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# Reactive Designs in Isabelle/UTP

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April 6, 2018

## Abstract

Reactive designs combine the UTP theories of reactive processes and designs to characterise reactive programs. Whereas sequential imperative programs are expected to run until termination, reactive programs pause at instances to allow interaction with the environment using abstract events, and often do not terminate at all. Thus, whereas a design describes the precondition and postcondition for a program, to characterise initial and final states, a reactive design also has a “pericondition”, which characterises intermediate quiescent observations. This gives rise to a notion of “reactive contract”, which specifies the assumptions a program makes of its environment, and the guarantees it will make of its own behaviour in both intermediate and final observations. This Isabelle/UTP document mechanises the UTP theory of reactive designs, including its healthiness conditions, signature, and a large library of algebraic laws of reactive programming.

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## 1 Introduction

This document contains a mechanisation in Isabelle/UTP [2] of our theory of reactive designs. Reactive designs form an important semantic foundation for reactive modelling languages such as Circus [3]. For more details of this work, please see our recent paper [1].

## 2 Reactive Designs Healthiness Conditions

```
theory utp-rdes-healths
  imports UTP-Reactive.utp-reactive
begin
```

### 2.1 Preliminaries

```
named-theorems rdes and rdes-def and RD-elim
```

```
type-synonym ('s,'t) rdes = ('s,'t,unit) hrel-rsp
```

```
translations
```

```
(type) ('s,'t) rdes <= (type) ('s, 't, unit) hrel-rsp
```

```
lemma R2-st-ex: R2 ( $\exists$  $st  $\cdot$  P) = ( $\exists$  $st  $\cdot$  R2(P))
  by (rel-auto)
```

```
lemma R2s-st'-eq-st:
  R2s($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R2c-st'-eq-st:
  R2c($st' =u $st) = ($st' =u $st)
  by (rel-auto)
```

```
lemma R1-des-lift-skip: R1( $\lceil$ II $\rceil_D$ ) =  $\lceil$ II $\rceil_D$ 
  by (rel-auto)
```

```
lemma R2-des-lift-skip:
  R2( $\lceil$ II $\rceil_D$ ) =  $\lceil$ II $\rceil_D$ 
  apply (rel-auto) using minus-zero-eq by blast
```

```
lemma R1-R2c-ex-st: R1 (R2c ( $\exists$  $st'  $\cdot$  Q1)) = ( $\exists$  $st'  $\cdot$  R1 (R2c Q1))
  by (rel-auto)
```

### 2.2 Identities

We define two identities fro reactive designs, which correspond to the regular and state-sensitive versions of reactive designs, respectively. The former is the one used in the UTP book and related publications for CSP.

```
definition skip-rea :: ('t::trace, 'α) hrel-rp (IIc) where
  skip-rea-def [urel-defs]: IIc = (II  $\vee$  ( $\neg$  $ok  $\wedge$  $tr  $\leq_u$  $tr'))
```

```
definition skip-srea :: ('s, 't::trace, 'α) hrel-rsp (IIR) where
```

*skip-srea-def* [*urel-defs*]:  $II_R = ((\exists \$st \cdot II_c) \triangleleft \$wait \triangleright II_c)$

**lemma** *skip-rea-R1-lemma*:  $II_c = R1(\$ok \Rightarrow II)$   
**by** (*rel-auto*)

**lemma** *skip-rea-form*:  $II_c = (II \triangleleft \$ok \triangleright R1(true))$   
**by** (*rel-auto*)

**lemma** *skip-srea-form*:  $II_R = ((\exists \$st \cdot II) \triangleleft \$wait \triangleright II) \triangleleft \$ok \triangleright R1(true)$   
**by** (*rel-auto*)

**lemma** *R1-skip-rea*:  $R1(II_c) = II_c$   
**by** (*rel-auto*)

**lemma** *R2c-skip-rea*:  $R2c II_c = II_c$   
**by** (*simp add: skip-rea-def R2c-and R2c-disj R2c-skip-r R2c-not R2c-ok R2c-tr'-ge-tr*)

**lemma** *R2-skip-rea*:  $R2(II_c) = II_c$   
**by** (*metis R1-R2c-is-R2 R1-skip-rea R2c-skip-rea*)

**lemma** *R2c-skip-srea*:  $R2c(II_R) = II_R$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *skip-srea-R1* [*closure*]:  $II_R$  is *R1*  
**by** (*rel-auto*)

**lemma** *skip-srea-R2c* [*closure*]:  $II_R$  is *R2c*  
**by** (*simp add: Healthy-def R2c-skip-srea*)

**lemma** *skip-srea-R2* [*closure*]:  $II_R$  is *R2*  
**by** (*metis Healthy-def' R1-R2c-is-R2 R2c-skip-srea skip-srea-R1*)

### 2.3 RD1: Divergence yields arbitrary traces

**definition** *RD1* ::  $(t::trace, \alpha, \beta) \text{ rel-rp} \Rightarrow (t, \alpha, \beta) \text{ rel-rp}$  **where**  
[*upred-defs*]:  $RD1(P) = (P \vee (\neg \$ok \wedge \$tr \leq_u \$tr'))$

*RD1* is essentially *H1* from the theory of designs, but viewed through the prism of reactive processes.

**lemma** *RD1-idem*:  $RD1(RD1(P)) = RD1(P)$   
**by** (*rel-auto*)

**lemma** *RD1-Idempotent*: *Idempotent RD1*  
**by** (*simp add: Idempotent-def RD1-idem*)

**lemma** *RD1-mono*:  $P \sqsubseteq Q \Longrightarrow RD1(P) \sqsubseteq RD1(Q)$   
**by** (*rel-auto*)

**lemma** *RD1-Monotonic*: *Monotonic RD1*  
**using** *mono-def RD1-mono* **by** *blast*

**lemma** *RD1-Continuous*: *Continuous RD1*  
**by** (*rel-auto*)

**lemma** *R1-true-RD1-closed* [*closure*]:  $R1(true)$  is *RD1*

by (rel-auto)

**lemma** *RD1-wait-false* [closure]:  $P$  is RD1  $\implies P\llbracket\text{false}/\text{\$wait}\rrbracket$  is RD1  
by (rel-auto)

**lemma** *RD1-wait'-false* [closure]:  $P$  is RD1  $\implies P\llbracket\text{false}/\text{\$wait}'\rrbracket$  is RD1  
by (rel-auto)

**lemma** *RD1-seq*:  $RD1(RD1(P) ;; RD1(Q)) = RD1(P) ;; RD1(Q)$   
by (rel-auto)

**lemma** *RD1-seq-closure* [closure]:  $\llbracket P$  is RD1;  $Q$  is RD1  $\rrbracket \implies P ;; Q$  is RD1  
by (metis Healthy-def' RD1-seq)

**lemma** *RD1-R1-commute*:  $RD1(R1(P)) = R1(RD1(P))$   
by (rel-auto)

**lemma** *RD1-R2c-commute*:  $RD1(R2c(P)) = R2c(RD1(P))$   
by (rel-auto)

**lemma** *RD1-via-R1*:  $R1(H1(P)) = RD1(R1(P))$   
by (rel-auto)

**lemma** *RD1-R1-cases*:  $RD1(R1(P)) = (R1(P) \triangleleft \text{\$ok} \triangleright R1(\text{true}))$   
by (rel-auto)

**lemma** *skip-rea-RD1-skip*:  $II_c = RD1(II)$   
by (rel-auto)

**lemma** *skip-srea-RD1* [closure]:  $II_R$  is RD1  
by (rel-auto)

**lemma** *RD1-algebraic-intro*:  
assumes  
   $P$  is R1  $(R1(\text{true}_h) ;; P) = R1(\text{true}_h) (II_c ;; P) = P$   
shows  $P$  is RD1  
proof –  
  have  $P = (II_c ;; P)$   
    by (simp add: assms(3))  
  also have  $\dots = (R1(\text{\$ok} \Rightarrow II) ;; P)$   
    by (simp add: skip-rea-R1-lemma)  
  also have  $\dots = (((\neg \text{\$ok} \wedge R1(\text{true})) ;; P) \vee P)$   
    by (metis (no-types, lifting) R1-def seqr-left-unit seqr-or-distl skip-rea-R1-lemma skip-rea-def utp-pred-laws.inf-top-left utp-pred-laws.sup-commute)  
  also have  $\dots = ((R1(\neg \text{\$ok}) ;; (R1(\text{true}_h) ;; P)) \vee P)$   
    using dual-order.trans by (rel-blast)  
  also have  $\dots = ((R1(\neg \text{\$ok}) ;; R1(\text{true}_h)) \vee P)$   
    by (simp add: assms(2))  
  also have  $\dots = (R1(\neg \text{\$ok}) \vee P)$   
    by (rel-auto)  
  also have  $\dots = RD1(P)$   
    by (rel-auto)  
  finally show ?thesis  
    by (simp add: Healthy-def)  
qed

**theorem** *RD1-left-zero*:  
**assumes** *P is R1 P is RD1*  
**shows**  $(R1(true) ;; P) = R1(true)$   
**proof** –  
**have**  $(R1(true) ;; R1(RD1(P))) = R1(true)$   
**by** *(rel-auto)*  
**thus** *?thesis*  
**by** *(simp add: Healthy-if assms(1) assms(2))*  
**qed**

**theorem** *RD1-left-unit*:  
**assumes** *P is R1 P is RD1*  
**shows**  $(II_c ;; P) = P$   
**proof** –  
**have**  $(II_c ;; R1(RD1(P))) = R1(RD1(P))$   
**by** *(rel-auto)*  
**thus** *?thesis*  
**by** *(simp add: Healthy-if assms(1) assms(2))*  
**qed**

**lemma** *RD1-alt-def*:  
**assumes** *P is R1*  
**shows**  $RD1(P) = (P \triangleleft \$ok \triangleright R1(true))$   
**proof** –  
**have**  $RD1(R1(P)) = (R1(P) \triangleleft \$ok \triangleright R1(true))$   
**by** *(rel-auto)*  
**thus** *?thesis*  
**by** *(simp add: Healthy-if assms)*  
**qed**

**theorem** *RD1-algebraic*:  
**assumes** *P is R1*  
**shows**  $P \text{ is } RD1 \iff (R1(true_h) ;; P) = R1(true_h) \wedge (II_c ;; P) = P$   
**using** *RD1-algebraic-intro RD1-left-unit RD1-left-zero assms* **by** *blast*

## 2.4 R3c and R3h: Reactive design versions of R3

**definition** *R3c* ::  $(t::trace, 'a) \text{ hrel-rp} \Rightarrow (t, 'a) \text{ hrel-rp}$  **where**  
 $[upred-defs]: R3c(P) = (II_c \triangleleft \$wait \triangleright P)$

**definition** *R3h* ::  $(s, t::trace, 'a) \text{ hrel-rsp} \Rightarrow (s, t, 'a) \text{ hrel-rsp}$  **where**  
 $R3h-def [upred-defs]: R3h(P) = ((\exists \$st \cdot II_c) \triangleleft \$wait \triangleright P)$

**lemma** *R3c-idem*:  $R3c(R3c(P)) = R3c(P)$   
**by** *(rel-auto)*

**lemma** *R3c-Idempotent*: *Idempotent R3c*  
**by** *(simp add: Idempotent-def R3c-idem)*

**lemma** *R3c-mono*:  $P \sqsubseteq Q \implies R3c(P) \sqsubseteq R3c(Q)$   
**by** *(rel-auto)*

**lemma** *R3c-Monotonic*: *Monotonic R3c*  
**by** *(simp add: mono-def R3c-mono)*

**lemma** *R3c-Continuous: Continuous R3c*  
**by** (*rel-auto*)

**lemma** *R3h-idem: R3h(R3h(P)) = R3h(P)*  
**by** (*rel-auto*)

**lemma** *R3h-Idempotent: Idempotent R3h*  
**by** (*simp add: Idempotent-def R3h-idem*)

**lemma** *R3h-mono: P ⊆ Q ⇒ R3h(P) ⊆ R3h(Q)*  
**by** (*rel-auto*)

**lemma** *R3h-Monotonic: Monotonic R3h*  
**by** (*simp add: mono-def R3h-mono*)

**lemma** *R3h-Continuous: Continuous R3h*  
**by** (*rel-auto*)

**lemma** *R3h-inf: R3h(P ⊓ Q) = R3h(P) ⊓ R3h(Q)*  
**by** (*rel-auto*)

**lemma** *R3h-UINF:*  
 $A \neq \{\} \implies R3h(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot R3h(P(i)))$   
**by** (*rel-auto*)

**lemma** *R3h-cond: R3h(P ◁ b ▷ Q) = (R3h(P) ◁ b ▷ R3h(Q))*  
**by** (*rel-auto*)

**lemma** *R3c-via-RD1-R3: RD1(R3(P)) = R3c(RD1(P))*  
**by** (*rel-auto*)

**lemma** *R3c-RD1-def: P is RD1 ⇒ R3c(P) = RD1(R3(P))*  
**by** (*simp add: Healthy-if R3c-via-RD1-R3*)

**lemma** *RD1-R3c-commute: RD1(R3c(P)) = R3c(RD1(P))*  
**by** (*rel-auto*)

**lemma** *R1-R3c-commute: R1(R3c(P)) = R3c(R1(P))*  
**by** (*rel-auto*)

**lemma** *R2c-R3c-commute: R2c(R3c(P)) = R3c(R2c(P))*  
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *R1-R3h-commute: R1(R3h(P)) = R3h(R1(P))*  
**by** (*rel-auto*)

**lemma** *R2c-R3h-commute: R2c(R3h(P)) = R3h(R2c(P))*  
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *RD1-R3h-commute: RD1(R3h(P)) = R3h(RD1(P))*  
**by** (*rel-auto*)

**lemma** *R3c-cancels-R3: R3c(R3(P)) = R3c(P)*  
**by** (*rel-auto*)



**lemma** *R3-cancels-R3c*:  $R3(R3c(P)) = R3(P)$

by (*rel-auto*)

**lemma** *R3h-cancels-R3c*:  $R3h(R3c(P)) = R3h(P)$

by (*rel-auto*)

**lemma** *R3c-semir-form*:

$(R3c(P) ;; R3c(R1(Q))) = R3c(P ;; R3c(R1(Q)))$

by (*rel-simp, safe, auto intro: order-trans*)

**lemma** *R3h-semir-form*:

$(R3h(P) ;; R3h(R1(Q))) = R3h(P ;; R3h(R1(Q)))$

by (*rel-simp, safe, auto intro: order-trans, blast+*)

**lemma** *R3c-seq-closure*:

assumes *P is R3c Q is R3c Q is R1*

shows  $(P ;; Q)$  is *R3c*

by (*metis Healthy-def' R3c-semir-form assms*)

**lemma** *R3h-seq-closure [closure]*:

assumes *P is R3h Q is R3h Q is R1*

shows  $(P ;; Q)$  is *R3h*

by (*metis Healthy-def' R3h-semir-form assms*)

**lemma** *R3c-R3-left-seq-closure*:

assumes *P is R3 Q is R3c*

shows  $(P ;; Q)$  is *R3c*

**proof** –

have  $(P ;; Q) = ((P ;; Q)[[true/\$wait]] \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis cond-var-split cond-var-subst-right in-var-uvar wait-vwb-lens*)

also have  $\dots = (((II \triangleleft \$wait \triangleright P) ;; Q)[[true/\$wait]] \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis Healthy-def' R3-def assms(1)*)

also have  $\dots = ((II[[true/\$wait]] ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$

by (*subst-tac*)

also have  $\dots = (((II \wedge \$wait') ;; Q) \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis (no-types, lifting) cond-def conj-pos-var-subst seqr-pre-var-out skip-var utp-pred-laws.inf-left-idem wait-vwb-lens*)

also have  $\dots = ((II[[true/\$wait']] ;; Q[[true/\$wait]]) \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)

also have  $\dots = ((II[[true/\$wait']] ;; (II_c \triangleleft \$wait \triangleright Q)[[true/\$wait]]) \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis Healthy-def' R3c-def assms(2)*)

also have  $\dots = ((II[[true/\$wait']] ;; II_c[[true/\$wait]]) \triangleleft \$wait \triangleright (P ;; Q))$

by (*subst-tac*)

also have  $\dots = (((II \wedge \$wait') ;; II_c) \triangleleft \$wait \triangleright (P ;; Q))$

by (*metis seqr-pre-transfer seqr-right-one-point true-alt-def uovar-convr upred-eq-true utp-rel.unrest-ouvar vwb-lens-mwb wait-vwb-lens*)

also have  $\dots = ((II ;; II_c) \triangleleft \$wait \triangleright (P ;; Q))$

by (*simp add: cond-def seqr-pre-transfer utp-rel.unrest-ouvar*)

also have  $\dots = (II_c \triangleleft \$wait \triangleright (P ;; Q))$

by (*simp*)

also have  $\dots = R3c(P ;; Q)$

by (*simp add: R3c-def*)

**finally show** *?thesis*

by (*simp add: Healthy-def'*)

qed

**lemma** *R3c-cases*:  $R3c(P) = ((II \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$   
by (*rel-auto*)

**lemma** *R3h-cases*:  $R3h(P) = (((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright P)$   
by (*rel-auto*)

**lemma** *R3h-form*:  $R3h(P) = II_R \triangleleft \$wait \triangleright P$   
by (*rel-auto*)

**lemma** *R3c-subst-wait*:  $R3c(P) = R3c(P_f)$   
by (*simp add: R3c-def cond-var-subst-right*)

**lemma** *R3h-subst-wait*:  $R3h(P) = R3h(P_f)$   
by (*simp add: R3h-cases cond-var-subst-right*)

**lemma** *skip-srea-R3h [closure]*:  $II_R$  is *R3h*  
by (*rel-auto*)

**lemma** *R3h-wait-true*:

assumes  $P$  is *R3h*

shows  $P_t = II_R t$

**proof** –

have  $P_t = (II_R \triangleleft \$wait \triangleright P)_t$

by (*metis Healthy-if R3h-form assms*)

also have  $\dots = II_R t$

by (*simp add: usubst*)

finally show *?thesis* .

qed

## 2.5 RD2: A reactive specification cannot require non-termination

**definition** *RD2* where

[*upred-defs*]:  $RD2(P) = H2(P)$

*RD2* is just *H2* since the type system will automatically have J identifying the reactive variables as required.

**lemma** *RD2-idem*:  $RD2(RD2(P)) = RD2(P)$   
by (*simp add: H2-idem RD2-def*)

**lemma** *RD2-Idempotent*: *Idempotent RD2*  
by (*simp add: Idempotent-def RD2-idem*)

**lemma** *RD2-mono*:  $P \sqsubseteq Q \implies RD2(P) \sqsubseteq RD2(Q)$   
by (*simp add: H2-def RD2-def seqr-mono*)

**lemma** *RD2-Monotonic*: *Monotonic RD2*  
using *mono-def RD2-mono* by *blast*

**lemma** *RD2-Continuous*: *Continuous RD2*  
by (*rel-auto*)

**lemma** *RD1-RD2-commute*:  $RD1(RD2(P)) = RD2(RD1(P))$   
by (*rel-auto*)

**lemma** *RD2-R3c-commute*:  $RD2(R3c(P)) = R3c(RD2(P))$   
 by (*rel-auto*)

**lemma** *RD2-R3h-commute*:  $RD2(R3h(P)) = R3h(RD2(P))$   
 by (*rel-auto*)

## 2.6 Major healthiness conditions

**definition** *RH* ::  $(t::trace, 'α) hrel-rp \Rightarrow (t, 'α) hrel-rp$  (**R**)  
 where [*upred-defs*]:  $RH(P) = R1(R2c(R3c(P)))$

**definition** *RHS* ::  $(s, t::trace, 'α) hrel-rsp \Rightarrow (s, t, 'α) hrel-rsp$  (**R<sub>s</sub>**)  
 where [*upred-defs*]:  $RHS(P) = R1(R2c(R3h(P)))$

**definition** *RD* ::  $(t::trace, 'α) hrel-rp \Rightarrow (t, 'α) hrel-rp$   
 where [*upred-defs*]:  $RD(P) = RD1(RD2(RP(P)))$

**definition** *SRD* ::  $(s, t::trace, 'α) hrel-rsp \Rightarrow (s, t, 'α) hrel-rsp$   
 where [*upred-defs*]:  $SRD(P) = RD1(RD2(RHS(P)))$

**lemma** *RH-comp*:  $RH = R1 \circ R2c \circ R3c$   
 by (*auto simp add: RH-def*)

**lemma** *RHS-comp*:  $RHS = R1 \circ R2c \circ R3h$   
 by (*auto simp add: RHS-def*)

**lemma** *RD-comp*:  $RD = RD1 \circ RD2 \circ RP$   
 by (*auto simp add: RD-def*)

**lemma** *SRD-comp*:  $SRD = RD1 \circ RD2 \circ RHS$   
 by (*auto simp add: SRD-def*)

**lemma** *RH-idem*:  $\mathbf{R}(\mathbf{R}(P)) = \mathbf{R}(P)$   
 by (*simp add: R1-R2c-commute R1-R3c-commute R1-idem R2c-R3c-commute R2c-idem R3c-idem RH-def*)

**lemma** *RH-Idempotent*: *Idempotent R*  
 by (*simp add: Idempotent-def RH-idem*)

**lemma** *RH-Monotonic*: *Monotonic R*  
 by (*metis (no-types, lifting) R1-Monotonic R2c-Monotonic R3c-mono RH-def mono-def*)

**lemma** *RH-Continuous*: *Continuous R*  
 by (*simp add: Continuous-comp R1-Continuous R2c-Continuous R3c-Continuous RH-comp*)

**lemma** *RHS-idem*:  $\mathbf{R}_s(\mathbf{R}_s(P)) = \mathbf{R}_s(P)$   
 by (*simp add: R1-R2c-is-R2 R1-R3h-commute R2-idem R2c-R3h-commute R3h-idem RHS-def*)

**lemma** *RHS-Idempotent [closure]*: *Idempotent R<sub>s</sub>*  
 by (*simp add: Idempotent-def RHS-idem*)

**lemma** *RHS-Monotonic*: *Monotonic R<sub>s</sub>*  
 by (*simp add: mono-def R1-R2c-is-R2 R2-mono R3h-mono RHS-def*)

**lemma** *RHS-mono*:  $P \sqsubseteq Q \implies \mathbf{R}_s(P) \sqsubseteq \mathbf{R}_s(Q)$

**using** *mono-def RHS-Monotonic* **by** *blast*

**lemma** *RHS-Continuous [closure]: Continuous*  $\mathbf{R}_s$   
**by** (*simp add: Continuous-comp R1-Continuous R2c-Continuous R3h-Continuous RHS-comp*)

**lemma** *RHS-inf:  $\mathbf{R}_s(P \sqcap Q) = \mathbf{R}_s(P) \sqcap \mathbf{R}_s(Q)$*   
**using** *Continuous-Disjunctous Disjunctuous-def RHS-Continuous* **by** *auto*

**lemma** *RHS-INF:*  
 $A \neq \{\} \implies \mathbf{R}_s(\bigsqcap i \in A \cdot P(i)) = (\bigsqcap i \in A \cdot \mathbf{R}_s(P(i)))$   
**by** (*simp add: RHS-def R3h-UINF R2c-USUP R1-USUP*)

**lemma** *RHS-sup:  $\mathbf{R}_s(P \sqcup Q) = \mathbf{R}_s(P) \sqcup \mathbf{R}_s(Q)$*   
**by** (*rel-auto*)

**lemma** *RHS-SUP:*  
 $A \neq \{\} \implies \mathbf{R}_s(\bigsqcup i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot \mathbf{R}_s(P(i)))$   
**by** (*rel-auto*)

**lemma** *RHS-cond:  $\mathbf{R}_s(P \triangleleft b \triangleright Q) = (\mathbf{R}_s(P) \triangleleft R2c\ b \triangleright \mathbf{R}_s(Q))$*   
**by** (*simp add: RHS-def R3h-cond R2c-condr R1-cond*)

**lemma** *RD-alt-def:  $RD(P) = RD1(RD2(\mathbf{R}(P)))$*   
**by** (*simp add: R3c-via-RD1-R3 RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RD1-RD2-commute RH-def RD-def RP-def*)

**lemma** *RD1-RH-commute:  $RD1(\mathbf{R}(P)) = \mathbf{R}(RD1(P))$*   
**by** (*simp add: RD1-R1-commute RD1-R2c-commute RD1-R3c-commute RH-def*)

**lemma** *RD2-RH-commute:  $RD2(\mathbf{R}(P)) = \mathbf{R}(RD2(P))$*   
**by** (*metis R1-H2-commute R2c-H2-commute RD2-R3c-commute RD2-def RH-def*)

**lemma** *RD-idem:  $RD(RD(P)) = RD(P)$*   
**by** (*simp add: RD-alt-def RD1-RH-commute RD2-RH-commute RD1-RD2-commute RD2-idem RD1-idem RH-idem*)

**lemma** *RD-Monotonic: Monotonic* *RD*  
**by** (*simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic RD-comp RP-Monotonic*)

**lemma** *RD-Continuous: Continuous* *RD*  
**by** (*simp add: Continuous-comp RD1-Continuous RD2-Continuous RD-comp RP-Continuous*)

**lemma** *R3-RD-RP:  $R3(RD(P)) = RP(RD1(RD2(P)))$*   
**by** (*metis (no-types, lifting) R1-R2c-is-R2 R2-R3-commute R3-cancels-R3c RD1-RH-commute RD2-RH-commute RD-alt-def RH-def RP-def*)

**lemma** *RD1-RHS-commute:  $RD1(\mathbf{R}_s(P)) = \mathbf{R}_s(RD1(P))$*   
**by** (*simp add: RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RHS-def*)

**lemma** *RD2-RHS-commute:  $RD2(\mathbf{R}_s(P)) = \mathbf{R}_s(RD2(P))$*   
**by** (*metis R1-H2-commute R2c-H2-commute RD2-R3h-commute RD2-def RHS-def*)

**lemma** *SRD-idem:  $SRD(SRD(P)) = SRD(P)$*   
**by** (*simp add: RD1-RD2-commute RD1-RHS-commute RD1-idem RD2-RHS-commute RD2-idem RHS-idem SRD-def*)

**lemma** *SRD-Idempotent* [*closure*]: *Idempotent SRD*  
 by (*simp add: Idempotent-def SRD-idem*)

**lemma** *SRD-Monotonic*: *Monotonic SRD*  
 by (*simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic RHS-Monotonic SRD-comp*)

**lemma** *SRD-Continuous* [*closure*]: *Continuous SRD*  
 by (*simp add: Continuous-comp RD1-Continuous RD2-Continuous RHS-Continuous SRD-comp*)

**lemma** *SRD-RHS-H1-H2*:  $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$   
 by (*rel-auto*)

**lemma** *SRD-healths* [*closure*]:  
 assumes *P is SRD*  
 shows *P is R1 P is R2 P is R3h P is RD1 P is RD2*  
 apply (*metis Healthy-def R1-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)  
 apply (*metis Healthy-def R1-R2c-is-R2 R2-idem RD1-RHS-commute RD2-RHS-commute RHS-def SRD-def assms*)  
 apply (*metis Healthy-def R1-R3h-commute R2c-R3h-commute R3h-idem RD1-R3h-commute RD2-R3h-commute RHS-def SRD-def assms*)  
 apply (*metis Healthy-def' RD1-idem SRD-def assms*)  
 apply (*metis Healthy-def' RD1-RD2-commute RD2-idem SRD-def assms*)  
 done

**lemma** *SRD-intro*:  
 assumes *P is R1 P is R2 P is R3h P is RD1 P is RD2*  
 shows *P is SRD*  
 by (*metis Healthy-def R1-R2c-is-R2 RHS-def SRD-def assms(2) assms(3) assms(4) assms(5)*)

**lemma** *SRD-ok-false* [*usubst*]:  $P \text{ is } SRD \implies P \llbracket \text{false}/\$ok \rrbracket = R1(\text{true})$   
 by (*metis (no-types, hide-lams) H1-H2-eq-design Healthy-def R1-ok-false RD1-R1-commute RD1-via-R1 RD2-def SRD-def SRD-healths(1) design-ok-false*)

