, these could also be developed from point equilibrium models through the kinds of methods described in Clark and Watling (2006, Section 4.4).

Therefore, in addition to the logit utility parameters, two new parameters are introduced (that can be estimated from data), namely one for the threshold that defines the indifference band, and one for the conditional probability of choosing alternative A_1 when the random utility difference is within the indifference band.

Let the alternatives be denoted A_1 and A_2, the random utilities U_1 and U_2, the systematic utilities V_1 and V_2, and let δ denote what Krishnan calls the minimum perceivable difference (MPD). The symbol $\succ$ denotes "is preferred to" and the symbol $\sim$ denotes "is equally preferred as". The preference model is then postulated as:

(i) $A_1 \succ A_2$ if $U_1 > U_2 + \delta$;
(ii) $A_2 \succ A_1$ if $U_2 > U_1 + \delta$;
(iii) $A_1 \sim A_2$ if $|U_1 - U_2| \leq \delta$.

Based on a RUM, we can then define the classification probabilities into the states (i), (ii) and (iii) above as

$$\pi_1 = \Pr(A_1 \succ A_2) = \Pr(U_1 > U_2 + \delta)$$

$$\pi_2 = \Pr(A_2 \succ A_1) = \Pr(U_2 > U_1 + \delta)$$

$$\pi_{12} = \Pr(A_1 \sim A_2) = \Pr(|U_1 - U_2| \leq \delta)$$

For the case of a binary logit model with scale parameter θ, we then have:

$$\pi_1 = (1 + \exp(\theta(\delta + V_2 - V_1)))^{-1}$$

$$\pi_2 = (1 + \exp(\theta(\delta + V_1 - V_2)))^{-1}$$

$$\pi_{12} = 1 - \pi_1 - \pi_2$$

It is further assumed that whenever the alternatives are in the indifference band, although there is no utility-based preference, a choice is still made (but the choice does not depend on the utilities). This is defined by incorporating an additional parameter α which is equal to the conditional probability of choosing alternative A_1 given that the utilities are within the indifference band. The postulated model is then operationalised by translating the preferences into choice probabilities as:

(i) if $A_1 \succ A_2$ then A_1 will be chosen with probability 1;
(ii) if $A_2 \succ A_1$ then A_2 will be chosen with probability 1;
(iii) if $A_1 \sim A_2$ then A_1 and A_2 will be chosen with probabilities α and $1 - \alpha$ respectively.

The final probabilities of choosing alternatives A_1 and A_2, denoted respectively by π_1^* and π_2^*, are then given by:

$$\pi_1^* = \pi_1 + \pi_{12}\alpha$$

$$\pi_2^* = \pi_2 + \pi_{12}(1 - \alpha) \ (= 1 - \pi_1^*)$$

Combining these together, what we effectively come up with is an alternative way of mapping the choice probability of alternative A_1 to the difference in *systematic* utility:

$$\pi_1^* = f_1(V_1 - V_2; \ \theta, \delta, \alpha)$$

where the smooth function f_1 is given by

$$f_1(x; \ \theta, \delta, \alpha) = (1 + \exp(\theta(\delta - x)))^{-1} + \alpha\left\{1 - (1 + \exp(\theta(\delta - x)))^{-1} - (1 + \exp(\theta(\delta + x)))^{-1}\right\} .$$

It should be noted that we write the function above as operating on the single dummy argument x in order to simplify the expression and to highlight the key elements of the function, rather than what is substituted into the function.

2.1.2. Threshold choice set model

Krishan's approach, as described above, does indeed incorporate the concept of a threshold into a RUM formulation, but what it ends up with is somewhat different from the original objectives of our study. As explained in Section 1, our motivation is to consider formulations that do not have the property of assigning non-zero probability to all available alternatives whatever the systematic utilities. In addition, its motivation is more closely related to bounded rationality than ours is, in that Krishan's model postulates that individuals are rather indifferent to the systematic utilities of the alternatives within the threshold, whereas our interest is in still using a conventional RUM to assign choice probabilities to alternatives. The way in which Krishnan presents the model assumptions is appealing, however, and in this sub-section and the following we propose two alternative models using a similar presentation style.

For our binary choice case, the particular features we aim to model are:

- a threshold is applied to the difference in *systematic* utilities;
- an option will only be in the choice set of considered alternatives if it is no more than a threshold amount worse in systematic utility than the other alternative, otherwise it is completely dominated by the other alternative (written $A_1 \succ\succ A_2$ or $A_2 \succ\succ A_1$);
- if both options are in the choice set (so the alternatives are competitive, which we denote $A_1 \curlywedge A_2$), then individuals choose based on a regular RUM.

The postulated preference model is thus:

(i) $A_1 \succ\succ A_2$ if $V_1 > V_2 + \delta$;
(ii) $A_2 \succ\succ A_1$ if $V_2 > V_1 + \delta$;
(iii) $A_1 \curlywedge A_2$ if $|V_1 - V_2| \leq \delta$.

The classification into the three states i), ii) and iii) above differs from Krishnan's in that it is based on systematic utility rather than random utility, and so it is not probabilistic. Thus in our case we obtain classification probabilities of

$$\pi_1 = \Pr(A_1 \succ\succ A_2) = \begin{cases} 1 & \text{if } V_1 > V_2 + \delta \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_2 = \Pr(A_2 \succ\succ A_1) = \begin{cases} 1 & \text{if } V_2 > V_1 + \delta \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_{12} = \Pr(A_1 \curlyvee A_2) = 1 - \pi_1 - \pi_2.$$

Since the threshold is applied to the systematic utility difference, it is still consistent to assume unbounded random utility differences, and so we may simply adopt a binary logit model for the $A_1 \curlywedge A_2$ case, to obtain the final choice probabilities of:

$$\pi_1^* = \pi_1 + \pi_{12}(1 + \exp(-\theta(V_1 - V_2)))^{-1}$$

$$\pi_2^* = 1 - \pi_1^*$$

which when all cases are combined gives:

$$\pi_1^* = \begin{cases} 1 & \text{if } V_1 > V_2 + \delta \\ 0 & \text{if } V_2 > V_1 + \delta \\ (1 + \exp(-\theta(V_1 - V_2)))^{-1} & \text{otherwise} \end{cases}$$

$$\pi_2^* = 1 - \pi_1^*.$$

As with the model in Section 2.1.1., we can therefore capture this model through the mapping of the choice probability of alternative A_1 to the difference in systematic utility (with one parameter fewer in this case):

$$\pi_1^* = f_2(V_1 - V_2; \ \theta, \delta)$$

where:

$$f_2(x; \ \theta, \delta) = \begin{cases} 0 & \text{if } x < -\delta \\ (1 + \exp(-\theta x))^{-1} & \text{if } |x| \leq \delta \\ 1 & \text{if } x > \delta \end{cases} .$$

2.1.3. Bounded choice model

One of the key differences between Krishnan's approach in Section 2.1.1 and our approach in Section 2.1.2 is that the former considers the random utilities for any threshold operation whereas the latter considers the systematic utilities. In the present section, we return to the behavioural principles of 2.1.2, but instead develop a model based on the random utilities as in 2.1.1. Thus unlike 2.1.1, we model the fact that some alternatives have zero choice probability, but unlike 2.1.2 we derive the model from Random Utility Theory (RUT) principles. This is equivalent, as we see below, to *bounding* the error distribution of random utility differences (between an alternative and a reference alternative), and for this reason we shall refer to it as a Bounded Choice model. Neither of the models presented earlier possess this property, since that in 2.1.1 adopts an unbounded distribution, and that in 2.1.2 is not derived from RUT. Thus, our terminology for the model described below is to refer to a 'bound' rather than a 'threshold' . We also believe this terminology to be appropriate from a functional perspective, in that a threshold is generally used as a (weaker) term to distinguish points at which the behaviour of some model/function changes (possibly in a discontinuous way), whereas our bounding approach can be viewed as adapting a single mathematical form by use of lower and upper bounding functions at choice probabilities of 0 and 1.

The particular features we aim to model are:

- a bound is applied to the difference in *random utility* between each given alternative A_i (with random utility U_i) and an imaginary "reference alternative" A^* with random utility U^* (we shall first derive a general case,[2] then a special case of this reference alternative);
- an option will only be in the choice set of considered alternatives if the difference between its random utility and the random utility of the reference alternative is within the bound; as a result, either both options will be in the choice set, or otherwise that option will be completely dominated by the other (real) alternative (i.e. in the 'otherwise' case, either $A_1 \succ\succ A_2$ if option 2 is outside the bound, or $A_2 \succ\succ A_1$ if option 1 is outside the bound);
- if both options are in the choice set ($A_1 \wedge A_2$), then individuals choose according to the odds associated with the two binary choice probabilities of A_1 versus A^* and A_2 versus A^*.

A key technical difference from the model in Section 2.1.2 is that in order to implement the choice for the $A_1 \wedge A_2$ case, we need to consider the *conditional* distribution of random utility differences, given that the bounding condition is met (in 2.1.2, the conditional and unconditional distributions coincide, since the threshold applies only to the systematic utilities). This technical feature is consistent with a behavioural assumption that travellers make decisions in two stages. Firstly they compare each alternative in a pairwise manner to the reference alternative, which through the bounding on the random utility difference provides an implicit mechanism for choice set generation. This first stage corresponds to the first two first bullet points in the 'features' listed above. Secondly, and corresponding to the third bullet point above, they compare the alternatives in the choice set from the first stage to make their choice, where importantly the proportional use of the alternatives is modelled using the log-odds from the first stage. This use of the log-odds in this way means that choice set generation and choice are handled *consistently* by the same underlying behavioural model.

The postulated preference model is thus:

(i) $A_1 \succ\succ A_2$ if $U^* > U_2 + \delta$ and $U^* \leq U_1 + \delta$;
(ii) $A_2 \succ\succ A_1$ if $U^* > U_1 + \delta$ and $U^* \leq U_2 + \delta$;
(iii) $A_1 \vee A_2$ if $U^* \leq U_2 + \delta$ and $U^* \leq U_1 + \delta$.

In order to operationalise such a preference model, we first present a derivation of the standard binary logit model that can then subsequently be adapted to deal with the bounding condition.

Now, let us suppose a specification of random utilities

$$U_j = \theta V_j + \varepsilon_j \ (j = 1, 2; \ \theta > 0).$$

According to RUT, the choice probabilities are given by

$$\Pr\left(\text{choose } A_1 \text{ from } \{A_1, A_2\}\right) = \Pr\left(U_1 \geq U_2\right) = \Pr\left(\theta V_1 + \varepsilon_1 \geq \theta V_2 + \varepsilon_2\right) = \Pr\left(\varepsilon_2 - \varepsilon_1 \leq \theta\left(V_1 - V_2\right)\right) = F\left(\theta\left(V_1 - V_2\right)\right)$$

where $F(\cdot)$ denotes the cumulative distribution function (c.d.f.) of the random variable $\varepsilon_2 - \varepsilon_1$. Thus, our random utility approach provides a model that ultimately maps the difference in systematic utility $V_1 - V_2$ to the choice probability through the c.d.f. of the random utility differences. In the special case that $(\varepsilon_1, \varepsilon_2)$ are i.i.d. Gumbel random variables, it is well known that we obtain the function $F(\theta x) \equiv f_0(x; \theta)$ given by the binary logit model:

$$f_0(x; \theta) = (1 + \exp(-\theta x))^{-1} \ (-\infty < x < \infty).$$

We now present an alternative way of deriving this same (standard logit) choice function, the advantage of this derivation being that it may be readily modified to incorporate a bound. This alternative formulation is derived by introducing an imaginary "reference alternative" A^* with random utility U^* given by:

$$U^* = \theta V^* + \varepsilon^*$$

where we now assume that $(\varepsilon_1, \varepsilon_2, \varepsilon^*)$ are i.i.d. Gumbel random variables. Then, by the same argument as above, for each j in turn:

$$\Pr\left(\text{choose } A_j \text{ from } \{A_j, A^*\}\right) = f_0\left(V_j - V^*; \theta\right) \ (j = 1, 2).$$

To make some notational simplification, let us now define:

$$x = V_1 - V_2 \quad y = V_1 - V^* \quad z = V_2 - V^*$$

where clearly $x = y - z$. The log-*odds* (log of the odds ratio) η_j for alternative A_j versus A^* is then ($j = 1, 2$):

$$\eta_1 = \ln\left(\frac{f_0(y; \theta)}{1 - f_0(y; \theta)}\right) = \ln\left(\frac{(1 + \exp(-\theta y))^{-1}}{1 - (1 + \exp(-\theta y))^{-1}}\right) = \theta y$$

$$\eta_2 = \ln\left(\frac{f_0(z; \theta)}{1 - f_0(z; \theta)}\right) = \theta z.$$

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[2] We suppose always that the imaginary alternative is defined so that at least one alternative is always in the choice set (so we can neglect the final possibility below that $U^* > U_1 + \delta$ and $U^* > U_2 + \delta$). This condition is automatically satisfied when we suppose (as we shall subsequently) that the imaginary alternative has a systematic utility equal to that of the alternative with highest systematic utility.

