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LR Based Pre-Coding Aided Spatial Modulation with Sub-optimal Detection for V2X Communications

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***Abstract*—In this manuscript, a novel transmit pre-coding matrix generation method and a linear decoding scheme are proposed for the generalised pre-coding aided spatial modulation (GPSM) system. The GPSM scheme proposed recently is a new multiple input multiple output (MIMO) transmission technique, which conveys information by activating a subset of receive antennas with the aid of transmit pre-coding. This scheme seems to be suitable in V2X since the base station only transmits data to certain users/vehicles. The proposed pre-coding matrix is based on the lattice reduction (LR) principle and provides significant performance improvement over the original pre-coding design in GPSM. Furthermore, as the GPSM in large-scale antenna systems, which is a trend in future communication systems, might be too complex to be implemented with the maximum likelihood (ML) detection, a linear decoding method is proposed to reduce the implementation complexity. Our studies show that the performance degradation caused by the linear detector can be compensated by the proposed LR based pre-coding. As a result, the LR based GPSM with linear detection is capable of achieving the comparable performance as that of the original GPSM scheme employing ML detection, while with significantly decreased detection complexity.**

I. INTRODUCTION

Using multiple antennas at both transmitter and receiver in wireless communications has led to significant performance improvement over single antenna systems. Owing to this, multiple input multiple output (MIMO) techniques, such as, vertical-Bell Laboratories layered space-time (V-BLAST) [1], space time block coding (STBC) [2], have been adopted in many standards for wireless communications, including 3GPP LTE and 802.11. Dedicated short range communications (DSRC) is a full duplex wireless communication dedicated for automotive use, which is part of the IEEE 802.11p wireless access for vehicular environments (WAVE) and architecture, security and message protocols standards. DSRC is utilised for the interoperability between vehicles (V2V) and the traffic signal infrastructure (V2I). Some of the DSRC characteristics are: high mobility performance, limited nominal range (around 300 m), extensive public key infrastructure, etc. [3]. This information exchange can be utilised for several applications such as host of safety, mobility and environmental, which include driver assistance and vehicle safety, speed adaptation and warning, emergency response, safety, traveller information, navigation, traffic operations and demand management, personal navigation, commercial fleet planning and payment transactions [4].

The spatial modulation (SM) proposed in [5] has established itself as an efficient transmission scheme, subsuming numerous MIMO systems family. The distinctive feature of SM is using some of the information to be transmitted to activate one or a subset of transmit antennas. In this way, some useful information is conveyed in the spatial domain, resulting in throughput increase. The novel fashions of SM include facilitating high data rate MIMO implementation with reduced signal processing, simplified channel estimation, reduced system complexity and avoiding the traditional problems associated with the existing MIMO schemes, such as, the inter-channel-interference (ICI) and inter-antenna-synchronisation (IAS) [6], etc.

The pre-coding and generalised pre-coding aided spatial modulation (GPSM) proposed in [7,8] represent the variations of the conventional SM [6]. In GPSM, in addition to the conventional amplitude and phase modulation (APM), transmitter pre-coding is employed to activate some receive antennas, whose indices are used to convey extra information. Assuming perfect channel state information at the transmitter (CSIT), the pre-coding in GPSM can be designed so as to proactively mitigate the ICI, allowing low-complexity maximum likelihood (ML) based detection at the receiver. Therefore, the GPSM is particularly suitable for downlink transmission. However, when viewing the GPSM's future potential applications in large scale antenna systems, the ML detector may still be too complicated for implementation and, hence, may not be viable. This motivates us to propose a sub-optimal detection scheme in order to further reduce the detection complexity. As expected, this sub-optimal detection introduces performance degradation in comparison with the ML detector. Therefore, to compensate for the performance loss, in this letter we propose a novel pre-coding design method based on lattice reduction (LR). Our studies show that, in comparison with the original GPSM [7,8], the bit error rate (BER) performance of the LR-assisted GPSM can be significantly improved, if the ML detection is applied, or similar BER can be attained, if the proposed sub-optimal detection is used to achieve significant complexity reduction.