**lemma** *SRD-ok-true-wait-true* [*usubst*]:  
 assumes *P is SRD*  
 shows  $P \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$   
**proof** –  
 have  $P = (\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P$   
 by (*metis Healthy-def R3h-cases SRD-healths(3) assms*)  
 moreover have  $((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1 \text{ true} \triangleleft \$wait \triangleright P) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$   
 by (*simp add: usubst*)  
 ultimately show *?thesis*  
 by (*simp*)  
**qed**

**lemma** *SRD-left-zero-1*:  $P \text{ is } SRD \implies R1(\text{true}) ;; P = R1(\text{true})$   
 by (*simp add: RD1-left-zero SRD-healths(1) SRD-healths(4)*)

**lemma** *SRD-left-zero-2*:  
 assumes *P is SRD*  
 shows  $(\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket ;; P = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$   
**proof** –  
 have  $(\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket ;; R3h(P) = (\exists \$st \cdot II) \llbracket \text{true}, \text{true}/\$ok, \$wait \rrbracket$

by (*rel-auto*)  
 thus *?thesis*  
 by (*simp add: Healthy-if SRD-healths(3) assms*)  
 qed

## 2.7 UTP theories

We create two theory objects: one for reactive designs and one for stateful reactive designs.

typedecl *RDES*  
 typedecl *SRDES*

abbreviation *RDES*  $\equiv$  *UTHY*(*RDES*, ('*t*::*trace*, ' $\alpha$ ) *rp*)  
 abbreviation *SRDES*  $\equiv$  *UTHY*(*SRDES*, ('*s*, '*t*::*trace*, ' $\alpha$ ) *rsp*)

### overloading

*rdes-hcond* == *utp-hcond* :: (*RDES*, ('*t*::*trace*, ' $\alpha$ ) *rp*) *uthy*  $\Rightarrow$  (('*t*, ' $\alpha$ ) *rp*  $\times$  ('*t*, ' $\alpha$ ) *rp*) *health*  
*srdes-hcond* == *utp-hcond* :: (*SRDES*, ('*s*, '*t*::*trace*, ' $\alpha$ ) *rsp*) *uthy*  $\Rightarrow$  (('*s*, '*t*, ' $\alpha$ ) *rsp*  $\times$  ('*s*, '*t*, ' $\alpha$ ) *rsp*) *health*

### begin

definition *rdes-hcond* :: (*RDES*, ('*t*::*trace*, ' $\alpha$ ) *rp*) *uthy*  $\Rightarrow$  (('*t*, ' $\alpha$ ) *rp*  $\times$  ('*t*, ' $\alpha$ ) *rp*) *health* **where**  
 [*upred-defs*]: *rdes-hcond* *T* = *RD*

definition *srdes-hcond* :: (*SRDES*, ('*s*, '*t*::*trace*, ' $\alpha$ ) *rsp*) *uthy*  $\Rightarrow$  (('*s*, '*t*, ' $\alpha$ ) *rsp*  $\times$  ('*s*, '*t*, ' $\alpha$ ) *rsp*) *health*  
**where**

[*upred-defs*]: *srdes-hcond* *T* = *SRD*

### end

interpretation *rdes-theory*: *utp-theory* *UTHY*(*RDES*, ('*t*::*trace*, ' $\alpha$ ) *rp*)

by (*unfold-locales*, *simp-all add: rdes-hcond-def RD-idem*)

interpretation *rdes-theory-continuous*: *utp-theory-continuous* *UTHY*(*RDES*, ('*t*::*trace*, ' $\alpha$ ) *rp*)

rewrites  $\bigwedge P. P \in \text{carrier } (\text{uthy-order } RDES) \longleftrightarrow P \text{ is } RD$

and  $\text{carrier } (\text{uthy-order } RDES) \rightarrow \text{carrier } (\text{uthy-order } RD) \equiv \llbracket RD \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

and  $le (\text{uthy-order } RDES) = op \sqsubseteq$

and  $eq (\text{uthy-order } RDES) = op =$

by (*unfold-locales*, *simp-all add: rdes-hcond-def RD-Continuous*)

interpretation *rdes-rea-galois*:

*galois-connection* (*RDES*  $\leftarrow$  (*RD1*  $\circ$  *RD2*, *R3*)  $\rightarrow$  *REA*)

**proof** (*simp add: mk-conn-def*, *rule galois-connectionI'*, *simp-all add: utp-partial-order rdes-hcond-def rea-hcond-def*)

show  $R3 \in \llbracket RD \rrbracket_H \rightarrow \llbracket RP \rrbracket_H$

by (*metis* (*no-types*, *lifting*) *Healthy-def' Pi-I R3-RD-RP RP-idem mem-Collect-eq*)

show  $RD1 \circ RD2 \in \llbracket RP \rrbracket_H \rightarrow \llbracket RD \rrbracket_H$

by (*simp add: Pi-iff Healthy-def*, *metis RD-def RD-idem*)

show *isotone* (*utp-order* *RD*) (*utp-order* *RP*) *R3*

by (*simp add: R3-Monotonic isotone-utp-orderI*)

show *isotone* (*utp-order* *RP*) (*utp-order* *RD*) (*RD1*  $\circ$  *RD2*)

by (*simp add: Monotonic-comp RD1-Monotonic RD2-Monotonic isotone-utp-orderI*)

fix *P* :: ('*a*, '*b*) *hrel-rp*

assume *P* is *RD*

thus  $P \sqsubseteq RD1 (RD2 (R3 P))$

by (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-iff*)

### next

fix *P* :: ('*a*, '*b*) *hrel-rp*

assume *a*: *P* is *RP*

**thus**  $R3 (RD1 (RD2 P)) \sqsubseteq P$   
**proof** –  
 have  $R3 (RD1 (RD2 P)) = RP (RD1 (RD2(P)))$   
 by (*metis Healthy-if R3-RD-RP RD-def a*)  
**moreover** have  $RD1(RD2(P)) \sqsubseteq P$   
 by (*rel-auto*)  
**ultimately show** *?thesis*  
 by (*metis Healthy-if RP-mono a*)  
**qed**  
**qed**

**interpretation** *rdes-rea-retract*:  
*retract (RDES  $\leftarrow$  (RD1  $\circ$  RD2, R3)  $\rightarrow$  REA)*  
**by** (*unfold-locales, simp-all add: mk-conn-def utp-partial-order rdes-hcond-def rea-hcond-def*)  
 (*metis Healthy-if R3-RD-RP RD-def RP-idem eq-refl*)

**interpretation** *srdes-theory*: *utp-theory UTHY(SRDES, ('s, 't::trace, 'α) rsp)*  
**by** (*unfold-locales, simp-all add: srdes-hcond-def SRD-idem*)

**interpretation** *srdes-theory-continuous*: *utp-theory-continuous UTHY(SRDES, ('s, 't::trace, 'α) rsp)*  
**rewrites**  $\bigwedge P. P \in \text{carrier (uthy-order SRDES)} \longleftrightarrow P \text{ is SRD}$   
**and**  $P \text{ is } \mathcal{H}_{SRDES} \longleftrightarrow P \text{ is SRD}$   
**and**  $(\mu X \cdot F (\mathcal{H}_{SRDES} X)) = (\mu X \cdot F (SRD X))$   
**and**  $\text{carrier (uthy-order SRDES)} \rightarrow \text{carrier (uthy-order SRDES)} \equiv \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$   
**and**  $\llbracket \mathcal{H}_{SRDES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{SRDES} \rrbracket_H \equiv \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$   
**and**  $le \text{ (uthy-order SRDES)} = op \sqsubseteq$   
**and**  $eq \text{ (uthy-order SRDES)} = op =$   
**by** (*unfold-locales, simp-all add: srdes-hcond-def SRD-Continuous*)

**declare** *srdes-theory-continuous.top-healthy* [*simp del*]  
**declare** *srdes-theory-continuous.bottom-healthy* [*simp del*]

**abbreviation** *Chaos* :: ('s, 't::trace, 'α) *hrel-rsp* **where**  
*Chaos*  $\equiv \perp_{SRDES}$

**abbreviation** *Miracle* :: ('s, 't::trace, 'α) *hrel-rsp* **where**  
*Miracle*  $\equiv \top_{SRDES}$

**thm** *srdes-theory-continuous.weak.bottom-lower*  
**thm** *srdes-theory-continuous.weak.top-higher*  
**thm** *srdes-theory-continuous.meet-bottom*  
**thm** *srdes-theory-continuous.meet-top*

**abbreviation** *srd-lfp* ( $\mu_R$ ) **where**  $\mu_R F \equiv \mu_{SRDES} F$

**abbreviation** *srd-gfp* ( $\nu_R$ ) **where**  $\nu_R F \equiv \nu_{SRDES} F$

**syntax**  
*-srd-mu* :: *pttrn*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* ( $\mu_R \cdot \cdot \cdot [0, 10] 10$ )  
*-srd-nu* :: *pttrn*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* ( $\nu_R \cdot \cdot \cdot [0, 10] 10$ )

**translations**  
 $\mu_R X \cdot P == \mu_R (\lambda X. P)$   
 $\nu_R X \cdot P == \nu_R (\lambda X. P)$

The reactive design weakest fixed-point can be defined in terms of relational calculus one.

**lemma** *srd-mu-equiv*:  
**assumes** *Monotonic F*  $F \in \llbracket SRD \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$   
**shows**  $(\mu_R X \cdot F(X)) = (\mu X \cdot F(SRD(X)))$   
**by** (*metis assms srdes-hcond-def srdes-theory-continuous.utp-lfp-def*)

**end**

### 3 Reactive Design Specifications

**theory** *utp-rdes-designs*  
**imports** *utp-rdes-healths*  
**begin**

#### 3.1 Reactive design forms

**lemma** *srdes-skip-def*:  $II_R = \mathbf{R}_s(\text{true} \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \lceil II \rceil_R))$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *Chaos-def*:  $\text{Chaos} = \mathbf{R}_s(\text{false} \vdash \text{true})$   
**proof** –  
**have**  $\text{Chaos} = SRD(\text{true})$   
**by** (*metis srdes-hcond-def srdes-theory-continuous.healthy-bottom*)  
**also have**  $\dots = \mathbf{R}_s(\mathbf{H}(\text{true}))$   
**by** (*simp add: SRD-RHS-H1-H2*)  
**also have**  $\dots = \mathbf{R}_s(\text{false} \vdash \text{true})$   
**by** (*metis H1-design H2-true design-false-pre*)  
**finally show** *?thesis* .  
**qed**

**lemma** *Miracle-def*:  $\text{Miracle} = \mathbf{R}_s(\text{true} \vdash \text{false})$   
**proof** –  
**have**  $\text{Miracle} = SRD(\text{false})$   
**by** (*metis srdes-hcond-def srdes-theory-continuous.healthy-top*)  
**also have**  $\dots = \mathbf{R}_s(\mathbf{H}(\text{false}))$   
**by** (*simp add: SRD-RHS-H1-H2*)  
**also have**  $\dots = \mathbf{R}_s(\text{true} \vdash \text{false})$   
**by** (*metis (no-types, lifting) H1-H2-eq-design p-imp-p subst-impl subst-not utp-pred-laws.compl-bot-eq utp-pred-laws.compl-top-eq*)  
**finally show** *?thesis* .  
**qed**

**lemma** *RD1-reactive-design*:  $RD1(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RD2-reactive-design*:  
**assumes**  $\$ok' \# P \ \$ok' \# Q$   
**shows**  $RD2(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
**using** *assms*  
**by** (*metis H2-design RD2-RH-commute RD2-def*)

**lemma** *RD1-st-reactive-design*:  $RD1(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$   
**by** (*rel-auto*)

**lemma** *RD2-st-reactive-design*:  
**assumes**  $\$ok' \# P \ \$ok' \# Q$



shows  $RD2(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$   
 using *assms*  
 by (*metis H2-design RD2-RHS-commute RD2-def*)

**lemma** *wait-false-design*:  
 $(P \vdash Q)_f = ((P_f) \vdash (Q_f))$   
 by (*rel-auto*)

**lemma** *RD-RH-design-form*:  
 $RD(P) = \mathbf{R}((\neg P^f_f) \vdash P^t_f)$

**proof** –

have  $RD(P) = RD1(RD2(R1(R2c(R3c(P)))))$   
 by (*simp add: RD-alt-def RH-def*)  
 also have  $\dots = RD1(H2(R1(R2s(R3c(P)))))$   
 by (*simp add: R1-R2s-R2c RD2-def*)  
 also have  $\dots = RD1(R1(H2(R2s(R3c(P)))))$   
 by (*simp add: R1-H2-commute*)  
 also have  $\dots = R1(H1(R1(H2(R2s(R3c(P))))))$   
 by (*simp add: R1-idem RD1-via-R1*)  
 also have  $\dots = R1(H1(H2(R2s(R3c(R1(P))))))$   
 by (*simp add: R1-H2-commute R1-R2c-commute R1-R2s-R2c R1-R3c-commute RD1-via-R1*)  
 also have  $\dots = R1(R2s(H1(H2(R3c(R1(P))))))$   
 by (*simp add: R2s-H1-commute R2s-H2-commute*)  
 also have  $\dots = R1(R2s(H1(R3c(H2(R1(P))))))$   
 by (*metis RD2-R3c-commute RD2-def*)  
 also have  $\dots = R2(R1(H1(R3c(H2(R1(P))))))$   
 by (*metis R1-R2-commute R1-idem R2-def*)  
 also have  $\dots = R2(R3c(R1(\mathbf{H}(R1(P)))))$   
 by (*simp add: R1-R3c-commute RD1-R3c-commute RD1-via-R1*)  
 also have  $\dots = RH(\mathbf{H}(R1(P)))$   
 by (*metis R1-R2s-R2c R1-R3c-commute R2-R1-form RH-def*)  
 also have  $\dots = RH(\mathbf{H}(P))$   
 by (*simp add: R1-H2-commute R1-R2c-commute R1-R3c-commute R1-idem RD1-via-R1 RH-def*)  
 also have  $\dots = RH((\neg P^f) \vdash P^t)$   
 by (*simp add: H1-H2-eq-design*)  
 also have  $\dots = \mathbf{R}((\neg P^f_f) \vdash P^t_f)$   
 by (*metis (no-types, lifting) R3c-subst-wait RH-def subst-not wait-false-design*)  
 finally show *?thesis* .

qed

**lemma** *RD-reactive-design*:  
 assumes *P is RD*  
 shows  $\mathbf{R}((\neg P^f_f) \vdash P^t_f) = P$   
 by (*metis RD-RH-design-form Healthy-def' assms*)

**lemma** *RD-RH-design*:  
 assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $RD(\mathbf{R}(P \vdash Q)) = \mathbf{R}(P \vdash Q)$   
 by (*simp add: RD1-reactive-design RD2-reactive-design RD-alt-def RH-idem assms(1) assms(2)*)

**lemma** *RH-design-is-RD*:  
 assumes  $\$ok' \# P \ \$ok' \# Q$   
 shows  $\mathbf{R}(P \vdash Q)$  *is RD*  
 by (*simp add: RD-RH-design Healthy-def' assms(1) assms(2)*)

**lemma** *SRD-RH-design-form*:

$SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

**proof** –

**have**  $SRD(P) = R1(R2c(R3h(RD1(RD2(R1(P))))))$

**by** (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-commute R1-R3h-commute R1-idem R2c-H2-commute RD1-R1-commute RD1-R2c-commute RD1-R3h-commute RD2-R3h-commute RD2-def RHS-def SRD-def*)

**also have**  $\dots = R1(R2s(R3h(\mathbf{H}(P))))$

**by** (*metis* (*no-types*, *lifting*) *R1-H2-commute R1-R2c-is-R2 R1-R3h-commute R2-R1-form RD1-via-R1 RD2-def*)

**also have**  $\dots = \mathbf{R}_s(\mathbf{H}(P))$

**by** (*simp add*: *R1-R2s-R2c RHS-def*)

**also have**  $\dots = \mathbf{R}_s((\neg P^f) \vdash P^t)$

**by** (*simp add*: *H1-H2-eq-design*)

**also have**  $\dots = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

**by** (*metis* (*no-types*, *lifting*) *R3h-subst-wait RHS-def subst-not wait-false-design*)

**finally show** *?thesis* .

**qed**

**lemma** *SRD-reactive-design*:

**assumes** *P is SRD*

**shows**  $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = P$

**by** (*metis* *SRD-RH-design-form Healthy-def' assms*)

**lemma** *SRD-RH-design*:

**assumes**  $\$ok' \# P \ \$ok' \# Q$

**shows**  $SRD(\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(P \vdash Q)$

**by** (*simp add*: *RD1-st-reactive-design RD2-st-reactive-design RHS-idem SRD-def assms(1) assms(2)*)

**lemma** *RHS-design-is-SRD*:

**assumes**  $\$ok' \# P \ \$ok' \# Q$

**shows**  $\mathbf{R}_s(P \vdash Q)$  *is SRD*

**by** (*simp add*: *Healthy-def' SRD-RH-design assms(1) assms(2)*)

**lemma** *SRD-RHS-H1-H2*:  $SRD(P) = \mathbf{R}_s(\mathbf{H}(P))$

**by** (*metis* (*no-types*, *lifting*) *H1-H2-eq-design R3h-subst-wait RHS-def SRD-RH-design-form subst-not wait-false-design*)

## 3.2 Auxiliary healthiness conditions

**definition** [*upred-defs*]:  $R3c\text{-pre}(P) = (true \triangleleft \$wait \triangleright P)$

**definition** [*upred-defs*]:  $R3c\text{-post}(P) = ([II]_D \triangleleft \$wait \triangleright P)$

**definition** [*upred-defs*]:  $R3h\text{-post}(P) = ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright P)$

**lemma** *R3c-pre-conj*:  $R3c\text{-pre}(P \wedge Q) = (R3c\text{-pre}(P) \wedge R3c\text{-pre}(Q))$

**by** (*rel-auto*)

**lemma** *R3c-pre-seq*:

$(true ;; Q) = true \implies R3c\text{-pre}(P ;; Q) = (R3c\text{-pre}(P) ;; Q)$

**by** (*rel-auto*)

**lemma** *unrest-ok-R3c-pre* [*unrest*]:  $\$ok \# P \implies \$ok \# R3c\text{-pre}(P)$

**by** (*simp add*: *R3c-pre-def cond-def unrest*)

**lemma** *unrest-ok'-R3c-pre* [*unrest*]:  $\$ok' \# P \implies \$ok' \# R3c\text{-pre}(P)$

by (simp add: R3c-pre-def cond-def unrest)

**lemma** *unrest-ok-R3c-post* [unrest]:  $\$ok \# P \implies \$ok \# R3c\text{-post}(P)$

by (simp add: R3c-post-def cond-def unrest)

**lemma** *unrest-ok-R3c-post'* [unrest]:  $\$ok' \# P \implies \$ok' \# R3c\text{-post}(P)$

by (simp add: R3c-post-def cond-def unrest)

**lemma** *unrest-ok-R3h-post* [unrest]:  $\$ok \# P \implies \$ok \# R3h\text{-post}(P)$

by (simp add: R3h-post-def cond-def unrest)

**lemma** *unrest-ok-R3h-post'* [unrest]:  $\$ok' \# P \implies \$ok' \# R3h\text{-post}(P)$

by (simp add: R3h-post-def cond-def unrest)

### 3.3 Composition laws

**theorem** *R1-design-composition*:

fixes  $P Q :: ('t::trace, 'α, 'β) \text{rel-rp}$

and  $R S :: ('t, 'β, 'γ) \text{rel-rp}$

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows

$(R1(P \vdash Q) ;; R1(R \vdash S)) =$

$R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$

**proof** –

**have**  $(R1(P \vdash Q) ;; R1(R \vdash S)) = (\exists ok_0 \cdot (R1(P \vdash Q)) \llbracket \llcorner ok_0 \gg / \$ok' \rrbracket ;; (R1(R \vdash S)) \llbracket \llcorner ok_0 \gg / \$ok \rrbracket)$

using *seqr-middle ok-vwb-lens* by *blast*

**also from** *assms* **have**  $\dots = (\exists ok_0 \cdot R1((\$ok \wedge P) \Rightarrow (\llcorner ok_0 \gg \wedge Q)) ;; R1((\llcorner ok_0 \gg \wedge R) \Rightarrow (\$ok' \wedge S)))$

by (simp add: *design-def R1-def usubst unrest*)

**also from** *assms* **have**  $\dots = ((R1((\$ok \wedge P) \Rightarrow (true \wedge Q)) ;; R1((true \wedge R) \Rightarrow (\$ok' \wedge S)))$

$\vee (R1((\$ok \wedge P) \Rightarrow (false \wedge Q)) ;; R1((false \wedge R) \Rightarrow (\$ok' \wedge S)))$

by (simp add: *false-alt-def true-alt-def*)

**also from** *assms* **have**  $\dots = ((R1((\$ok \wedge P) \Rightarrow Q) ;; R1(R \Rightarrow (\$ok' \wedge S)))$

$\vee (R1(\neg (\$ok \wedge P)) ;; R1(true)))$

by *simp*

**also from** *assms* **have**  $\dots = ((R1(\neg \$ok \vee \neg P \vee Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$

$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (simp add: *impl-alt-def utp-pred-laws.sup.assoc*)

**also from** *assms* **have**  $\dots = (((R1(\neg \$ok \vee \neg P) \vee R1(Q)) ;; R1(\neg R \vee (\$ok' \wedge S)))$

$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (simp add: *R1-disj utp-pred-laws.disj-assoc*)

**also from** *assms* **have**  $\dots = ((R1(\neg \$ok \vee \neg P) ;; R1(\neg R \vee (\$ok' \wedge S)))$

$\vee (R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$

$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (simp add: *seqr-or-distl utp-pred-laws.sup.assoc*)

**also from** *assms* **have**  $\dots = ((R1(Q) ;; R1(\neg R \vee (\$ok' \wedge S)))$

$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (*rel-blast*)

**also from** *assms* **have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$

$\vee (R1(\neg \$ok \vee \neg P) ;; R1(true)))$

by (simp add: *R1-disj R1-extend-conj utp-pred-laws.inf-commute*)

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$

$\vee ((R1(\neg \$ok) :: ('t, 'α, 'β) \text{rel-rp}) ;; R1(true)))$

$\vee (R1(\neg P) ;; R1(true)))$

by (simp add: *R1-disj seqr-or-distl*)

**also have**  $\dots = ((R1(Q) ;; (R1(\neg R) \vee R1(S) \wedge \$ok'))$

$\vee (R1(\neg \$ok))$   
 $\vee (R1(\neg P) ;; R1(true))$

**proof** –

**have**  $((R1(\neg \$ok) ;; ('t, 'α, 'β) rel-rp) ;; R1(true)) =$   
      $(R1(\neg \$ok) ;; ('t, 'α, 'γ) rel-rp)$   
   **by** *(rel-auto)*  
   **thus** *?thesis*  
   **by** *simp*  
**qed**

**also have**  $... = ((R1(Q) ;; (R1(\neg R) \vee (R1(S \wedge \$ok'))))$   
      $\vee R1(\neg \$ok)$   
      $\vee (R1(\neg P) ;; R1(true)))$   
   **by** *(simp add: R1-extend-conj)*

**also have**  $... = ( (R1(Q) ;; (R1(\neg R)))$   
      $\vee (R1(Q) ;; (R1(S \wedge \$ok')))$   
      $\vee R1(\neg \$ok)$   
      $\vee (R1(\neg P) ;; R1(true)))$   
   **by** *(simp add: seqr-or-distr utp-pred-laws.sup.assoc)*

**also have**  $... = R1( (R1(Q) ;; (R1(\neg R)))$   
      $\vee (R1(Q) ;; (R1(S \wedge \$ok')))$   
      $\vee (\neg \$ok)$   
      $\vee (R1(\neg P) ;; R1(true)))$   
   **by** *(simp add: R1-disj R1-seqr)*

**also have**  $... = R1( (R1(Q) ;; (R1(\neg R)))$   
      $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$   
      $\vee (\neg \$ok)$   
      $\vee (R1(\neg P) ;; R1(true)))$   
   **by** *(rel-blast)*

**also have**  $... = R1(\neg(\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$   
      $\vee ((R1(Q) ;; R1(S)) \wedge \$ok')$   
   **by** *(rel-blast)*

**also have**  $... = R1((\$ok \wedge \neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; (R1(\neg R))))$   
      $\Rightarrow (\$ok' \wedge (R1(Q) ;; R1(S))))$   
   **by** *(simp add: impl-alt-def utp-pred-laws.inf-commute)*

**also have**  $... = R1((\neg (R1(\neg P) ;; R1(true)) \wedge \neg (R1(Q) ;; R1(\neg R))) \vdash (R1(Q) ;; R1(S)))$   
   **by** *(simp add: design-def)*  
**finally show** *?thesis* .  
**qed**

**theorem** *R1-design-composition-RR:*

**assumes** *P is RR Q is RR R is RR S is RR*

**shows**

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) wp_r false \wedge Q wp_r R) \vdash (Q ;; S))$

**apply** *(subst R1-design-composition)*

**apply** *(simp-all add: assms unrest wp-rea-def Healthy-if closure)*

**apply** *(rel-auto)*

**done**

**theorem** *R1-design-composition-RC:*

**assumes** *P is RC Q is RR R is RR S is RR*

**shows**

$(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q wp_r R) \vdash (Q ;; S))$

**by** *(simp add: R1-design-composition-RR assms unrest Healthy-if closure wp)*

**lemma** *R2s-design:*  $R2s(P \vdash Q) = (R2s(P) \vdash R2s(Q))$

by (simp add: R2s-def design-def usubst)

**lemma** *R2c-design*:  $R2c(P \vdash Q) = (R2c(P) \vdash R2c(Q))$

by (simp add: design-def impl-alt-def R2c-disj R2c-not R2c-ok R2c-and R2c-ok')

**lemma** *R1-R3c-design*:

$R1(R3c(P \vdash Q)) = R1(R3c-pre(P) \vdash R3c-post(Q))$

by (rel-auto)

**lemma** *R1-R3h-design*:

$R1(R3h(P \vdash Q)) = R1(R3c-pre(P) \vdash R3h-post(Q))$

by (rel-auto)

**lemma** *R3c-R1-design-composition*:

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows  $(R3c(R1(P \vdash Q)) ;; R3c(R1(R \vdash S))) =$

$R3c(R1((\neg (R1(\neg P)) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$   
 $\vdash (R1(Q) ;; ([II]_D \triangleleft \$wait \triangleright R1(S))))$

**proof** –

have 1:  $(\neg (R1(\neg R3c-pre P) ;; R1 true)) = (R3c-pre(\neg (R1(\neg P) ;; R1 true)))$

by (rel-auto)

have 2:  $(\neg (R1(R3c-post Q) ;; R1(\neg R3c-pre R))) = R3c-pre(\neg ((R1 Q \wedge \neg \$wait') ;; R1(\neg R)))$

by (rel-auto, blast+)

have 3:  $(R1(R3c-post Q) ;; R1(R3c-post S)) = R3c-post(R1 Q ;; ([II]_D \triangleleft \$wait \triangleright R1 S))$

by (rel-auto)

show ?thesis

apply (simp add: R3c-semir-form R1-R3c-commute[THEN sym] R1-R3c-design unrest)

apply (subst R1-design-composition)

apply (simp-all add: unrest assms R3c-pre-conj 1 2 3)

done

qed

**lemma** *R3h-R1-design-composition*:

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows  $(R3h(R1(P \vdash Q)) ;; R3h(R1(R \vdash S))) =$

$R3h(R1((\neg (R1(\neg P)) ;; R1(true)) \wedge \neg ((R1(Q) \wedge \neg \$wait') ;; R1(\neg R)))$   
 $\vdash (R1(Q) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(S))))$

**proof** –

have 1:  $(\neg (R1(\neg R3c-pre P) ;; R1 true)) = (R3c-pre(\neg (R1(\neg P) ;; R1 true)))$

by (rel-auto)

have 2:  $(\neg (R1(R3h-post Q) ;; R1(\neg R3c-pre R))) = R3c-pre(\neg ((R1 Q \wedge \neg \$wait') ;; R1(\neg R)))$

by (rel-auto, blast+)

have 3:  $(R1(R3h-post Q) ;; R1(R3h-post S)) = R3h-post(R1 Q ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 S))$

by (rel-auto, blast+)

show ?thesis

apply (simp add: R3h-semir-form R1-R3h-commute[THEN sym] R1-R3h-design unrest)

apply (subst R1-design-composition)

apply (simp-all add: unrest assms R3c-pre-conj 1 2 3)

done

qed

**lemma** *R2-design-composition*:

assumes  $\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$

shows  $(R2(P \vdash Q) ;; R2(R \vdash S)) =$

$R2((\neg (R1(\neg R2c P) ;; R1 true)) \wedge \neg (R1(R2c Q) ;; R1(\neg R2c R))) \vdash (R1(R2c Q) ;; R1$