The choice probability for alternative A_1 from $\{A_1, A_2\}$ is then given by:

$$\frac{\exp(\eta_1)}{\exp(\eta_1) + \exp(\eta_2)} = \frac{\exp(\theta y)}{\exp(\theta y) + \exp(\theta z)} = (1 + \exp(-\theta(y - z)))^{-1}$$

$$= f_0(y - z; \theta) = f_0(x; \theta) \text{ (since } x = y - z\text{)}.$$

In this case, as could have been forecasted from the well-known IIA property, the choice probabilities between alternatives A_1 and A_2 are independent of the utility of the reference alternative.

This alternative derivation of the standard logit choice function is now modified to incorporate a bound. Again we introduce an imaginary reference alternative A^*. The key difference is that the choice probability for alternative A_1 from $\{A_1, A^*\}$ is now supposed to be given by

$$\Pr(\text{choose } A_1 \text{ from } \{A_1, A^*\}) = g(y; \ \theta, \delta) = \begin{cases} 0 & \text{if } y \leq -\delta \\ \dfrac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(-\delta; \theta)} & \text{if } y > -\delta \end{cases}.$$

The logic of this function is that, since y measures the extent to which alternative A_1 is worse than the reference alternative, the break-point $y = -\delta$ corresponds to a case where A_1 is perceived to be exactly $\delta \geq 0$ utility units worse than the reference alternative. For $y \leq -\delta$, our proposed bounding condition asserts that alternative A_1 will be unused. For $y > -\delta$, on the other hand, the proportional share between A_1 and A^* is given by a linear transformation of the logit choice function. This linear transformation is trivially derived from the basis of RUT, in which g is a cumulative distribution function of random utility differences, and the transformation simply scales the logit choice function so that it is a proper c.d.f. (i.e. it sums to one, notably unlike the model proposed in Section 2.1.2). The second branch of the choice function is the conditional distribution of utility differences, given that $y > -\delta$. As well as preserving the theoretical link to RUT, we may also note the important property (for later analysis) that g is continuous, even at $y = -\delta$.

Similarly, the choice probability for alternative A_2 from $\{A_2, A^*\}$ is supposed to be $g(z; \theta, \delta)$. Under the bounded model, the log-odds λ_j for alternative A_j versus A^* is then given by (for $j = 1, 2$):

$$\lambda_1 = \begin{cases} -\infty & \text{if } y \leq -\delta \\ \ln\left(\dfrac{g(y; \ \theta, \delta)}{1 - g(y; \ \theta, \delta)}\right) & \text{if } y > -\delta \end{cases} \qquad \lambda_2 = \begin{cases} -\infty & \text{if } z \leq -\delta \\ \ln\left(\dfrac{g(z; \ \theta, \delta)}{1 - g(z; \ \theta, \delta)}\right) & \text{if } z > -\delta \end{cases}.$$

Now, for $y > -\delta$:

$$\frac{g(y; \ \theta, \delta)}{1 - g(y; \ \theta, \delta)} = \frac{\dfrac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(-\delta; \theta)}}{1 - \dfrac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(-\delta; \theta)}} = \frac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(y; \theta)}.$$

Under this bounded model, the choice probability for alternative A_1 from $\{A_1, A_2\}$ is thus given by:

$$\frac{\exp(\lambda_1)}{\exp(\lambda_1) + \exp(\lambda_2)} = \begin{cases} 0 & \text{if } y \leq -\delta \\ \dfrac{\dfrac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(y; \theta)}}{\dfrac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(y; \theta)} + \dfrac{f_0(z; \theta) - f_0(-\delta; \theta)}{1 - f_0(z; \theta)}} & \text{if } y > -\delta \text{ and } z > -\delta = h(y, z; \theta, \delta) \\ 1 & \text{if } z \leq -\delta \end{cases}$$

Now,

$$\frac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(y; \theta)} = \frac{f_0(y; \theta)}{1 - f_0(y; \theta)} - \frac{(1 + \exp(\theta\delta))^{-1}}{1 - (1 + \exp(-\theta y))^{-1}}$$

and as previously noted:

$$\frac{f_0(y; \theta)}{1 - f_0(y; \theta)} = \exp(\theta y).$$

As for the second term above,

$$\frac{(1 + \exp(\theta\delta))^{-1}}{1 - (1 + \exp(-\theta y))^{-1}} = \frac{(1 + \exp(\theta\delta))^{-1}}{(1 + \exp(\theta y))^{-1}} = \frac{1 + \exp(\theta y)}{1 + \exp(\theta\delta)}.$$

Hence:

$$\frac{f_0(y; \theta) - f_0(-\delta; \theta)}{1 - f_0(y; \theta)} = \exp(\theta y) - \frac{1 + \exp(\theta y)}{1 + \exp(\theta\delta)} = \frac{\exp(\theta(y + \delta)) - 1}{1 + \exp(\theta\delta)}.$$

Therefore the general choice probability model for alternative A_1 versus A_2 given reference alternative A^* which is better than A_1 and A_2 by y and z utility units respectively, and given bound $\delta > 0$, is:

$$h(y, z; \theta, \delta) = \begin{cases} 0 & \text{if } y \leq -\delta \\ \dfrac{\exp(\theta(y+\delta)) - 1}{(\exp(\theta(y+\delta)) - 1) + (\exp(\theta(z+\delta)) - 1)} & \text{if } y > -\delta \text{ and } z > -\delta \\ 1 & \text{if } z \leq -\delta \end{cases}.$$

Again, it is notable that h is continuous. We avoid further simplifying the expression above, i.e we avoid dividing throughout by $\exp(\theta(y+\delta)) - 1$, since at $y = -\delta$ this term would be zero, in order that in the form given we may equally consider the boundary cases within the second term (e.g. for easier comparison with earlier-presented models, as well as making the continuity more evident) as:

$$h(y, z; \theta, \delta) = \begin{cases} 0 & \text{if } y < -\delta \\ \dfrac{\exp(\theta(y+\delta)) - 1}{(\exp(\theta(y+\delta)) - 1) + (\exp(\theta(z+\delta)) - 1)} & \text{if } y \geq -\delta \text{ and } z \geq -\delta \\ 1 & \text{if } z < -\delta \end{cases}.$$

The above form is the one that we shall henceforth take forward as the main functional form for h.

We may also remark that this general model is clearly *not* independent of the utility of the reference alternative; in particular, it cannot be written as a function of $y - z (= x)$ only, as in the standard binary logit model.

With a slight rearrangement, we have a third possible way of expressing this function

$$h(y, z; \theta, \delta) = \begin{cases} 0 & \text{if } y < -\delta \\ \dfrac{\exp(\theta y) - \exp(-\theta\delta)}{(\exp(\theta y) - \exp(-\theta\delta)) + (\exp(\theta z) - \exp(-\theta\delta))} & \text{if } y \geq -\delta \text{ and } z \geq -\delta \\ 1 & \text{if } z < -\delta \end{cases}$$

and in this form it is evident that, as $\delta \to \infty$, so $\exp(-\theta\delta) \to 0$ for $\theta > 0$, and so in the limit we recover the standard binary logit model, as might be expected.

Now, we shall consider a special case of this model in which the reference alternative has systematic utility:

$$V^* = \max(V_1, V_2).$$

We now consider two possible cases:

i) When $V_1 \geq V_2$ (i.e. when $x \geq 0$), it follows that $V^* = V_1$. Hence $y = V_1 - V^* = 0$ and $z = V_2 - V^* = V_2 - V_1 = -x$. That is: $\geq 0 \Rightarrow y = 0$ and $z = -x$.

ii) When $V_2 \geq V_1$ (i.e. when $x \leq 0$), it follows that $V^* = V_2$. Hence $y = V_1 - V^* = V_1 - V_2 = x$ and $z = V_2 - V^* = 0$. That is: $\leq 0 \Rightarrow y = x$ and $z = 0$.

Now in case i), with $y = 0$, we can never have $y < -\delta$ satisfied if (as it is assumed) $\delta \geq 0$, and so in this case we obtain (with $y = 0$ and $z = -x$ for $x \geq 0$ substituted in the 'main functional form' for h noted above):

$$h(0, -x; \theta, \delta) = \begin{cases} \dfrac{\exp(\theta\delta) - 1}{(\exp(\theta\delta) - 1) + (\exp(\theta(-x+\delta)) - 1)} & \text{if } 0 \leq x \leq \delta \\ 1 & \text{if } x > \delta \end{cases}.$$

In case ii), since $z = 0$, we can never have $z < -\delta$ satisfied with $\delta \geq 0$, and so in this case we obtain (with $y = x$ and $z = 0$ for $x \leq 0$):

$$h(x, 0; \theta, \delta) = \begin{cases} 0 & \text{if } x < -\delta \\ \dfrac{\exp(\theta(x+\delta)) - 1}{(\exp(\theta(x+\delta)) - 1) + (\exp(\theta\delta) - 1)} & \text{if } -\delta \leq x \leq 0 \end{cases}.$$

Combining these cases we obtain a choice probability model for alternative A_1 versus A_2 which is independent of the reference alternative A^*, depending only on the utility difference x between A_1 and A_2 and on the bound $\delta > 0$ relative to

the maximum utility alternative:

$$
f_3(x;\ \theta, \delta) = \begin{cases} 0 & \text{if } x < -\delta \\[2mm] \dfrac{\exp\left(\theta\left(x+\delta\right)\right) - 1}{\left(\exp\left(\theta\left(x+\delta\right)\right) - 1\right) + \left(\exp\left(\theta\delta\right) - 1\right)} & \text{if } -\delta \leq x \leq 0 \\[4mm] \dfrac{\exp\left(\theta\delta\right) - 1}{\left(\exp\left(\theta\delta\right) - 1\right) + \left(\exp\left(\theta\left(-x+\delta\right)\right) - 1\right)} & \text{if } 0 \leq x \leq \delta \\[4mm] 1 & \text{if } x > \delta \end{cases}.
$$

A rather obvious, but important, remark to make is that while the general form of our bounded model h presented earlier is not expressible only as a function of the utility difference x, clearly the special case considered to derive f_3 above is.

2.1.4. Comparison and evaluation of candidate approaches

From the analysis in Sections 2.1.1–2.1.3, we have three candidate approaches to consider, alongside the 'null' choice of a standard binary logit model:

Binary Logit Model:

$$
f_0(x;\theta) = \left(1 + \exp(-\theta x)\right)^{-1} \quad (-\infty < x < \infty)
$$

Threshold Indifference Model:

$$
f_1(x;\ \theta, \delta, \alpha) = \left(1 + \exp(\theta(\delta - x))\right)^{-1} + \alpha\left\{1 - \left(1 + \exp(\theta(\delta - x))\right)^{-1} - \left(1 + \exp(\theta(\delta + x))\right)^{-1}\right\}
$$

Threshold Choice Set Model:

$$
f_2(x;\ \theta, \delta) = \begin{cases} 0 & \text{if } x < -\delta \\ \left(1 + \exp(-\theta x)\right)^{-1} & \text{if } |x| \leq \delta \\ 1 & \text{if } x > \delta \end{cases}
$$

Bounded Choice Model:

$$
f_3(x;\ \theta, \delta) = \begin{cases} 0 & \text{if } x < -\delta \\[2mm] \dfrac{\exp\left(\theta\left(x+\delta\right)\right) - 1}{\left(\exp\left(\theta\left(x+\delta\right)\right) - 1\right) + \left(\exp\left(\theta\delta\right) - 1\right)} & \text{if } -\delta \leq x \leq 0 \\[4mm] \dfrac{\exp\left(\theta\delta\right) - 1}{\left(\exp\left(\theta\delta\right) - 1\right) + \left(\exp\left(\theta\left(-x+\delta\right)\right) - 1\right)} & \text{if } 0 \leq x \leq \delta \\[4mm] 1 & \text{if } x > \delta \end{cases}.
$$

These four different approaches are illustrated in Fig. 3, for an example with $\theta = 0.5$, $\delta = 4$ and $\alpha = 0.5$.

An obvious feature of f_1 is that — like f_0 — it is smooth and asymptotically approaches 0 at its lower end and 1 at its upper end; thus, both alternatives always have a non-zero probability of being chosen. Model f_2 is not only non-smooth but it is discontinuous at $x = \pm \delta$. Model f_3, while non-smooth, is a continuous mapping over its range. As shown later, these will be important properties in our subsequent analysis.

Models f_2 and f_3 are appealing in the context of the issues identified in the Introduction, since they allow for the possibility of an alternative to be completely unused (i.e. to have zero probability). Model f_1 shares more in common with the concept of bounded rationality, as implemented in BRUE models for example (see §1), since it posits that individuals are relatively indifferent to choices with utilities that differ by less than the threshold (in the case of f_1, 'relatively' means relative to the choice probabilities suggested by a binary logit model). In practice, this is achieved by a probability-weighting of the random utility being inside or outside the threshold, providing a smooth 'flattening' of the original binary logit model. In comparison with a standard binary logit model, the particular feature of model f_1 therefore is to make a special treatment of alternatives lying *within* the threshold. Model f_2, on the other hand, is concerned with defining a point (on the utility difference scale) at which a choice becomes so unattractive relative to its competitor(s) that there is zero probability of its use; in this sense its contribution is more concerned with what happens *outside* the threshold. Model f_3 is similar to f_2 in this respect, but additionally adjusts choice probabilities as the bound is approached in order to provide a continuous mapping. Intuitively, then, it may be better to think of the models operating on thresholds or bounds that differ markedly in meaning and scale: f_1 is a "small threshold" model, interested in identifying relatively indistinguishable options, while f_2 and f_3 are "large threshold/bound" models, interested in identifying options that are so relatively poor compared with alternatives that (for modelling purposes) we might neglect that they may be chosen.