The rest of the paper is organised as follows. Section II introduces the principles of the original GPSM and its ML detector. In Section III, the LR based GPSM described and the linear detector are addressed. Section IV presents simulation results and, finally, the paper is concluded in Section V.

Notations: Italicised symbols denote scalar values while bold lower variable denotes vectors and upper case symbols matrices. $(\cdot)^H$ and $(\cdot)^*$ denote Hermitian and conjugate, respectively. $\|\cdot\|$ represents the 2-norm of a vector/matrix and $\mathcal{CN}(n,\sigma^2)$ means the complex Gaussian distribution of a random variable, having independent Gaussian distributed real and imaginary parts denoted by $\mathcal{N}\left(n,\frac{\sigma^2}{2}\right)$, with mean n and variance $\frac{\sigma^2}{2}$. $\boldsymbol{I}_N$ denotes the $N\times N$ identity matrix and $\mathbb{C}$ is the complex set. $\mathcal{O}(\cdot)$ indicates the computational complexity in terms of number of arithmetic operations.

II. GENERALISED PRE-CODING AIDED SPATIAL MODULATION

As our proposed pre-coding and detection schemes are designed to improve the GPSM presented in [8] (although the pre-coding method may also be applied in other scenarios), for the sake of completeness, below we briefly introduce the principles of the GPSM.

A. Principles of GPSM

The system model for the GPSM consists of a MIMO link with N_t transmit and N_r receive antennas. GPSM mitigates the ICI at the transmitter side with the aid of a pre-coder $\boldsymbol{P}\in\mathbb{C}^{N_t\times N_r}$ deigned under the assumption of the CSIT [7]. In the GPSM, a symbol to be transmitted is divided into two sub-blocks: the first sub-block determines the indices of the n_r out of N_r receive antennas to be activated, and the second sub-block is mapped to an APM symbol, which is sent by all the transmit antennas. In mathematics, a GPSM symbol $\boldsymbol{s}_m^c\in\mathbb{C}^{N_r\times 1}$ can be written as $\boldsymbol{s}_m^c=\boldsymbol{\Omega}_c\boldsymbol{b}_m$, $\boldsymbol{b}_m$ is a n_r-length vector, which contains APM symbols determined by the second sub-block of the GPSM symbol, while $\boldsymbol{\Omega}_c\in\mathbb{R}^{N_r\times n_r}$ is a matrix, which only contains elements non-zero at its diagonal, where the diagonal means which c^{th} receive active antennas are activated. Each pattern of active receivers is determined by the first sub-block of the GPSM symbol and $c=1,...,\mathcal{B}$ with $\mathcal{B}$ containing all the possible combinations of choosing n_r out of N_r receive antennas.

In [7], the transmitter pre-coder is designed based on either the zero forcing (ZFP) or the minimum mean-square error (MMSEP) principles. Specifically, when the ZFP is considered, the pre-coding matrix $\mathbf{P}$ is designed to make the transmitted signals only present on the selected receive antennas. Assume that $N_t\geq N_r$. Then, the pre-coding matrix can be expressed as

$$\boldsymbol{P}=\gamma\boldsymbol{H}^H(\boldsymbol{H}\boldsymbol{H}^H)^{-1}, \tag{1}$$

where $\boldsymbol{H}\in\mathbb{C}^{N_r\times N_t}$ represents the channel matrix, each entry of which is assumed to obey independent flat Rayleigh fading, $\gamma=\sqrt{N_t}$ is a normalising factor. Finally, the signals $\boldsymbol{x}\in\mathbb{C}^{N_t\times 1}$ transmitted from N_t transmit antennas can be expressed as