( $R2c\ S$ ))

**apply** (*simp add: R2-R2c-def R2c-design R1-design-composition assms unrest R2c-not R2c-and R2c-disj R1-R2c-commute*[*THEN sym*] *R2c-idem R2c-R1-seq*)

**apply** (*metis (no-types, lifting) R2c-R1-seq R2c-not R2c-true*)  
**done**

**lemma** *RH-design-composition:*

**assumes**  $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$

**shows**  $(RH(P \vdash Q) ;; RH(R \vdash S)) =$

$RH((\neg (R1 (\neg R2s P) ;; R1\ true) \wedge \neg ((R1 (R2s Q) \wedge (\neg \$wait')) ;; R1 (\neg R2s R))) \vdash$   
 $(R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))))$

**proof** –

**have** 1:  $R2c (R1 (\neg R2s P) ;; R1\ true) = (R1 (\neg R2s P) ;; R1\ true)$

**proof** –

**have** 1:  $(R1 (\neg R2s P) ;; R1\ true) = (R1(R2 (\neg P) ;; R2\ true))$

**by** (*rel-auto*)

**have**  $R2c(R1(R2 (\neg P) ;; R2\ true)) = R2c(R1(R2 (\neg P) ;; R2\ true))$

**using** *R2c-not* **by** *blast*

**also have**  $\dots = R2(R2 (\neg P) ;; R2\ true)$

**by** (*metis R1-R2c-commute R1-R2c-is-R2*)

**also have**  $\dots = (R2 (\neg P) ;; R2\ true)$

**by** (*simp add: R2-seqr-distribute*)

**also have**  $\dots = (R1 (\neg R2s P) ;; R1\ true)$

**by** (*simp add: R2-def R2s-not R2s-true*)

**finally show** *?thesis*

**by** (*simp add: 1*)

**qed**

**have** 2:  $R2c ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$

**proof** –

**have**  $((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = R1 (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$

**by** (*rel-auto*)

**hence**  $R2c ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R)) = (R2 (Q \wedge \neg \$wait') ;; R2 (\neg R))$

**by** (*metis R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute*)

**also have**  $\dots = ((R1 (R2s Q) \wedge \neg \$wait') ;; R1 (\neg R2s R))$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

**have** 3:  $R2c((R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S)))) = (R1 (R2s Q) ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S)))$

**proof** –

**have**  $R2c(((R1 (R2s Q))\lceil true/\$wait' \rceil ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\lceil true/\$wait \rceil))$

$= ((R1 (R2s Q))\lceil true/\$wait' \rceil ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\lceil true/\$wait \rceil)$

**proof** –

**have**  $R2c(((R1 (R2s Q))\lceil true/\$wait' \rceil ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\lceil true/\$wait \rceil)) =$

$R2c(R1 (R2s (Q\lceil true/\$wait' \rceil))) ;; \lceil II \rceil_D \lceil true/\$wait \rceil$

**by** (*simp add: usubst cond-unit-T R1-def R2s-def*)

**also have**  $\dots = R2c(R2(Q\lceil true/\$wait' \rceil)) ;; R2(\lceil II \rceil_D \lceil true/\$wait \rceil)$

**by** (*metis R2-def R2-des-lift-skip R2-subst-wait-true*)

**also have**  $\dots = (R2(Q\lceil true/\$wait' \rceil)) ;; R2(\lceil II \rceil_D \lceil true/\$wait \rceil)$

**using** *R2c-seq* **by** *blast*

**also have**  $\dots = ((R1 (R2s Q))\lceil true/\$wait' \rceil ;; (\lceil II \rceil_D \triangleleft \$wait \triangleright R1 (R2s S))\lceil true/\$wait \rceil)$

**apply** (*simp add: usubst R2-des-lift-skip*)

**apply** (*metis R2-def R2-des-lift-skip R2-subst-wait'-true R2-subst-wait-true*)  
**done**  
**finally show** *?thesis* .  
**qed**  
**moreover have**  $R2c(((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket))$   
 $= ((R1 (R2s Q))\llbracket false/\$wait' \rrbracket ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s S))\llbracket false/\$wait \rrbracket)$   
**by** (*simp add: usubst cond-unit-F*)  
*(metis (no-types, hide-lams) R1-wait'-false R1-wait-false R2-def R2-subst-wait'-false R2-subst-wait-false*  
*R2c-seq)*  
**ultimately show** *?thesis*  
**proof** –  
**have**  $[II]_D \triangleleft \$wait \triangleright R1 (R2s S) = R2 ([II]_D \triangleleft \$wait \triangleright S)$   
**by** (*simp add: R1-R2c-is-R2 R1-R2s-R2c R2-condr' R2-des-lift-skip R2s-wait*)  
**then show** *?thesis*  
**by** (*simp add: R1-R2c-is-R2 R1-R2s-R2c R2c-seq*)  
**qed**  
**qed**  
**have**  $(R1(R2s(R3c(P \vdash Q))) ;; R1(R2s(R3c(R \vdash S)))) =$   
 $((R3c(R1(R2s(P) \vdash R2s(Q)))) ;; R3c(R1(R2s(R) \vdash R2s(S))))$   
**by** (*metis (no-types, hide-lams) R1-R2s-R2c R1-R3c-commute R2c-R3c-commute R2s-design*)  
**also have**  $\dots = R3c (R1 ((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait' ) ;; R1 (\neg R2s R))) \vdash$   
 $(R1 (R2s Q) ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s S))))$   
**by** (*simp add: R3c-R1-design-composition assms unrest*)  
**also have**  $\dots = R3c(R1(R2c((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge \neg \$wait' ) ;; R1 (\neg R2s R))) \vdash$   
 $(R1 (R2s Q) ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s S))))$   
**by** (*simp add: R2c-design R2c-and R2c-not 1 2 3*)  
**finally show** *?thesis*  
**by** (*simp add: R1-R2s-R2c R1-R3c-commute R2c-R3c-commute RH-def*)  
**qed**  
**lemma** *RHS-design-composition:*  
**assumes**  $\$ok' \# P \$ok' \# Q \$ok \# R \$ok \# S$   
**shows**  $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$   
 $\mathbf{R}_s((\neg (R1 (\neg R2s P) ;; R1 true) \wedge \neg ((R1 (R2s Q) \wedge (\neg \$wait' ) ;; R1 (\neg R2s R))) \vdash$   
 $(R1 (R2s Q) ;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright R1 (R2s S))))$   
**proof** –  
**have**  $1: R2c (R1 (\neg R2s P) ;; R1 true) = (R1 (\neg R2s P) ;; R1 true)$   
**proof** –  
**have**  $1: (R1 (\neg R2s P) ;; R1 true) = (R1(R2 (\neg P) ;; R2 true))$   
**by** (*rel-auto, blast*)  
**have**  $R2c(R1(R2 (\neg P) ;; R2 true)) = R2c(R1(R2 (\neg P) ;; R2 true))$   
**using** *R2c-not* **by** *blast*  
**also have**  $\dots = R2(R2 (\neg P) ;; R2 true)$   
**by** (*metis R1-R2c-commute R1-R2c-is-R2*)  
**also have**  $\dots = (R2 (\neg P) ;; R2 true)$   
**by** (*simp add: R2-seqr-distribute*)  
**also have**  $\dots = (R1 (\neg R2s P) ;; R1 true)$   
**by** (*simp add: R2-def R2s-not R2s-true*)  
**finally show** *?thesis*  
**by** (*simp add: 1*)  
**qed**

**have**  $2:R2c((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg R2s R)) = ((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg R2s R))$

**proof** –

**have**  $((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg R2s R)) = R1(R2(Q \wedge \neg \$wait')) ;; R2(\neg R)$   
**by** *(rel-auto, blast+)*

**hence**  $R2c((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg R2s R)) = (R2(Q \wedge \neg \$wait')) ;; R2(\neg R)$   
**by** *(metis (no-types, lifting) R1-R2c-commute R1-R2c-is-R2 R2-seqr-distribute)*

**also have**  $\dots = ((R1(R2s Q) \wedge \neg \$wait') ;; R1(\neg R2s R))$   
**by** *(rel-auto, blast+)*

**finally show** *?thesis* .

**qed**

**have**  $3:R2c((R1(R2s Q) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S)))) = (R1(R2s Q) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S)))$

**proof** –

**have**  $R2c(((R1(R2s Q))[true/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[true/\$wait])) = ((R1(R2s Q))[true/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[true/\$wait])$

**proof** –

**have**  $R2c(((R1(R2s Q))[true/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[true/\$wait])) = R2c(R1(R2s(Q[true/\$wait']))) ;; (\exists \$st \cdot [II]_D)[true/\$wait]$

**by** *(simp add: usubst cond-unit-T R1-def R2s-def)*

**also have**  $\dots = R2c(R2(Q[true/\$wait'])) ;; R2((\exists \$st \cdot [II]_D)[true/\$wait])$

**by** *(metis (no-types, lifting) R2-def R2-des-lift-skip R2-subst-wait-true R2-st-ex)*

**also have**  $\dots = (R2(Q[true/\$wait'])) ;; R2((\exists \$st \cdot [II]_D)[true/\$wait])$

**using** *R2c-seq* **by** *blast*

**also have**  $\dots = ((R1(R2s Q))[true/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[true/\$wait])$   
**apply** *(simp add: usubst R2-des-lift-skip)*

**apply** *(metis (no-types) R2-def R2-des-lift-skip R2-st-ex R2-subst-wait'-true R2-subst-wait-true)*

**done**

**finally show** *?thesis* .

**qed**

**moreover have**  $R2c(((R1(R2s Q))[false/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[false/\$wait])) = ((R1(R2s Q))[false/\$wait'] ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))[false/\$wait])$

**by** *(simp add: usubst)*

*(metis (no-types, lifting) R1-wait'-false R1-wait-false R2-R1-form R2-subst-wait'-false R2-subst-wait-false*

*R2c-seq)*

**ultimately show** *?thesis*

**by** *(smt R2-R1-form R2-condr' R2-des-lift-skip R2-st-ex R2c-seq R2s-wait)*

**qed**

**have**  $(R1(R2s(R3h(P \vdash Q))) ;; R1(R2s(R3h(R \vdash S)))) =$

$((R3h(R1(R2s(P) \vdash R2s(Q)))) ;; R3h(R1(R2s(R) \vdash R2s(S))))$

**by** *(metis (no-types, hide-lams) R1-R2s-R2c R1-R3h-commute R2c-R3h-commute R2s-design)*

**also have**  $\dots = R3h(R1((\neg(R1(\neg R2s P) ;; R1 true) \wedge \neg((R1(R2s Q) \wedge \neg \$wait')) ;; R1(\neg R2s R))) \vdash$

$(R1(R2s Q) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))))$

**by** *(simp add: R3h-R1-design-composition assms unrest)*

**also have**  $\dots = R3h(R1(R2c((\neg(R1(\neg R2s P) ;; R1 true) \wedge \neg((R1(R2s Q) \wedge \neg \$wait')) ;; R1(\neg R2s R))) \vdash$

$(R1(R2s Q) ;; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1(R2s S))))$

**by** *(simp add: R2c-design R2c-and R2c-not 1 2 3)*

**finally show** *?thesis*

**by** *(simp add: R1-R2s-R2c R1-R3h-commute R2c-R3h-commute RHS-def)*

**qed**



**lemma** *RHS-R2s-design-composition:*

**assumes**

$\$ok' \# P \ \$ok' \# Q \ \$ok \# R \ \$ok \# S$   
*P is R2s Q is R2s R is R2s S is R2s*

**shows**  $(\mathbf{R}_s(P \vdash Q) ;; \mathbf{R}_s(R \vdash S)) =$

$\mathbf{R}_s((\neg (R1 \ (\neg P) ;; R1 \ true) \wedge \neg ((R1 \ Q \ \wedge \ \neg \ \$wait') ;; R1 \ (\neg R))) \vdash$   
 $(R1 \ Q ;; ((\exists \ \$st \cdot [II]_D) \triangleleft \ \$wait \triangleright R1 \ S)))$

**proof** –

**have**  $f1: R2s \ P = P$

**by** (*meson Healthy-def assms(5)*)

**have**  $f2: R2s \ Q = Q$

**by** (*meson Healthy-def assms(6)*)

**have**  $f3: R2s \ R = R$

**by** (*meson Healthy-def assms(7)*)

**have**  $R2s \ S = S$

**by** (*meson Healthy-def assms(8)*)

**then show** *?thesis*

**using**  $f3 \ f2 \ f1$  **by** (*simp add: RHS-design-composition assms(1) assms(2) assms(3) assms(4)*)

**qed**

**lemma** *RH-design-export-R1:*  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R1(Q))$

**by** (*rel-auto*)

**lemma** *RH-design-export-R2s:*  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2s(Q))$

**by** (*rel-auto*)

**lemma** *RH-design-export-R2c:*  $\mathbf{R}(P \vdash Q) = \mathbf{R}(P \vdash R2c(Q))$

**by** (*rel-auto*)

**lemma** *RHS-design-export-R1:*  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R1(Q))$

**by** (*rel-auto*)

**lemma** *RHS-design-export-R2s:*  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2s(Q))$

**by** (*rel-auto*)

**lemma** *RHS-design-export-R2c:*  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2c(Q))$

**by** (*rel-auto*)

**lemma** *RHS-design-export-R2:*  $\mathbf{R}_s(P \vdash Q) = \mathbf{R}_s(P \vdash R2(Q))$

**by** (*rel-auto*)

**lemma** *R1-design-R1-pre:*

$\mathbf{R}_s(R1(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$

**by** (*rel-auto*)

**lemma** *RHS-design-ok-wait:*  $\mathbf{R}_s(P \llbracket true, false / \$ok, \$wait \rrbracket \vdash Q \llbracket true, false / \$ok, \$wait \rrbracket) = \mathbf{R}_s(P \vdash Q)$

**by** (*rel-auto*)

**lemma** *RHS-design-neg-R1-pre:*

$\mathbf{R}_s((\neg R1 \ P) \vdash R) = \mathbf{R}_s((\neg P) \vdash R)$

**by** (*rel-auto*)

**lemma** *RHS-design-conj-neg-R1-pre:*

$\mathbf{R}_s(((\neg R1 \ P) \wedge Q) \vdash R) = \mathbf{R}_s(((\neg P) \wedge Q) \vdash R)$

**by** (*rel-auto*)

**lemma** *RHS-pre-lemma*:  $(\mathbf{R}_s P)^f_f = R1(R2c(P^f_f))$   
 by (*rel-auto*)

**lemma** *RHS-design-R2c-pre*:  
 $\mathbf{R}_s(R2c(P) \vdash Q) = \mathbf{R}_s(P \vdash Q)$   
 by (*rel-auto*)

### 3.4 Refinement introduction laws

**lemma** *R1-design-refine*:

**assumes**

$P_1$  is R1  $P_2$  is R1  $Q_1$  is R1  $Q_2$  is R1  
 $\$ok \# P_1 \ \$ok' \# P_1 \ \$ok \# P_2 \ \$ok' \# P_2$   
 $\$ok \# Q_1 \ \$ok' \# Q_1 \ \$ok \# Q_2 \ \$ok' \# Q_2$

**shows**  $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

**proof** –

**have**  $R1((\exists \$ok; \$ok' \cdot P_1) \vdash (\exists \$ok; \$ok' \cdot P_2)) \sqsubseteq R1((\exists \$ok; \$ok' \cdot Q_1) \vdash (\exists \$ok; \$ok' \cdot Q_2))$   
 $\longleftrightarrow 'R1(\exists \$ok; \$ok' \cdot P_1) \Rightarrow R1(\exists \$ok; \$ok' \cdot Q_1)' \wedge 'R1(\exists \$ok; \$ok' \cdot P_1) \wedge R1(\exists \$ok; \$ok'$   
 $\cdot Q_2) \Rightarrow R1(\exists \$ok; \$ok' \cdot P_2)'$

by (*rel-auto, meson+*)

**thus** *?thesis*

by (*simp-all add: ex-unrest ex-plus Healthy-if assms*)

**qed**

**lemma** *R1-design-refine-RR*:

**assumes**  $P_1$  is RR  $P_2$  is RR  $Q_1$  is RR  $Q_2$  is RR

**shows**  $R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

by (*simp add: R1-design-refine assms unrest closure*)

**lemma** *RHS-design-refine*:

**assumes**

$P_1$  is R1  $P_2$  is R1  $Q_1$  is R1  $Q_2$  is R1  
 $P_1$  is R2c  $P_2$  is R2c  $Q_1$  is R2c  $Q_2$  is R2c  
 $\$ok \# P_1 \ \$ok' \# P_1 \ \$ok \# P_2 \ \$ok' \# P_2$   
 $\$ok \# Q_1 \ \$ok' \# Q_1 \ \$ok \# Q_2 \ \$ok' \# Q_2$   
 $\$wait \# P_1 \ \$wait \# P_2 \ \$wait \# Q_1 \ \$wait \# Q_2$

**shows**  $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

**proof** –

**have**  $\mathbf{R}_s(P_1 \vdash P_2) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2) \longleftrightarrow R1(R3h(R2c(P_1 \vdash P_2))) \sqsubseteq R1(R3h(R2c(Q_1 \vdash Q_2)))$

by (*simp add: R2c-R3h-commute RHS-def*)

**also have**  $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \sqsubseteq R1(R3h(Q_1 \vdash Q_2))$

by (*simp add: Healthy-if R2c-design assms*)

**also have**  $\dots \longleftrightarrow R1(R3h(P_1 \vdash P_2)) \llbracket false/\$wait \rrbracket \sqsubseteq R1(R3h(Q_1 \vdash Q_2)) \llbracket false/\$wait \rrbracket$

by (*rel-auto, metis+*)

**also have**  $\dots \longleftrightarrow R1(P_1 \vdash P_2) \llbracket false/\$wait \rrbracket \sqsubseteq R1(Q_1 \vdash Q_2) \llbracket false/\$wait \rrbracket$

by (*rel-auto*)

**also have**  $\dots \longleftrightarrow R1(P_1 \vdash P_2) \sqsubseteq R1(Q_1 \vdash Q_2)$

by (*simp add: usubst assms closure unrest*)

**also have**  $\dots \longleftrightarrow 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2'$

by (*simp add: R1-design-refine assms*)

**finally show** *?thesis* .

**qed**

**lemma** *srdes-refine-intro*:

**assumes**  $'P_1 \Rightarrow P_2' \ 'P_1 \wedge Q_2 \Rightarrow Q_1'$

shows  $\mathbf{R}_s(P_1 \vdash Q_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2)$   
 by (*simp add: RHS-mono assms design-refine-intro*)

### 3.5 Distribution laws

**lemma** *RHS-design-choice*:  $\mathbf{R}_s(P_1 \vdash Q_1) \sqcap \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \vee Q_2))$   
 by (*metis RHS-inf design-choice*)

**lemma** *RHS-design-sup*:  $\mathbf{R}_s(P_1 \vdash Q_1) \sqcup \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \vee P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$   
 by (*metis RHS-sup design-inf*)

**lemma** *RHS-design-USUP*:

assumes  $A \neq \{\}$

shows  $(\prod i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\prod i \in A \cdot P(i)) \vdash (\prod i \in A \cdot Q(i)))$

by (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms*)

end

## 4 Reactive Design Triples

**theory** *utp-rdes-triples*

**imports** *utp-rdes-designs*

**begin**

### 4.1 Diamond notation

**definition** *wait'-cond* ::

$(t::\text{trace}, \alpha, \beta) \text{ rel-rp} \Rightarrow (t, \alpha, \beta) \text{ rel-rp} \Rightarrow (t, \alpha, \beta) \text{ rel-rp}$  (**infixr**  $\diamond$  65) **where**  
 [*upred-defs*]:  $P \diamond Q = (P \triangleleft \$wait' \triangleright Q)$

**lemma** *wait'-cond-unrest* [*unrest*]:

$\llbracket \text{out-var } wait' \bowtie x; x \# P; x \# Q \rrbracket \Longrightarrow x \# (P \diamond Q)$

by (*simp add: wait'-cond-def unrest*)

**lemma** *wait'-cond-subst* [*usubst*]:

$\$wait' \# \sigma \Longrightarrow \sigma \dagger (P \diamond Q) = (\sigma \dagger P) \diamond (\sigma \dagger Q)$

by (*simp add: wait'-cond-def usubst unrest*)

**lemma** *wait'-cond-left-false*:  $\text{false} \diamond P = (\neg \$wait' \wedge P)$

by (*rel-auto*)

**lemma** *wait'-cond-seq*:  $((P \diamond Q) ;; R) = ((P ;; (\$wait' \wedge R)) \vee (Q ;; (\neg \$wait' \wedge R)))$

by (*simp add: wait'-cond-def cond-def seqr-or-distl, rel-blast*)

**lemma** *wait'-cond-true*:  $(P \diamond Q \wedge \$wait') = (P \wedge \$wait')$

by (*rel-auto*)

**lemma** *wait'-cond-false*:  $(P \diamond Q \wedge (\neg \$wait')) = (Q \wedge (\neg \$wait'))$

by (*rel-auto*)

**lemma** *wait'-cond-idem*:  $P \diamond P = P$

by (*rel-auto*)

**lemma** *wait'-cond-conj-exchange*:

$((P \diamond Q) \wedge (R \diamond S)) = (P \wedge R) \diamond (Q \wedge S)$

by (*rel-auto*)

**lemma** *subst-wait'-cond-true* [*usubst*]:  $(P \diamond Q)[[true/\$wait']] = P[[true/\$wait']]$   
by (*rel-auto*)

**lemma** *subst-wait'-cond-false* [*usubst*]:  $(P \diamond Q)[[false/\$wait']] = Q[[false/\$wait']]$   
by (*rel-auto*)

**lemma** *subst-wait'-left-subst*:  $(P[[true/\$wait']] \diamond Q) = (P \diamond Q)$   
by (*rel-auto*)

**lemma** *subst-wait'-right-subst*:  $(P \diamond Q[[false/\$wait']]) = (P \diamond Q)$   
by (*rel-auto*)

**lemma** *wait'-cond-split*:  $P[[true/\$wait']] \diamond P[[false/\$wait']] = P$   
by (*simp add: wait'-cond-def cond-var-split*)

**lemma** *wait-cond'-assoc* [*simp*]:  $P \diamond Q \diamond R = P \diamond R$   
by (*rel-auto*)

**lemma** *wait-cond'-shadow*:  $(P \diamond Q) \diamond R = P \diamond Q \diamond R$   
by (*rel-auto*)

**lemma** *wait-cond'-conj* [*simp*]:  $P \diamond (Q \wedge (R \diamond S)) = P \diamond (Q \wedge S)$   
by (*rel-auto*)

**lemma** *R1-wait'-cond*:  $R1(P \diamond Q) = R1(P) \diamond R1(Q)$   
by (*rel-auto*)

**lemma** *R2s-wait'-cond*:  $R2s(P \diamond Q) = R2s(P) \diamond R2s(Q)$   
by (*simp add: wait'-cond-def R2s-def R2s-def usubst*)

**lemma** *R2-wait'-cond*:  $R2(P \diamond Q) = R2(P) \diamond R2(Q)$   
by (*simp add: R2-def R2s-wait'-cond R1-wait'-cond*)

**lemma** *wait'-cond-R1-closed* [*closure*]:  
 $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies P \diamond Q \text{ is } R1$   
by (*simp add: Healthy-def R1-wait'-cond*)

**lemma** *wait'-cond-R2c-closed* [*closure*]:  $\llbracket P \text{ is } R2c; Q \text{ is } R2c \rrbracket \implies P \diamond Q \text{ is } R2c$   
by (*simp add: R2c-condr wait'-cond-def Healthy-def, rel-auto*)

## 4.2 Export laws

**lemma** *RH-design-peri-R1*:  $\mathbf{R}(P \vdash R1(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
by (*metis (no-types, lifting) R1-idem R1-wait'-cond RH-design-export-R1*)

**lemma** *RH-design-post-R1*:  $\mathbf{R}(P \vdash Q \diamond R1(R)) = \mathbf{R}(P \vdash Q \diamond R)$   
by (*metis R1-wait'-cond RH-design-export-R1 RH-design-peri-R1*)

**lemma** *RH-design-peri-R2s*:  $\mathbf{R}(P \vdash R2s(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
by (*metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s*)

**lemma** *RH-design-post-R2s*:  $\mathbf{R}(P \vdash Q \diamond R2s(R)) = \mathbf{R}(P \vdash Q \diamond R)$   
by (*metis (no-types, lifting) R2s-idem R2s-wait'-cond RH-design-export-R2s*)

**lemma** *RH-design-peri-R2c*:  $\mathbf{R}(P \vdash R2c(Q) \diamond R) = \mathbf{R}(P \vdash Q \diamond R)$   
 by (*metis R1-R2s-R2c RH-design-peri-R1 RH-design-peri-R2s*)

**lemma** *RHS-design-peri-R1*:  $\mathbf{R}_s(P \vdash R1(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (*metis (no-types, lifting) R1-idem R1-wait'-cond RHS-design-export-R1*)

**lemma** *RHS-design-post-R1*:  $\mathbf{R}_s(P \vdash Q \diamond R1(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (*metis R1-wait'-cond RHS-design-export-R1 RHS-design-peri-R1*)

**lemma** *RHS-design-peri-R2s*:  $\mathbf{R}_s(P \vdash R2s(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (*metis (no-types, lifting) R2s-idem R2s-wait'-cond RHS-design-export-R2s*)

**lemma** *RHS-design-post-R2s*:  $\mathbf{R}_s(P \vdash Q \diamond R2s(R)) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (*metis R2s-wait'-cond RHS-design-export-R2s RHS-design-peri-R2s*)

**lemma** *RHS-design-peri-R2c*:  $\mathbf{R}_s(P \vdash R2c(Q) \diamond R) = \mathbf{R}_s(P \vdash Q \diamond R)$   
 by (*metis R1-R2s-R2c RHS-design-peri-R1 RHS-design-peri-R2s*)

**lemma** *RH-design-lemma1*:  
 $RH(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RH(P \vdash (Q \vee R) \diamond S)$   
 by (*metis (no-types, lifting) R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RH-design-peri-R1 RH-design-peri-R2s*)

**lemma** *RHS-design-lemma1*:  
 $RHS(P \vdash (R1(R2c(Q)) \vee R) \diamond S) = RHS(P \vdash (Q \vee R) \diamond S)$   
 by (*metis (no-types, lifting) R1-R2c-is-R2 R1-R2s-R2c R2-R1-form R2-disj R2c-idem RHS-design-peri-R1 RHS-design-peri-R2s*)

### 4.3 Pre-, peri-, and postconditions

#### 4.3.1 Definitions

**abbreviation**  $pre_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s false, \$wait \mapsto_s false]$

**abbreviation**  $cmt_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false]$

**abbreviation**  $peri_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s true]$

**abbreviation**  $post_s \equiv [\$ok \mapsto_s true, \$ok' \mapsto_s true, \$wait \mapsto_s false, \$wait' \mapsto_s false]$

**abbreviation**  $npre_R(P) \equiv pre_s \dagger P$

**definition** [*upred-defs*]:  $pre_R(P) = (\neg_r npre_R(P))$

**definition** [*upred-defs*]:  $cmt_R(P) = R1(cmt_s \dagger P)$

**definition** [*upred-defs*]:  $peri_R(P) = R1(peri_s \dagger P)$

**definition** [*upred-defs*]:  $post_R(P) = R1(post_s \dagger P)$

#### 4.3.2 Unrestriction laws

**lemma** *ok-pre-unrest* [*unrest*]:  $\$ok \# pre_R P$   
 by (*simp add: pre\_R-def unrest usubst*)

**lemma** *ok-peri-unrest* [*unrest*]:  $\$ok \# peri_R P$   
 by (*simp add: peri\_R-def unrest usubst*)

**lemma** *ok-post-unrest* [*unrest*]:  $\$ok \# post_R P$   
 by (*simp add: post\_R-def unrest usubst*)

**lemma** *ok-cmt-unrest* [*unrest*]:  $\$ok \# cmt_R P$

by (simp add: cmt<sub>R</sub>-def unrest usubst)