Aside from the different behavioural objectives of these models, they all provide opportunities for implementation within a network equilibrium context. However, based on the requirements identified in Section 1, we shall select only one of them for further development:
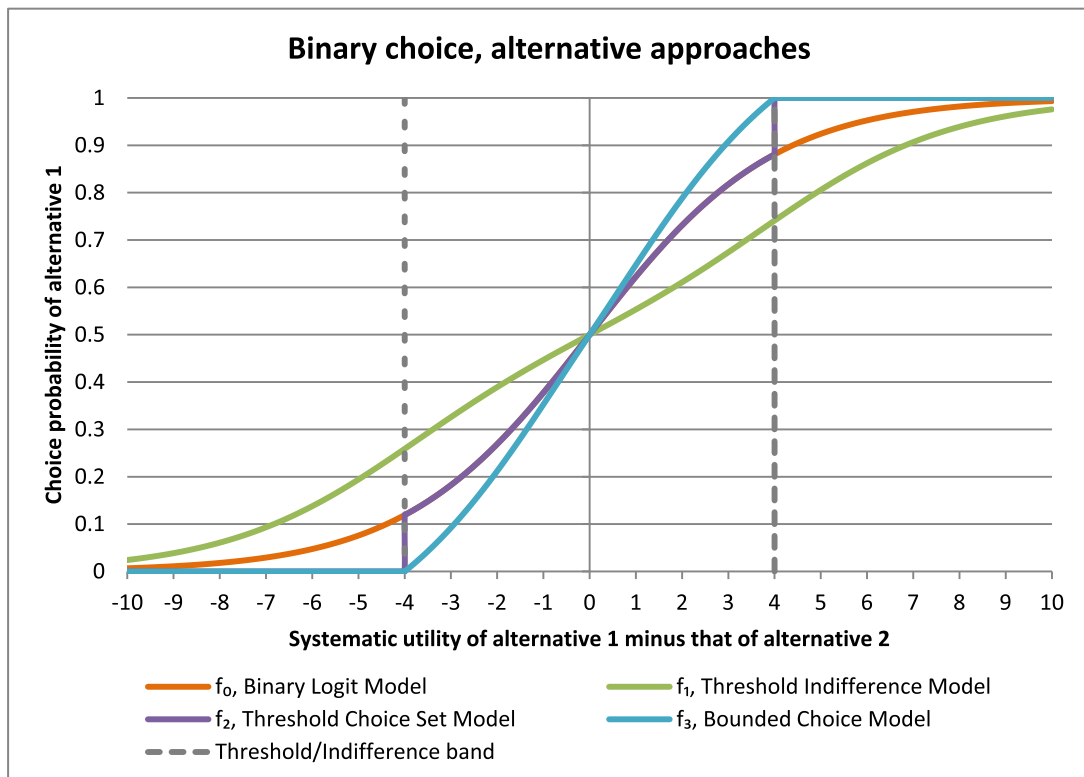
Fig. 3. Choice probability of choosing alternative 1 as function of difference in systematic utility between two alternatives, various approaches.

- *Threshold Indifference Model:* Krishnan's model f_1 is seemingly attractive to explore, as a kind of RUM counterpart to BRUE. The fact that the mapping f_1 is continuous should mean that, under standard assumptions on other elements of the model, such equilibria can be theoretically guaranteed to exist. However, some more work would be required on finding an efficient, tractable implementation of this approach for more than two alternatives. Lioukas (1984) provides a formulation for three alternatives, but this is already rather complex given that it involves a combinatorial problem due to the different orderings of the alternatives that can occur within the indifference band. We shall not, however, further study Krishnan's model in the present paper, since it does not accord with our requirement in Section 1 to eliminate some potential alternatives from the choice set.
- *Threshold Choice Set Model:* Model f_2 is attractive in that it satisfies the requirement to distinguish a route as unused, while adopting RUT. This model is in fact the basis of an approach that was generalised and implemented within an equilibrium framework in Rasmussen et al., (2017). The major disadvantage to the approach is the discontinuity of the choice model; examples can be constructed to show that this leads to non-existence of equilibria in some circumstances. Therefore this does not accord with the requirement that the approach should have guaranteed existence/uniqueness properties, and so it is not considered further.
- *Bounded Choice Model:* The newly proposed model f_3 is the most attractive in that it satisfies the requirement to distinguish a route as unused, while at the same time providing a continuous mapping of the choice model. Thus it provides promise at least that existence of equilibria may be proven.

In conclusion, then, the Bounded Choice Model is selected as the approach to consider further in the context of the present study, given the requirements set out in Section 1.

2.2. Problems with multiple alternatives

In the present section, we propose a generalisation of the two-alternative Bounded Choice Model presented in Section 2.1.3 to the case of $n \geq 2$ alternatives. This is a relatively straightforward extension of the binary case, and we will re-use many of the derivations and intermediate functions defined there.

The model features are that:

- a bound is applied to the difference in *random* utility between a given alternative and an imaginary reference alternative A^*;

- an option will only be in the choice set of considered alternatives if the difference between its random utility and the random utility of the reference alternative is within the bound;
- for the subset of considered alternatives, indexed by $\mathcal{J} \subseteq \{1, 2, \ldots, n\}$, the choice probabilities are given by the odds associated with the binary choice probabilities of A_j versus A^* for $j \in \mathcal{J}$.

Suppose a specification of random utilities for the real and imaginary alternatives of:

$$U_j = \theta V_j + \varepsilon_j \ (j = 1, 2 \ldots, n) \qquad U^* = \theta V^* + \varepsilon^*$$

where $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon^*)$ are i.i.d. Gumbel,[3] and define:

$$y_j = V_j - V^* \quad (j = 1, 2, \ldots, n).$$

Then, combining RUT with a bounding condition gives us (following the same rationale as for the binary case):

$$\Pr\left(\text{choose } A_j \text{ from } \{A_j, A^*\}\right) = g(y_j; \ \theta, \delta) = \begin{cases} 0 & \text{if } y_j \leq -\delta \\ \dfrac{f_0(y_j; \theta) - f_0(-\delta; \theta)}{1 - f_0(-\delta; \theta)} & \text{if } y_j > -\delta \end{cases} \quad (j = 1, 2, \ldots, n).$$

with the log-odds λ_j for alternative A_j versus A^* of:

$$\lambda_j = \begin{cases} -\infty & \text{if } y_j \leq -\delta \\ \ln\left(\dfrac{g(y_j; \ \theta, \delta)}{1 - g(y_j; \ \theta, \delta)}\right) & \text{if } y_j > -\delta \end{cases} \quad (j = 1, 2, \ldots n).$$

In the binary case we showed that we can write:

$$\frac{g(y; \ \theta, \delta)}{1 - g(y; \ \theta, \delta)} = \frac{\exp\left(\theta(y + \delta)\right) - 1}{1 + \exp(\theta \delta)} \ (y > -\delta)$$

and so the *odds* $\exp(\lambda_j)$ for alternative A_j versus A^* is:

$$\exp(\lambda_j) = \frac{1}{1 + \exp(\theta \delta)} \begin{cases} 0 & \text{if } y_j \leq -\delta \\ \exp\left(\theta(y_j + \delta)\right) - 1 & \text{if } y_j > -\delta \end{cases} \ (j = 1, 2, \ldots n)$$

which can also be written:

$$\exp(\lambda_j) = \frac{1}{1 + \exp(\theta \delta)} \left(\exp\left(\theta(y_j + \delta)\right) - 1\right)_+ \ (j = 1, 2, \ldots n).$$

where the $+$ operator above is such that $(x)_+ = \max(x, 0)$.

We thus obtain a choice probability model for alternative A_j from $\{A_1, A_2, \ldots A_n\}$, given reference alternative A^* which is better than A_k by y_k utility units ($k = 1, 2, \ldots, n$), and given bound $\delta > 0$:

$$h_j(\mathbf{y}; \theta, \delta) = \frac{\exp(\lambda_j)}{\sum_{k=1}^{n} \exp(\lambda_k)} = \frac{\left(\exp\left(\theta(y_j + \delta)\right) - 1\right)_+}{\sum_{k=1}^{n} \left(\exp\left(\theta(y_k + \delta)\right) - 1\right)_+} \ (j = 1, 2, \ldots, n)$$

where $\mathbf{y}$ is the n-vector $(y_1, y_2, \ldots, y_n)$.

The final step is, as in the binary case, where we select a particular (logical) specification of the imaginary reference alternative, whereby:

$$\mathbf{V}^* = \max(V_1, V_2, \ldots, V_n)$$

and then resubstituting for $y_j = V_j - V^*$ ($j = 1, 2, \ldots, n$) we obtain a choice probability model for alternative A_j from $\{A_1, A_2, \ldots A_n\}$ given systematic utilities $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ and bound $\delta > 0$:

$$p_j(\mathbf{V}; \theta, \delta) = \frac{\left(\exp\left(\theta(V_j - \max(V_1, V_2, \ldots, V_n) + \delta)\right) - 1\right)_+}{\sum_{k=1}^{n} \left(\exp\left(\theta(V_k - \max(V_1, V_2, \ldots, V_n) + \delta)\right) - 1\right)_+} \ (j = 1, 2, \ldots, n).$$

As in the binary case, with a slight re-writing of the expression, it is easily seen that, in the limit as $\delta \to \infty$, this model approaches the conventional logit model, as would be expected:

$$p_j(\mathbf{V}; \ \theta, \delta) \to \frac{\exp(\theta V_j)}{\sum_{k=1}^{n} \exp(\theta V_k)} \text{ as } \delta \to \infty.$$

In the other limit as $\delta \to 0$, the minimum cost routes become dominant and routes only slightly more costly become unused (the domination of the minimum cost routes can also be obtained by increasing θ as in a traditional MNL model, however in such cases all routes with a cost below the cost bound would still be used). Namely, we have a choice model that if used for route choices in their extreme cases approaches *either* a SUE with a probability that all routes are chosen no-matter how circuitous they might be, *or* a DUE where only the minimum cost routes are chosen.

---

[3] Note that the bounded choice model we shall derive does not itself have Gumbel error terms, due to the transformation implied by the bounding condition.

3. Bounded SUE: Definition, existence and uniqueness

We now move on to consider how the Bounded Choice Model introduced in Section 2 may be implemented in a network equilibrium context and, as a first stage, we set up the basic notation.

We consider a network as a directed graph consisting of links indexed by a ($a = 1, 2, ..., A$) and origin-destination (OD) pairs indexed by m ($m = 1, 2, ..., M$). We define the demand d_m for OD-pair m composing a non-negative M-dimensional vector $\mathbf{d}$, the index set R_m of all simple paths (without cycles) for each OD-pair m, the number N_m of paths in R_m and the union R of the sets R_m over all OD-pairs. The route index sets are constructed so that $R = \{1,2,...,N\}$, where $N = \sum_{m=1}^{M} N_m$.

Denote the flow on path $r \in R_m$ between OD-pair m as x_{mr} and let $\mathbf{x}$ be the N-dimensional flow-vector on the universal choice set across all M OD-pairs, so that the notation x_{mr} refers to element number $r + \sum_{k=1}^{m-1} N_k$ in the N-dimensional vector $\mathbf{x}$. Denote the flow on link a ($a = 1,2,...,A$) as f_a and let $\mathbf{f} = (f_1, f_2,..., f_a,...,f_A)$ be the A-dimensional link flow-vector where f_a refers to element number a in $\mathbf{f}$. The convex set G of demand-feasible non-negative path flows is given by:

$$G = \left\{ \mathbf{x} \in \mathbb{R}_+^N : \sum_{r=1}^{N_m} x_{mr} = d_m, \ m = 1, 2, \ldots, M \right\} \tag{1}$$

where $\mathbb{R}_+^N$ denotes N-dimensional, non-negative Euclidean space. Next, define δ_{amr} equal to 1 if link a is part of path r for OD-pair m and zero otherwise. Then the convex set of demand-feasible link flows is:

$$F = \left\{ \mathbf{f} \in \mathbb{R}_+^A : f_a = \sum_{m=1}^{M} \sum_{r=1}^{N_m} \delta_{amr} x_{mr}, \ a = 1, 2, \ldots, A, \mathbf{x} \in G \right\} \tag{2}$$

In vector/matrix notation, let $\mathbf{x}$ and $\mathbf{f}$ be column vectors, and define $\boldsymbol{\Delta}$ as the $A \times N$-dimensional link-path incidence matrix. Then the relationship between link and path flows may be written as $\mathbf{f} = \boldsymbol{\Delta}\mathbf{x}$. We suppose that the travel cost on path r for OD-pair m is additive in the link travel costs of the utilised links:

$$c_{mr}(\mathbf{x}) = \sum_{a=1}^{A} \delta_{amr} \cdot t_a(\boldsymbol{\Delta}\mathbf{x}) \ (r \in R_m; m = 1, 2, \ldots, M; \mathbf{x} \in G) \tag{3}$$

Define $\mathbf{t}(\mathbf{f})$ ($\mathbf{t}: \mathbb{R}_+^A \to \mathbb{R}_+^A$) as the vector of generalised link travel cost functions, and $\mathbf{c}(\mathbf{x})$ ($\mathbf{c}: \mathbb{R}_+^N \to \mathbb{R}_+^N$) as the vector of generalised route travel cost functions. Then the relationships between path and link flows, and between link and path costs, may be succinctly written as:

$$\mathbf{f} = \boldsymbol{\Delta}\mathbf{x} \text{ and } \mathbf{c}(\mathbf{x}) = \boldsymbol{\Delta}^{\mathrm{T}} \mathbf{t}(\boldsymbol{\Delta}\mathbf{x}). \tag{4}$$

On the demand side, for each OD movement $m = 1, 2, ..., M$, we let $P_{mr}(\mathbf{c})$ denote the proportion of drivers on that movement that would choose path $r \in R_m$ when the vector of path costs is $\mathbf{c} \in \mathbb{R}^N$.