$$\boldsymbol{x}=\boldsymbol{P}\boldsymbol{s}_m^c. \tag{2}$$

B. GPSM Detection

The signal observed at the N_r receive antennas can be written as

$$\boldsymbol{y}=\boldsymbol{H}\boldsymbol{x}+\ \boldsymbol{v}=\boldsymbol{H}\boldsymbol{P}\boldsymbol{s}_m^c+\ \boldsymbol{v}, \tag{3}$$

where $\boldsymbol{v}=[v_1,v_2,\cdots,v_{N_r}]^T$ is the additive white Gaussian noise (AWGN) vector observed at the receive antennas, which is distributed with zero mean and a covariance matrix $E[\boldsymbol{v}\boldsymbol{v}^H]=\sigma_v^2\boldsymbol{I}_{N_r}$, here σ_v^2 is the noise variance.

The spatial and APM symbols conveyed by the GPSM may be detected jointly or separated in the ML principles [8]. As the joint ML detector has high complexity, in [8] a separated ML detection is proposed. In this case, the detection process is divided into two stages. The first stage detects the spatial symbol by identifying which n_r of the N_r receive antennas are activated, described as:

$$\hat{k}=\underset{w\in[1,\mathcal{B}]}{\arg\max}\left\{||\boldsymbol{y}_w||\right\}, \tag{4}$$

where $\boldsymbol{y}_w$ is a vector extracted from $\boldsymbol{y}$ by taking n_r according to the wth activation pattern of selecting n_r out of the N_r receive antennas. After the detection of the spatial symbol, the second stage estimates the APM symbols by solving the ML problem:

$$\hat{\boldsymbol{b}}_m=\underset{m'\in\mathcal{A}}{\arg\min}\left\{||\boldsymbol{y}_{\hat{k}}/\gamma-\boldsymbol{b}_{m'}||\right\}, \tag{5}$$

where $\mathcal{A}$ denotes the alphabet of a particular M-ary PSK/QAM scheme.

As shown in [9], this method of the ML separation detection has the complexity of $\mathcal{O}(\mathcal{B}(3N_r+n_r)+7Mn_r)$ number of FLOPS, with 1 FLOP for addition and 3 FLOPS for multiplication.

III. PROPOSED LR-GPSM SYSTEM MODEL

The GPSM employs a range of attractive features [7,8], which may constitute a promising transmission scheme for application in future wireless systems, such as, V2X communication systems and/or massive MIMO. In V2X the aim is to exchange information from a vehicle to any entity and/or access point (AP), and vice versa. Additionally, massive MIMO systems may also be utilised in this type of communications. Then, usually the usage of big numbers of transmit and receive antennas make even the separate ML detection in GPSM impractical. We therefore consider a linear detection algorithm in this letter, which has much lower complexity than the ML detectors. However, as linear detectors are in general unable to perform as well as the ML detector, for compensation of the performance loss, we propose a novel pre-coder based on the LR algorithm [10]. Recently, the LR-aided linear equalisers have been utilised for the detection in MIMO systems, and the studies show that they are capable of achieving the performance comparable to that achieved by the ML but with lower computational complexity [11,12]. In principle, the LR-aided detection can be briefly explained as follows. Considering the MIMO equation as shown in (3), where $\boldsymbol{x} \in \mathbb{C}^{N_t \times 1}$ needs to be estimated, $\boldsymbol{Hx}$ forms a lattice space spanned by $\boldsymbol{H}$ [13]. Therefore, in order to estimate $\mathbf{x}$, the detector only needs to identify the lattice point closest to $\mathbf{y}$. In the LR-aided detection, the main purpose of lattice basis reduction is to find a good basis for a given lattice. A basis is considered to be good, when the vectors forming the basis are nearly orthogonal. Therefore, given a lattice, the objective of LR is to find a reduced set of basis vectors, which have the required properties, such as, short and nearly orthogonal vectors [14].