**lemma** *ok'-pre-unrest* [unrest]: \$ok' # pre<sub>R</sub> P  
by (simp add: pre<sub>R</sub>-def unrest usubst)

**lemma** *ok'-peri-unrest* [unrest]: \$ok' # peri<sub>R</sub> P  
by (simp add: peri<sub>R</sub>-def unrest usubst)

**lemma** *ok'-post-unrest* [unrest]: \$ok' # post<sub>R</sub> P  
by (simp add: post<sub>R</sub>-def unrest usubst)

**lemma** *ok'-cmt-unrest* [unrest]: \$ok' # cmt<sub>R</sub> P  
by (simp add: cmt<sub>R</sub>-def unrest usubst)

**lemma** *wait-pre-unrest* [unrest]: \$wait # pre<sub>R</sub> P  
by (simp add: pre<sub>R</sub>-def unrest usubst)

**lemma** *wait-peri-unrest* [unrest]: \$wait # peri<sub>R</sub> P  
by (simp add: peri<sub>R</sub>-def unrest usubst)

**lemma** *wait-post-unrest* [unrest]: \$wait # post<sub>R</sub> P  
by (simp add: post<sub>R</sub>-def unrest usubst)

**lemma** *wait-cmt-unrest* [unrest]: \$wait # cmt<sub>R</sub> P  
by (simp add: cmt<sub>R</sub>-def unrest usubst)

**lemma** *wait'-peri-unrest* [unrest]: \$wait' # peri<sub>R</sub> P  
by (simp add: peri<sub>R</sub>-def unrest usubst)

**lemma** *wait'-post-unrest* [unrest]: \$wait' # post<sub>R</sub> P  
by (simp add: post<sub>R</sub>-def unrest usubst)

### 4.3.3 Substitution laws

**lemma** *pre<sub>s</sub>-design*: pre<sub>s</sub> † (P † Q) = (¬ pre<sub>s</sub> † P)  
by (simp add: design-def pre<sub>R</sub>-def usubst)

**lemma** *peri<sub>s</sub>-design*: peri<sub>s</sub> † (P † Q ◊ R) = peri<sub>s</sub> † (P ⇒ Q)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** *post<sub>s</sub>-design*: post<sub>s</sub> † (P † Q ◊ R) = post<sub>s</sub> † (P ⇒ R)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** *cmt<sub>s</sub>-design*: cmt<sub>s</sub> † (P † Q) = cmt<sub>s</sub> † (P ⇒ Q)  
by (simp add: design-def usubst wait'-cond-def)

**lemma** *pre<sub>s</sub>-R1* [usubst]: pre<sub>s</sub> † R1(P) = R1(pre<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** *pre<sub>s</sub>-R2c* [usubst]: pre<sub>s</sub> † R2c(P) = R2c(pre<sub>s</sub> † P)  
by (simp add: R2c-def R2s-def usubst)

**lemma** *peri<sub>s</sub>-R1* [usubst]: peri<sub>s</sub> † R1(P) = R1(peri<sub>s</sub> † P)  
by (simp add: R1-def usubst)

**lemma** *peri<sub>s</sub>-R2c* [usubst]: peri<sub>s</sub> † R2c(P) = R2c(peri<sub>s</sub> † P)

by (simp add: R2c-def R2s-def usubst)

**lemma** *post<sub>s</sub>-R1* [usubst]:  $post_s \dagger R1(P) = R1(post_s \dagger P)$   
by (simp add: R1-def usubst)

**lemma** *post<sub>s</sub>-R2c* [usubst]:  $post_s \dagger R2c(P) = R2c(post_s \dagger P)$   
by (simp add: R2c-def R2s-def usubst)

**lemma** *cmt<sub>s</sub>-R1* [usubst]:  $cmt_s \dagger R1(P) = R1(cmt_s \dagger P)$   
by (simp add: R1-def usubst)

**lemma** *cmt<sub>s</sub>-R2c* [usubst]:  $cmt_s \dagger R2c(P) = R2c(cmt_s \dagger P)$   
by (simp add: R2c-def R2s-def usubst)

**lemma** *pre-wait-false*:  
 $pre_R(P[[false/\$wait]]) = pre_R(P)$   
by (rel-auto)

**lemma** *cmt-wait-false*:  
 $cmt_R(P[[false/\$wait]]) = cmt_R(P)$   
by (rel-auto)

**lemma** *rea-pre-RHS-design*:  $pre_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(pre_s \dagger P))$   
by (simp add: RHS-def usubst R3h-def pre\_R-def pre\_s-design R1-negate-R1 R2c-not rea-not-def)

**lemma** *rea-cmt-RHS-design*:  $cmt_R(\mathbf{R}_s(P \vdash Q)) = R1(R2c(cmt_s \dagger (P \Rightarrow Q)))$   
by (simp add: RHS-def usubst R3h-def cmt\_R-def cmt\_s-design R1-idem)

**lemma** *rea-peri-RHS-design*:  $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(peri_s \dagger (P \Rightarrow_r Q)))$   
by (simp add: RHS-def usubst peri\_R-def R3h-def peri\_s-design, rel-auto)

**lemma** *rea-post-RHS-design*:  $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = R1(R2c(post_s \dagger (P \Rightarrow_r R)))$   
by (simp add: RHS-def usubst post\_R-def R3h-def post\_s-design, rel-auto)

**lemma** *peri-cmt-def*:  $peri_R(P) = (cmt_R(P))[[true/\$wait']]$   
by (rel-auto)

**lemma** *post-cmt-def*:  $post_R(P) = (cmt_R(P))[[false/\$wait']]$   
by (rel-auto)

**lemma** *rdes-export-cmt*:  $\mathbf{R}_s(P \vdash cmt_s \dagger Q) = \mathbf{R}_s(P \vdash Q)$   
by (rel-auto)

**lemma** *rdes-export-pre*:  $\mathbf{R}_s((P[[true,false/\$ok,\$wait]]) \vdash Q) = \mathbf{R}_s(P \vdash Q)$   
by (rel-auto)

#### 4.3.4 Healthiness laws

**lemma** *wait'-unrest-pre-SRD* [unrest]:  
 $\$wait' \# pre_R(P) \Longrightarrow \$wait' \# pre_R(SRD P)$   
apply (rel-auto)  
using least-zero apply blast+  
done

**lemma** *R1-R2s-cmt-SRD*:  
assumes  $P$  is SRD

**shows**  $R1(R2s(cmt_R(P))) = cmt_R(P)$   
**by** (*metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-idem R2c-idem SRD-reactive-design*  
*assms rea-cmt-RHS-design*)

**lemma** *R1-R2s-peri-SRD*:

**assumes** *P is SRD*

**shows**  $R1(R2s(peri_R(P))) = peri_R(P)$

**by** (*metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R2-def R2-idem RHS-def SRD-RH-design-form*  
*assms R1-idem peri\_R-def peri\_s-R1 peri\_s-R2c*)

**lemma** *R1-peri-SRD*:

**assumes** *P is SRD*

**shows**  $R1(peri_R(P)) = peri_R(P)$

**proof** –

**have**  $R1(peri_R(P)) = R1(R1(R2s(peri_R(P))))$

**by** (*simp add: R1-R2s-peri-SRD assms*)

**also have**  $\dots = peri_R(P)$

**by** (*simp add: R1-idem, simp add: R1-R2s-peri-SRD assms*)

**finally show** *?thesis* .

**qed**

**lemma** *periR-SRD-R1 [closure]*:  $P \text{ is SRD} \implies peri_R(P) \text{ is R1}$

**by** (*simp add: Healthy-def' R1-peri-SRD*)

**lemma** *R1-R2c-peri-RHS*:

**assumes** *P is SRD*

**shows**  $R1(R2c(peri_R(P))) = peri_R(P)$

**by** (*metis R1-R2s-R2c R1-R2s-peri-SRD assms*)

**lemma** *R1-R2s-post-SRD*:

**assumes** *P is SRD*

**shows**  $R1(R2s(post_R(P))) = post_R(P)$

**by** (*metis (no-types, hide-lams) Healthy-def R1-R2s-R2c R1-idem R2-def R2-idem RHS-def SRD-RH-design-form*  
*assms post\_R-def post\_s-R1 post\_s-R2c*)

**lemma** *R2c-peri-SRD*:

**assumes** *P is SRD*

**shows**  $R2c(peri_R(P)) = peri_R(P)$

**by** (*metis R1-R2c-commute R1-R2c-peri-RHS R1-peri-SRD assms*)

**lemma** *R1-post-SRD*:

**assumes** *P is SRD*

**shows**  $R1(post_R(P)) = post_R(P)$

**proof** –

**have**  $R1(post_R(P)) = R1(R1(R2s(post_R(P))))$

**by** (*simp add: R1-R2s-post-SRD assms*)

**also have**  $\dots = post_R(P)$

**by** (*simp add: R1-idem, simp add: R1-R2s-post-SRD assms*)

**finally show** *?thesis* .

**qed**

**lemma** *R2c-post-SRD*:

**assumes** *P is SRD*

**shows**  $R2c(post_R(P)) = post_R(P)$

**by** (*metis R1-R2c-commute R1-R2s-R2c R1-R2s-post-SRD R1-post-SRD assms*)



**lemma** *postR-SRD-R1 [closure]*:  $P$  is SRD  $\implies$   $\text{post}_R(P)$  is R1  
 by (*simp add: Healthy-def' R1-post-SRD*)

**lemma** *R1-R2c-post-RHS*:  
 assumes  $P$  is SRD  
 shows  $R1(R2c(\text{post}_R(P))) = \text{post}_R(P)$   
 by (*metis R1-R2s-R2c R1-R2s-post-SRD assms*)

**lemma** *R2-cmt-conj-wait'*:  
 $P$  is SRD  $\implies R2(\text{cmt}_R P \wedge \neg \$\text{wait}' ) = (\text{cmt}_R P \wedge \neg \$\text{wait}' )$   
 by (*simp add: R2-def R2s-conj R2s-not R2s-wait' R1-extend-conj R1-R2s-cmt-SRD*)

**lemma** *R2c-preR*:  
 $P$  is SRD  $\implies R2c(\text{pre}_R(P)) = \text{pre}_R(P)$   
 by (*metis (no-types, lifting) R1-R2c-commute R2c-idem SRD-reactive-design rea-pre-RHS-design*)

**lemma** *preR-R2c-closed [closure]*:  $P$  is SRD  $\implies \text{pre}_R(P)$  is R2c  
 by (*simp add: Healthy-def' R2c-preR*)

**lemma** *R2c-periR*:  
 $P$  is SRD  $\implies R2c(\text{peri}_R(P)) = \text{peri}_R(P)$   
 by (*metis (no-types, lifting) R1-R2c-commute R1-R2s-R2c R1-R2s-peri-SRD R2c-idem*)

**lemma** *periR-R2c-closed [closure]*:  $P$  is SRD  $\implies \text{peri}_R(P)$  is R2c  
 by (*simp add: Healthy-def R2c-peri-SRD*)

**lemma** *R2c-postR*:  
 $P$  is SRD  $\implies R2c(\text{post}_R(P)) = \text{post}_R(P)$   
 by (*metis (no-types, hide-lams) R1-R2c-commute R1-R2c-is-R2 R1-R2s-post-SRD R2-def R2s-idem*)

**lemma** *postR-R2c-closed [closure]*:  $P$  is SRD  $\implies \text{post}_R(P)$  is R2c  
 by (*simp add: Healthy-def R2c-post-SRD*)

**lemma** *periR-RR [closure]*:  $P$  is SRD  $\implies \text{peri}_R(P)$  is RR  
 by (*rule RR-intro, simp-all add: closure unrest*)

**lemma** *postR-RR [closure]*:  $P$  is SRD  $\implies \text{post}_R(P)$  is RR  
 by (*rule RR-intro, simp-all add: closure unrest*)

**lemma** *wpR-trace-ident-pre [wp]*:  
 $(\$tr' =_u \$tr \wedge [II]_R) \text{wp}_r \text{pre}_R P = \text{pre}_R P$   
 by (*rel-auto*)

**lemma** *R1-preR [closure]*:  
 $\text{pre}_R(P)$  is R1  
 by (*rel-auto*)

**lemma** *trace-ident-left-periR*:  
 $(\$tr' =_u \$tr \wedge [II]_R) ;; \text{peri}_R(P) = \text{peri}_R(P)$   
 by (*rel-auto*)

**lemma** *trace-ident-left-postR*:  
 $(\$tr' =_u \$tr \wedge [II]_R) ;; \text{post}_R(P) = \text{post}_R(P)$   
 by (*rel-auto*)

**lemma** *trace-ident-right-postR*:  
 $post_R(P) ;; (\$tr' =_u \$tr \wedge [II]_R) = post_R(P)$   
 by (*rel-auto*)

**lemma** *preR-R2-closed [closure]*:  $P$  is SRD  $\implies pre_R(P)$  is R2  
 by (*simp add: R2-comp-def Healthy-comp closure*)

**lemma** *periR-R2-closed [closure]*:  $P$  is SRD  $\implies peri_R(P)$  is R2  
 by (*simp add: Healthy-def' R1-R2c-peri-RHS R2-R2c-def*)

**lemma** *postR-R2-closed [closure]*:  $P$  is SRD  $\implies post_R(P)$  is R2  
 by (*simp add: Healthy-def' R1-R2c-post-RHS R2-R2c-def*)

### 4.3.5 Calculation laws

**lemma** *wait'-cond-peri-post-cmt [rdes]*:  
 $cmt_R P = peri_R P \diamond post_R P$   
 by (*rel-auto*)

**lemma** *preR-rdes [rdes]*:  
 assumes  $P$  is RR  
 shows  $pre_R(\mathbf{R}_s(P \vdash Q \diamond R)) = P$   
 by (*simp add: rea-pre-RHS-design unrest usubst assms Healthy-if RR-implies-R2c RR-implies-R1*)

**lemma** *periR-rdes [rdes]*:  
 assumes  $P$  is RR  $Q$  is RR  
 shows  $peri_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r Q)$   
 by (*simp add: rea-peri-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

**lemma** *postR-rdes [rdes]*:  
 assumes  $P$  is RR  $R$  is RR  
 shows  $post_R(\mathbf{R}_s(P \vdash Q \diamond R)) = (P \Rightarrow_r R)$   
 by (*simp add: rea-post-RHS-design unrest usubst assms Healthy-if RR-implies-R2c closure*)

**lemma** *preR-Chaos [rdes]*:  $pre_R(\text{Chaos}) = \text{false}$   
 by (*simp add: Chaos-def, rel-simp*)

**lemma** *periR-Chaos [rdes]*:  $peri_R(\text{Chaos}) = \text{true}_r$   
 by (*simp add: Chaos-def, rel-simp*)

**lemma** *postR-Chaos [rdes]*:  $post_R(\text{Chaos}) = \text{true}_r$   
 by (*simp add: Chaos-def, rel-simp*)

**lemma** *preR-Miracle [rdes]*:  $pre_R(\text{Miracle}) = \text{true}_r$   
 by (*simp add: Miracle-def, rel-auto*)

**lemma** *periR-Miracle [rdes]*:  $peri_R(\text{Miracle}) = \text{false}$   
 by (*simp add: Miracle-def, rel-auto*)

**lemma** *postR-Miracle [rdes]*:  $post_R(\text{Miracle}) = \text{false}$   
 by (*simp add: Miracle-def, rel-auto*)

**lemma** *preR-srdes-skip [rdes]*:  $pre_R(II_R) = \text{true}_r$   
 by (*rel-auto*)

**lemma** *periR-srdes-skip* [rdes]:  $peri_R(II_R) = false$   
**by** (*rel-auto*)

**lemma** *postR-srdes-skip* [rdes]:  $post_R(II_R) = (\$tr' =_u \$tr \wedge [II]_R)$   
**by** (*rel-auto*)

**lemma** *preR-INF* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap A) = (\bigwedge P \in A \cdot pre_R(P))$   
**by** (*rel-auto*)

**lemma** *periR-INF* [rdes]:  $peri_R(\sqcap A) = (\bigvee P \in A \cdot peri_R(P))$   
**by** (*rel-auto*)

**lemma** *postR-INF* [rdes]:  $post_R(\sqcap A) = (\bigvee P \in A \cdot post_R(P))$   
**by** (*rel-auto*)

**lemma** *preR-UINF* [rdes]:  $pre_R(\sqcap i \cdot P(i)) = (\bigsqcup i \cdot pre_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *periR-UINF* [rdes]:  $peri_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot peri_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *postR-UINF* [rdes]:  $post_R(\sqcap i \cdot P(i)) = (\sqcap i \cdot post_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap i \in A \cdot P(i)) = (\bigsqcup i \in A \cdot pre_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member-2* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap (i,j) \in A \cdot P i j) = (\bigsqcup (i,j) \in A \cdot pre_R(P i j))$   
**by** (*rel-auto*)

**lemma** *preR-UINF-member-3* [rdes]:  $A \neq \{\} \implies pre_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\bigsqcup (i,j,k) \in A \cdot pre_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member* [rdes]:  $peri_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot peri_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member-2* [rdes]:  $peri_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot peri_R(P i j))$   
**by** (*rel-auto*)

**lemma** *periR-UINF-member-3* [rdes]:  $peri_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot peri_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member* [rdes]:  $post_R(\sqcap i \in A \cdot P(i)) = (\sqcap i \in A \cdot post_R(P(i)))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member-2* [rdes]:  $post_R(\sqcap (i,j) \in A \cdot P i j) = (\sqcap (i,j) \in A \cdot post_R(P i j))$   
**by** (*rel-auto*)

**lemma** *postR-UINF-member-3* [rdes]:  $post_R(\sqcap (i,j,k) \in A \cdot P i j k) = (\sqcap (i,j,k) \in A \cdot post_R(P i j k))$   
**by** (*rel-auto*)

**lemma** *preR-inf* [rdes]:  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*rel-auto*)

**lemma** *periR-inf* [rdes]:  $peri_R(P \sqcap Q) = (peri_R(P) \vee peri_R(Q))$   
 by (*rel-auto*)

**lemma** *postR-inf* [rdes]:  $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$   
 by (*rel-auto*)

**lemma** *preR-SUP* [rdes]:  $pre_R(\bigsqcup A) = (\bigvee P \in A \cdot pre_R(P))$   
 by (*rel-auto*)

**lemma** *periR-SUP* [rdes]:  $A \neq \{\} \implies peri_R(\bigsqcup A) = (\bigwedge P \in A \cdot peri_R(P))$   
 by (*rel-auto*)

**lemma** *postR-SUP* [rdes]:  $A \neq \{\} \implies post_R(\bigsqcup A) = (\bigwedge P \in A \cdot post_R(P))$   
 by (*rel-auto*)

#### 4.4 Formation laws

**lemma** *srdes-skip-tri-design* [rdes-def]:  $II_R = \mathbf{R}_s(true_r \vdash false \diamond II_r)$   
 by (*simp add: srdes-skip-def, rel-auto*)

**lemma** *Chaos-tri-def* [rdes-def]:  $Chaos = \mathbf{R}_s(false \vdash false \diamond false)$   
 by (*simp add: Chaos-def design-false-pre*)

**lemma** *Miracle-tri-def* [rdes-def]:  $Miracle = \mathbf{R}_s(true_r \vdash false \diamond false)$   
 by (*simp add: Miracle-def R1-design-R1-pre wait'-cond-idem*)

**lemma** *RHS-tri-design-form*:

assumes  $P_1$  is *RR*  $P_2$  is *RR*  $P_3$  is *RR*

shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge P_1) \Rightarrow_r (\$ok' \wedge (P_2 \diamond P_3))))$

**proof** –

have  $\mathbf{R}_s(RR(P_1) \vdash RR(P_2) \diamond RR(P_3)) = (II_R \triangleleft \$wait \triangleright ((\$ok \wedge RR(P_1)) \Rightarrow_r (\$ok' \wedge (RR(P_2) \diamond RR(P_3))))$

apply (*rel-auto*) using *minus-zero-eq* by *blast*

thus *?thesis*

by (*simp add: Healthy-if assms*)

**qed**

**lemma** *RHS-design-pre-post-form*:

$\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$

**proof** –

have  $\mathbf{R}_s((\neg P^f_f) \vdash P^t_f) = \mathbf{R}_s((\neg P^f_f)[[true/\$ok]] \vdash P^t_f[[true/\$ok]])$

by (*simp add: design-subst-ok*)

also have  $\dots = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$

by (*simp add: pre\_R-def cmt\_R-def usubst, rel-auto*)

finally show *?thesis* .

**qed**

**lemma** *SRD-as-reactive-design*:

$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash cmt_R(P))$

by (*simp add: RHS-design-pre-post-form SRD-RH-design-form*)

**lemma** *SRD-reactive-design-alt*:

assumes  $P$  is *SRD*

shows  $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = P$

**proof** –

have  $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f)$

by (*simp add: RHS-design-pre-post-form*)  
 thus *?thesis*  
 by (*simp add: SRD-reactive-design assms*)  
 qed

**lemma** *SRD-reactive-tri-design-lemma:*

$SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket true/\$wait' \rrbracket \diamond P^t_f \llbracket false/\$wait' \rrbracket)$   
 by (*simp add: SRD-RH-design-form wait'-cond-split*)

**lemma** *SRD-as-reactive-tri-design:*

$SRD(P) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

**proof** –

**have**  $SRD(P) = \mathbf{R}_s((\neg P^f_f) \vdash P^t_f \llbracket true/\$wait' \rrbracket \diamond P^t_f \llbracket false/\$wait' \rrbracket)$

  by (*simp add: SRD-RH-design-form wait'-cond-split*)

**also have**  $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$

**apply** (*simp add: usubst*)

**apply** (*subst design-subst-ok-ok'[THEN sym]*)

**apply** (*simp add: pre\_R-def peri\_R-def post\_R-def usubst unrest*)

**apply** (*rel-auto*)

**done**

**finally show** *?thesis* .

qed

**lemma** *SRD-reactive-tri-design:*

**assumes** *P is SRD*

**shows**  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$

by (*metis Healthy-if SRD-as-reactive-tri-design assms*)

**lemma** *SRD-elim [RD-elim]:*  $\llbracket P \text{ is SRD}; Q(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))) \rrbracket \implies Q(P)$

by (*simp add: SRD-reactive-tri-design*)

**lemma** *RHS-tri-design-is-SRD [closure]:*

**assumes**  $\$ok' \# P \ \$ok' \# Q \ \$ok' \# R$

**shows**  $\mathbf{R}_s(P \vdash Q \diamond R)$  *is SRD*

by (*rule RHS-design-is-SRD, simp-all add: unrest assms*)

**lemma** *SRD-rdes-intro [closure]:*

**assumes** *P is RR Q is RR R is RR*

**shows**  $\mathbf{R}_s(P \vdash Q \diamond R)$  *is SRD*

by (*rule RHS-tri-design-is-SRD, simp-all add: unrest closure assms*)

**lemma** *USUP-R1-R2s-cmt-SRD:*

**assumes**  $A \subseteq \llbracket SRD \rrbracket_H$

**shows**  $(\bigsqcup P \in A \cdot R1 (R2s (cmt_R P))) = (\bigsqcup P \in A \cdot cmt_R P)$

by (*rule USUP-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms*)

**lemma** *UINF-R1-R2s-cmt-SRD:*

**assumes**  $A \subseteq \llbracket SRD \rrbracket_H$

**shows**  $(\bigsqcap P \in A \cdot R1 (R2s (cmt_R P))) = (\bigsqcap P \in A \cdot cmt_R P)$

by (*rule UINF-cong[of A], metis (mono-tags, lifting) Ball-Collect R1-R2s-cmt-SRD assms*)

#### 4.4.1 Order laws

**lemma** *preR-antitone:*  $P \sqsubseteq Q \implies pre_R(Q) \sqsubseteq pre_R(P)$

by (*rel-auto*)

**lemma** *periR-monotone*:  $P \sqsubseteq Q \implies \text{peri}_R(P) \sqsubseteq \text{peri}_R(Q)$   
 by (*rel-auto*)

**lemma** *postR-monotone*:  $P \sqsubseteq Q \implies \text{post}_R(P) \sqsubseteq \text{post}_R(Q)$   
 by (*rel-auto*)

## 4.5 Composition laws

**theorem** *RH-tri-design-composition*:

**assumes**  $\$ok' \# P \ \$ok' \# Q_1 \ \$ok' \# Q_2 \ \$ok \# R \ \$ok \# S_1 \ \$ok \# S_2$   
 $\$wait' \# Q_2 \ \$wait \# S_1 \ \$wait \# S_2$

**shows**  $(RH(P \vdash Q_1 \diamond Q_2) ;; RH(R \vdash S_1 \diamond S_2)) =$   
 $RH((\neg (R1 (\neg R2s P) ;; R1 \text{ true}) \wedge \neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R))) \vdash$   
 $((Q_1 \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

**proof** –

**have**  $1: (\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$   
 $(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$

by (*metis (no-types, hide-lams) R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)

**have**  $2: (R1 (R2s (Q_1 \diamond Q_2)) ;; ([II]_D \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$   
 $((R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2)))$

**proof** –

**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) \wedge \$wait')$

**proof** –

**have**  $(R1 (R2s Q_1) ;; (\$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) ;; (\$wait \wedge [II]_D))$

by (*rel-auto*)

**also have**  $\dots = ((R1 (R2s Q_1) ;; [II]_D) \wedge \$wait')$

by (*rel-auto*)

**also from** *assms(2)* **have**  $\dots = ((R1 (R2s Q_1)) \wedge \$wait')$

by (*simp add: lift-des-skip-dr-unit-unrest unrest*)

**finally show** *?thesis* .

**qed**

**moreover have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

**proof** –

**have**  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ([II]_D \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_2) ;; (\neg \$wait \wedge (R1 (R2s S_1) \diamond R1 (R2s S_2))))$

by (*metis (no-types, lifting) cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb*)

**also have**  $\dots = ((R1 (R2s Q_2)) \llbracket \text{false} / \$wait \rrbracket ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)) \llbracket \text{false} / \$wait \rrbracket)$   
 by (*metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens*)

**also have**  $\dots = ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

by (*simp add: wait'-cond-def usubst unrest assms*)

**finally show** *?thesis* .

**qed**

**moreover**

**have**  $((R1 (R2s Q_1) \wedge \$wait') \vee ((R1 (R2s Q_2) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2)))$

by (*simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest*)

**ultimately show** *?thesis*  
 by (*simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq*)  
 qed

**show** *?thesis*  
 apply (*subst RH-design-composition*)  
 apply (*simp-all add: assms*)  
 apply (*simp add: assms wait'-cond-def unrest*)  
 apply (*simp add: assms wait'-cond-def unrest*)  
 apply (*simp add: 1 2*)  
 apply (*simp add: R1-R2s-R2c RH-design-lemma1*)  
 done  
 qed

**theorem** *R1-design-composition-RR:*  
 assumes *P is RR Q is RR R is RR S is RR*  
 shows  
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$   
 apply (*subst R1-design-composition*)  
 apply (*simp-all add: assms unrest wp-rea-def Healthy-if closure*)  
 apply (*rel-auto*)  
 done

**theorem** *R1-design-composition-RC:*  
 assumes *P is RC Q is RR R is RR S is RR*  
 shows  
 $(R1(P \vdash Q) ;; R1(R \vdash S)) = R1((P \wedge Q \text{ wp}_r R) \vdash (Q ;; S))$   
 by (*simp add: R1-design-composition-RR assms unrest Healthy-if closure wp*)

**theorem** *RHS-tri-design-composition:*  
 assumes  $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$   
 $\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$   
 shows  $(\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)) =$   
 $\mathbf{R}_s((\neg (R1 (\neg R2s P) ;; R1 \text{ true}) \wedge \neg (R1(R2s Q_2) ;; R1 (\neg R2s R))) \vdash$   
 $((\exists \$st' \cdot Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2))))$

**proof** –  
 have 1:  $(\neg ((R1 (R2s (Q_1 \diamond Q_2)) \wedge \neg \$wait') ;; R1 (\neg R2s R))) =$   
 $(\neg ((R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R)))$   
 by (*metis (no-types, hide-lams) R1-extend-conj R2s-conj R2s-not R2s-wait' wait'-cond-false*)  
 have 2:  $(R1 (R2s (Q_1 \diamond Q_2)) ;; ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s (S_1 \diamond S_2)))) =$   
 $((\exists \$st' \cdot R1 (R2s Q_1)) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond (R1 (R2s Q_2) ;; R1 (R2s S_2))$