Having introduced the network notation, we now turn to propose a new equilibrium model. We do this by using the Bounded Choice Model defined in Section 2.2 within a conventional SUE framework, setting the systematic utility used in Section 2.2 equal to the negative of the travel cost.

In order to accommodate the possibility of either assuming an absolute or relative bound, we consider two cases:

(1) *Relative Cost Model:* Variable bounds $\boldsymbol{\delta}(\mathbf{c}) = (\delta_1(\mathbf{c}), \delta_2(\mathbf{c}),..., \delta_M(\mathbf{c}))$ where

$$\delta_m(\mathbf{c}) = (\tau - 1) \min \{c_{ms} : s \in R_m\} \quad (m = 1, 2, \ldots, M; \ \tau > 1).$$

(2) *Absolute Cost Model:* Constant OD-specific bounds, $\boldsymbol{\delta}(\mathbf{c}) \equiv \boldsymbol{\delta} = (\delta_1, \delta_2,..., \delta_M)$ where $\delta_m > 0$ for m = 1,2,...,M.

This yields the following definition:

Definition 1. *Bounded Stochastic User Equilibrium (Bounded(δ) SUE)*

The route flow $\mathbf{x} \in G$ *is a Bounded SUE if and only if it is both an SUE:*

$$x_{mr} = d_m P_{mr}(\mathbf{c}(\mathbf{x})) \ (r \in R_m; \ m = 1, 2, \ldots, M)$$

and the choice model is given by the Bounded Choice Model form based on bounds $\delta(\mathbf{c})$:

$$P_{mr}(\mathbf{c}) = \frac{(\exp(-\theta(c_{mr} - \min(c_{ms} : \ s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\sum_{t \in R_m}(\exp(-\theta(c_{mt} - \min(c_{ms} : \ s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+} \ (r \in R_m; \ m = 1, 2, \ldots, M).$$

We first consider the issue of whether any flow allocation might exist satisfying the Bounded SUE conditions.

Proposition 1. *Existence of Bounded SUE Solutions*

Suppose that for each m = 1, 2, ..., M, the OD demand d_m is positive and finite, and that the feasible route set $R_m \neq \emptyset$. Suppose that $\mathbf{c}(\cdot)$ is continuous and bounded over the convex set G (defined by Eq. (1)) and that $\delta(\cdot)$ is continuous and bounded over $\mathbf{c}(G)$. Then at least one Bounded SUE solution exists.

Proof.

In general, if $f(\mathbf{x})$ and $g(\mathbf{x})$ are continuous in $\mathbf{x} \in \mathbb{R}^n$, then both $\min(f(\mathbf{x}), g(\mathbf{x}))$ and $(f(\mathbf{x}))_+ = \max(f(\mathbf{x}), 0)$ are also continuous. Hence, by the hypotheses made on $\mathbf{c}(\cdot)$ and $\boldsymbol{\delta}(\cdot)$, it follows that $P_{mr}(\mathbf{c}(\mathbf{x}))$, as a continuous composition of continuous functions, is itself a continuous function of $\mathbf{x} \in G$, for all $r \in R_m$ and $m = 1, 2, \ldots, M$. Now consider the *demand map* $\boldsymbol{\Phi}(\mathbf{x})$ with elements defined as the following functions:

$$\Phi_{mr}(\mathbf{x}) = d_m P_{mr}(\mathbf{c}(\mathbf{x})) \quad (r \in R_m; \ m = 1, 2, \ldots, M).$$

By construction of the bounded choice model, the route choice probabilities sum to one, and so $\Phi(\cdot)$ maps into G. From this point we can apply the existence theorem presented by Cantarella & Cascetta (1995, p. 314), which we outline in brief for completeness. In particular, the assumed boundedness of the demand flows implies that G is closed and bounded, and so compact, and the assumption that the feasible route sets are non-empty implies G is non-empty. Hence, summing up, $\Phi(\cdot)$ is a continuous function from a non-empty, compact and convex set G into G, which implies (by Brouwer's fixed point theorem) that there is at least one solution $\mathbf{x} \in G$ to the fixed point problem $\mathbf{x} = \Phi(\mathbf{x})$, i.e. at least one Bounded SUE solution exists.

[Proof complete].

Note that the hypothesis that $\boldsymbol{\delta}(\cdot)$ is continuous means that the existence result holds for both the case of absolute bounds (where the function is constant) and relative bounds (where the function is continuous by virtue of the continuity of the minimum operator). We also note that the proof has been given with respect to route flow solutions, but with such models (unlike with deterministic equilibria) we always have a unique mapping between any route flow solution $\mathbf{x}$ and a corresponding link flow solution $\boldsymbol{\Delta}\Phi(\mathbf{x})$ where $\Phi(\cdot)$ is the demand map defined in the proof.

Having established hypotheses under which Bounded SUE solutions exist, the next question is whether sufficient conditions exist to ensure a *unique* Bounded SUE solution. In order to do this, a first key step is to establish a mathematical property of the proposed choice model.

Proposition 2. *Monotonicity of the BCM*

The BCM, given by:

$$p_j(\mathbf{c}) = \frac{\left(\exp\left(-\theta\left(c_j - \min\{c_1, c_2, \ldots, c_n\} - \delta\right)\right) - 1\right)_+}{\sum_{k=1}^{n} \left(\exp\left(-\theta\left(c_k - \min\{c_1, c_2, \ldots, c_n\} - \delta\right)\right) - 1\right)_+}$$

satisfies the monotonicity condition:

$$\Omega(\mathbf{c}, \mathbf{d}) = \sum_{j=1}^{n} \left(c_j - d_j\right)\left(p_j(\mathbf{c}) - p_j(\mathbf{d})\right) \leq 0 \quad \forall \mathbf{c}, \mathbf{d} \in \mathbb{R}^n.$$

Proof.

First define:

$$\min(\mathbf{c}) = \min\{c_k : k = 1, 2, \ldots, n\}$$

$$\tilde{c}_j = c_j - \min(\mathbf{c}) \quad (j = 1, 2, \ldots, n)$$

$$\tilde{d}_j = d_j - \min(\mathbf{d}) \quad (j = 1, 2, \ldots, n)$$

with $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$ the corresponding vectors, and define

$$\tilde{p}_j(\tilde{\mathbf{c}}) = \frac{\left(\exp\left(-\theta\left(\tilde{c}_j - \delta\right)\right) - 1\right)_+}{\sum_{k=1}^{n} \left(\exp\left(-\theta\left(\tilde{c}_k - \delta\right)\right) - 1\right)_+} \quad (j = 1, 2, \ldots n).$$

Then consider:

$$\sum_{j=1}^{n} \left(\tilde{c}_j - \tilde{d}_j\right)\left(\tilde{p}_j(\tilde{\mathbf{c}}) - \tilde{p}_j(\tilde{\mathbf{d}})\right)$$

$$= \sum_{j=1}^{n} \left(c_j - \min(\mathbf{c}) - d_j + \min(\mathbf{d})\right)\left(p_j(\mathbf{c}) - p_j(\mathbf{d})\right)$$

$$= \sum_{j=1}^{n} \left(c_j - d_j\right)\left(p_j(\mathbf{c}) - p_j(\mathbf{d})\right) - (\min(\mathbf{c}) - \min(\mathbf{d}))\left(\sum_{j=1}^{n} p_j(\mathbf{c}) - \sum_{j=1}^{n} p_j(\mathbf{d})\right)$$

$$= \sum_{j=1}^{n} \left(c_j - d_j\right)\left(p_j(\mathbf{c}) - p_j(\mathbf{d})\right) = \Omega(\mathbf{c}, \mathbf{d}).$$

Hence, we may equivalently show that:

$$\tilde{\Omega}\big(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}\big) = \sum_{j=1}^{n} \big(\tilde{c}_j - \tilde{d}_j\big)\big(\tilde{p}_j(\tilde{\mathbf{c}}) - \tilde{p}_j(\tilde{\mathbf{d}})\big) \leq 0 \ \forall \ \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \in \mathcal{C}$$

where:

$$\mathcal{C} = \big\{\tilde{\mathbf{c}} : \tilde{c}_j = c_j - \min(\mathbf{c})(j = 1, 2, \ldots, n) \ \text{where} \ \mathbf{c} \in \mathbb{R}^n\big\} .$$

We now note that the modified choice probability function $\tilde{p}_j(\tilde{\mathbf{c}})$, while non-smooth, is piecewise differentiable except when $\tilde{c}_k = \delta$ for some k, and directionally differentiable everywhere in $\mathcal{C}$.

Let us write:

$$\tilde{p}_j(\tilde{\mathbf{c}}) = \frac{h(\tilde{c}_j)}{\sum_{r=1}^{n} h(\tilde{c}_r)} \ (j = 1, 2, \ldots n)$$

where:

$$h(x) = \big(\exp\big(-\theta(x - \delta)\big) - 1\big)_+$$

and the directional derivative of h at $x = x_0$ in the direction of $x = x^*$ is:

$$h'(x_0, x^*) = \begin{cases} -\theta\big(\exp\big(-\theta(x_0 - \delta)\big) - 1\big) & \text{if } x_0 < \delta \text{ or } (x_0 = \delta \text{ and } x^* < \delta) \\ 0 & \text{if } x_0 > \delta \text{ or } (x_0 = \delta \text{ and } x^* > \delta) \end{cases} .$$

Then, the directional partial derivative of $\tilde{p}_j(.)$ with respect to $\tilde{c}_k$ at the point $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}_0$ in the direction of $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}^*$, denoted $\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)$ is given by:

$$\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = \begin{cases} \dfrac{h'\big(\tilde{c}_{0j}, c_j^*\big) \sum_{r=1}^{n} h(\tilde{c}_{0r}) - h\big(\tilde{c}_{0j}\big)h'\big(\tilde{c}_{0j}, c_j^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} & \text{if } j = k \\[4ex] \dfrac{-h\big(\tilde{c}_{0j}\big)h'\big(\tilde{c}_{0k}, c_k^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} & \text{if } j \neq k \end{cases} \quad (j = 1, 2, \ldots, n; \ k = 1, 2, \ldots, n).$$

The negative of this matrix has elements that can be written as:

$$-\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = \begin{cases} -\dfrac{h'\big(\tilde{c}_{0j}, c_j^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} \sum_{\substack{r=1 \\ r \neq j}}^{n} h(\tilde{c}_{0r}) & \text{if } j = k \\[4ex] \dfrac{h\big(\tilde{c}_{0j}\big)h'\big(\tilde{c}_{0k}, c_k^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} & \text{if } j \neq k \end{cases} \quad (j = 1, 2, \ldots, n; \ k = 1, 2, \ldots, n).$$

We now aim to show that this matrix is diagonally dominant. Thus, consider a generic row j of this (negated) matrix for some $j = 1, 2, \ldots n$, and in particular consider the difference $D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)$ between the absolute value of the diagonal element and the sum of the absolute values of the off-diagonal elements:

$$D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = \big|-\tilde{p}'_{jj}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)\big| - \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \big|-\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)\big|\right) = \left|-\frac{h'\big(\tilde{c}_{0j}, c_j^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} \sum_{\substack{r=1 \\ r \neq j}}^{n} h(\tilde{c}_{0r})\right| - \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \left|\frac{h\big(\tilde{c}_{0j}\big)h'\big(\tilde{c}_{0k}, c_k^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2}\right|\right) .$$

Since everywhere the functions $h \geq 0$ and $h' \leq 0$, it follows that:

$$\begin{aligned} D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) &= -\frac{h'\big(\tilde{c}_{0j}, c_j^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2} \sum_{\substack{r=1 \\ r \neq j}}^{n} h(\tilde{c}_{0r}) + \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{h\big(\tilde{c}_{0j}\big)h'\big(\tilde{c}_{0k}, c_k^*\big)}{\left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^2}\right) \\ &= \left(\sum_{r=1}^{n} h(\tilde{c}_{0r})\right)^{-2} \left(-h'\big(\tilde{c}_{0j}, c_j^*\big) \sum_{\substack{k=1 \\ k \neq j}}^{n} h(\tilde{c}_{0k}) - h\big(\tilde{c}_{0j}\big) \sum_{\substack{k=1 \\ k \neq j}}^{n} \big(-h'(\tilde{c}_{0k}, c_k^*)\big)\right) . \end{aligned}$$