A. *Transmission*

In order to attain good estimation performance using linear detectors, the lattice basis must be orthogonal or at least near orthogonal. As it was mentioned in the previous section, LR has been used to improve the performance of linear detectors to achieve performance similar to that of ML. Therefore, we are using LR to enhance the power at the receiver side to detect the indices of the active receivers. In [10], the authors have proposed the LLL algorithm, which is a very popular in the LR-aided detection. The LLL algorithm can guarantee to obtain a lattice basis with the vectors near orthogonal. The details of this algorithm can be found in [10].

The input of the LLL algorithm is the *QR decomposition* of the channel matrix $\boldsymbol{H}$. A parameter δ is selected randomly from the range $(\frac{1}{2}, 1)$ to control the performance trade-off of the LLL algorithm. Based on δ, *QR decomposition* and iterative operations, the LLL generates an unimodular matrix $\boldsymbol{T}$, which has two properties: it is formed only by integers and its determinant is ± 1. The $\tilde{\boldsymbol{H}} = \boldsymbol{HT}$ channel matrix is close to be orthogonal. Fig. 1 shows an example of the transmission scheme of the proposal scheme. The figures illustrates the way of how the symbol is divided and mapping in both parts: the receive antenna indices and the APM symbol.

In the proposed work, we can obtain a lattice basis given by $\tilde{\boldsymbol{H}}$ using LLL, which is utilised to generate the pre-coder. In this case, when the ZFP is considered, the pre-coding matrix can be designed as

$$\tilde{\boldsymbol{P}} = \gamma \tilde{\boldsymbol{H}}^H \left(\tilde{\boldsymbol{H}} \tilde{\boldsymbol{H}}^H \right)^{-1}. \tag{6}$$

Correspondingly, the transmitted GPSM signal $\boldsymbol{x} \in \mathbb{C}^{N_t \times 1}$ is given by

$$\boldsymbol{x} = \tilde{\boldsymbol{P}} \boldsymbol{s}_m^c. \tag{7}$$

Note that this pre-coder is based on a more orthogonal channel matrix $\tilde{\boldsymbol{H}}$, so the energy delivered to the n_r active receive antennas can be near maximum, which improves the detection accuracy of the active antennas' indices without requiring the complicated ML detector.

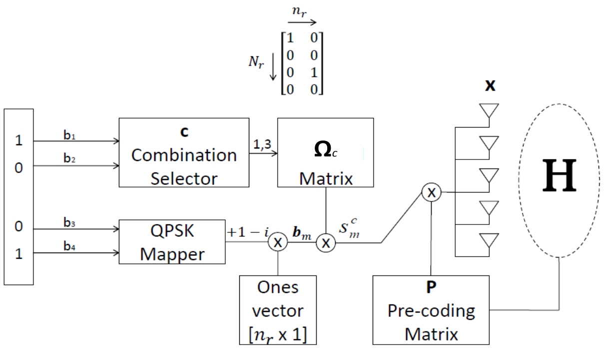

Fig. 1: Proposed transmission system model.

B. *Detection*

The received signal observed is written as

$$\boldsymbol{y} = \boldsymbol{H}\tilde{\boldsymbol{P}}\boldsymbol{s}_m^c + \boldsymbol{v} = \boldsymbol{Hx} + \boldsymbol{v}. \tag{8}$$

The proposed detection method can be divided into two steps. First, the rows of the estimated vector $\boldsymbol{z}$ are sorted in descending order based on their power magnitude as follows:

$$\begin{aligned} \boldsymbol{z} &= sort\{\boldsymbol{y}\}. \\ &= [z_{\hat{\lambda}_1}, z_{\hat{\lambda}_2}, \ldots, z_{\hat{\lambda}_{n_r-1}}, z_{\hat{\lambda}_{n_r}}, z_{\hat{\lambda}_{n_r+1}}, \ldots, z_{\hat{\lambda}_{Nr}}]^T, \end{aligned} \tag{9}$$

where $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{N_r}$ are the indices and the first n_r elements in (9) are the estimation of the indices of the n_r active receive antennas. If it is correct, the indices of the n_r elements should be identical to one of the selection patterns. This can be detected by comparing with all possible patterns of selecting n_r out of N_r. It can be noticed that the possibility to detect corrected set of n_r is high due to the utilisation of the proposed decoder. If the combination is not found in the selection patterns, the $z_{\hat{\lambda}_{n_r}}$ element is replaced by the $z_{\hat{\lambda}_{n_r+1}}$ element, $z_{\hat{\lambda}_{n_r+2}}$ and so on. This process is repeated until the first block is de-mapped.