**proof** –  
 have  $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$

**proof** –  
 have  $(R1 (R2s Q_1) ;; (\$wait \wedge ((\exists \$st \cdot \lceil II \rceil_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) ;; (\$wait \wedge (\exists \$st \cdot \lceil II \rceil_D)))$   
 by (*rel-auto, blast+*)  
 also have ... =  $((R1 (R2s Q_1) ;; (\exists \$st \cdot \lceil II \rceil_D)) \wedge \$wait')$   
 by (*rel-auto*)  
 also from *assms(2)* have ... =  $(\exists \$st' \cdot ((R1 (R2s Q_1)) \wedge \$wait'))$   
 by (*rel-auto, blast*)  
 finally show *?thesis* .

qed

moreover have  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$

$$= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$$

proof –

have  $(R1 (R2s Q_2) ;; (\neg \$wait \wedge ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_2) ;; (\neg \$wait \wedge (R1 (R2s S_1) \diamond R1 (R2s S_2))))$

by *(metis (no-types, lifting) cond-def conj-disj-not-abs utp-pred-laws.double-compl utp-pred-laws.inf.left-idem utp-pred-laws.sup-assoc utp-pred-laws.sup-inf-absorb)*

also have ...  $= ((R1 (R2s Q_2))\llbracket false/\$wait' \rrbracket ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))\llbracket false/\$wait' \rrbracket)$

by *(metis false-alt-def seqr-right-one-point upred-eq-false wait-vwb-lens)*

also have ...  $= ((R1 (R2s Q_2)) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2)))$

by *(simp add: wait'-cond-def usubst unrest assms)*

finally show *?thesis* .

qed

moreover

have  $((R1 (R2s Q_1) \wedge \$wait') \vee ((R1 (R2s Q_2) ;; (R1 (R2s S_1) \diamond R1 (R2s S_2))))$   
 $= (R1 (R2s Q_1) \vee (R1 (R2s Q_2) ;; R1 (R2s S_1))) \diamond ((R1 (R2s Q_2) ;; R1 (R2s S_2)))$

by *(simp add: wait'-cond-def cond-seq-right-distr cond-and-T-integrate unrest)*

ultimately show *?thesis*

by *(simp add: R2s-wait'-cond R1-wait'-cond wait'-cond-seq ex-conj-contr-right unrest)*  
*(simp add: cond-and-T-integrate cond-seq-right-distr unrest-var wait'-cond-def)*

qed

from *assms(7,8)* have  $\exists: (R1 (R2s Q_2) \wedge \neg \$wait') ;; R1 (\neg R2s R) = R1 (R2s Q_2) ;; R1 (\neg R2s R)$

by *(rel-auto, blast, meson)*

show *?thesis*

apply *(subst RHS-design-composition)*

apply *(simp-all add: assms)*

apply *(simp add: assms wait'-cond-def unrest)*

apply *(simp add: assms wait'-cond-def unrest)*

apply *(simp add: 1 2 3)*

apply *(simp add: R1-R2s-R2c RHS-design-lemma1)*

apply *(metis R1-R2c-ex-st RHS-design-lemma1)*

done

qed

theorem *RHS-tri-design-composition-wp*:

assumes  $\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

$P$  is R2c  $Q_1$  is R1  $Q_1$  is R2c  $Q_2$  is R1  $Q_2$  is R2c

$R$  is R2c  $S_1$  is R1  $S_1$  is R2c  $S_2$  is R1  $S_2$  is R2c

shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$  (is *?lhs =*

*?rhs*)

proof –

have *?lhs*  $= \mathbf{R}_s ((\neg R1 (\neg P) ;; R1 \text{ true} \wedge \neg Q_2 ;; R1 (\neg R)) \vdash ((\exists \$st' \cdot Q_1) \sqcap Q_2 ;; S_1) \diamond Q_2 ;; S_2)$



by (*simp add: RHS-tri-design-composition assms Healthy-if R2c-healthy-R2s disj-upred-def*)  
 (*metis (no-types, hide-lams) R1-negate-R1 R2c-healthy-R2s assms(11,16)*)  
 also have ... = ?rhs  
 by (*rel-auto*)  
 finally show ?thesis .  
 qed

**theorem** *RHS-tri-design-composition-RR-wp:*

assumes *P is RR Q<sub>1</sub> is RR Q<sub>2</sub> is RR*  
*R is RR S<sub>1</sub> is RR S<sub>2</sub> is RR*

shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s(((\neg_r P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$  (*is ?lhs =*  
 ?rhs)

by (*simp add: RHS-tri-design-composition-wp add: closure assms unrest RR-implies-R2c*)

**lemma** *RHS-tri-normal-design-composition:*

assumes

$\$ok' \# P \$ok' \# Q_1 \$ok' \# Q_2 \$ok \# R \$ok \# S_1 \$ok \# S_2$

$\$wait \# R \$wait' \# Q_2 \$wait \# S_1 \$wait \# S_2$

*P is R2c Q<sub>1</sub> is R1 Q<sub>1</sub> is R2c Q<sub>2</sub> is R1 Q<sub>2</sub> is R2c*

*R is R2c S<sub>1</sub> is R1 S<sub>1</sub> is R2c S<sub>2</sub> is R1 S<sub>2</sub> is R2c*

*R1 ( $\neg P$ ) ;; R1(true) = R1( $\neg P$ ) \$st' # Q<sub>1</sub>*

shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

have  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((R1 (\neg P) \text{ wp}_r \text{ false} \wedge Q_2 \text{ wp}_r R) \vdash (((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2)))$

by (*simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest*)

also have ... =  $\mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

by (*simp add: assms wp-rea-def ex-unrest, rel-auto*)

finally show ?thesis .

qed

**lemma** *RHS-tri-normal-design-composition' [rdes-def]:*

assumes *P is RC Q<sub>1</sub> is RR \$st' # Q<sub>1</sub> Q<sub>2</sub> is RR R is RR S<sub>1</sub> is RR S<sub>2</sub> is RR*

shows  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{ wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

have *R1 ( $\neg P$ ) ;; R1 true = R1( $\neg P$ )*

using *RC-implies-RC1[OF assms(1)]*

by (*simp add: Healthy-def RC1-def rea-not-def*)

(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)

thus ?thesis

by (*simp add: RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)

qed

**lemma** *RHS-tri-design-right-unit-lemma:*

assumes  $\$ok' \# P \$ok' \# Q \$ok' \# R \$wait' \# R$

shows  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash (((\exists \$st' \cdot Q) \diamond R)))$

**proof** –

have  $\mathbf{R}_s(P \vdash Q \diamond R) ;; II_R = \mathbf{R}_s(P \vdash Q \diamond R) ;; \mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [II]_R))$

by (*simp add: srdes-skip-tri-design, rel-auto*)

also have ... =  $\mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{true}) \vdash (\exists \$st' \cdot Q) \diamond (R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))))$

by (*simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)

also have ... =  $\mathbf{R}_s ((\neg R1 (\neg R2s P) ;; R1 \text{ true}) \vdash (\exists \$st' \cdot Q) \diamond R1 (R2s R))$

proof –

from  $assms(3,4)$  have  $(R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 (R2s R)$

by  $(rel\text{-}auto, metis (no\text{-}types, lifting) minus\text{-}zero\text{-}eq, meson order\text{-}reft trace\text{-}class.diff\text{-}cancel)$

thus  $?thesis$

by  $simp$

qed

also have ... =  $\mathbf{R}_s((\neg (\neg P) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$

by  $(metis (no\text{-}types, lifting) R1\text{-}R2s\text{-}R1\text{-}true\text{-}lemma R1\text{-}R2s\text{-}R2c R2c\text{-}not RHS\text{-}design\text{-}R2c\text{-}pre RHS\text{-}design\text{-}neg\text{-}R1\text{-}pre RHS\text{-}design\text{-}post\text{-}R1 RHS\text{-}design\text{-}post\text{-}R2s)$

also have ... =  $\mathbf{R}_s((\neg_r (\neg_r P) ;; true_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$

by  $(rel\text{-}auto)$

finally show  $?thesis$  .

qed

lemma  $SRD\text{-}composition\text{-}wp$ :

assumes  $P$  is  $SRD$   $Q$  is  $SRD$

shows  $(P ;; Q) = \mathbf{R}_s (((\neg_r pre_R P) wp_r \text{ false} \wedge post_R P wp_r pre_R Q) \vdash$

$(\exists \$st' \cdot peri_R P) \vee (post_R P ;; peri_R Q) \diamond (post_R P ;; post_R Q))$

(is  $?lhs = ?rhs$ )

proof –

have  $(P ;; Q) = (\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) ;; \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q)))$

by  $(simp \text{ add: } SRD\text{-}reactive\text{-}tri\text{-}design \text{ assms}(1) \text{ assms}(2))$

also from  $assms$

have ... =  $?rhs$

by  $(simp \text{ add: } RHS\text{-}tri\text{-}design\text{-}composition\text{-}wp \text{ disj\text{-}upred\text{-}def unrest \text{ assms } closure)$

finally show  $?thesis$  .

qed

## 4.6 Refinement introduction laws

lemma  $RHS\text{-}tri\text{-}design\text{-}refine$ :

assumes  $P_1$  is  $RR$   $P_2$  is  $RR$   $P_3$  is  $RR$   $Q_1$  is  $RR$   $Q_2$  is  $RR$   $Q_3$  is  $RR$

shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \iff 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \Rightarrow P_2' \wedge 'P_1 \wedge Q_3 \Rightarrow P_3'$

(is  $?lhs = ?rhs$ )

proof –

have  $?lhs \iff 'P_1 \Rightarrow Q_1' \wedge 'P_1 \wedge Q_2 \diamond Q_3 \Rightarrow P_2 \diamond P_3'$

by  $(simp \text{ add: } RHS\text{-}design\text{-}refine \text{ assms } closure RR\text{-}implies\text{-}R2c \text{ unrest } ex\text{-}unrest)$

also have ...  $\iff 'P_1 \Rightarrow Q_1' \wedge '(P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3'$

by  $(rel\text{-}auto)$

also have ...  $\iff 'P_1 \Rightarrow Q_1' \wedge '((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[true/\$wait'] \wedge '((P_1 \wedge Q_2) \diamond (P_1 \wedge Q_3) \Rightarrow P_2 \diamond P_3)[false/\$wait']$

by  $(rel\text{-}auto, metis)$

also have ...  $\iff ?rhs$

by  $(simp \text{ add: } usubst \text{ unrest } assms)$

finally show  $?thesis$  .

qed

lemma  $srdes\text{-}tri\text{-}refine\text{-}intro$ :

assumes  $'P_1 \Rightarrow P_2' 'P_1 \wedge Q_2 \Rightarrow Q_1' 'P_1 \wedge R_2 \Rightarrow R_1'$

shows  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$

using  $assms$

by  $(rule\text{-}tac \text{ srdes\text{-}refine\text{-}intro, simp\text{-}all, rel\text{-}auto)$

lemma  $srdes\text{-}tri\text{-}eq\text{-}intro$ :

**assumes**  $P_1 = Q_1 \ P_2 = Q_2 \ P_3 = Q_3$   
**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) = \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
**using** *assms* **by** (*simp*)

**lemma** *srdes-tri-refine-intro'*:

**assumes**  $P_2 \sqsubseteq P_1 \ Q_1 \sqsubseteq (P_1 \wedge Q_2) \ R_1 \sqsubseteq (P_1 \wedge R_2)$   
**shows**  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \sqsubseteq \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2)$   
**using** *assms*  
**by** (*rule-tac srdes-tri-refine-intro, simp-all add: refBy-order*)

**lemma** *SRD-peri-under-pre*:

**assumes**  $P$  is SRD  $\$wait' \ \# \ pre_R(P)$   
**shows**  $(pre_R(P) \Rightarrow_r peri_R(P)) = peri_R(P)$

**proof** –

**have**  $peri_R(P) =$   
 $peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$   
**by** (*simp add: SRD-reactive-tri-design assms*)  
**also have**  $\dots = (pre_R P \Rightarrow_r peri_R P)$   
**by** (*simp add: rea-pre-RHS-design rea-peri-RHS-design assms*  
 $unrest \ usubst \ R1\text{-}peri\text{-}SRD \ R2c\text{-}preR \ R1\text{-}rea\text{-}impl \ R2c\text{-}rea\text{-}impl \ R2c\text{-}periR$ )

**finally show** *?thesis ..*

**qed**

**lemma** *SRD-post-under-pre*:

**assumes**  $P$  is SRD  $\$wait' \ \# \ pre_R(P)$   
**shows**  $(pre_R(P) \Rightarrow_r post_R(P)) = post_R(P)$

**proof** –

**have**  $post_R(P) =$   
 $post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$   
**by** (*simp add: SRD-reactive-tri-design assms*)  
**also have**  $\dots = (pre_R P \Rightarrow_r post_R P)$   
**by** (*simp add: rea-pre-RHS-design rea-post-RHS-design assms*  
 $unrest \ usubst \ R1\text{-}post\text{-}SRD \ R2c\text{-}preR \ R1\text{-}rea\text{-}impl \ R2c\text{-}rea\text{-}impl \ R2c\text{-}postR$ )

**finally show** *?thesis ..*

**qed**

**lemma** *SRD-refine-intro*:

**assumes**  
 $P$  is SRD  $Q$  is SRD  
 $'pre_R(P) \Rightarrow pre_R(Q)'$   $'pre_R(P) \wedge peri_R(Q) \Rightarrow peri_R(P)'$   $'pre_R(P) \wedge post_R(Q) \Rightarrow post_R(P)'$   
**shows**  $P \sqsubseteq Q$   
**by** (*metis SRD-reactive-tri-design assms(1) assms(2) assms(3) assms(4) assms(5) srdes-tri-refine-intro*)

**lemma** *SRD-refine-intro'*:

**assumes**  
 $P$  is SRD  $Q$  is SRD  
 $'pre_R(P) \Rightarrow pre_R(Q)'$   $peri_R(P) \sqsubseteq (pre_R(P) \wedge peri_R(Q))$   $post_R(P) \sqsubseteq (pre_R(P) \wedge post_R(Q))$   
**shows**  $P \sqsubseteq Q$   
**using** *assms* **by** (*rule-tac SRD-refine-intro, simp-all add: refBy-order*)

**lemma** *SRD-eq-intro*:

**assumes**  
 $P$  is SRD  $Q$  is SRD  $pre_R(P) = pre_R(Q)$   $peri_R(P) = peri_R(Q)$   $post_R(P) = post_R(Q)$   
**shows**  $P = Q$   
**by** (*metis SRD-reactive-tri-design assms*)

## 4.7 Closure laws

**lemma** *SRD-srdes-skip* [closure]:  $\Pi_R$  is SRD  
 by (*simp add: srdes-skip-def RHS-design-is-SRD unrest*)

**lemma** *SRD-seqr-closure* [closure]:  
 assumes  $P$  is SRD  $Q$  is SRD  
 shows  $(P ;; Q)$  is SRD

**proof** –

have  $(P ;; Q) = \mathbf{R}_s (((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q) \vdash$   
 $((\exists \$st' \cdot \text{peri}_R P) \vee \text{post}_R P ;; \text{peri}_R Q) \diamond \text{post}_R P ;; \text{post}_R Q)$   
 by (*simp add: SRD-composition-wp assms(1) assms(2)*)  
 also have ... is SRD  
 by (*rule RHS-design-is-SRD, simp-all add: wp-rea-def unrest*)  
 finally show ?thesis .

qed

**lemma** *SRD-power-Suc* [closure]:  $P$  is SRD  $\implies P^\wedge(\text{Suc } n)$  is SRD

**proof** (*induct n*)

case 0

then show ?case

by (*simp*)

next

case ( $\text{Suc } n$ )

then show ?case

using *SRD-seqr-closure* by (*simp add: SRD-seqr-closure upred-semiring.power-Suc*)

qed

**lemma** *SRD-power-comp* [closure]:  $P$  is SRD  $\implies P ;; P^\wedge n$  is SRD

by (*metis SRD-power-Suc upred-semiring.power-Suc*)

**lemma** *uplus-SRD-closed* [closure]:  $P$  is SRD  $\implies P^+$  is SRD

by (*simp add: uplus-power-def closure*)

**lemma** *SRD-Sup-closure* [closure]:

assumes  $A \subseteq \llbracket \text{SRD} \rrbracket_H A \neq \{\}$

shows  $(\sqcap A)$  is SRD

**proof** –

have  $\text{SRD } (\sqcap A) = (\sqcap (\text{SRD } A))$

by (*simp add: ContinuousD SRD-Continuous assms(2)*)

also have ... =  $(\sqcap A)$

by (*simp only: Healthy-carrier-image assms*)

finally show ?thesis by (*simp add: Healthy-def*)

qed

## 4.8 Distribution laws

**lemma** *RHS-tri-design-choice* [rdes-def]:

$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcap \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (P_2 \vee Q_2) \diamond (P_3 \vee Q_3))$

apply (*simp add: RHS-design-choice*)

apply (*rule cong[of  $\mathbf{R}_s$   $\mathbf{R}_s$ ]*)

apply (*simp*)

apply (*rel-auto*)

done

**lemma** *RHS-tri-design-sup* [rdes-def]:

$\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqcup \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$   
 by (*simp add: RHS-design-sup, rel-auto*)

**lemma** *RHS-tri-design-conj* [*rdes-def*]:

$(\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \wedge \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)) = \mathbf{R}_s((P_1 \vee Q_1) \vdash ((P_1 \Rightarrow_r P_2) \wedge (Q_1 \Rightarrow_r Q_2)) \diamond ((P_1 \Rightarrow_r P_3) \wedge (Q_1 \Rightarrow_r Q_3)))$   
 by (*simp add: RHS-tri-design-sup conj-upred-def*)

**lemma** *SRD-UINF* [*rdes-def*]:

**assumes**  $A \neq \{\}$   $A \subseteq \llbracket \text{SRD} \rrbracket_H$   
**shows**  $\prod A = \mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R(P)) \vdash (\bigvee P \in A \cdot \text{peri}_R(P)) \diamond (\bigvee P \in A \cdot \text{post}_R(P)))$

**proof** –

**have**  $\prod A = \mathbf{R}_s(\text{pre}_R(\prod A) \vdash \text{peri}_R(\prod A) \diamond \text{post}_R(\prod A))$

by (*metis SRD-as-reactive-tri-design assms srdes-hcond-def*  
*srdes-theory-continuous.healthy-inf srdes-theory-continuous.healthy-inf-def*)

**also have**  $\dots = \mathbf{R}_s((\bigwedge P \in A \cdot \text{pre}_R(P)) \vdash (\bigvee P \in A \cdot \text{peri}_R(P)) \diamond (\bigvee P \in A \cdot \text{post}_R(P)))$

by (*simp add: preR-INF periR-INF postR-INF assms*)

**finally show** *?thesis* .

**qed**

**lemma** *RHS-tri-design-USUP* [*rdes-def*]:

**assumes**  $A \neq \{\}$

**shows**  $(\prod i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i) \diamond R(i))) = \mathbf{R}_s((\bigwedge i \in A \cdot P(i)) \vdash (\prod i \in A \cdot Q(i)) \diamond (\prod i \in A \cdot R(i)))$

by (*subst RHS-INF[OF assms, THEN sym], simp add: design-UINF-mem assms, rel-auto*)

**lemma** *SRD-UINF-mem*:

**assumes**  $A \neq \{\}$   $\bigwedge i. P i$  is SRD

**shows**  $(\prod i \in A \cdot P i) = \mathbf{R}_s((\bigwedge i \in A \cdot \text{pre}_R(P i)) \vdash (\bigvee i \in A \cdot \text{peri}_R(P i)) \diamond (\bigvee i \in A \cdot \text{post}_R(P i)))$   
 (*is ?lhs = ?rhs*)

**proof** –

**have**  $?lhs = (\prod (P \text{ ' } A))$

by (*rel-auto*)

**also have**  $\dots = \mathbf{R}_s((\bigwedge Pa \in P \text{ ' } A \cdot \text{pre}_R Pa) \vdash (\prod Pa \in P \text{ ' } A \cdot \text{peri}_R Pa) \diamond (\prod Pa \in P \text{ ' } A \cdot \text{post}_R Pa))$

by (*subst rdes-def, simp-all add: assms image-subsetI*)

**also have**  $\dots = ?rhs$

by (*rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *RHS-tri-design-UINF-ind* [*rdes-def*]:

$(\prod i \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) = \mathbf{R}_s((\bigwedge i \cdot P_1 i) \vdash (\bigvee i \cdot P_2(i)) \diamond (\bigvee i \cdot P_3(i)))$

by (*rel-auto*)

**lemma** *cond-srea-form* [*rdes-def*]:

$\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$

**proof** –

**have**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft b \triangleright_R \mathbf{R}_s(R \vdash S_1 \diamond S_2) = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2) \triangleleft R2c(\llbracket b \rrbracket_{S<}) \triangleright \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

by (*pred-auto*)

**also have**  $\dots = \mathbf{R}_s(P \vdash Q_1 \diamond Q_2 \triangleleft b \triangleright_R R \vdash S_1 \diamond S_2)$

by (*simp add: RHS-cond lift-cond-srea-def*)

also have ... =  $\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \diamond Q_2 \triangleleft b \triangleright_R S_1 \diamond S_2))$   
 by (*simp add: design-condr lift-cond-srea-def*)  
 also have ... =  $\mathbf{R}_s((P \triangleleft b \triangleright_R R) \vdash (Q_1 \triangleleft b \triangleright_R S_1) \diamond (Q_2 \triangleleft b \triangleright_R S_2))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 finally show *?thesis* .  
 qed

**lemma** *SRD-cond-srea [closure]*:

assumes *P is SRD Q is SRD*  
 shows  $P \triangleleft b \triangleright_R Q$  *is SRD*

**proof** –

have  $P \triangleleft b \triangleright_R Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \triangleleft b \triangleright_R \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$   
 by (*simp add: SRD-reactive-tri-design assms*)

also have ... =  $\mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond (\text{post}_R P \triangleleft b \triangleright_R \text{post}_R Q))$

by (*simp add: cond-srea-form*)

also have ... *is SRD*

by (*simp add: RHS-tri-design-is-SRD lift-cond-srea-def unrest*)

finally show *?thesis* .

qed

## 4.9 Algebraic laws

**lemma** *SRD-left-unit*:

assumes *P is SRD*

shows  $\text{II}_R ;; P = P$

by (*simp add: SRD-composition-wp closure rdes wp C1 R1-negate-R1 R1-false rpred trace-ident-left-periR trace-ident-left-postR SRD-reactive-tri-design assms*)

**lemma** *skip-srea-self-unit [simp]*:

$\text{II}_R ;; \text{II}_R = \text{II}_R$

by (*simp add: SRD-left-unit closure*)

**lemma** *SRD-right-unit-tri-lemma*:

assumes *P is SRD*

shows  $P ;; \text{II}_R = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P)$

by (*simp add: SRD-composition-wp closure rdes wp rpred trace-ident-right-postR assms*)

**lemma** *Miracle-left-zero*:

assumes *P is SRD*

shows *Miracle* ;;  $P = \text{Miracle}$

**proof** –

have  $\text{Miracle} ;; P = \mathbf{R}_s(\text{true} \vdash \text{false}) ;; \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (*simp add: Miracle-def SRD-reactive-design-alt assms*)

also have ... =  $\mathbf{R}_s(\text{true} \vdash \text{false})$

by (*simp add: RHS-design-composition unrest R1-false R2s-false R2s-true*)

also have ... = *Miracle*

by (*simp add: Miracle-def*)

finally show *?thesis* .

qed

**lemma** *Chaos-left-zero*:

assumes *P is SRD*

shows (*Chaos* ;;  $P$ ) = *Chaos*

**proof** –

have  $\text{Chaos} ;; P = \mathbf{R}_s(\text{false} \vdash \text{true}) ;; \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P))$

by (*simp add: Chaos-def SRD-reactive-design-alt assms*)  
 also have ... =  $\mathbf{R}_s ((\neg R1 \text{ true} \wedge \neg (R1 \text{ true} \wedge \neg \$wait')) ; R1 (\neg R2s (pre_R P))) \vdash$   
 $R1 \text{ true} ; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s (cmt_R P)))$   
 by (*simp add: RHS-design-composition unrest R2s-false R2s-true R1-false*)  
 also have ... =  $\mathbf{R}_s ((false \wedge \neg (R1 \text{ true} \wedge \neg \$wait')) ; R1 (\neg R2s (pre_R P))) \vdash$   
 $R1 \text{ true} ; ((\exists \$st \cdot [II]_D) \triangleleft \$wait \triangleright R1 (R2s (cmt_R P)))$   
 by (*simp add: RHS-design-conj-neg-R1-pre*)  
 also have ... =  $\mathbf{R}_s(true)$   
 by (*simp add: design-false-pre*)  
 also have ... =  $\mathbf{R}_s(false \vdash true)$   
 by (*simp add: design-def*)  
 also have ... = *Chaos*  
 by (*simp add: Chaos-def*)  
 finally show *?thesis* .  
 qed

**lemma** *SRD-right-Chaos-tri-lemma:*

assumes *P is SRD*  
 shows  $P ; Chaos = \mathbf{R}_s (((\neg_r pre_R P) wp_r false \wedge post_R P wp_r false) \vdash (\exists \$st' \cdot peri_R P) \diamond false)$   
 by (*simp add: SRD-composition-wp closure rdes assms wp, rel-auto*)

**lemma** *SRD-right-Miracle-tri-lemma:*

assumes *P is SRD*  
 shows  $P ; Miracle = \mathbf{R}_s (((\neg_r pre_R P) wp_r false \vdash (\exists \$st' \cdot peri_R P) \diamond false)$   
 by (*simp add: SRD-composition-wp closure rdes assms wp, rel-auto*)

Stateful reactive designs are left unital

**overloading**

*srdes-unit == utp-unit :: (SRDES, ('s, 't::trace, 'α) rsp) uthy ⇒ ('s, 't, 'α) hrel-rsp*

**begin**

**definition** *srdes-unit :: (SRDES, ('s, 't::trace, 'α) rsp) uthy ⇒ ('s, 't, 'α) hrel-rsp where*  
*srdes-unit T = II<sub>R</sub>*

**end**

**interpretation** *srdes-left-unital: utp-theory-left-unital SRDES*

by (*unfold-locales, simp-all add: srdes-hcond-def srdes-unit-def SRD-seqr-closure SRD-srdes-skip SRD-left-unit*)

## 4.10 Recursion laws

**lemma** *mono-srd-iter:*

assumes *mono F F ∈ [[SRD]]<sub>H</sub> → [[SRD]]<sub>H</sub>*  
 shows *mono (λX.  $\mathbf{R}_s(pre_R(F X) \vdash peri_R(F X) \diamond post_R(F X))$ )*  
 apply (*rule monoI*)  
 apply (*rule srdes-tri-refine-intro'*)  
 apply (*meson assms(1) monoE preR-antitone utp-pred-laws.le-infI2*)  
 apply (*meson assms(1) monoE periR-monotone utp-pred-laws.le-infI2*)  
 apply (*meson assms(1) monoE postR-monotone utp-pred-laws.le-infI2*)  
 done