Then, we have two cases to consider. In the first case, for all $k \neq j$ the following condition holds:

$$\tilde{c}_{0k} > \delta \text{ or } (\tilde{c}_{0k} = \delta \text{ and } c_k^* > \delta)$$

in which event simultaneously $\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} h(\tilde{c}_{0k}) = \sum\limits_{\substack{k=1 \\ k \neq j}}^{n} (-h'(\tilde{c}_{0k}, c_k^*)) = 0$, which implies $D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = 0$. In the second case, for some $k \neq j$ the following condition holds:

$$\tilde{c}_{0k} < \delta \text{ or } (\tilde{c}_{0k} = \delta \text{ and } c_k^* < \delta)$$

in which event $\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} h(\tilde{c}_{0k}) > 0$ and $\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} (-h'(\tilde{c}_{0k}, c_k^*)) > 0$. In such an event, it is clear from the definitions of h and h' that the following holds:

$$\frac{-h'(\tilde{c}_{0j}, c_j^*)}{\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} (-h'(\tilde{c}_{0k}, c_k^*))} = \frac{h(\tilde{c}_{0j})}{\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} h(\tilde{c}_{0k})} = \frac{\exp\left(-\theta\left(\tilde{c}_{0j} - \delta\right)\right) - 1}{\sum\limits_{\substack{k=1 \\ k \neq j}}^{n} (\exp\left(-\theta(\tilde{c}_{0k} - \delta)\right) - 1)}$$

When rearranged, this implies:

$$-h'(\tilde{c}_{0j}, c_j^*) \sum\limits_{\substack{k=1 \\ k \neq j}}^{n} h(\tilde{c}_{0k}) = h(\tilde{c}_{0j}) \sum\limits_{\substack{k=1 \\ k \neq j}}^{n} \left(-h'(\tilde{c}_{0k}, c_k^*)\right)$$

and so again we find that $D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = 0$.

So in both cases $D_j(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*) = 0$, which is the boundary case for the matrix with elements $-\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)$ to be (weakly) diagonally dominant. This matrix is also symmetric. Now any symmetric, diagonally dominant, real matrix with non-negative diagonal entries is positive semi-definite. Thus, the negative of this (the matrix with elements $\tilde{p}'_{jk}(\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}^*)$ representing the directional partial derivative matrix of $\tilde{p}_j(\cdot)$ with respect to $\tilde{c}_k$ at the point $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}_0$ in the direction of $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}^*$) is everywhere negative semi-definite. As noted by Cantarella & Cascetta (1995, p. 315), and applied to our current case, this is a sufficient condition for the following monotonicity condition to hold:

$$\tilde{\Omega}(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) = \sum\limits_{j=1}^{n} (\tilde{c}_j - \tilde{d}_j)(\tilde{p}_j(\tilde{\mathbf{c}}) - \tilde{p}_j(\tilde{\mathbf{d}})) \leq 0 \quad \forall \, \tilde{\mathbf{c}}, \, \tilde{\mathbf{d}} \in \mathcal{C}$$

and as established at the outset, this is equivalent to establishing the required monotonicity condition in the statement of the Proposition.

[Proof complete].

This monotonicity allows us to establish directly a sufficient condition for uniqueness of a Bounded SUE solution (i.e. a SUE solution based on the BCM choice model).

Corollary 1. *Uniqueness of Bounded SUE Solution*

Suppose that the conditions of Proposition 1 hold, and that additionally the link cost-flow functions satisfy the strict monotonicity condition:

$$\sum\limits_{a=1}^{A} (f_a - g_a)(t_a(\mathbf{f}) - t_a(\mathbf{g})) > 0 \,\, \forall \, \mathbf{f}, \,\, \mathbf{g} \in F.$$

Then there exists exactly one Bounded SUE solution.

Proof.

We aim for a proof by contradiction, following that set out in the Fixed Point Uniqueness Theorem of Cantarella & Cascetta (1995), which we outline here for completeness. Consider the *route demand map* $\mathbf{\Phi} : R_+^N \to R_+^N$ with components given by:

$$\Phi_{mr}(\mathbf{c}) = d_m P_{mr}(\mathbf{c}) \,\, (r \in R_m; m = 1, 2, \ldots, M)$$

and the *link demand map* $\mathbf{\phi} : R_+^A \to R_+^A$ given by:

$$\mathbf{\phi}(\mathbf{t}) = \mathbf{\Delta} \mathbf{\Phi}(\mathbf{\Delta}^T \mathbf{t}).$$

Then,

$$\left(\mathbf{\Phi}(\mathbf{c}') - \mathbf{\Phi}(\mathbf{c}'')\right)^T (\mathbf{c}' - \mathbf{c}'') = \left(\mathbf{\Phi}(\mathbf{\Delta}^T \mathbf{t}') - \mathbf{\Phi}(\mathbf{\Delta}^T \mathbf{t}'')\right)^T (\mathbf{\Delta}^T \mathbf{t}' - \mathbf{\Delta}^T \mathbf{t}'')$$

and following some algebraic manipulation we find:

$$\left(\boldsymbol{\Phi}(\mathbf{c}') - \boldsymbol{\Phi}(\mathbf{c}'')\right)^{\mathrm{T}}(\mathbf{c}' - \mathbf{c}'') = \left(\boldsymbol{\Delta}\boldsymbol{\Phi}(\Delta^{\mathrm{T}}\mathbf{t}') - \boldsymbol{\Delta}\boldsymbol{\Phi}(\Delta^{\mathrm{T}}\mathbf{t}'')\right)^{\mathrm{T}}(\mathbf{t}' - \mathbf{t}'')$$

yielding:

$$\left(\boldsymbol{\Phi}(\mathbf{c}') - \boldsymbol{\Phi}(\mathbf{c}'')\right)^{\mathrm{T}}(\mathbf{c}' - \mathbf{c}'') = \left(\boldsymbol{\phi}(\mathbf{t}') - \boldsymbol{\phi}(\mathbf{t}'')\right)^{\mathrm{T}}(\mathbf{t}' - \mathbf{t}'') \ .$$

From Proposition 2, $(\boldsymbol{\Phi}(\mathbf{c}') - \boldsymbol{\Phi}(\mathbf{c}''))^{\mathrm{T}}(\mathbf{c}' - \mathbf{c}'')$ is a sum of everywhere non-positive elements so is itself non-positive, and so therefore also we have:

$$\left(\boldsymbol{\phi}(\mathbf{t}') - \boldsymbol{\phi}(\mathbf{t}'')\right)^{\mathrm{T}}(\mathbf{t}' - \mathbf{t}'') \le 0 \ .$$

Now suppose that two different equilibrium flow solutions $\mathbf{f}'$ and $\mathbf{f}''$ exist, with corresponding link costs $\mathbf{t}' = \mathbf{t}(\mathbf{f}')$ and $\mathbf{t}'' = \mathbf{t}(\mathbf{f}'')$, and so that $\mathbf{f}' = \boldsymbol{\phi}(\mathbf{t}')$ and $\mathbf{f}'' = \boldsymbol{\phi}(\mathbf{t}'')$. Then applying the result above, it follows that:

$$\left(\boldsymbol{\phi}(\mathbf{t}') - \boldsymbol{\phi}(\mathbf{t}'')\right)^{\mathrm{T}}(\mathbf{t}' - \mathbf{t}'') = \left(\mathbf{f}' - \mathbf{f}''\right)^{\mathrm{T}}\left(\mathbf{t}(\mathbf{f}') - \mathbf{t}(\mathbf{f}'')\right) \le 0$$

and this contradicts the monotonicity assumption of Corollary 1, which asserts that:

$$\left(\mathbf{f}' - \mathbf{f}''\right)^{\mathrm{T}}\left(\mathbf{t}(\mathbf{f}') - \mathbf{t}(\mathbf{f}'')\right) > 0.$$

The contradiction thus asserts that our original hypothesis of multiple solutions must have been false, yet we know by the Existence Theorem that at least one solution exists, thus (taken together) establishing that a unique solution exists.
[Proof complete].

4. Solution algorithm and numerical experiments

In this section we present the results of applications of various instances of the Bounded SUE model to two cases: a simple single OD-network with three parallel routes, and the well-known Sioux Falls Network. With these experiments we wish to highlight particular characteristics of the new model proposed, e.g. related to the composition of the equilibrated choice sets and flow allocation among the paths. For the Sioux Falls case, we also compare to the MNL SUE as well as DUE flow solution.

Equilibrium can easily be found in the parallel route case, whereas the Sioux Falls network requires an iterative approach to identify equilibrium. Section 4.1 describes this iterative approach, while Section 4.2 presents a measure to monitor convergence of any Bounded SUE algorithm. In Section 4.3-4.4 the results of the numerical experiments are presented.

4.1. Solution algorithm for Bounded SUE

A main characteristic of the Bounded SUE is that it assigns zero probability to paths violating the cost bound, thus leaving these completely unused. This we aim to utilise in the development of a solution algorithm, namely that a large part of the universal choice set remains unused for a given OD-relation (unlike SUE), and so it should avoid enumerating all such paths while ensuring that all paths with a cost below the cost bound has been identified at convergence. This can be considered as a 'scalability' property of the Bounded SUE that it shares with the DUE; adding new, highly costly paths to a previously equilibrated network leaves the solution unchanged, unlike for SUE.

Below we present a generic solution algorithm that ensures the fulfilment of the Bounded(δ) SUE conditions at convergence. The algorithm is an adaptation of the generic algorithm we presented in Rasmussen et al., (2015). The adaptation consists of the addition of a *Bound condition phase* which checks for the fulfilment of the cost bound among used paths, and removes flow from any violating paths. Since the algorithm is specified in a highly generic way, it is not possible to give a formal proof of convergence. Relying on our promising numerical results, convergence to the corresponding Bounded SUE however seems to be obtained if all phases of the algorithm are specified appropriately.

An iteration of the proposed solution algorithm consists of 4 steps, namely the *Column generation phase*, the *Restricted master problem phase,* the *Network loading phase* and the *Bound condition phase*, as shown in Table 1.

In the above specification the path flow vector is denoted by $\mathbf{X}$ rather than $\mathbf{x}$. This is to emphasise that in practical implementations it is not possible/practical to operate with the vector $\mathbf{x}$, as this requires enumerating the universal choice set for all OD-pairs to obtain the dimension of the $\mathbf{x}$ vector. Rather, in practical implementations, the dimension of the flow vector is not pre-specified, but it is allowed to increase as the algorithm progresses. The same occurs for the path cost vector $\mathbf{c}(\mathbf{x})$, which we have denoted $\mathbf{C}(\mathbf{X})$ to highlight that this might grow as the algorithm progresses. The elements x_{mr} and c_{mr} thus refer to the vectors $\mathbf{X}$ and $\mathbf{C}$, respectively. The algorithm is very flexible in its specification, and in the following subsections we discuss in more detail the *Column generation phase*, the *Restricted master problem phase* and the *Bound condition phase*.

Table 1
Proposed solution algorithm for Bounded SUE.

Algorithm	
Step 0	*Initialisation.* Iteration $n = 1$. Perform deterministic all-or-nothing assignment for all $m = 1, 2,..., M$ OD-pairs and obtain the flow vector for all utilised paths $\mathbf{X}_n$. Perform network loading, compute link travel costs $\mathbf{t}(\mathbf{f}_n)$ on all network links $a = 1, 2,..., A$, and compute generalised path travel costs $\mathbf{C}(\mathbf{X}_n)$. Set $n = 2$.
Step 1	*Column generation phase.* Let $k_{m,n-1}$ denote the number of unique paths in the choice set of used paths for OD-pair $m = 1, 2,..., M$ in iteration *n-1*. For each OD-pair $m = 1, 2,..., M$, based on actual link travel costs $\mathbf{t}(\mathbf{f}_{n-1})$, check for new routes to add to the choice set $R_{m,n}$ of used paths by applying some path generation method which supports the fulfilment that all paths with a cost below the cost bound should be used. If for any OD-relation $m = 1, 2,..., M$ new unique paths are generated, add them to the choice set and let $x_{mr,n-1} = 0$ for any such path r added.
Step 2	*Restricted master problem phase.* Given the choice sets $R_{m,n}$ for all $m = 1, 2,..., M$, apply the selected *Inner assignment component* and *Averaging scheme* to find the new flow solution $\mathbf{X}_n$.
Step 3	*Network loading phase.* Perform the network loading to obtain $\mathbf{f}_n$ from $\mathbf{X}_n$. Compute the link travel costs $\mathbf{t}(\mathbf{f}_n)$ and the generalised path travel costs $\mathbf{C}(\mathbf{X}_n)$.
Step 4	*Bound condition phase.* Given the choice sets $R_{m,n}$ for all $m = 1, 2,..., M$, check whether the condition $C_{mr}(\mathbf{X}_n) \leq \min (C_{ms}(\mathbf{X}_n): s \in R_{m,n}) + \delta_m(\mathbf{C})$ is violated for any $r \in R_{m,n}$ for $m = 1, 2,..., M$. Remove relevant routes and redistribute the flow on routes removed among the remaining routes in the respective choice sets, ensuring OD demands are still satisfied. If no routes have been removed for any of the M OD-relations, continue. Else, perform the network loading, compute the link travel costs $\mathbf{t}(\mathbf{f}_n)$ and the generalised path travel costs $\mathbf{C}(\mathbf{X}_n)$.
Step 5	*Convergence evaluation phase.* If all three gap measures are below their pre-specified convergence value, Stop[a]. Else, set $n = n + 1$ and return to *Step 1*.