The resulting vector with the n_r elements is defined as $\boldsymbol{y}_{\hat{k}}$. Then, the APM symbol $\hat{m}$ can be estimated as:

$$\hat{m} = \mathcal{Q}\Big(\sum_{p=1}^{n_r} \boldsymbol{y}_{\hat{k}}(p) \Big), \tag{10}$$

where $\mathcal{Q}()$ is the symbol quantiser to the APM constellation set and $\boldsymbol{y}_{\hat{k}}$ includes the n_r maximum values from the vector $\boldsymbol{z}$.

Table I illustrates the decoding algorithm proposed for GPSM. This table shows how (9) and (10) work to estimate the receiver indices and the APM symbol, respectively. It is worth noticing that (9) is repeated until a pattern can be estimated. The n_r^{th} element is replaced because it is the extremely likely

TABLE I: Proposed Decoding Algorithm

```
INPUT: $\boldsymbol{y}$, $c$; OUTPUT: $\boldsymbol{y}_{\hat{k}}$, $\hat{m}$

$\boldsymbol{z} = sort\ \{\ \boldsymbol{y}\ \}$
$\lambda_{ind} = 1 \rightarrow \mathcal{B}$
$\boldsymbol{y}_{\hat{k}} = \boldsymbol{z}_{\lambda_{ind}}$
while $\boldsymbol{z}_{\lambda_{ind}} \cup c$ false do
  $\boldsymbol{y}_{\hat{k}} = \boldsymbol{z}_{\lambda_{ind+1}}$
end while
$\hat{m} = \mathcal{Q}(\sum \boldsymbol{y}_{\hat{k}})$
```

that the elements with the lowest power may be the idle receivers. However, the simulation results show that it is most likely that the indices of the first n_r elements represent the right active receivers, particularly at moderate or high SNRs. Therefore, the possibility of repeating the comparison is very low, and hence this procedure has a very low complexity.

The computational complexity of the proposed decoding method is $\mathcal{O}(n_r\mathcal{B}(N_r + M))$. This complexity is assuming that the pattern c is found until comparing with all the possible combinations, which is unlikely. Thus, taking N_r operations is the most pessimistic scenario. However, with the original GPSM/ML detection method, the power of all elements are calculated and the highest elements are selected and compared with the power of all the possible patterns. In addition, the proposed detection method utilises linear combination to detect the APM symbol. On the other hand, with original GPSM, APM is detected by comparing with all the possible APM symbols, which may severely increase the complexity for higher APM order schemes.

Fig. 2 shows the comparisons between the original GPSM/ML and the proposed LR-GPSM using linear decoding scheme (LR-GPSM/LD) in terms of computational complexity. Both systems (GPSM/ML and LR-GPSM/LD) are analysed using different number of active receivers n_r. The proposed decoding scheme requires less operations because it only utilises a matrix multiplication, a sort function and the quantisation operator. On the other hand, the GPSM/ML decoding method uses the ML algorithm at both steps. For this reason, the decoding process of LR-GPSM/LD is faster than GPSM/ML. Another aspect to highlight is that the complexity is higher when the number of possible patterns c increases because the decoder has to compare all the possible combinations. This has a bigger impact on the ML scheme than a linear scheme. It should be emphasised that low complexity detection is significantly important for the potential application in future wireless systems such as V2X communications, where the latency must be low for "safety of life" applications [4].