**lemma** *mu-srd-SRD:*

assumes *mono F F ∈ [[SRD]]<sub>H</sub> → [[SRD]]<sub>H</sub>*  
 shows *(μ X ·  $\mathbf{R}_s(pre_R(F X) \vdash peri_R(F X) \diamond post_R(F X))$ ) is SRD*  
 apply (*subst gfp-unfold*)  
 apply (*simp add: mono-srd-iter assms*)  
 apply (*rule RHS-tri-design-is-SRD*)  
 apply (*simp-all add: unrest*)

done

lemma *mu-srd-iter*:

assumes  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
shows  $(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X)))) = F(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$   
apply (*subst gfp-unfold*)  
apply (*simp add: mono-srd-iter assms*)  
apply (*subst SRD-as-reactive-tri-design [THEN sym]*)  
using *Healthy-func assms(1) assms(2) mu-srd-SRD* apply *blast*  
done

lemma *mu-srd-form*:

assumes  $\text{mono } F \ F \in \llbracket \text{SRD} \rrbracket_H \rightarrow \llbracket \text{SRD} \rrbracket_H$   
shows  $\mu_R \ F = (\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))))$   
proof –  
have 1:  $F(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X)))$  is *SRD*  
by (*simp add: Healthy-apply-closed assms(1) assms(2) mu-srd-SRD*)  
have 2: *Mono<sub>uthy-order</sub> SRDES*  $F$   
by (*simp add: assms(1) mono-Monotone-utp-order*)  
hence 3:  $\mu_R \ F = F(\mu_R \ F)$   
by (*simp add: srdes-theory-continuous.LFP-unfold [THEN sym] assms*)  
hence  $\mathbf{R}_s(\text{pre}_R(F(F(\mu_R \ F))) \vdash \text{peri}_R(F(F(\mu_R \ F))) \diamond \text{post}_R(F(F(\mu_R \ F)))) = \mu_R \ F$   
using *SRD-reactive-tri-design* by *force*  
hence  $(\mu \ X \cdot \mathbf{R}_s(\text{pre}_R(F(X)) \vdash \text{peri}_R(F(X)) \diamond \text{post}_R(F(X))) \sqsubseteq F(\mu_R \ F)$   
by (*simp add: 2 srdes-theory-continuous.weak.LFP-lemma3 gfp-upperbound assms*)  
thus *?thesis*  
using *assms 1 3 srdes-theory-continuous.weak.LFP-lowerbound eq-iff mu-srd-iter*  
by (*metis (mono-tags, lifting)*)

qed

lemma *Monotonic-SRD-comp [closure]*: *Monotonic (op ;; P o SRD)*

by (*simp add: mono-def R1-R2c-is-R2 R2-mono R3h-mono RD1-mono RD2-mono RHS-def SRD-def seqr-mono*)

end

## 5 Normal Reactive Designs

theory *utp-rdes-normal*

imports

*utp-rdes-triples*

*UTP-KAT.utp-kleene*

begin

This additional healthiness condition is analogous to H3

definition *RD3* where

[*upred-defs*]:  $\text{RD3}(P) = P \ ;\; \text{II}_R$

lemma *RD3-idem*:  $\text{RD3}(\text{RD3}(P)) = \text{RD3}(P)$

proof –

have *a*:  $\text{II}_R \ ;\; \text{II}_R = \text{II}_R$

by (*simp add: SRD-left-unit SRD-srdes-skip*)

show *?thesis*

by (*simp add: RD3-def seqr-assoc a*)



qed

**lemma** *RD3-Idempotent [closure]: Idempotent RD3*  
by (*simp add: Idempotent-def RD3-idem*)

**lemma** *RD3-continuous: RD3( $\prod A$ ) = ( $\prod P \in A. RD3(P)$ )*  
by (*simp add: RD3-def seq-Sup-distr*)

**lemma** *RD3-Continuous [closure]: Continuous RD3*  
by (*simp add: Continuous-def RD3-continuous*)

**lemma** *RD3-right-subsumes-RD2: RD2(RD3(P)) = RD3(P)*

**proof** –

have  $a:II_R$  ;;  $J = II_R$

by (*rel-auto*)

show *?thesis*

by (*metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc*)

qed

**lemma** *RD3-left-subsumes-RD2: RD3(RD2(P)) = RD3(P)*

**proof** –

have  $a:J$  ;;  $II_R = II_R$

by (*rel-simp, safe, blast+*)

show *?thesis*

by (*metis (no-types, hide-lams) H2-def RD2-def RD3-def a seqr-assoc*)

qed

**lemma** *RD3-implies-RD2: P is RD3  $\implies$  P is RD2*

by (*metis Healthy-def RD3-right-subsumes-RD2*)

**lemma** *RD3-intro-pre:*

assumes  $P$  is SRD  $(\neg_r \text{pre}_R(P))$  ;;  $\text{true}_r = (\neg_r \text{pre}_R(P)) \ \$st' \ \# \ \text{peri}_R(P)$

shows  $P$  is RD3

**proof** –

have  $RD3(P) = \mathbf{R}_s ((\neg_r \text{pre}_R P) \ \text{wp}_r \ \text{false} \ \vdash \ (\exists \ \$st' \cdot \ \text{peri}_R P) \ \diamond \ \text{post}_R P)$

by (*simp add: RD3-def SRD-right-unit-tri-lemma assms*)

also have  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \ \text{wp}_r \ \text{false} \ \vdash \ \text{peri}_R P \ \diamond \ \text{post}_R P)$

by (*simp add: assms(3) ex-unrest*)

also have  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \ \text{wp}_r \ \text{false} \ \vdash \ \text{cmt}_R P)$

by (*simp add: wait'-cond-peri-post-cmt*)

also have  $\dots = \mathbf{R}_s (\text{pre}_R P \ \vdash \ \text{cmt}_R P)$

by (*simp add: assms(2) rpred wp-rea-def R1-preR*)

finally show *?thesis*

by (*metis Healthy-def SRD-as-reactive-design assms(1)*)

qed

**lemma** *RHS-tri-design-right-unit-lemma:*

assumes  $\$ok' \ \# \ P \ \$ok' \ \# \ Q \ \$ok' \ \# \ R \ \$wait' \ \# \ R$

shows  $\mathbf{R}_s(P \ \vdash \ Q \ \diamond \ R)$  ;;  $II_R = \mathbf{R}_s((\neg_r (\neg_r P))$  ;;  $\text{true}_r) \ \vdash \ ((\exists \ \$st' \cdot Q) \ \diamond \ R)$

**proof** –

have  $\mathbf{R}_s(P \ \vdash \ Q \ \diamond \ R)$  ;;  $II_R = \mathbf{R}_s(P \ \vdash \ Q \ \diamond \ R)$  ;;  $\mathbf{R}_s(\text{true} \ \vdash \ \text{false} \ \diamond \ (\$tr' =_u \ \$tr \ \wedge \ [II]_R))$

by (*simp add: srdes-skip-tri-design, rel-auto*)

also have  $\dots = \mathbf{R}_s ((\neg R1 \ (\neg R2s P))$  ;;  $R1 \ \text{true}) \ \vdash \ (\exists \ \$st' \cdot Q) \ \diamond \ (R1 \ (R2s R))$  ;;  $R1 \ (R2s (\$tr' =_u \ \$tr \ \wedge \ [II]_R))$

by (*simp-all add: RHS-tri-design-composition assms unrest R2s-true R1-false R2s-false*)

also have ... =  $\mathbf{R}_s((\neg R1 (\neg R2s P) ;; R1 \text{ true}) \vdash (\exists \$st' \cdot Q) \diamond R1 (R2s R))$   
**proof** –  
 from *assms*(3,4) **have**  $(R1 (R2s R) ;; R1 (R2s (\$tr' =_u \$tr \wedge [II]_R))) = R1 (R2s R)$   
 by (*rel-auto*, *metis* (*no-types*, *lifting*) *minus-zero-eq*, *meson* *order-refl* *trace-class.diff-cancel*)  
 thus ?thesis  
 by *simp*  
**qed**  
 also have ... =  $\mathbf{R}_s((\neg (\neg P) ;; R1 \text{ true}) \vdash ((\exists \$st' \cdot Q) \diamond R))$   
 by (*metis* (*no-types*, *lifting*) *R1-R2s-R1-true-lemma* *R1-R2s-R2c* *R2c-not* *RHS-design-R2c-pre* *RHS-design-neg-R1-pre* *RHS-design-post-R1* *RHS-design-post-R2s*)  
 also have ... =  $\mathbf{R}_s((\neg_r (\neg_r P) ;; \text{true}_r) \vdash ((\exists \$st' \cdot Q) \diamond R))$   
 by (*rel-auto*)  
**finally show** ?thesis .  
**qed**

**lemma** *RHS-tri-design-RD3-intro*:

**assumes**  
 $\$ok' \# P \ \$ok' \# Q \ \$ok' \# R \ \$st' \# Q \ \$wait' \# R$   
 $P \text{ is } R1 (\neg_r P) ;; \text{true}_r = (\neg_r P)$   
**shows**  $\mathbf{R}_s(P \vdash Q \diamond R)$  is *RD3*  
**apply** (*simp* *add: Healthy-def RD3-def*)  
**apply** (*subst* *RHS-tri-design-right-unit-lemma*)  
**apply** (*simp-all* *add:assms* *ex-unrest* *rpred*)  
**done**

RD3 reactive designs are those whose assumption can be written as a conjunction of a precondition on (undashed) program variables, and a negated statement about the trace. The latter allows us to state that certain events must not occur in the trace – which are effectively safety properties.

**lemma** *R1-right-unit-lemma*:

$\llbracket \text{out}\alpha \# b; \text{out}\alpha \# e \rrbracket \implies (\neg_r b \vee \$tr \hat{=}^u e \leq_u \$tr') ;; R1(\text{true}) = (\neg_r b \vee \$tr \hat{=}^u e \leq_u \$tr')$   
 by (*rel-auto*, *blast*, *metis* (*no-types*, *lifting*) *dual-order.trans*)

**lemma** *RHS-tri-design-RD3-intro-form*:

**assumes**  
 $\text{out}\alpha \# b \ \text{out}\alpha \# e \ \$ok' \# Q \ \$st' \# Q \ \$ok' \# R \ \$wait' \# R$   
**shows**  $\mathbf{R}_s((b \wedge \neg_r \$tr \hat{=}^u e \leq_u \$tr') \vdash Q \diamond R)$  is *RD3*  
**apply** (*rule* *RHS-tri-design-RD3-intro*)  
**apply** (*simp-all* *add: assms* *unrest* *closure* *rpred*)  
**apply** (*subst* *R1-right-unit-lemma*)  
**apply** (*simp-all* *add: assms* *unrest*)  
**done**

**definition** *NSRD* ::  $(\prime s, \prime t :: \text{trace}, \prime \alpha) \text{ hrel-rsp} \Rightarrow (\prime s, \prime t, \prime \alpha) \text{ hrel-rsp}$   
**where** [*upred-defs*]:  $\text{NSRD} = \text{RD1} \circ \text{RD3} \circ \text{RHS}$

**lemma** *RD1-RD3-commute*:  $\text{RD1}(\text{RD3}(P)) = \text{RD3}(\text{RD1}(P))$   
 by (*rel-auto*, *blast+*)

**lemma** *NSRD-is-SRD* [*closure*]:  $P \text{ is } \text{NSRD} \implies P \text{ is } \text{SRD}$

by (*simp* *add: Healthy-def* *NSRD-def* *SRD-def*, *metis* *Healthy-def* *RD1-RD3-commute* *RD2-RHS-commute* *RD3-def* *RD3-right-subsumes-RD2* *SRD-def* *SRD-idem* *SRD-seqr-closure* *SRD-srdes-skip*)

**lemma** *NSRD-elim* [*RD-elim*]:

$\llbracket P \text{ is } \text{NSRD}; Q(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P))) \rrbracket \implies Q(P)$

by (simp add: RD-elim closure)

**lemma** *NSRD-Idempotent* [closure]: *Idempotent NSRD*

by (clarsimp simp add: Idempotent-def NSRD-def, metis (no-types, hide-lams) Healthy-def RD1-RD3-commute RD3-def RD3-idem RD3-left-subsumes-RD2 SRD-def SRD-idem SRD-seqr-closure SRD-srdes-skip)

**lemma** *NSRD-Continuous* [closure]: *Continuous NSRD*

by (simp add: Continuous-comp NSRD-def RD1-Continuous RD3-Continuous RHS-Continuous)

**lemma** *NSRD-form*:

$NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$

**proof** –

have  $NSRD(P) = RD3(SRD(P))$

by (metis (no-types, lifting) NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-def)

also have  $\dots = RD3(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)))$

by (simp add: SRD-as-reactive-tri-design)

also have  $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) ;; II_R$

by (simp add: RD3-def)

also have  $\dots = \mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$

by (simp add: RHS-tri-design-right-unit-lemma unrest)

finally show ?thesis .

qed

**lemma** *NSRD-healthy-form*:

assumes  $P$  is NSRD

shows  $\mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P))) = P$

by (metis Healthy-def NSRD-form assms)

**lemma** *NSRD-Sup-closure* [closure]:

assumes  $A \subseteq \llbracket NSRD \rrbracket_H$   $A \neq \{\}$

shows  $\sqcap A$  is NSRD

**proof** –

have  $NSRD (\sqcap A) = (\sqcap (NSRD 'A))$

by (simp add: ContinuousD NSRD-Continuous assms(2))

also have  $\dots = (\sqcap A)$

by (simp only: Healthy-carrier-image assms)

finally show ?thesis by (simp add: Healthy-def)

qed

**lemma** *intChoice-NSRD-closed* [closure]:

assumes  $P$  is NSRD  $Q$  is NSRD

shows  $P \sqcap Q$  is NSRD

using *NSRD-Sup-closure*[of  $\{P, Q\}$ ] by (simp add: assms)

**lemma** *NRSD-SUP-closure* [closure]:

$\llbracket \bigwedge i. i \in A \implies P(i) \text{ is NSRD}; A \neq \{\} \rrbracket \implies (\sqcap i \in A. P(i)) \text{ is NSRD}$

by (rule NSRD-Sup-closure, auto)

**lemma** *NSRD-neg-pre-unit*:

assumes  $P$  is NSRD

shows  $(\neg_r pre_R(P)) ;; true_r = (\neg_r pre_R(P))$

**proof** –

have  $(\neg_r pre_R(P)) = (\neg_r pre_R(\mathbf{R}_s((\neg_r (\neg_r pre_R(P)) ;; R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P))))$

by (simp add: NSRD-healthy-form assms)

also have  $\dots = R1 (R2c ((\neg_r pre_R P) ;; R1 true))$

**by** (*simp add: rea-pre-RHS-design R1-negate-R1 R1-idem R1-rea-not' R2c-rea-not usubst rpred unrest closure*)  
**also have** ... =  $(\neg_r \text{pre}_R P) ;; R1 \text{ true}$   
**by** (*simp add: R1-R2c-seqr-distribute closure assms*)  
**finally show** ?thesis  
**by** (*simp add: rea-not-def*)  
**qed**

**lemma** *NSRD-neg-pre-left-zero*:  
**assumes** *P is NSRD Q is R1 Q is RD1*  
**shows**  $(\neg_r \text{pre}_R(P)) ;; Q = (\neg_r \text{pre}_R(P))$   
**by** (*metis (no-types, hide-lams) NSRD-neg-pre-unit RD1-left-zero assms(1) assms(2) assms(3) seqr-assoc*)

**lemma** *NSRD-st'-unrest-peri* [*unrest*]:  
**assumes** *P is NSRD*  
**shows**  $\$st' \# \text{peri}_R(P)$   
**proof** –  
**have**  $\text{peri}_R(P) = \text{peri}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true})) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$   
**by** (*simp add: NSRD-healthy-form assms*)  
**also have** ... =  $R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true}) \Rightarrow_r (\exists \$st' \cdot \text{peri}_R P))$   
**by** (*simp add: rea-peri-RHS-design usubst unrest*)  
**also have**  $\$st' \# \dots$   
**by** (*simp add: R1-def R2c-def unrest*)  
**finally show** ?thesis .  
**qed**

**lemma** *NSRD-wait'-unrest-pre* [*unrest*]:  
**assumes** *P is NSRD*  
**shows**  $\$wait' \# \text{pre}_R(P)$   
**proof** –  
**have**  $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true})) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$   
**by** (*simp add: NSRD-healthy-form assms*)  
**also have** ... =  $(R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true})))$   
**by** (*simp add: rea-pre-RHS-design usubst unrest*)  
**also have**  $\$wait' \# \dots$   
**by** (*simp add: R1-def R2c-def unrest*)  
**finally show** ?thesis .  
**qed**

**lemma** *NSRD-st'-unrest-pre* [*unrest*]:  
**assumes** *P is NSRD*  
**shows**  $\$st' \# \text{pre}_R(P)$   
**proof** –  
**have**  $\text{pre}_R(P) = \text{pre}_R(\mathbf{R}_s((\neg_r (\neg_r \text{pre}_R(P)) ;; R1 \text{ true})) \vdash ((\exists \$st' \cdot \text{peri}_R(P)) \diamond \text{post}_R(P)))$   
**by** (*simp add: NSRD-healthy-form assms*)  
**also have** ... =  $R1 (R2c (\neg_r (\neg_r \text{pre}_R P) ;; R1 \text{ true}))$   
**by** (*simp add: rea-pre-RHS-design usubst unrest*)  
**also have**  $\$st' \# \dots$   
**by** (*simp add: R1-def R2c-def unrest*)  
**finally show** ?thesis .  
**qed**

**lemma** *NSRD-alt-def*:  $\text{NSRD}(P) = \text{RD3}(\text{SRD}(P))$   
**by** (*metis NSRD-def RD1-RD3-commute RD3-left-subsumes-RD2 SRD-def comp-eq-dest-lhs*)

**lemma** *preR-RR* [*closure*]:  $P$  is NSRD  $\implies$   $pre_R(P)$  is RR  
 by (rule *RR-intro*, *simp-all add: closure unrest*)

**lemma** *NSRD-neg-pre-RC* [*closure*]:  
 assumes  $P$  is NSRD  
 shows  $pre_R(P)$  is RC  
 by (rule *RC-intro*, *simp-all add: closure assms NSRD-neg-pre-unit rpred*)

**lemma** *NSRD-intro*:  
 assumes  $P$  is SRD  $(\neg_r pre_R(P))$  ;;  $true_r = (\neg_r pre_R(P)) \$st' \# peri_R(P)$   
 shows  $P$  is NSRD

**proof** –

have  $NSRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R(P))$  ;;  $R1 true) \vdash ((\exists \$st' \cdot peri_R(P)) \diamond post_R(P)))$   
 by (*simp add: NSRD-form*)

also have  $\dots = \mathbf{R}_s(pre_R P \vdash peri_R P \diamond post_R P)$   
 by (*simp add: assms ex-unrest rpred closure*)

also have  $\dots = P$   
 by (*simp add: SRD-reactive-tri-design assms(1)*)

**finally show** *?thesis*

using *Healthy-def* by *blast*

**qed**

**lemma** *NSRD-intro'*:  
 assumes  $P$  is R2  $P$  is R3h  $P$  is RD1  $P$  is RD3  
 shows  $P$  is NSRD  
 by (*metis (no-types, hide-lams) Healthy-def NSRD-def R1-R2c-is-R2 RHS-def assms comp-apply*)

**lemma** *NSRD-RC-intro*:  
 assumes  $P$  is SRD  $pre_R(P)$  is RC  $\$st' \# peri_R(P)$   
 shows  $P$  is NSRD  
 by (*metis Healthy-def NSRD-form SRD-reactive-tri-design assms(1) assms(2) assms(3) ex-unrest rea-not-false wp-rea-RC-false wp-rea-def*)

**lemma** *NSRD-rdes-intro* [*closure*]:  
 assumes  $P$  is RC  $Q$  is RR  $R$  is RR  $\$st' \# Q$   
 shows  $\mathbf{R}_s(P \vdash Q \diamond R)$  is NSRD  
 by (rule *NSRD-RC-intro*, *simp-all add: rdes closure assms unrest*)

**lemma** *SRD-RD3-implies-NSRD*:  
 $\llbracket P$  is SRD;  $P$  is RD3  $\rrbracket \implies P$  is NSRD  
 by (*metis (no-types, lifting) Healthy-def NSRD-def RHS-idem SRD-healths(4) SRD-reactive-design comp-apply*)

**lemma** *NSRD-iff*:  
 $P$  is NSRD  $\iff ((P$  is SRD)  $\wedge$   $(\neg_r pre_R(P))$  ;;  $R1(true) = (\neg_r pre_R(P)) \wedge (\$st' \# peri_R(P))$ )  
 by (*meson NSRD-intro NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri*)

**lemma** *NSRD-is-RD3* [*closure*]:  
 assumes  $P$  is NSRD  
 shows  $P$  is RD3  
 by (*simp add: NSRD-is-SRD NSRD-neg-pre-unit NSRD-st'-unrest-peri RD3-intro-pre assms*)

**lemma** *NSRD-refine-elim*:  
 assumes

$P \sqsubseteq Q$   $P$  is NSRD  $Q$  is NSRD  
 $\llbracket \text{'pre}_R(P) \Rightarrow \text{pre}_R(Q) \text{'}; \text{'pre}_R(P) \wedge \text{peri}_R(Q) \Rightarrow \text{peri}_R(P) \text{'}; \text{'pre}_R(P) \wedge \text{post}_R(Q) \Rightarrow \text{post}_R(P) \text{' } \rrbracket$   
 $\Rightarrow R$   
**shows**  $R$   
**proof** –  
**have**  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)) \sqsubseteq \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond \text{post}_R(Q))$   
**by** (*simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) assms(2) assms(3)*)  
**hence**  $1: \text{'pre}_R P \Rightarrow \text{pre}_R Q \text{'}$  **and**  $2: \text{'pre}_R P \wedge \text{peri}_R Q \Rightarrow \text{peri}_R P \text{'}$  **and**  $3: \text{'pre}_R P \wedge \text{post}_R Q \Rightarrow \text{post}_R P \text{'}$   
**by** (*simp-all add: RHS-tri-design-refine assms closure*)  
**with**  $\text{assms}(4)$  **show** *?thesis*  
**by** *simp*  
**qed**

**lemma** *NSRD-right-unit: P is NSRD  $\Rightarrow P ;; II_R = P$*   
**by** (*metis Healthy-if NSRD-is-RD3 RD3-def*)

**lemma** *NSRD-composition-wp:*  
**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $P ;; Q =$   
 $\mathbf{R}_s((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q) \vdash (\text{peri}_R P \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; \text{post}_R Q))$   
**by** (*simp add: SRD-composition-wp assms NSRD-is-SRD wp-rea-def NSRD-neg-pre-unit NSRD-st'-unrest-peri R1-negate-R1 R1-preR ex-unrest rpred*)

**lemma** *preR-NSRD-seq-lemma:*  
**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $R1 (R2c(\text{post}_R P ;; (\neg_r \text{pre}_R Q))) = \text{post}_R P ;; (\neg_r \text{pre}_R Q)$

**proof** –  
**have**  $\text{post}_R P ;; (\neg_r \text{pre}_R Q) = R1(R2c(\text{post}_R P)) ;; R1(R2c(\neg_r \text{pre}_R Q))$   
**by** (*simp add: NSRD-is-SRD R1-R2c-post-RHS R1-rea-not R2c-preR R2c-rea-not assms(1) assms(2)*)  
**also have**  $\dots = R1 (R2c(\text{post}_R P ;; (\neg_r \text{pre}_R Q)))$   
**by** (*simp add: R1-seqr R2c-R1-seq calculation*)  
**finally show** *?thesis ..*  
**qed**

**lemma** *preR-NSRD-seq [rdes]:*  
**assumes**  $P$  is NSRD  $Q$  is SRD  
**shows**  $\text{pre}_R(P ;; Q) = (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q)$   
**by** (*simp add: NSRD-composition-wp assms rea-pre-RHS-design usubst unrest wp-rea-def R2c-disj R1-disj R2c-and R2c-preR R1-R2c-commute[THEN sym] R1-extend-conj' R1-idem R2c-not closure*)  
*(metis (no-types, lifting) Healthy-def Healthy-if NSRD-is-SRD R1-R2c-commute R1-R2c-seqr-distribute R1-seqr-closure assms(1) assms(2) postR-R2c-closed postR-SRD-R1 preR-R2c-closed rea-not-R1 rea-not-R2c)*

**lemma** *periR-NSRD-seq [rdes]:*  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $\text{peri}_R(P ;; Q) = ((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q) \Rightarrow_r (\text{peri}_R P \vee (\text{post}_R P ;; \text{peri}_R Q)))$   
**by** (*simp add: NSRD-composition-wp assms closure rea-peri-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not R1-peri-SRD*)

**lemma** *postR-NSRD-seq [rdes]:*  
**assumes**  $P$  is NSRD  $Q$  is NSRD  
**shows**  $\text{post}_R(P ;; Q) = ((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R Q) \Rightarrow_r (\text{post}_R P ;; \text{post}_R Q))$

by (*simp add: NSRD-composition-wp assms closure rea-post-RHS-design usubst unrest wp-rea-def R1-extend-conj' R1-disj R1-R2c-seqr-distribute R2c-disj R2c-and R2c-rea-impl R1-rea-impl' R2c-preR R2c-periR R1-rea-not' R2c-rea-not*)

**lemma** *NSRD-seqr-closure* [*closure*]:

**assumes** *P is NSRD Q is NSRD*

**shows**  $(P ;; Q)$  *is NSRD*

**proof** –

**have**  $(\neg_r \text{post}_R P \text{wp}_r \text{pre}_R Q) ;; \text{true}_r = (\neg_r \text{post}_R P \text{wp}_r \text{pre}_R Q)$

by (*simp add: wp-rea-def rpred assms closure seqr-assoc NSRD-neg-pre-unit*)

**moreover have**  $\$st' \# \text{pre}_R P \wedge \text{post}_R P \text{wp}_r \text{pre}_R Q \Rightarrow_r \text{peri}_R P \vee \text{post}_R P ;; \text{peri}_R Q$

by (*simp add: unrest assms wp-rea-def*)

**ultimately show** *?thesis*

by (*rule-tac NSRD-intro, simp-all add: seqr-or-distl NSRD-neg-pre-unit assms closure rdes unrest*)

**qed**

**lemma** *RHS-tri-normal-design-composition*:

**assumes**

$\$ok' \# P \ \$ok' \# Q_1 \ \$ok' \# Q_2 \ \$ok \# R \ \$ok \# S_1 \ \$ok \# S_2$

$\$wait \# R \ \$wait' \# Q_2 \ \$wait \# S_1 \ \$wait \# S_2$

*P is R2c Q1 is R1 Q1 is R2c Q2 is R1 Q2 is R2c*

*R is R2c S1 is R1 S1 is R2c S2 is R1 S2 is R2c*

$R1 (\neg P) ;; R1(\text{true}) = R1(\neg P) \ \$st' \# Q_1$

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2) =$

$\mathbf{R}_s((R1 (\neg P) \text{wp}_r \text{false} \wedge Q_2 \text{wp}_r R) \vdash ((\exists \$st' \cdot Q_1) \sqcap (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

by (*simp-all add: RHS-tri-design-composition-wp rea-not-def assms unrest*)

**also have**  $\dots = \mathbf{R}_s((P \wedge Q_2 \text{wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

by (*simp add: assms wp-rea-def ex-unrest, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *RHS-tri-normal-design-composition'* [*rdes-def*]:

**assumes** *P is RC Q1 is RR \$st' # Q1 Q2 is RR R is RR S1 is RR S2 is RR*

**shows**  $\mathbf{R}_s(P \vdash Q_1 \diamond Q_2) ;; \mathbf{R}_s(R \vdash S_1 \diamond S_2)$

$= \mathbf{R}_s((P \wedge Q_2 \text{wp}_r R) \vdash (Q_1 \vee (Q_2 ;; S_1)) \diamond (Q_2 ;; S_2))$

**proof** –

**have**  $R1 (\neg P) ;; R1 \text{true} = R1(\neg P)$

using *RC-implies-RC1[OF assms(1)]*

by (*simp add: Healthy-def RC1-def rea-not-def*)

(*metis R1-negate-R1 R1-seqr utp-pred-laws.double-compl*)

**thus** *?thesis*

by (*simp add: RHS-tri-normal-design-composition assms closure unrest RR-implies-R2c*)

**qed**

If a normal reactive design has postcondition false, then it is a left zero for sequential composition.