[a] See Section 4.2 for the definition of the three gap measures

4.1.1. Column generation phase

The algorithm allows for various approaches in the *Column generation phase*, as long as it supports the fulfilment of the underlying Bounded SUE conditions at convergence. The phase should thus solve the subproblem that, at convergence, at least all paths which have a cost below the cost bound should be identified (in the last and/or some earlier iteration). Note that some flexibility is allowed in that, as long as all paths with a cost bound are identified at convergence, it is not problematic if the approach applied in the *Column generation phase* (at some iteration or the last) identifies paths violating the cost bound – such paths will be allocated zero flow by the BCM choice model and/or removed by the *Bound condition phase*.

For example, a Constrained Enumeration approach can be adopted (e.g. Prato & Bekhor, 2006; Hoogendoorn-Lanser et al., 2007). In each iteration, the approach could enumerate all paths with a cost below the current cost bound, based on the link travel costs resulting from the previous iteration. Subsequently, a comparison to the choice set $\tilde{R}_{m,n-1}$ could be performed and any new unique paths identified added to the choice set. Such an approach ensures that at convergence all paths with a cost below the cost bound are included in the choice set.

Performing a constrained enumeration path search is relatively computationally expensive and does not scale linearly with the bound δ and the network size. Moreover, the speed depends exponentially on the network depth (Prato & Bekhor, 2006) and also non-linearly on δ as exemplified in Section 4.4. For applications to very large-scale networks with large bounds, the approach outlined above may thus not be feasible within reasonable computation time. One possibility to reduce the computation time of the solution algorithm could be not to perform the search (*Column generation phase*) in each iteration, and instead identify all paths with a cost less than some amount more than the cost bound of the choice model when conducted. The choice model would not allocate any flow to paths violating the bound in the subsequent *Restricted master problem phase* of the same iteration (BCM will assign zero probability to violating paths), but due to the redistribution among used paths such violating paths might become attractive to allocate flow in some subsequent iteration before the next iteration in which the column generation phase is conducted. Even such an approach scales non-linearly with network size, and may prove infeasible for large networks. Therefore, other consistent methods for the column generation phase may need to be developed.

4.1.2. Restricted master problem phase

The *Restricted master problem phase* is flexible in its specification, as long as it is specified in a way which ensures that flow is allocated among paths in the choice set according to the BCM at convergence of the algorithm. Standard path-based SUE solution methods can be used. Thus, the closed-form BCM choice probability expression could be used directly to find a feasible auxiliary solution and the well-known method of successive averages (MSA, Robbins & Monro, 1951) to determine the step-size from the current solution $\mathbf{X}_{n-1}$ towards the auxiliary solution. Such an approach has been shown to converge for the MNL SUE, and is expected to also converge for the Bounded SUE (Sheffi, 1985).

Table 2

Proposed *Bound condition phase* in solution algorithm for Bounded SUE.

Step 4.0	Set $m = 1$
Step 4.1	For each route r in the choice set $R_{m,n}$, check whether the condition $C_{mr}(\mathbf{X}_n) \leq (\min (C_{ms}(\mathbf{X}): s \in R_{m,n}) + \delta_m(\mathbf{C})$ is violated. If any route r violates this condition, then flag the route.
Step 4.2	If no route is flagged by Step 4.1 and if $m < M$, set $m = m + 1$ and return to Step 4.1. If no routes are flagged by Step 4.1 and if $m = M$, continue to Step 4.3. If a route r is flagged by Step 4.1, remove the route from the choice set and redistribute flow $X_{mr,n}$ among the remaining currently-used routes s according to the following: $$x_{ms,n} = x_{ms,n} + x_{mr,n} \cdot \frac{(\exp(-\theta(c_{ms,n} - \min(c_{mu,n}: u \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\sum_{t \in R_m} (\exp(-\theta(c_{mt,n} - \min(c_{mu,n}: u \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}. \text{ If } m < M, \text{ set}$$ $m = m + 1$ and return to Step 4.1. If $m = M$, continue.
Step 4.3	If no routes have been removed for any of the M OD-relations, continue. Else, perform the network loading, compute the link travel costs $\mathbf{t}(\mathbf{f}_n)$ and the generalised path travel costs $C(\mathbf{X}_n)$.

4.1.3. Bound condition phase

The algorithm is also flexible in the specification of the method taken to ensure that the cost bound is not violated at termination of the algorithm. One possible approach to the *Bound condition phase* is introduced in Table 2. In this approach, flow on violating paths is reassigned among the other paths according to their BCM choice probabilities. This ensures that flow is only reassigned to paths with costs below the cost bound.

4.2. Monitoring convergence

In this section we propose a three-part gap measure to monitor convergence of any Bounded SUE solution algorithm to a solution satisfying the Bounded SUE conditions. Two parts monitor convergence of the equilibrated choice sets and one part monitors the convergence of the allocation of flow to fulfil the underlying choice model. Each of the three parts is defined for iteration n below:

$$Rel.gap^{unused\ below\ bound} = \frac{\sum_{m=1}^{M} d_m \cdot max_{r \in R_m, x_{mr}=0}(\min(c_{ms}(\mathbf{x}_n): s \in R_m) + \delta - c_{mr}(\mathbf{x}_n))_+}{\delta \cdot \sum_{m=1}^{M} d_m} \tag{5}$$

$$Rel.gap^{used\ above\ bound} = \frac{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot (c_{mr}(\mathbf{x}_n) - \min(c_{ms}(\mathbf{x}_n): s \in R_m) - \delta)_+}{\sum_{m=1}^{M} \sum_{r \in R_m} x_{mr,n} \cdot c_{mr}(\mathbf{x}_n)} \tag{6}$$

$$Rel.gap^{used\ below\ bound} = \frac{\sum_{m=1}^{M} \sum_{r \in R_m, x_{mr}>0} x_{mr,n} \cdot (\tilde{c}_{mr}(\mathbf{x}_n) - \tilde{c}_{m,min}(\mathbf{x}_n))}{\sum_{m=1}^{M} \sum_{r \in R_m, x_{mr}>0} x_{mr,n} \cdot \tilde{c}_{mr}(\mathbf{x}_n)} \tag{7}$$

where $\tilde{c}_{mr}(\mathbf{x}_n)$ shall be defined in (8) and $\tilde{c}_{m,min}(\mathbf{x}_n)$ refers to the lowest value of $\tilde{c}_{mr}(\mathbf{x}_n)$ across all utilised paths for OD-relation m in iteration n.

The first part (5) relates to ensuring that no paths with a cost *below* the cost bound are *unused*. Moreover, for each OD-relation, it measures the average relative violation of the bound for the unused path violating the bound the most. When across all OD-relations, no unused paths have cost below the bound, then the gap is zero.

The second part (6) relates to ensuring that no paths with a cost *above* the cost bound are *used*. Moreover, it measures the total violation relative to the total costs across all paths. When across all OD-relations, no used paths have cost above the bound, then the gap is zero.

The third part (7) relates to ensuring that flow is allocated across used paths in a way that fulfils the underlying choice model. For this we extend an idea in Rasmussen et al., (2015), where we defined a transformed cost measure $\tilde{c}_{mr}(\mathbf{x}_n)$ for the standard MNL choice model and proved that (7) is zero at convergence when using this. For the BCM, we define $\tilde{c}_{mr}(\mathbf{x}_n)$ to be used in (7) instead to be:

$$\tilde{c}_{mr,n} = x_{mr,n} \frac{1}{(\exp(-\theta(c_{mr,n} - \min(c_{ms,n}: s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+} \tag{8}$$

At equilibrium, $\tilde{c}_{mr}(\mathbf{x}_n)$ will be the same across all utilised paths for OD-relation m, and the gap is zero. Below we show that if (8) is the same across all used paths, then flow is allocated according to the BCM:

$$\left. \begin{array}{rcl} \tilde{c}_{mr} & = & \tilde{c}_{ms}, \quad \forall (r, s) \in R_m, x_{mr} > 0, x_{ms} > 0 \\ x_{mr} & = & \tilde{c}_{mr} \cdot (\exp(-\theta(c_{mr,n} - \min(c_{ms,n}: s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+ \\ P_{mr}(\mathbf{c}(\mathbf{x})) & = & \frac{x_{mr}}{d_m} = \frac{x_{mr}}{\sum_{s \in R_m, x_{ms}>0} x_{ms}} \end{array} \right\} \implies$$

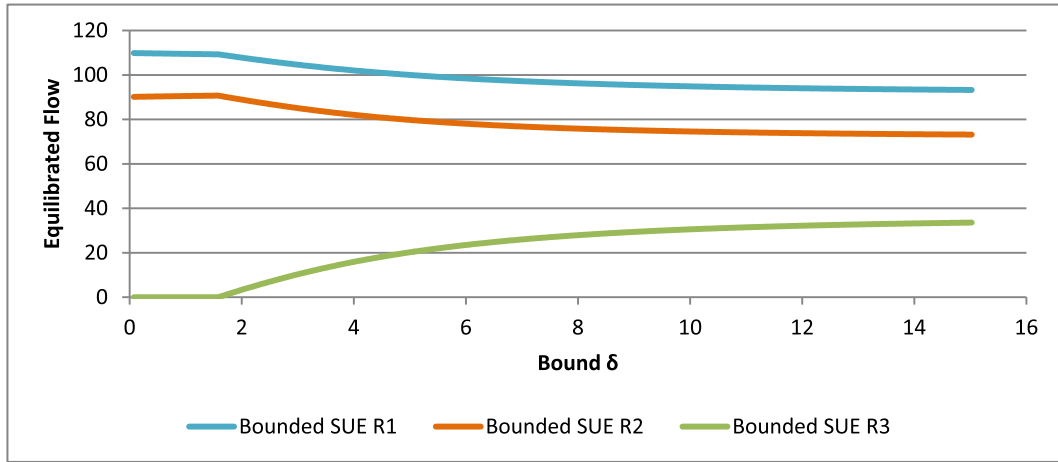**Fig. 4.** Flow share across three paths as function of value δ of the bound, Bounded(δ) SUE with $\theta = 0.2$.

$$P_{mr}(\mathbf{c}(\mathbf{x})) = \frac{\tilde{c}_{mr} \cdot (\exp(-\theta(c_{mr,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\sum_{s \in R_m, x_{ms} > 0} \tilde{c}_{ms} \cdot (\exp(-\theta(c_{ms,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}$$

$$= \frac{\tilde{c}_{mr} \cdot (\exp(-\theta(c_{mr,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\sum_{s \in R_m, x_{ms} > 0} \tilde{c}_{mr} \cdot (\exp(-\theta(c_{ms,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}$$

$$= \frac{\tilde{c}_{mr} \cdot (\exp(-\theta(c_{mr,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\tilde{c}_{mr} \cdot \sum_{s \in R_m, x_{ms} > 0} (\exp(-\theta(c_{ms,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}$$

$$= \frac{(\exp(-\theta(c_{mr,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}{\sum_{s \in R_m, x_{ms} > 0} (\exp(-\theta(c_{ms,n} - \min(c_{ms,n} : s \in R_m) - \delta_m(\mathbf{c}))) - 1)_+}$$

4.3. Parallel route network

Consider a network with three independent routes linking a single OD-pair with a demand of 200 vehicles. The routes (links) have the following BPR cost-flow relation:

$$c_i(\mathbf{x}) = c_i(x_i) = t_{0i} \cdot \left(1 + 0.3 \cdot \left(\frac{x_i}{100}\right)^4\right), \text{ for } i = 1, 2, 3$$

In the following we evaluate the effect of varying the bound δ and the free-flow travel cost t_{01} on the solution of the Bounded(δ) SUE. With these simple examples we wish to highlight issues related to existence, route exclusion and properties of limiting cases.

In the following, assume $t_{01} = 15$, $t_{02} = 18$, $t_{03} = 23$ and $\theta = 0.2$. Fig. 4 illustrates the equilibrated flows on the three routes as a function of δ.

As can be seen, a solution exists across the whole range of δ. This is expected since the example satisfies the sufficient conditions for a solution to exist (Proposition 1). It can also be observed that, as the bound δ decreases towards zero, the Bounded(δ) SUE approaches the DUE which has only paths 1 and 2 used with $x_1 = 109.9$ and $x_2 = 90.1$. In the other limit as the bound δ increases towards infinity, the Bounded(δ) SUE approaches the MNL SUE solution with $x_1 = 92.4$, $x_2 = 72.5$ and $x_3 = 35.2$.