IV. SIMULATION RESULTS

In this section, we present the proposed LR-GPSM results in comparison with the original GPSM scheme [8]. Monte Carlo simulations are performed at least 10^6 channel realizations. For comparison, we also assume flat Rayleigh fading channel. We target $\eta = 4$ bits/s/Hz transmission, QPSK modulation and with the following antenna configuration: $N_t = 8$, $N_r = 4$ and $n_r = 3$.

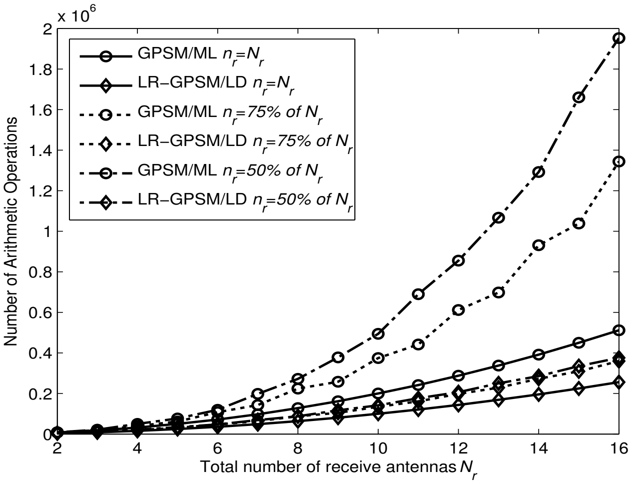

Fig. 2: Decoding processing computational time of GPSM/ML vs LR-GPSM/LD.

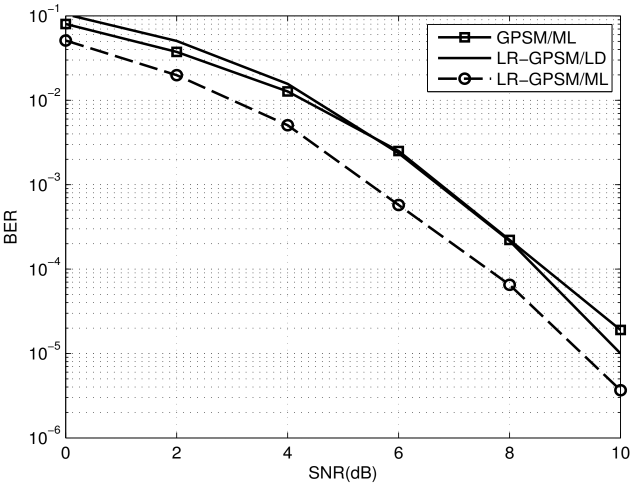

Fig. 3: BER performance of GPSM versus SNR, for $N_t = 8$, $N_r = 4$ and $n_r = 3$.

Fig. 3 shows the BER performance of the GPSM/ML [8] in comparison with the proposed LR-GPSM using ML detection (LR-GPSM/ML) and LR-GPSM/LD.

As seen in the figure, LR-GPSM/ML produces a gain of more than 1 dB over the GPSM/ML in terms of BER, showing the effectiveness of LR-based pre-coding. The LR-GPSM/LD has a performance comparable to that of the GPSM/ML but with significant lower complexity. At low SNR the GPSM/ML performs sightly better than LR-GPSM/LD. This is because the impact of noise at low SNRs is bigger and will lead to wrong estimation of the indices and worse performance than that of the ML detector.

V. CONCLUSION

In this manuscript, we derive a novel LR-based method to design the pre-coding matrix for the pre-coding aided spatial modulation schemes and a sub-optimal detector. The simulation results demonstrate the performance gain due to the new pre-coder if the ML detection is used as in the original GPSM and show that LR-GPSM/LD provides a performance comparable to that of the original GPSM [8] but with a significant complexity reduction at the decoding. Therefore, as in V2X communications, performance and reliability are extremely important to cope the system requirements. The proposed scheme can be a good candidate for being utilised in this wireless systems.

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