**lemma** *NSRD-seq-post-false*:

**assumes** *P is NSRD Q is SRD post<sub>R</sub>(P) = false*

**shows**  $P ;; Q = P$

**apply** (*simp add: NSRD-composition-wp assms wp rpred closure*)

**using** *NSRD-is-SRD SRD-reactive-tri-design assms(1,3)* **apply** *fastforce*

**done**

**lemma** *NSRD-srd-skip* [closure]:  $II_R$  is NSRD  
 by (rule *NSRD-intro*, *simp-all* add: *rdes closure unrest*)

**lemma** *NSRD-Chaos* [closure]: *Chaos* is NSRD  
 by (rule *NSRD-intro*, *simp-all* add: *closure rdes unrest*)

**lemma** *NSRD-Miracle* [closure]: *Miracle* is NSRD  
 by (rule *NSRD-intro*, *simp-all* add: *closure rdes unrest*)

Post-composing a miracle filters out the non-terminating behaviours

**lemma** *NSRD-right-Miracle-tri-lemma*:  
 assumes  $P$  is NSRD  
 shows  $P ;; \text{Miracle} = \mathbf{R}_s (pre_R P \vdash peri_R P \diamond false)$   
 by (*simp* add: *NSRD-composition-wp closure assms rdes wp rpred*)

The set of non-terminating behaviours is a subset

**lemma** *NSRD-right-Miracle-refines*:  
 assumes  $P$  is NSRD  
 shows  $P \sqsubseteq P ;; \text{Miracle}$   
**proof** –  
 have  $\mathbf{R}_s (pre_R P \vdash peri_R P \diamond post_R P) \sqsubseteq \mathbf{R}_s (pre_R P \vdash peri_R P \diamond false)$   
 by (rule *srdes-tri-refine-intro*, *rel-auto+*)  
 thus ?thesis  
 by (*simp* add: *NSRD-elim NSRD-right-Miracle-tri-lemma assms*)  
**qed**

**lemma** *upower-Suc-NSRD-closed* [closure]:  
 $P$  is NSRD  $\implies P \wedge \text{Suc } n$  is NSRD  
**proof** (*induct*  $n$ )  
 case 0  
 then show ?case  
 by (*simp*)  
**next**  
 case (*Suc*  $n$ )  
 then show ?case  
 by (*simp* add: *NSRD-seqr-closure upred-semiring.power-Suc*)  
**qed**

**lemma** *NSRD-power-Suc* [closure]:  
 $P$  is NSRD  $\implies P ;; P \wedge n$  is NSRD  
 by (*metis upower-Suc-NSRD-closed upred-semiring.power-Suc*)

**lemma** *uplus-NSRD-closed* [closure]:  $P$  is NSRD  $\implies P^+$  is NSRD  
 by (*simp* add: *uplus-power-def closure*)

**lemma** *preR-power*:  
 assumes  $P$  is NSRD  
 shows  $pre_R(P ;; P \wedge n) = (\bigsqcup_{i \in \{0..n\}} (post_R(P) \wedge i) wp_r (pre_R(P)))$   
**proof** (*induct*  $n$ )  
 case 0  
 then show ?case  
 by (*simp* add: *wp closure*)  
**next**  
 case (*Suc*  $n$ ) **note** *hyp = this*



**have**  $pre_R (P \wedge (Suc\ n + 1)) = pre_R (P ;; P \wedge (n+1))$   
**by** (*simp add: upred-semiring.power-Suc*)  
**also have** ... =  $(pre_R P \wedge post_R P wp_r pre_R (P \wedge (Suc\ n)))$   
**using** *NSRD-iff assms preR-NSRD-seq upower-Suc-NSRD-closed* **by** *fastforce*  
**also have** ... =  $(pre_R P \wedge post_R P wp_r (\bigsqcup i \in \{0..n\}. post_R P \wedge i wp_r pre_R P))$   
**by** (*simp add: hyp upred-semiring.power-Suc*)  
**also have** ... =  $(pre_R P \wedge (\bigsqcup i \in \{0..n\}. post_R P wp_r (post_R P \wedge i wp_r pre_R P)))$   
**by** (*simp add: wp*)  
**also have** ... =  $(pre_R P \wedge (\bigsqcup i \in \{0..n\}. (post_R P \wedge (i+1) wp_r pre_R P)))$   
**proof** –  
**have**  $\bigwedge i. R1 (post_R P \wedge i ;; (\neg_r pre_R P)) = (post_R P \wedge i ;; (\neg_r pre_R P))$   
**by** (*induct-tac i, simp-all add: closure Healthy-if assms*)  
**thus** *?thesis*  
**by** (*simp add: wp-rea-def upred-semiring.power-Suc seqr-assoc rpred closure assms*)  
**qed**  
**also have** ... =  $(post_R P \wedge 0 wp_r pre_R P \wedge (\bigsqcup i \in \{0..n\}. (post_R P \wedge (i+1) wp_r pre_R P)))$   
**by** (*simp add: wp assms closure*)  
**also have** ... =  $(post_R P \wedge 0 wp_r pre_R P \wedge (\bigsqcup i \in \{1..Suc\ n\}. (post_R P \wedge i wp_r pre_R P)))$   
**proof** –  
**have**  $(\bigsqcup i \in \{0..n\}. (post_R P \wedge (i+1) wp_r pre_R P)) = (\bigsqcup i \in \{1..Suc\ n\}. (post_R P \wedge i wp_r pre_R P))$   
**by** (*rule cong[of Inf], simp-all add: fun-eq-iff*)  
*(metis (no-types, lifting) image-Suc-atLeastAtMost image-cong image-image)*  
**thus** *?thesis* **by** *simp*  
**qed**  
**also have** ... =  $(\bigsqcup i \in insert\ 0\ \{1..Suc\ n\}. (post_R P \wedge i wp_r pre_R P))$   
**by** (*simp add: conj-upred-def*)  
**also have** ... =  $(\bigsqcup i \in \{0..Suc\ n\}. post_R P \wedge i wp_r pre_R P)$   
**by** (*simp add: atLeast0-atMost-Suc-eq-insert-0*)  
**finally show** *?case* **by** (*simp add: upred-semiring.power-Suc*)  
**qed**

**lemma** *preR-power' [rdes]*:  
**assumes** *P is NSRD*  
**shows**  $pre_R(P ;; P^n) = (\bigsqcup i \in \{0..n\}. (post_R(P) \wedge i) wp_r (pre_R(P)))$   
**by** (*simp add: preR-power assms USUP-as-Inf[THEN sym]*)

**lemma** *preR-power-Suc [rdes]*:  
**assumes** *P is NSRD*  
**shows**  $pre_R(P \wedge (Suc\ n)) = (\bigsqcup i \in \{0..n\}. (post_R(P) \wedge i) wp_r (pre_R(P)))$   
**by** (*simp add: upred-semiring.power-Suc rdes assms*)

**declare** *upred-semiring.power-Suc [simp]*

**lemma** *periR-power*:  
**assumes** *P is NSRD*  
**shows**  $peri_R(P ;; P^n) = (pre_R(P \wedge (Suc\ n)) \Rightarrow_r (\prod i \in \{0..n\}. post_R(P) \wedge i) ;; peri_R(P))$   
**proof** (*induct n*)  
**case** *0*  
**then show** *?case*  
**by** (*simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-peri-under-pre assms*)  
**next**  
**case**  $(Suc\ n)$  **note** *hyp = this*  
**have**  $peri_R(P \wedge (Suc\ n + 1)) = peri_R(P ;; P \wedge (n+1))$   
**by** (*simp*)  
**also have** ... =  $(pre_R(P \wedge (Suc\ n + 1)) \Rightarrow_r (peri_R P \vee post_R P ;; peri_R(P ;; P \wedge n)))$

```

  by (simp add: closure assms rdes)
  also have ... = (pre_R(P ^ (Suc n + 1)) ⇒_r (peri_R P ∨ post_R P ;; (pre_R (P ^ (Suc n)) ⇒_r (∏ i∈{0..n}.
post_R P ^ i) ;; peri_R P)))
  by (simp only: hyp)
  also
  have ... = (pre_R P ⇒_r peri_R P ∨ (post_R P wp_r pre_R (P ;; P ^ n) ⇒_r post_R P ;; (pre_R (P ;; P ^ n)
⇒_r (∏ i∈{0..n}. post_R P ^ i) ;; peri_R P)))
  by (simp add: rdes closure assms, rel-blast)
  also
  have ... = (pre_R P ⇒_r peri_R P ∨ (post_R P wp_r pre_R (P ;; P ^ n) ⇒_r post_R P ;; ((∏ i∈{0..n}. post_R
P ^ i) ;; peri_R P)))
  proof -
    have (∏ i∈{0..n}. post_R P ^ i) is R1
    by (simp add: NSRD-is-SRD R1-Continuous R1-power Sup-Continuous-closed assms postR-SRD-R1)
    hence 1:(∏ i∈{0..n}. post_R P ^ i) ;; peri_R P is R1
    by (simp add: closure assms)
    hence (pre_R (P ;; P ^ n) ⇒_r (∏ i∈{0..n}. post_R P ^ i) ;; peri_R P) is R1
    by (simp add: closure)
    hence (post_R P wp_r pre_R (P ;; P ^ n) ⇒_r post_R P ;; (pre_R (P ;; P ^ n) ⇒_r (∏ i∈{0..n}. post_R P
^ i) ;; peri_R P))
    = (post_R P wp_r pre_R (P ;; P ^ n) ⇒_r R1(post_R P) ;; R1(pre_R (P ;; P ^ n) ⇒_r (∏ i∈{0..n}.
post_R P ^ i) ;; peri_R P))
    by (simp add: Healthy-if R1-post-SRD assms closure)
    thus ?thesis
    by (simp only: wp-rea-impl-lemma, simp add: Healthy-if 1, simp add: R1-post-SRD assms closure)
  qed
  also
  have ... = (pre_R P ∧ post_R P wp_r pre_R (P ;; P ^ n) ⇒_r peri_R P ∨ post_R P ;; ((∏ i∈{0..n}. post_R
P ^ i) ;; peri_R P))
  by (pred-auto)
  also
  have ... = (pre_R P ∧ post_R P wp_r pre_R (P ;; P ^ n) ⇒_r peri_R P ∨ ((∏ i∈{0..n}. post_R P ^ (Suc
i)) ;; peri_R P))
  by (simp add: seq-Sup-distl seqr-assoc[THEN sym])
  also
  have ... = (pre_R P ∧ post_R P wp_r pre_R (P ;; P ^ n) ⇒_r peri_R P ∨ ((∏ i∈{1..Suc n}. post_R P ^ i)
;; peri_R P))
  proof -
    have (∏ i∈{0..n}. post_R P ^ Suc i) = (∏ i∈{1..Suc n}. post_R P ^ i)
    apply (rule cong[of Sup], auto)
    apply (metis atLeast0AtMost atMost-iff image-Suc-atLeastAtMost rev-image-eqI upred-semiring.power-Suc)
    using Suc-le-D apply fastforce
  done
  thus ?thesis by simp
  qed
  also
  have ... = (pre_R P ∧ post_R P wp_r pre_R (P ;; P ^ n) ⇒_r ((∏ i∈{0..Suc n}. post_R P ^ i) ;; peri_R P)
  by (simp add: SUP-atLeastAtMost-first uinf-or seqr-or-distl seqr-or-distr)
  also
  have ... = (pre_R(P ^ (Suc (Suc n)))) ⇒_r ((∏ i∈{0..Suc n}. post_R P ^ i) ;; peri_R P)
  by (simp add: rdes closure assms)
  finally show ?case by (simp)
  qed

```

lemma periR-power' [rdes]:

**assumes**  $P$  is NSRD  
**shows**  $\text{peri}_R(P ;; P^n) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r (\prod_{i \in \{0..n\}} \cdot \text{post}_R(P) \wedge i) ;; \text{peri}_R(P))$   
**by** (*simp add: periR-power assms UINF-as-Sup[THEN sym]*)

**lemma** *periR-power-Suc* [rdes]:  
**assumes**  $P$  is NSRD  
**shows**  $\text{peri}_R(P^\wedge(\text{Suc } n)) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r (\prod_{i \in \{0..n\}} \cdot \text{post}_R(P) \wedge i) ;; \text{peri}_R(P))$   
**by** (*simp add: rdes assms*)

**lemma** *postR-power* [rdes]:  
**assumes**  $P$  is NSRD  
**shows**  $\text{post}_R(P ;; P^n) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r \text{post}_R(P) \wedge \text{Suc } n)$   
**proof** (*induct n*)  
**case** 0  
**then show** ?case  
**by** (*simp add: NSRD-is-SRD NSRD-wait'-unrest-pre SRD-post-under-pre assms*)

**next**  
**case** ( $\text{Suc } n$ ) **note** *hyp = this*  
**have**  $\text{post}_R(P \wedge (\text{Suc } n + 1)) = \text{post}_R(P ;; P \wedge (n+1))$   
**by** (*simp*)  
**also have**  $\dots = (\text{pre}_R(P \wedge (\text{Suc } n + 1)) \Rightarrow_r (\text{post}_R P ;; \text{post}_R(P \wedge n)))$   
**by** (*simp add: closure assms rdes*)  
**also have**  $\dots = (\text{pre}_R(P \wedge (\text{Suc } n + 1)) \Rightarrow_r (\text{post}_R P ;; (\text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } n)))$   
**by** (*simp only: hyp*)  
**also**  
**have**  $\dots = (\text{pre}_R P \Rightarrow_r (\text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P ;; (\text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } n)))$   
**by** (*simp add: rdes closure assms, pred-auto*)  
**also**  
**have**  $\dots = (\text{pre}_R P \Rightarrow_r (\text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P ;; \text{post}_R P \wedge \text{Suc } n))$   
**by** (*metis (no-types, lifting) Healthy-if NSRD-is-SRD NSRD-power-Suc R1-power assms hyp postR-SRD-R1 upred-semiring.power-Suc wp-rea-impl-lemma*)  
**also**  
**have**  $\dots = (\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{pre}_R(P \wedge \text{Suc } n) \Rightarrow_r \text{post}_R P \wedge \text{Suc } (\text{Suc } n))$   
**by** (*pred-auto*)  
**also have**  $\dots = (\text{pre}_R(P^\wedge(\text{Suc } (\text{Suc } n))) \Rightarrow_r \text{post}_R P \wedge \text{Suc } (\text{Suc } n))$   
**by** (*simp add: rdes closure assms*)  
**finally show** ?case **by** (*simp*)

**qed**

**lemma** *postR-power-Suc* [rdes]:  
**assumes**  $P$  is NSRD  
**shows**  $\text{post}_R(P^\wedge(\text{Suc } n)) = (\text{pre}_R(P^\wedge(\text{Suc } n)) \Rightarrow_r \text{post}_R(P) \wedge \text{Suc } n)$   
**by** (*simp add: rdes assms*)

**lemma** *power-rdes-def* [rdes-def]:  
**assumes**  $P$  is RC  $Q$  is RR  $R$  is RR  $\$st' \# Q$   
**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))^\wedge(\text{Suc } n)$   
 $= \mathbf{R}_s((\prod_{i \in \{0..n\}} \cdot (R \wedge i)) \text{ wp}_r P) \vdash ((\prod_{i \in \{0..n\}} \cdot R \wedge i) ;; Q) \diamond (R \wedge \text{Suc } n)$   
**proof** (*induct n*)  
**case** 0  
**then show** ?case  
**by** (*simp add: wp assms closure*)

**next**  
**case** ( $\text{Suc } n$ )

have 1:  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot R \text{ wp}_r (R \hat{=} i \text{ wp}_r P))) = (\bigsqcup i \in \{0..Suc\ n\} \cdot R \hat{=} i \text{ wp}_r P)$   
(is ?lhs = ?rhs)

proof –

have ?lhs =  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot (R \hat{=} Suc\ i \text{ wp}_r P)))$   
by (simp add: wp closure assms)  
also have ... =  $(P \wedge (\bigsqcup i \in \{0..n\} \cdot (R \hat{=} Suc\ i \text{ wp}_r P)))$   
by (simp only: USUP-as-Inf-collect)  
also have ... =  $(P \wedge (\bigsqcup i \in \{1..Suc\ n\} \cdot (R \hat{=} i \text{ wp}_r P)))$   
by (metis (no-types, lifting) INF-cong One-nat-def image-Suc-atLeastAtMost image-image)  
also have ... =  $(\bigsqcup i \in \text{insert } 0\ \{1..Suc\ n\} \cdot (R \hat{=} i \text{ wp}_r P))$   
by (simp add: wp assms closure conj-upred-def)  
also have ... =  $(\bigsqcup i \in \{0..Suc\ n\} \cdot (R \hat{=} i \text{ wp}_r P))$   
by (simp add: atLeastAtMost-insertL)  
finally show ?thesis  
by (simp add: USUP-as-Inf-collect)

qed

have 2:  $(Q \vee R ;; (\prod i \in \{0..n\} \cdot R \hat{=} i) ;; Q) = (\prod i \in \{0..Suc\ n\} \cdot R \hat{=} i) ;; Q$   
(is ?lhs = ?rhs)

proof –

have ?lhs =  $(Q \vee (\prod i \in \{0..n\} \cdot R \hat{=} Suc\ i) ;; Q)$   
by (simp add: seqr-assoc[THEN sym] seq-UINF-distl)  
also have ... =  $(Q \vee (\prod i \in \{0..n\} \cdot R \hat{=} Suc\ i) ;; Q)$   
by (simp only: UINF-as-Sup-collect)  
also have ... =  $(Q \vee (\prod i \in \{1..Suc\ n\} \cdot R \hat{=} i) ;; Q)$   
by (metis One-nat-def image-Suc-atLeastAtMost image-image)  
also have ... =  $((\prod i \in \text{insert } 0\ \{1..Suc\ n\} \cdot R \hat{=} i) ;; Q)$   
by (simp add: disj-upred-def[THEN sym] seqr-or-distl)  
also have ... =  $((\prod i \in \{0..Suc\ n\} \cdot R \hat{=} i) ;; Q)$   
by (simp add: atLeastAtMost-insertL)  
finally show ?thesis  
by (simp add: UINF-as-Sup-collect)

qed

have 3:  $(\prod i \in \{0..n\} \cdot R \hat{=} i) ;; Q$  is RR

proof –

have  $(\prod i \in \{0..n\} \cdot R \hat{=} i) ;; Q = (\prod i \in \{0..n\} \cdot R \hat{=} i) ;; Q$   
by (simp add: UINF-as-Sup-collect)  
also have ... =  $(\prod i \in \text{insert } 0\ \{1..n\} \cdot R \hat{=} i) ;; Q$   
by (simp add: atLeastAtMost-insertL)  
also have ... =  $(Q \vee (\prod i \in \{1..n\} \cdot R \hat{=} i) ;; Q)$   
by (metis (no-types, lifting) SUP-insert disj-upred-def seqr-left-unit seqr-or-distl upred-semiring.power-0)  
also have ... =  $(Q \vee (\prod i \in \{0..<n\} \cdot R \hat{=} Suc\ i) ;; Q)$   
by (metis One-nat-def atLeastLessThanSuc-atLeastAtMost image-Suc-atLeastLessThan image-image)  
also have ... =  $(Q \vee (\prod i \in \{0..<n\} \cdot R \hat{=} Suc\ i) ;; Q)$   
by (simp add: UINF-as-Sup-collect)  
also have ... is RR  
by (simp-all add: closure assms)  
finally show ?thesis .

qed

from 1 2 3 Suc show ?case

by (simp add: Suc RHS-tri-normal-design-composition' closure assms wp)

qed

**declare** *upred-semiring.power-Suc* [simp del]

**theorem** *uplus-rdes-def* [rdes-def]:

**assumes** *P is RC Q is RR R is RR \$st' # Q*

**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))^+ = \mathbf{R}_s(R^{*r} \text{ wp}_r P \vdash R^{*r} ;; Q \diamond R^+)$

**proof** –

**have**  $1:(\prod i \cdot R \hat{=} i) ;; Q = R^{*r} ;; Q$

**by** (*metis (no-types) RA1 assms(2) rea-skip-unit(2) rrel-thy.Star-def ustar-alt-def*)

**show** *?thesis*

**by** (*simp add: uplus-power-def seq-UINF-distr wp closure assms rdes-def*)  
(*metis 1 seq-UINF-distr'*)

**qed**

## 5.1 UTP theory

**typedecl** *NSRDES*

**abbreviation**  $NSRDES \equiv UTHY(NSRDES, ('s, 't::trace, 'α) \text{ rsp})$

**overloading**

*nsrdes-hcond* == *utp-hcond* ::  $(NSRDES, ('s, 't::trace, 'α) \text{ rsp}) \text{ uthy} \Rightarrow ((('s, 't, 'α) \text{ rsp} \times ('s, 't, 'α) \text{ rsp})$

*health*

*nsrdes-unit* == *utp-unit* ::  $(NSRDES, ('s, 't::trace, 'α) \text{ rsp}) \text{ uthy} \Rightarrow ('s, 't, 'α) \text{ hrel-rsp}$

**begin**

**definition** *nsrdes-hcond* ::  $(NSRDES, ('s, 't::trace, 'α) \text{ rsp}) \text{ uthy} \Rightarrow ((('s, 't, 'α) \text{ rsp} \times ('s, 't, 'α) \text{ rsp})$

*health* **where**

[*upred-defs*]: *nsrdes-hcond* *T* = *NSRD*

**definition** *nsrdes-unit* ::  $(NSRDES, ('s, 't::trace, 'α) \text{ rsp}) \text{ uthy} \Rightarrow ('s, 't, 'α) \text{ hrel-rsp}$  **where**

[*upred-defs*]: *nsrdes-unit* *T* = *II<sub>R</sub>*

**end**

**interpretation** *nsrd-thy*: *utp-theory-kleene*  $UTHY(NSRDES, ('s, 't::trace, 'α) \text{ rsp})$

**rewrites**  $\bigwedge P. P \in \text{carrier (uthy-order } NSRDES) \iff P \text{ is } NSRD$

**and**  $P \text{ is } \mathcal{H}_{NSRDES} \iff P \text{ is } NSRD$

**and**  $(\mu X \cdot F (\mathcal{H}_{NSRDES} X)) = (\mu X \cdot F (NSRD X))$

**and**  $\text{carrier (uthy-order } NSRDES) \rightarrow \text{carrier (uthy-order } NSRDES) \equiv \llbracket NSRD \rrbracket_H \rightarrow \llbracket NSRD \rrbracket_H$

**and**  $\llbracket \mathcal{H}_{NSRDES} \rrbracket_H \rightarrow \llbracket \mathcal{H}_{NSRDES} \rrbracket_H \equiv \llbracket NSRD \rrbracket_H \rightarrow \llbracket NSRD \rrbracket_H$

**and**  $\top_{NSRDES} = \text{Miracle}$

**and**  $\mathcal{II}_{NSRDES} = II_R$

**and**  $le \text{ (uthy-order } NSRDES) = op \sqsubseteq$

**proof** –

**interpret** *lat*: *utp-theory-continuous*  $UTHY(NSRDES, ('s, 't, 'α) \text{ rsp})$

**by** (*unfold-locales, simp-all add: nsrdes-hcond-def nsrdes-unit-def closure Healthy-if*)

**show**  $1: \top_{NSRDES} = (\text{Miracle} :: ('s, 't, 'α) \text{ hrel-rsp})$

**by** (*metis NSRD-Miracle NSRD-is-SRD lat.top-healthy lat.utp-theory-continuous-axioms nsrdes-hcond-def sdes-theory-continuous.meet-top upred-semiring.add-commute utp-theory-continuous.meet-top*)

**thus** *utp-theory-kleene*  $UTHY(NSRDES, ('s, 't, 'α) \text{ rsp})$

**by** (*unfold-locales, simp-all add: nsrdes-hcond-def nsrdes-unit-def closure Healthy-if Miracle-left-zero SRD-left-unit NSRD-right-unit*)

**qed** (*simp-all add: nsrdes-hcond-def nsrdes-unit-def closure Healthy-if*)

**declare** *nsrd-thy.top-healthy* [simp del]

**declare** *nsrd-thy.bottom-healthy* [simp del]

**abbreviation** *TestR* (*test<sub>R</sub>*) **where**

$test_R P \equiv utest\ NSRDES\ P$

**abbreviation**  $StarR :: ('s, 't::trace, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp\ (-^{*R}\ [999]\ 999)$  **where**  
 $StarR\ P \equiv P\star\ NSRDES$

**lemma**  $StarR\ rdes-def\ [rdes-def]$ :

**assumes**  $P$  is RC  $Q$  is RR  $R$  is RR  $\$st' \# Q$

**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))^{*R} = \mathbf{R}_s((R^{*r}\ wp_r\ P) \vdash R^{*r} ;; Q \diamond R^{*r})$

**by** (*simp add: rrel-thy.Star-alt-def nsrd-thy.Star-alt-def assms closure rdes-def unrest rpred disj-upred-def*)

**end**

## 6 Syntax for reactive design contracts

**theory**  $utp-rdes-contracts$

**imports**  $utp-rdes-normal$

**begin**

We give an experimental syntax for reactive design contracts  $[P \vdash Q | R]_R$ , where  $P$  is a precondition on undashed state variables only,  $Q$  is a pericondition that can refer to the trace and before state but not the after state, and  $R$  is a postcondition. Both  $Q$  and  $R$  can refer only to the trace contribution through a HOL variable  $trace$  which is bound to  $\&tt$ .

**definition**  $mk-RD :: 's\ upred \Rightarrow ('t::trace \Rightarrow 's\ upred) \Rightarrow ('t \Rightarrow 's\ hrel) \Rightarrow ('s, 't, 'a) hrel-rsp$  **where**  
 $mk-RD\ P\ Q\ R = \mathbf{R}_s(\lceil P \rceil_{S<} \vdash \lceil Q(x) \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket \diamond \lceil R(x) \rceil_S \llbracket x \rightarrow \&tt \rrbracket)$

**definition**  $trace-pred :: ('t::trace \Rightarrow 's\ upred) \Rightarrow ('s, 't, 'a) hrel-rsp$  **where**  
 $[upred-defs]: trace-pred\ P = \lceil (P\ x) \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket$

**syntax**

$-trace-var :: logic$

$-mk-RD :: logic \Rightarrow logic \Rightarrow logic \Rightarrow logic\ (\lceil - / \vdash - / \mid - \rceil_R)$

$-trace-pred :: logic \Rightarrow logic\ (\lceil - \rceil_t)$

**parse-translation**  $\ll$

*let*

$fun\ trace-var-tr\ [] = Syntax.free\ trace$   
 $\mid\ trace-var-tr\ - = raise\ Match;$

*in*

$\lceil (@\{syntax-const\ -trace-var\}, K\ trace-var-tr) \rceil$

*end*

$\gg$

**translations**

$\lceil P \vdash Q \mid R \rceil_R \Rightarrow CONST\ mk-RD\ P\ (\lambda\ -trace-var.\ Q)\ (\lambda\ -trace-var.\ R)$

$\lceil P \vdash Q \mid R \rceil_R \leq CONST\ mk-RD\ P\ (\lambda\ x.\ Q)\ (\lambda\ y.\ R)$

$\lceil P \rceil_t \Rightarrow CONST\ trace-pred\ (\lambda\ -trace-var.\ P)$

$\lceil P \rceil_t \leq CONST\ trace-pred\ (\lambda\ t.\ P)$

**lemma**  $SRD-mk-RD\ [closure]: [P \vdash Q(trace) \mid R(trace)]_R$  is SRD

**by** (*simp add: mk-RD-def closure unrest*)

**lemma**  $preR-mk-RD\ [rdes]: pre_R(\lceil P \vdash Q(trace) \mid R(trace) \rceil_R) = R1(\lceil P \rceil_{S<})$

**by** (*simp add: mk-RD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre*)

**lemma** *trace-pred-RR-closed* [*closure*]:

$[P \text{ trace}]_t$  is *RR*

by (*rel-auto*)

**lemma** *unrest-trace-pred-st'* [*unrest*]:

$\$st' \# [P \text{ trace}]_t$

by (*rel-auto*)

**lemma** *R2c-msubst-tt*:  $R2c \ (msubst \ (\lambda x. \lceil Q \ x \rceil_S) \ \&tt) = (msubst \ (\lambda x. \lceil Q \ x \rceil_S) \ \&tt)$

by (*rel-auto*)

**lemma** *periR-mk-RD* [*rdes*]:  $peri_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil Q(\text{trace}) \rceil_{S<})[\text{trace} \rightarrow \&tt]))$

by (*simp add: mk-RD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre R2c-disj R2c-msubst-tt R1-disj R2c-rea-impl R1-rea-impl*)

**lemma** *postR-mk-RD* [*rdes*]:  $post_R([P \vdash Q(\text{trace}) \mid R(\text{trace})]_R) = (\lceil P \rceil_{S<} \Rightarrow_r R1((\lceil R(\text{trace}) \rceil_S)[\text{trace} \rightarrow \&tt]))$

by (*simp add: mk-RD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R2c-rea-impl R1-rea-impl*)

Refinement introduction law for contracts

**lemma** *RD-contract-refine*:

**assumes**

$Q$  is *SRD* ‘ $\lceil P_1 \rceil_{S<} \Rightarrow pre_R Q$ ’

‘ $\lceil P_1 \rceil_{S<} \wedge peri_R Q \Rightarrow \lceil P_2 \ x \rceil_{S<} \llbracket x \rightarrow \&tt \rrbracket$ ’

‘ $\lceil P_1 \rceil_{S<} \wedge post_R Q \Rightarrow \lceil P_3 \ x \rceil_S \llbracket x \rightarrow \&tt \rrbracket$ ’

**shows**  $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq Q$

**proof** –

**have**  $[P_1 \vdash P_2(\text{trace}) \mid P_3(\text{trace})]_R \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$

**using** *assms*

by (*simp add: mk-RD-def, rule-tac srdes-tri-refine-intro, simp-all*)

**thus** *?thesis*

by (*simp add: SRD-reactive-tri-design assms(1)*)

qed

end

## 7 Reactive design tactics

**theory** *utp-rdes-tactics*

**imports** *utp-rdes-triples*

**begin**

Theorems for normalisation

**lemmas** *rdes-rel-norms* =

*prod.case-eq-if*

*conj-assoc*

*disj-assoc*

*conj-UINF-dist*

*conj-UINF-ind-dist*

*seqr-or-distl*

*seqr-or-distr*

*seq-UINF-distl*

*seq-UINF-distl'*

*seq-UINF-distr*

*seq-UINF-distr'*

The following tactic can be used to simply and evaluate reactive predicates.