We now analyse the effect of varying a network attribute, the free-flow travel time on one of the routes. In the following, it is assumed $t_{02} = 18$, $t_{03} = 20$, $\theta = 0.2$ and $\delta = 4$. Figure 5 illustrates the equilibrated flows on the three routes as a function of t_{01}. A unique Bounded(4) SUE solution exists on the whole range of t_{01} (in accordance with Proposition 1 and Corollary 1). Paths 2 and 3 are 'activated' (allocated flow) when $c_1(\mathbf{x}) + \delta$ increases to t_{02} and t_{03}, respectively. For $t_{01} > 19.75$, path 1 is the costliest (allocated least flow), and for $t_{01} > 28.6$ then $c_1(\mathbf{x}) > \min(c_2(\mathbf{x}), c_3(\mathbf{x})) + \delta$, and path 1 is thus not used.

Overall, the tests on the parallel route network illustrate existence (as expected from Proposition 1) and help to exemplify how the continuity of the choice function maps onto a continuous flow allocation, where paths are activated and allocated a non-zero flow as their costs move (by changing the bound δ or free-flow travel time on a path) from being outside the cost bound to within the cost bound, and *vice versa*.
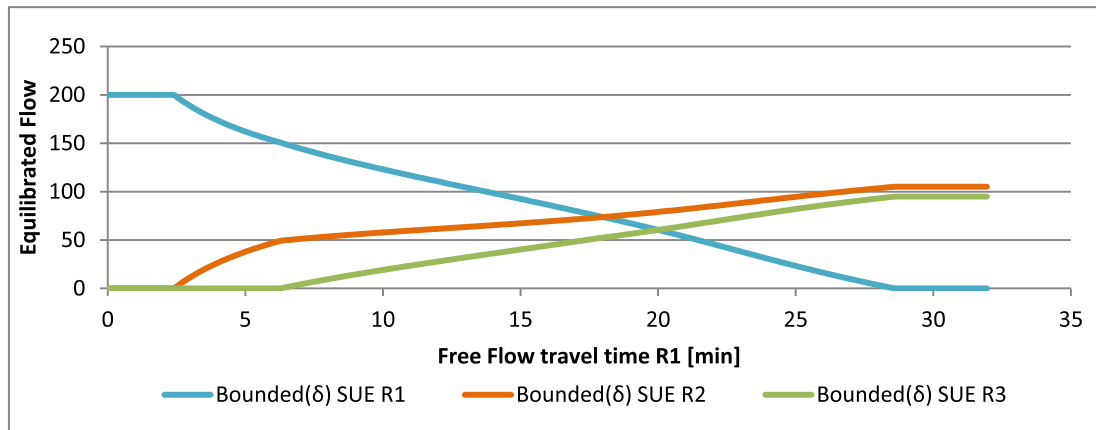
Fig. 5. Equilibrated flow share across three paths as function of free-flow travel time on path 1 (t_{01}), Bounded(4) SUE with $\theta = 0.2$.

4.4. Sioux falls application

The Sioux Falls network contains 76 links and 528 OD-relations between which there is non-zero demand.[4] In this section we demonstrate the applicability of the Bounded(δ) SUE, and highlight issues related to the size and composition of the equilibrated choice sets. We also illustrate the flow distribution across paths for a single OD-relation and compare this to corresponding allocations for the DUE and corresponding MNL SUE. All tests were performed by coding the Bounded SUE in JAVA.

4.4.1. Algorithm configuration

In the Sioux Falls application we adapt the Constrained Enumeration approach with searches for paths with cost below the cost bound in every iteration, as described in Section 4.1.1. In the *Restricted master problem phase* we apply the choice probability expression of Definition 1 to compute an auxiliary solution and then move towards this solution using the method of successive weighted averages proposed by Liu et al., (2009) (with step-size parameter $d = 2$), i.e.:

$$X_{mr,n} = \left(1 - \frac{n^2}{\sum_{k=1..n} k^2} \right) \cdot X_{mr,n-1} + \frac{n^2}{\sum_{k=1..n} k^2} \cdot d_m \cdot P_{mr}(\mathbf{C}(\mathbf{X}_{n-1}))$$

We conduct the *Bound condition phase* as outlined in Table 2.

4.4.2. Convergence and computation time

In the analysis, we have used (5), (6) and (7) to monitor convergence of the algorithm to ensure that a solution satisfying the underlying conditions was indeed found at termination. We assume convergence when the choice sets are fully converged (i.e. (5) and (6) both zero) and (7) is below 0.00005. Figure 6 shows the convergence pattern of the Bounded(15) SUE with $\theta = 0.2$. The composition of the choice sets converges faster than the flow allocation across used paths, which then subsequently converges rapidly. This is typical of the convergence patterns we have observed with this algorithm. With tighter bounds we found that, after having converged fully, (5) and (6) might increase slightly during a few iterations and then converge fully – this happens when the reallocation of flow among used paths causes the costs of some used/unused paths to increase/decrease above/below the cost bound.

The number of iterations needed to reach convergence and the computation time depends on δ, as described in Section 4.1. The numerical experiments support this as highlighted by Table 3, which reports the number of iterations needed to converge and the average calculation time per iteration for various values of θ and δ.

As can be seen, the algorithm converges within a limited number of iterations and computation time across all tested combinations of θ and δ, e.g. with a total convergence time of 1.04 seconds for Bounded(15) SUE with $\theta = 0.2$ ($106 ite \cdot 9.8 \frac{ms}{ite} / 1000 \frac{ms}{s}$). The computation time and required number of iterations generally increases with increasing θ. This seems reasonable, since the choice probabilities are more sensitive to cost differences and thus cause larger between-iteration fluctuations in the probabilities and thus network flows. The computation time per iteration also increases considerably when increasing δ, as expected. This is partly due to the computational costs of the constrained enumeration algorithm used in the *Column generation phase* which increases non-linearly. The computational costs of the other phases of the algorithm also increases, as e.g. the computation of choice probabilities in the *Restricted master problem phase* depends on the number of alternatives. This is highlighted in Table 4, which reports the computation time for the *Column generation phase* and

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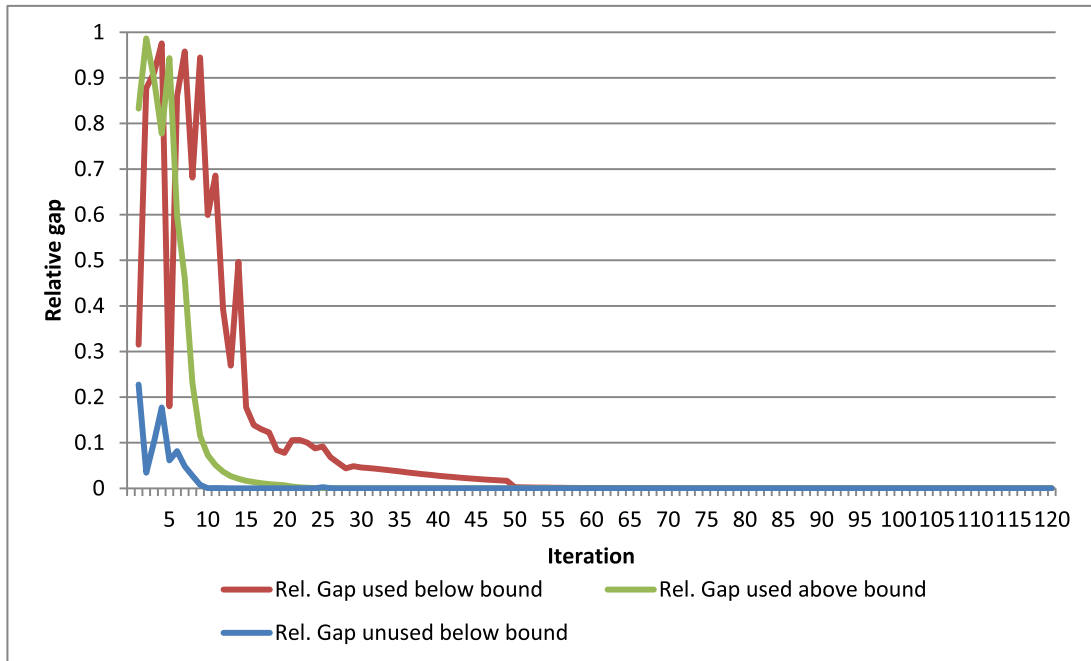[4] A detailed description of the network can be found in Bar-Gera (2013).

Fig. 6. Convergence pattern of the application of the Bounded(15) SUE with $\theta = 0.2$, Sioux-Falls.

Table 3
Number of iterations required to obtain convergence and average calculation time (milliseconds per iteration) for various configurations of Bounded(δ) SUE.

		θ		
		0.05	0.2	1.0
	5	431/4.7	334/4.5	434/4.5
δ	15	85/8.9	106/9.8	222/11.6
	30	86/18.1	169/27.9	236/42.6

Table 4
Calculation time in milliseconds per iteration for various values of bound δ, Bounded(δ) SUE with $\theta = 0.2$.

		Column generation phase	Other phases	Total
	5	3.3	1.2	4.5
	15	7.7	2.1	9.8
	30	22.7	5.2	27.9
	45	63.8	12.8	76.5
	60	148	40.8	188.8
	75	216	140.9	356.9
δ	90	507.9	350.3	858.3
	120	1730.2	1102.8	2833
	150	3875	1770.7	5645.7
	180	6027.7	2050.7	8078.5
	200	6796.9	2147.1	8944.1
	250	6887.2	2204.1	9091.3
	300	6876.4	2196.1	9072.5

other phases separately for various applications of the Bounded(δ) SUE with $\theta = 0.2$. Note that the computation time for the *Column generation phase* is almost constant across very large values of δ, corresponding to the case in which the universal choice set has to be enumerated for all OD-relations in all iterations.

Table 5

Average/maximum choice set size for various specifications of θ and δ for the Bounded(δ) SUE, Sioux-Falls.

		δ		
		5	15	30
θ	0.05	2.1/8	4.1/16	8.3/33
	0.2	2.2/9	4.5/18	13.1/54
	1.0	2.2/10	5.9/26	21.3/87

4.4.3. Results

We evaluate the effect of varying the scale parameter θ and the bound value δ on the average/maximum choice set size at equilibrium. The model was applied for 3 different specifications of each of the parameters δ and θ, yielding a total of 9 specifications (3 × 3). Table 5 reports the average and maximum choice set size generated by the different specifications.

The choice sets generated for the parameter settings tested can be seen to be highly dependent on the values of the parameters. The table indicates that the choice set size decreases with decreasing δ. This seems reasonable, since a tighter bound should lower the number of alternatives with a cost lower than the cost bound. Overall, the equilibrated choice sets are much smaller than those from the corresponding MNL SUE model, which allocates flow to all routes in the universal choice set (which we explicitly pre-generated in this case, for comparison purposes), giving an average/maximum choice set size of 3092.5/4787 paths.

We now turn to a more disaggregate analysis focusing on a single OD-relation, with the aim of illustrating the equilibrated choice set composition and the network link flows resulting from the path flow allocation. Furthermore, we also compare to the DUE and MNL SUE flow solutions. In the analysis we use $\delta = 15$ and $\theta = 0.2$ for the Bounded SUE and $\theta = 0.2$ for the MNL SUE. We have downloaded a highly converged DUE link flow solution from Bar-Gera (2013), and using the corresponding link travel costs we have generated the (DUE) path costs for the universal choice set.

In the analysis below we focus on the OD-relation connecting node 1 to node 17, as illustrated in Fig. 7. This OD-relation was chosen for a particular reason. Namely, it is an example of what seems to be a quite common case in the Sioux Falls network, an OD-relation for which, at the DUE link costs, there exists only a *single* minimum-cost path.[5] In relation to the DUE solution, note that in general networks/cases we cannot guarantee unique OD-specific link flow nor unique path flows. However, for any OD-relation with a single minimum-cost path, we can infer unique OD-specific link-flows and path-flows from the DUE link costs; this allows a more detailed comparison with our Bounded SUE model than would otherwise be possible. At equilibrium, the Bounded SUE in fact allocates flow across 12 paths for the OD-relation 1–17, which results in the OD-specific link flow allocation shown in Fig. 7A. Fig. 7B illustrates the corresponding DUE OD-specific link flow allocation, whereas Fig. 7C shows the MNL SUE OD-specific link flows. In relation to the MNL SUE solution, notice that we only consider acyclic paths and therefore a few links are not allocated any flow for this OD-relation.

The Bounded SUE link usage and link flow solutions indicate that clearly paths which have an equilibrated cost within the cost bound use links primarily located in the upper part of the network, leaving, unlike the MNL SUE, a large share of the links unused. Comparing the link flow shares across the three models indicates that the links used in the DUE solution are also the links which carry the largest flow shares in the Bounded SUE. However, for these links the Bounded SUE and MNL SUE link flow shares are more similar than the Bounded SUE and DUE flow shares.

Fig. 8 illustrates the flow share allocation to the paths as a function of their equilibrated costs. The Bounded SUE allocates the largest share of the flow to the cheapest paths, while longer paths within the cost bound are allocated a low share of the flow. Furthermore, we clearly see the continuity of the BCM choice probabilities across the bound, with no paths violating the bound being used. The figure also illustrates the equilibrated flow share on paths for the DUE and MNL SUE. While the figure does not directly compare flows on individual paths, it clearly highlights the differences in flow allocation. The MNL SUE especially differs from the Bounded SUE for costs close to and above the bound, where all paths are assigned a non-zero flow, no matter how costly they are. Note that in the MNL SUE, flow is assigned to 4739 paths, with the most costly having a cost of 237.4.

While both the BCM and MNL SUE reasonably assign flow to paths which have costs slightly higher than the cheapest alternative, the DUE has a strict cut-off; all flow is assigned to a single path, even though the second-cheapest path is only slightly more costly (a cost of 43.92 versus a cost of 42.24 for the cheapest path). When comparing the path flow as well as link flow solutions across the three models, it can be seen that the Bounded SUE solution lies between the DUE and MNL SUE. Moreover, unlike DUE, it uses paths slightly more costly than the cheapest, while, unlike SUE, it hinders the use of paths with unreasonably long detours compared to the cheapest. These characteristics of the Bounded SUE lies well in line with the first requirement for the model set out in the Introduction.

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[5] For 386 of the 528 OD-relations the second cheapest path is at least 0.01 cost units higher than the minimum; assuming that the 'highly converged' DUE solution is able to distinguish path costs to this accuracy, we can claim that these OD-relations only have a single used path at equilibrium. For the particular OD-relation considered, from node 1 to node 17, the cost difference is 1.68 between the minimum-cost path (42.24) and the second-cheapest path (43.92), which seems unlikely to be caused by convergence error in the downloaded solution.
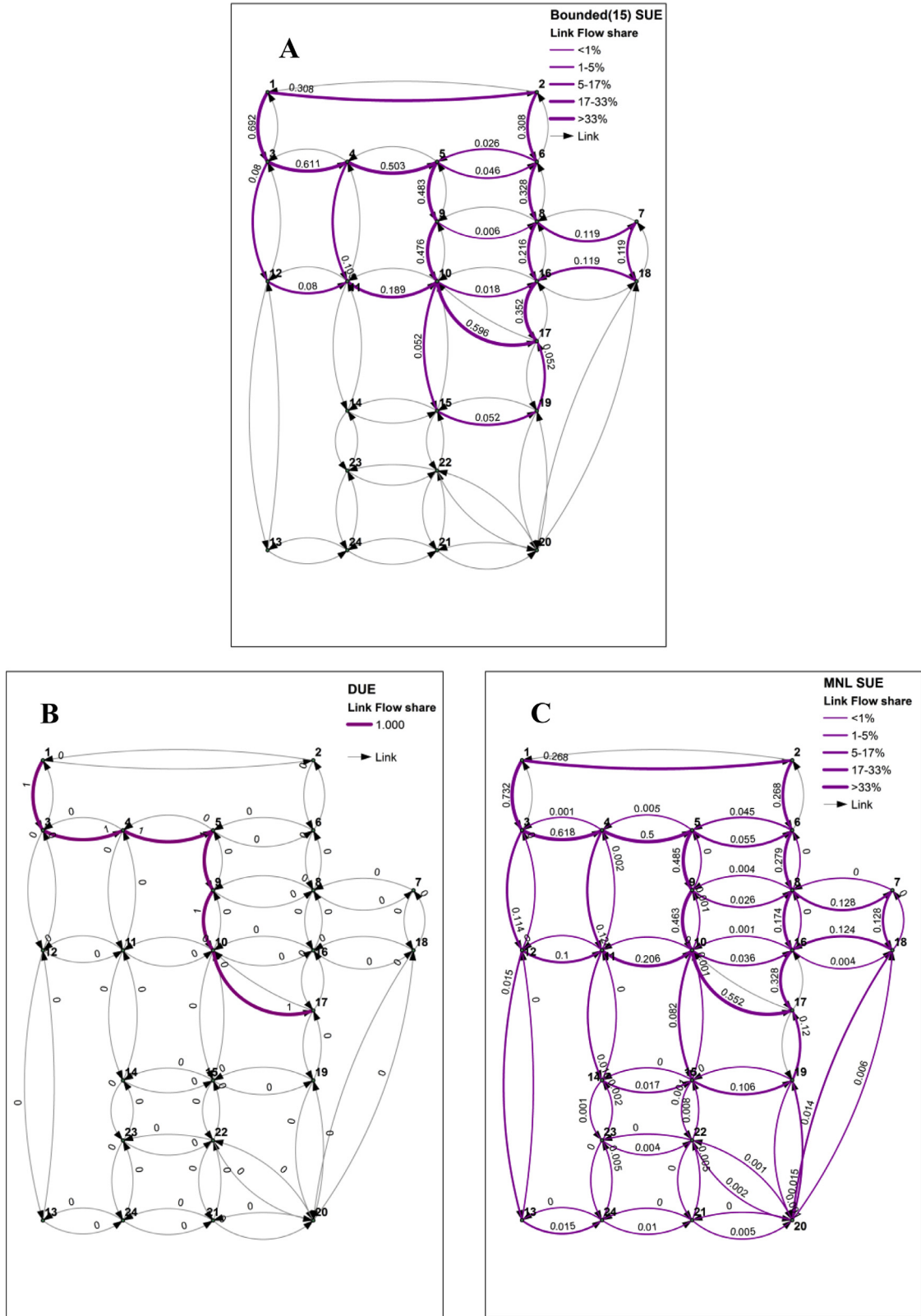
Fig. 7. A: Equilibrated Bounded(15) SUE link flow shares for single OD-relation (Origin: 1, Destination: 17), $\theta = 0.2$. B: Equilibrated MNL SUE link flow shares for single OD-relation (Origin: 1, Destination: 17), $\theta = 0.2$. C: Equilibrated DUE link flow shares for single OD-relation (Origin: 1, Destination: 17).
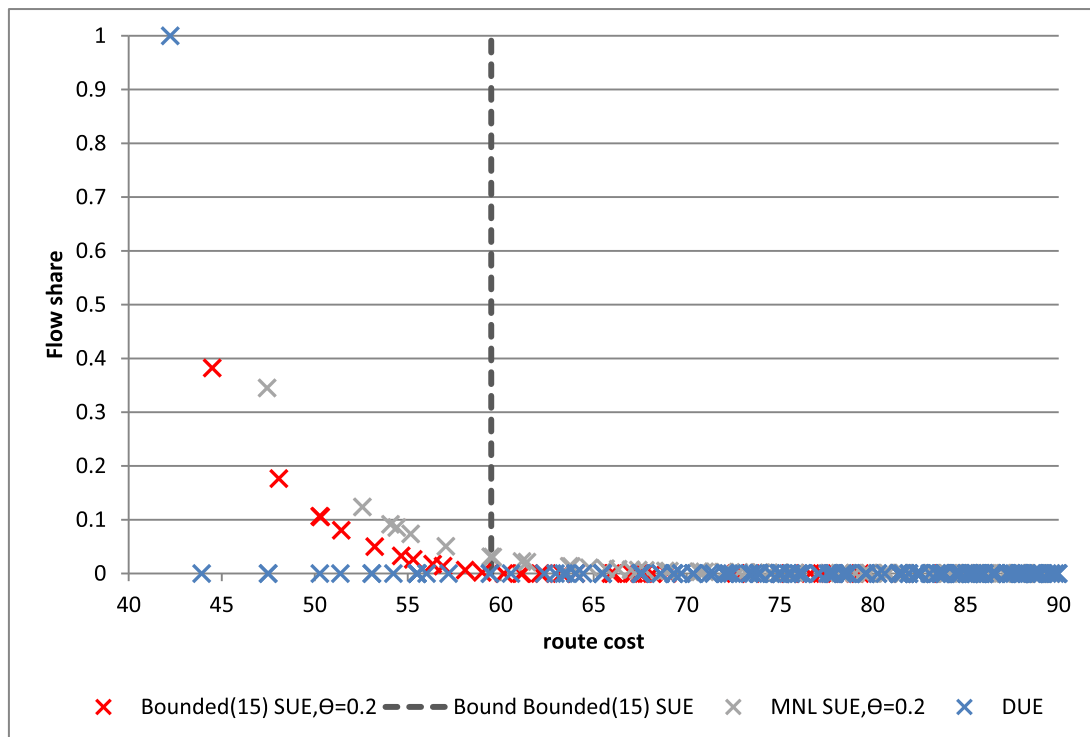
Fig. 8. Equilibrated flow share on paths as function of their costs for a single OD-relation (Origin: 1, Destination: 17), Bounded(15) SUE, MNL SUE and DUE.

The numerical experiments support the theoretical properties of the BCM established in earlier sections, namely that a solution exists and is unique when the sufficient conditions are fulfilled, and that the Bounded SUE approximates the MNL SUE as the bound approaches infinity. The numerical results of the application to both networks also find that the Bounded SUE provides a solution that allocates most flow to the cheapest paths, less flow to more expensive paths within the bound, while leaving unused unattractive paths violating the bound. Thereby, the Bounded SUE solution lies between the MNL SUE and DUE in terms of path- and link flow solutions, which is well in line with the motivation of our research and the requirements we defined for the new model in the Introduction.

5. Conclusions and further research

In the Introduction we reviewed the strengths and limitations of existing network models and identified five desirable requirements for a model to satisfy. We have proposed in this study a novel choice model, the Bounded Choice Model, and a corresponding novel network equilibrium model Bounded SUE. The proposed Bounded SUE is unique in the network equilibrium field in that it satisfies all five requirements, notably: (i) it neither excludes all sub-optimal routes nor includes all available ones; (ii) it provides a point estimate of equilibrium; (iii) it equilibrates the choice set of used alternatives simultaneously with flow equilibration; (iv) it draws on Random Utility Theory; and (v) it allows the establishment of properties to guarantee equilibrium existence and uniqueness.

To our knowledge, the formulation of the BCM is novel in the choice literature, and the network conditions defining the Bounded SUE are novel in the traffic assignment literature. Moreover, this study not only proposes a novel Bounded SUE model for traffic assignment, but also presents a corresponding solution algorithm. Numerical results of the application of the Bounded SUE model to a three-route network and the Sioux Falls network using different model specifications are illustrated. The experiments highlight the applicability and existence of the Bounded SUE according to the theoretical properties established when introducing both the choice and the assignment model.

Also, the numerical results illustrate that the Bounded SUE solution lies between the two extremes of the DUE (when the bound is equal to zero, and only the minimum cost routes are used) and the MNL SUE (when the bound is equal to infinity, and all route are used no-matter how costly they are), to further prove that we have delivered the model in line with our initial motivation for this piece of research.

Further research could look into the specification of the bound. A naïve approach could be to test a range of bounds, from zero (equal to a DUE-model) to a very large one (approaching SUE) and compare each solution with traffic counts to find the best Bounded SUE solution. Another possibility would be to specify the value based on a stated preference survey designed in a way in which respondents reveal their bound. Finally, another option would be to utilise the rapidly growing

body of evidence of observed route choices tracked passively through GPS or mobile phone devices. Comparisons to the travel costs of the shortest paths between trip start and end points would reveal the range of detours that travellers accept, and their distribution could be used in the specification of the bound.

Further research could explore alternative algorithms used for logit-based SUE for a given choice set, such as gradient projection and disaggregated simplicial decomposition (DSD). In particular, there could be merit in exploring their computational efficiency as part of a sub-problem for solving the Bounded SUE model. As a step to suggesting suitable algorithms, a first step may be to explore alternative *formulations* of the Bounded SUE model, such as the dual formulations recently proposed for SUE by Xie and Waller (2012), or by adapting the recent variational inequality formulation of SUE proposed by Smith and Watling (2016). The need to find efficient algorithms is particularly relevant for studying Bounded SUE in very large-scale networks, as there is then a particular challenge in enumerating and storing in memory all paths with costs below the bound, especially if the bound is large (as indicated in Table 4). Future research should also be devoted to the development of a formal proof of convergence of solution algorithms for the Bounded SUE.

We acknowledge that the BCM is a model type inspired from logit as we intended to maintain a convenient closed-form that allows for the definition of the network conditions. It should be noted that the BCM does not have restrictions in the definition of the deterministic part of the utility function, which makes it possible to extend to methods for considering similarity across alternative routes within the equilibrated choice sets. Further research could extend the BCM to alternative approaches to consider the similarity across alternative routes within the error structure of the model, and especially those that still guarantee existence and uniqueness. As an alternative direction, an additional dimension to add would be that of demand elasticity, for which there has been some notable recent attention in the SUE-oriented literature (Meng and Liu, 2012; Yu et al., 2014). Lastly, it would be interesting to explore the properties of the Bounded SUE model under wider behavioural assumptions, such as those more recently proposed for the BRUE model, including the concept of aspiration level proposed by Zhao and Huang (2016), additional constraints on unused routes proposed by Sun et al., (2016), and including asymmetric preferences and context-dependent values-of-time as considered by Hongli et al., (2017).

Acknowledgements

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