**method** *rpred-simp* = (*uepr-simp* *simps*: *rpred* *usubst* *closure* *unrest*)

Tactic to expand out healthy reactive design predicates into the syntactic triple form.

**method** *rdes-expand* **uses** *cls* = (*insert* *cls*, (*erule* *RD-elim*)<sup>+</sup>)

Tactic to simplify the definition of a reactive design

**method** *rdes-simp* **uses** *cls* *cong* *simps* =  
(*(rdes-expand* *cls*: *cls*)?, (*simp* *add*: *rdes-def* *rdes-rel-norms* *rdes* *rpred* *cls* *closure* *alpha* *usubst* *unrest*  
*wp* *simps* *cong*: *cong*))

Tactic to split a refinement conjecture into three POs

**method** *rdes-refine-split* **uses** *cls* *cong* *simps* =  
(*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; *rule-tac* *srdes-tri-refine-intro'*)

Tactic to split an equality conjecture into three POs

**method** *rdes-eq-split* **uses** *cls* *cong* *simps* =  
(*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*rule-tac* *srdes-tri-eq-intro*))

Tactic to prove a refinement

**method** *rdes-refine* **uses** *cls* *cong* *simps* =  
(*rdes-refine-split* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*insert* *cls*; *rel-auto*))

Tactics to prove an equality

**method** *rdes-eq* **uses** *cls* *cong* *simps* =  
(*rdes-eq-split* *cls*: *cls* *cong*: *cong* *simps*: *simps*; *rel-auto*)

Via antisymmetry

**method** *rdes-eq-anti* **uses** *cls* *cong* *simps* =  
(*rdes-simp* *cls*: *cls* *cong*: *cong* *simps*: *simps*; (*rule-tac* *antisym*; (*rule-tac* *srdes-tri-refine-intro*; *rel-auto*)))

Tactic to calculate pre/peri/postconditions from reactive designs

**method** *rdes-calc* = (*simp* *add*: *rdes* *rpred* *closure* *alpha* *usubst* *unrest* *wp* *prod.case-eq-if*)

The following tactic attempts to prove a reactive design refinement by calculation of the pre-, peri-, and postconditions and then showing three implications between them using *rel-blast*.

**method** *rdspl-refine* =  
(*rule-tac* *SRD-refine-intro*; (*simp* *add*: *closure* *rdes* *unrest* *usubst* ; *rel-blast*?))

The following tactic combines antisymmetry with the previous tactic to prove an equality.

**method** *rdspl-eq* =  
(*rule-tac* *antisym*, *rdes-refine*, *rdes-refine*)

**end**

## 8 Reactive design parallel-by-merge

**theory** *utp-rdes-parallel*

**imports**

*utp-rdes-normal*



*utp-rdes-tactics*

**begin**

R3h implicitly depends on RD1, and therefore it requires that both sides be RD1. We also require that both sides are R3c, and that  $wait_m$  is a quasi-unit, and  $div_m$  yields divergence.

**lemma** *st-U0-alpha*:  $\lceil \exists \$st \cdot II \rceil_0 = (\exists \$st \cdot \lceil II \rceil_0)$   
**by** (*rel-auto*)

**lemma** *st-U1-alpha*:  $\lceil \exists \$st \cdot II \rceil_1 = (\exists \$st \cdot \lceil II \rceil_1)$   
**by** (*rel-auto*)

**definition** *skip-rm* :: (*'s, 't::trace, 'α*) *rsp merge* ( $II_{RM}$ ) **where**  
 $[upred-defs]: II_{RM} = (\exists \$st_{<} \cdot skip_m \vee (\neg \$ok_{<} \wedge \$tr_{<} \leq_u \$tr'))$

**definition**  $[upred-defs]: R3hm(M) = (II_{RM} \triangleleft \$wait_{<} \triangleright M)$

**lemma** *R3hm-idem*:  $R3hm(R3hm(P)) = R3hm(P)$   
**by** (*rel-auto*)

**lemma** *R3h-par-by-merge* [*closure*]:  
**assumes**  $P$  is R3h  $Q$  is R3h  $M$  is R3hm  
**shows**  $(P \parallel_M Q)$  is R3h

**proof** –

**have**  $(P \parallel_M Q) = (((P \parallel_M Q)[true/\$ok] \triangleleft \$ok \triangleright (P \parallel_M Q)[false/\$ok])[true/\$wait] \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*simp add: cond-var-subst-left cond-var-subst-right*)

**also have**  $\dots = (((P \parallel_M Q)[true, true/\$ok, \$wait] \triangleleft \$ok \triangleright (P \parallel_M Q)[false, true/\$ok, \$wait]) \triangleleft \$wait \triangleright (P \parallel_M Q))$

**by** (*rel-auto*)

**also have**  $\dots = (((\exists \$st \cdot II)[true, true/\$ok, \$wait] \triangleleft \$ok \triangleright (P \parallel_M Q)[false, true/\$ok, \$wait]) \triangleleft \$wait \triangleright (P \parallel_M Q))$

**proof** –

**have**  $(P \parallel_M Q)[true, true/\$ok, \$wait] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<} =_u \$v) ;; R3hm(M))[true, true/\$ok, \$wait]$

**by** (*simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)

**also have**  $\dots = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<} =_u \$v) ;; (\exists \$st_{<} \cdot \$v' =_u \$v_{<}))[true, true/\$ok, \$wait]$

**by** (*rel-blast*)

**also have**  $\dots = ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$v_{<} =_u \$v) ;; (\exists \$st_{<} \cdot \$v' =_u \$v_{<}))[true, true/\$ok, \$wait]$

**by** (*simp add: assms Healthy-if*)

**also have**  $\dots = (\exists \$st \cdot II)[true, true/\$ok, \$wait]$

**by** (*rel-auto*)

**finally show** *?thesis* **by** (*simp add: closure assms unrest*)

**qed**

**also have**  $\dots = (((\exists \$st \cdot II)[true, true/\$ok, \$wait] \triangleleft \$ok \triangleright (R1(true))[false, true/\$ok, \$wait]) \triangleleft \$wait \triangleright (P \parallel_M Q))$

**proof** –

**have**  $(P \parallel_M Q)[false, true/\$ok, \$wait] = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<} =_u \$v) ;; R3hm(M))[false, true/\$ok, \$wait]$

**by** (*simp add: par-by-merge-def U0-as-alpha U1-as-alpha assms Healthy-if*)

**also have**  $\dots = ((\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$v_{<} =_u \$v) ;; (\$tr_{<} \leq_u \$tr'))[false, true/\$ok, \$wait]$

**by** (*rel-blast*)

**also have**  $\dots = ((\lceil R3h(P) \rceil_0 \wedge \lceil R3h(Q) \rceil_1 \wedge \$v_{<} =_u \$v) ;; (\$tr_{<} \leq_u \$tr'))[false, true/\$ok, \$wait]$

**by** (*simp add: assms Healthy-if*)

**also have**  $\dots = (R1(true))[false, true/\$ok, \$wait]$

**by** (*rel-blast*)

**finally show** *?thesis* **by** *simp*

**qed**

**also have** ... =  $((\exists \$st \cdot II) \triangleleft \$ok \triangleright R1(true)) \triangleleft \$wait \triangleright (P \parallel_M Q)$   
**by** (*rel-auto*)  
**also have** ... =  $R3h(P \parallel_M Q)$   
**by** (*simp add: R3h-cases*)  
**finally show** *?thesis*  
**by** (*simp add: Healthy-def*)  
**qed**

**definition** [*upred-defs*]:  $RD1m(M) = (M \vee \neg \$ok_{<} \wedge \$tr_{<} \leq_u \$tr')$

**lemma** *RD1-par-by-merge* [*closure*]:

**assumes**  $P$  is  $R1$   $Q$  is  $R1$   $M$  is  $R1m$   $P$  is  $RD1$   $Q$  is  $RD1$   $M$  is  $RD1m$   
**shows**  $(P \parallel_M Q)$  is  $RD1$

**proof** –

**have** 1:  $(RD1(R1(P)) \parallel_{RD1m(R1m(M))} RD1(R1(Q))) \llbracket false/\$ok \rrbracket = R1(true)$

**by** (*rel-blast*)

**have**  $(P \parallel_M Q) = (P \parallel_M Q) \llbracket true/\$ok \rrbracket \triangleleft \$ok \triangleright (P \parallel_M Q) \llbracket false/\$ok \rrbracket$

**by** (*simp add: cond-var-split*)

**also have** ... =  $R1(P \parallel_M Q) \triangleleft \$ok \triangleright R1(true)$

**by** (*metis 1 Healthy-if R1-par-by-merge assms calculation*)

*cond-idem cond-var-subst-right in-var-uvar ok-vwb-lens*)

**also have** ... =  $RD1(P \parallel_M Q)$

**by** (*simp add: Healthy-if R1-par-by-merge RD1-alt-def assms(3)*)

**finally show** *?thesis*

**by** (*simp add: Healthy-def*)

**qed**

**lemma** *RD2-par-by-merge* [*closure*]:

**assumes**  $M$  is  $RD2$

**shows**  $(P \parallel_M Q)$  is  $RD2$

**proof** –

**have**  $(P \parallel_M Q) = ((P \parallel_s Q) ;; M)$

**by** (*simp add: par-by-merge-def*)

**also from** *assms* **have** ... =  $((P \parallel_s Q) ;; (M ;; J))$

**by** (*simp add: Healthy-def' RD2-def H2-def*)

**also from** *assms* **have** ... =  $((P \parallel_s Q) ;; M) ;; J$

**by** (*simp add: seqr-assoc*)

**also from** *assms* **have** ... =  $RD2(P \parallel_M Q)$

**by** (*simp add: RD2-def H2-def par-by-merge-def*)

**finally show** *?thesis*

**by** (*simp add: Healthy-def'*)

**qed**

**lemma** *SRD-par-by-merge*:

**assumes**  $P$  is  $SRD$   $Q$  is  $SRD$   $M$  is  $R1m$   $M$  is  $R2m$   $M$  is  $R3hm$   $M$  is  $RD1m$   $M$  is  $RD2$

**shows**  $(P \parallel_M Q)$  is  $SRD$

**by** (*rule SRD-intro, simp-all add: assms closure SRD-healths*)

**definition** *nmerge-rd0* ( $N_0$ ) **where**

[*upred-defs*]:  $N_0(M) = (\$wait' =_u (\$0-wait \vee \$1-wait) \wedge \$tr_{<} \leq_u \$tr' \wedge (\exists \$0-ok; \$1-ok; \$ok_{<}; \$ok'; \$0-wait; \$1-wait; \$wait_{<}; \$wait' \cdot M))$

**definition** *nmerge-rd1* ( $N_1$ ) **where**

[*upred-defs*]:  $N_1(M) = (\$ok' =_u (\$0-ok \wedge \$1-ok) \wedge N_0(M))$

**definition** *nmerge-rd* ( $N_R$ ) **where**

[*upred-defs*]:  $N_R(M) = ((\exists \$st_{<} \cdot \$v' =_u \$v_{<}) \triangleleft \$wait_{<} \triangleright N_1(M)) \triangleleft \$ok_{<} \triangleright (\$tr_{<} \leq_u \$tr')$

**definition** *merge-rd1* ( $M_1$ ) **where**

[*upred-defs*]:  $M_1(M) = (N_1(M) ;; II_R)$

**definition** *merge-rd* ( $M_R$ ) **where**

[*upred-defs*]:  $M_R(M) = N_R(M) ;; II_R$

**abbreviation** *rdes-par* ( $- \parallel_R -$  [85,0,86] 85) **where**

$P \parallel_{RM} Q \equiv P \parallel_{M_R(M)} Q$

Healthiness condition for reactive design merge predicates

**definition** [*upred-defs*]:  $RDM(M) = R2m(\exists \$0-ok; \$1-ok; \$ok_{<}; \$ok'; \$0-wait; \$1-wait; \$wait_{<}; \$wait' \cdot M)$

**lemma** *nmerge-rd-is-R1m* [*closure*]:

$N_R(M)$  is *R1m*

**by** (*rel-blast*)

**lemma** *R2m-nmerge-rd*:  $R2m(N_R(R2m(M))) = N_R(R2m(M))$

**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *nmerge-rd-is-R2m* [*closure*]:

$M$  is *R2m*  $\implies N_R(M)$  is *R2m*

**by** (*metis Healthy-def' R2m-nmerge-rd*)

**lemma** *nmerge-rd-is-R3hm* [*closure*]:  $N_R(M)$  is *R3hm*

**by** (*rel-blast*)

**lemma** *nmerge-rd-is-RD1m* [*closure*]:  $N_R(M)$  is *RD1m*

**by** (*rel-blast*)

**lemma** *merge-rd-is-RD3*:  $M_R(M)$  is *RD3*

**by** (*metis Healthy-Idempotent RD3-Idempotent RD3-def merge-rd-def*)

**lemma** *merge-rd-is-RD2*:  $M_R(M)$  is *RD2*

**by** (*simp add: RD3-implies-RD2 merge-rd-is-RD3*)

**lemma** *par-rdes-NSRD* [*closure*]:

**assumes**  $P$  is *SRD*  $Q$  is *SRD*  $M$  is *RDM*

**shows**  $P \parallel_{RM} Q$  is *NSRD*

**proof** –

**have** ( $P \parallel_{N_R} M Q ;; II_R$ ) is *NSRD*

**by** (*rule NSRD-intro'*, *simp-all add: SRD-healths closure assms*)

(*metis (no-types, lifting) Healthy-def R2-par-by-merge R2-seqr-closure R2m-nmerge-rd RDM-def SRD-healths(2) assms skip-srea-R2*)

,*metis Healthy-Idempotent RD3-Idempotent RD3-def*)

**thus** *?thesis*

**by** (*simp add: merge-rd-def par-by-merge-def seqr-assoc*)

**qed**

**lemma** *RDM-intro*:

**assumes**  $M$  is *R2m*  $\$0-ok \# M \ \$1-ok \# M \ \$ok_{<} \# M \ \$ok' \# M$

$\$0-wait \# M \ \$1-wait \# M \ \$wait_{<} \# M \ \$wait' \# M$

shows  $M$  is RDM  
 using *assms*  
 by (*simp add: Healthy-def RDM-def ex-unrest unrest*)

**lemma** *RDM-unrests* [*unrest*]:

assumes  $M$  is RDM  
 shows  $\$0-ok \# M \$1-ok \# M \$ok_< \# M \$ok' \# M$   
 $\$0-wait \# M \$1-wait \# M \$wait_< \# M \$wait' \# M$   
 by (*subst Healthy-if[OF assms, THEN sym], simp-all add: RDM-def unrest, rel-auto*)+

**lemma** *RDM-R1m* [*closure*]:  $M$  is RDM  $\implies M$  is R1m

by (*metis (no-types, hide-lams) Healthy-def R1m-idem R2m-def RDM-def*)

**lemma** *RDM-R2m* [*closure*]:  $M$  is RDM  $\implies M$  is R2m

by (*metis (no-types, hide-lams) Healthy-def R2m-idem RDM-def*)

**lemma** *ex-st'-R2m-closed* [*closure*]:

assumes  $P$  is R2m  
 shows  $(\exists \$st' \cdot P)$  is R2m

**proof** –

have  $R2m(\exists \$st' \cdot R2m P) = (\exists \$st' \cdot R2m P)$   
 by (*rel-auto*)

thus *?thesis*

by (*metis Healthy-def' assms*)

**qed**

**lemma** *parallel-RR-closed*:

assumes  $P$  is RR  $Q$  is RR  $M$  is R2m  
 $\$ok_< \# M \$wait_< \# M \$ok' \# M \$wait' \# M$

shows  $P \parallel_M Q$  is RR

by (*rule RR-R2-intro, simp-all add: unrest assms RR-implies-R2 closure*)

**lemma** *parallel-ok-cases*:

$((P \parallel_s Q) ;; M) = ($   
 $((P^t \parallel_s Q^t) ;; (M \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^f \parallel_s Q^t) ;; (M \llbracket false, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^t \parallel_s Q^f) ;; (M \llbracket true, false / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^f \parallel_s Q^f) ;; (M \llbracket false, false / \$0-ok, \$1-ok \rrbracket))$   
 $)$

**proof** –

have  $((P \parallel_s Q) ;; M) = (\exists ok_0 \cdot (P \parallel_s Q) \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket ;; M \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket$ )  
 by (*subst seqr-middle[of left-uvar ok], simp-all*)

also have  $\dots = (\exists ok_0 \cdot \exists ok_1 \cdot ((P \parallel_s Q) \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket \llbracket \llbracket ok_1 \rrbracket / \$1-ok \rrbracket ;; (M \llbracket \llbracket ok_0 \rrbracket / \$0-ok \rrbracket \llbracket \llbracket ok_1 \rrbracket / \$1-ok \rrbracket$ )

by (*subst seqr-middle[of right-uvar ok], simp-all*)  
 also have  $\dots = (\exists ok_0 \cdot \exists ok_1 \cdot (P \llbracket \llbracket ok_0 \rrbracket / \$ok' \rrbracket \parallel_s Q \llbracket \llbracket ok_1 \rrbracket / \$ok' \rrbracket ;; (M \llbracket \llbracket ok_0 \rrbracket, \llbracket ok_1 \rrbracket / \$0-ok, \$1-ok \rrbracket$ )

by (*rel-auto robust*)

also have  $\dots = ($   
 $((P^t \parallel_s Q^t) ;; (M \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^f \parallel_s Q^t) ;; (M \llbracket false, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^t \parallel_s Q^f) ;; (M \llbracket true, false / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((P^f \parallel_s Q^f) ;; (M \llbracket false, false / \$0-ok, \$1-ok \rrbracket))$   
 $)$

by (*simp add: true-alt-def[THEN sym] false-alt-def[THEN sym] disj-assoc*  
*utp-pred-laws.sup.left-commute utp-pred-laws.sup-commute usubst*)

finally show *?thesis* .

**qed**

**lemma** *skip-srea-ok-f* [*usubst*]:

$II_R^f = R1(\neg \$ok)$   
**by** (*rel-auto*)

**lemma** *nmerge0-rd-unrest* [*unrest*]:

$\$0-ok \# N_0 M \ \$1-ok \# N_0 M$   
**by** (*pred-auto*)<sup>+</sup>

**lemma** *parallel-assm-lemma*:

**assumes** *P* is *RD2*

**shows**  $pre_s \dagger (P \parallel_{M_R(M)} Q) = (((pre_s \dagger P) \parallel_{N_0(M)} ;; R1(true) (cmt_s \dagger Q))$   
 $\vee ((cmt_s \dagger P) \parallel_{N_0(M)} ;; R1(true) (pre_s \dagger Q)))$

**proof** –

**have**  $pre_s \dagger (P \parallel_{M_R(M)} Q) = pre_s \dagger ((P \parallel_s Q) ;; M_R(M))$

**by** (*simp add: par-by-merge-def*)

**also have**  $\dots = ((P \parallel_s Q) \llbracket true, false / \$ok, \$wait \rrbracket ;; N_R M ;; R1(\neg \$ok))$

**by** (*simp add: merge-rd-def usubst, rel-auto*)

**also have**  $\dots = ((P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_s Q \llbracket true, false / \$ok, \$wait \rrbracket) ;; N_1(M) ;; R1(\neg \$ok))$

**by** (*rel-auto robust, metis*)<sup>+</sup>

**also have**  $\dots = (($

$((P \llbracket true, false / \$ok, \$wait \rrbracket)^t \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^t) ;; ((N_1 M) \llbracket true, true / \$0-ok, \$1-ok \rrbracket)$   
 $;; R1(\neg \$ok)) \vee$

$((P \llbracket true, false / \$ok, \$wait \rrbracket)^f \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^t) ;; ((N_1 M) \llbracket false, true / \$0-ok, \$1-ok \rrbracket)$   
 $;; R1(\neg \$ok)) \vee$

$((P \llbracket true, false / \$ok, \$wait \rrbracket)^t \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^f) ;; ((N_1 M) \llbracket true, false / \$0-ok, \$1-ok \rrbracket)$   
 $;; R1(\neg \$ok)) \vee$

$((P \llbracket true, false / \$ok, \$wait \rrbracket)^f \parallel_s (Q \llbracket true, false / \$ok, \$wait \rrbracket)^f) ;; ((N_1 M) \llbracket false, false / \$0-ok, \$1-ok \rrbracket)$   
 $;; R1(\neg \$ok)) \vee$

$(is - = (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4))$

**by** (*subst parallel-ok-cases, subst-tac*)

**also have**  $\dots = (?C2 \vee ?C3)$

**proof** –

**have**  $?C1 = false$

**by** (*rel-auto*)

**moreover have** ‘ $?C4 \Rightarrow ?C3$ ’ (**is** ‘ $(?A ;; ?B) \Rightarrow (?C ;; ?D)$ ’)

**proof** –

**from** *assms* **have** ‘ $P^f \Rightarrow P^t$ ’

**by** (*metis RD2-def H2-equivalence Healthy-def*)

**hence**  $P: P^f_f \Rightarrow P^t_f$

**by** (*rel-auto*)

**have** ‘ $?A \Rightarrow ?C$ ’

**using**  $P$  **by** (*rel-auto*)

**moreover have** ‘ $?B \Rightarrow ?D$ ’

**by** (*rel-auto*)

**ultimately show**  $?thesis$

**by** (*simp add: impl-seqr-mono*)

**qed**

**ultimately show**  $?thesis$

**by** (*simp add: subsumption2*)

**qed**

**also have**  $\dots = ($

$((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((N_0 M) ;; R1(true))) \vee$

$((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((N_0 M) ;; R1(true)))$

**by** (*rel-auto, metis*)<sup>+</sup>

**also have**  $\dots = ($

$((pre_s \dagger P) \parallel_{N_0} M ;; R1(true) (cmt_s \dagger Q)) \vee$   
 $((cmt_s \dagger P) \parallel_{N_0} M ;; R1(true) (pre_s \dagger Q))$   
**by** (*simp add: par-by-merge-def*)  
**finally show** *?thesis* .  
**qed**

**lemma** *pre\_s-SRD*:  
**assumes** *P is SRD*  
**shows**  $pre_s \dagger P = (\neg_r pre_R(P))$   
**proof** –  
**have**  $pre_s \dagger P = pre_s \dagger \mathbf{R}_s(pre_R P \vdash peri_R P \diamond post_R P)$   
**by** (*simp add: SRD-reactive-tri-design assms*)  
**also have**  $\dots = R1(R2c(\neg pre_s \dagger pre_R P))$   
**by** (*simp add: RHS-def usubst R3h-def pre\_s-design*)  
**also have**  $\dots = R1(R2c(\neg pre_R P))$   
**by** (*rel-auto*)  
**also have**  $\dots = (\neg_r pre_R P)$   
**by** (*simp add: R2c-not R2c-preR assms rea-not-def*)  
**finally show** *?thesis* .  
**qed**

**lemma** *parallel-assm*:  
**assumes** *P is SRD Q is SRD*  
**shows**  $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r ((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q)) \wedge$   
 $\neg_r (cmt_R(P) \parallel_{N_0(M)} ;; R1(true) (\neg_r pre_R(Q))))$   
**(is** *?lhs = ?rhs*)  
**proof** –  
**have**  $pre_R(P \parallel_{M_R(M)} Q) = (\neg_r (pre_s \dagger P) \parallel_{N_0} M ;; R1 true (cmt_s \dagger Q) \wedge$   
 $\neg_r (cmt_s \dagger P) \parallel_{N_0} M ;; R1 true (pre_s \dagger Q))$   
**by** (*simp add: pre\_R-def parallel-assm-lemma assms SRD-healths R1-conj rea-not-def[THEN sym]*)  
**also have**  $\dots = ?rhs$   
**by** (*simp add: pre\_s-SRD assms cmt\_R-def Healthy-if closure unrest*)  
**finally show** *?thesis* .  
**qed**

**lemma** *parallel-assm-unrest-wait'* [*unrest*]:  
 $\llbracket P \text{ is SRD}; Q \text{ is SRD} \rrbracket \implies \$wait' \# pre_R(P \parallel_{M_R(M)} Q)$   
**by** (*simp add: parallel-assm, simp add: par-by-merge-def unrest*)

**lemma** *JL1*:  $(M_1 M)^t \llbracket false, true / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$   
**by** (*rel-blast*)

**lemma** *JL2*:  $(M_1 M)^t \llbracket true, false / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$   
**by** (*rel-blast*)

**lemma** *JL3*:  $(M_1 M)^t \llbracket false, false / \$0-ok, \$1-ok \rrbracket = N_0(M) ;; R1(true)$   
**by** (*rel-blast*)

**lemma** *JL4*:  $(M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket = (\$ok' \wedge N_0 M) ;; II_R^t$   
**by** (*simp add: merge-rd1-def usubst nmerge-rd1-def unrest*)

**lemma** *parallel-commitment-lemma-1*:

**assumes**  $P$  is  $RD2$   
**shows**  $cmt_s \dagger (P \parallel_{M_R(M)} Q) = ($   
 $((cmt_s \dagger P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t (cmt_s \dagger Q)) \vee$   
 $((pre_s \dagger P) \parallel_{N_0(M)} ;; R1(true) (cmt_s \dagger Q)) \vee$   
 $((cmt_s \dagger P) \parallel_{N_0(M)} ;; R1(true) (pre_s \dagger Q)))$   
**proof** –  
**have**  $cmt_s \dagger (P \parallel_{M_R(M)} Q) = (P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_{(M_1(M))^t} Q \llbracket true, false / \$ok, \$wait \rrbracket)$   
**by** (*simp add: usubst, rel-auto*)  
**also have**  $\dots = ((P \llbracket true, false / \$ok, \$wait \rrbracket \parallel_s Q \llbracket true, false / \$ok, \$wait \rrbracket) ;; (M_1 M)^t)$   
**by** (*simp add: par-by-merge-def*)  
**also have**  $\dots = ($   
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket false, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, false / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; ((M_1 M)^t \llbracket false, false / \$0-ok, \$1-ok \rrbracket)))$   
**by** (*subst parallel-ok-cases, subst-tac*)  
**also have**  $\dots = ($   
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1 M)^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$   
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$   
 $((pre_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)))$   
 $(\mathbf{is} - = (?C1 \vee_p ?C2 \vee_p ?C3 \vee_p ?C4))$   
**by** (*simp add: JL1 JL2 JL3*)  
**also have**  $\dots = ($   
 $((cmt_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; ((M_1(M))^t \llbracket true, true / \$0-ok, \$1-ok \rrbracket)) \vee$   
 $((pre_s \dagger P) \parallel_s (cmt_s \dagger Q)) ;; (N_0(M) ;; R1(true)) \vee$   
 $((cmt_s \dagger P) \parallel_s (pre_s \dagger Q)) ;; (N_0(M) ;; R1(true)))$   
**proof** –  
**from** *assms* **have**  $\langle P^f \Rightarrow P^t \rangle$   
**by** (*metis RD2-def H2-equivalence Healthy-def*)  
**hence**  $P: \langle P^f_f \Rightarrow P^t_f \rangle$   
**by** (*rel-auto*)  
**have**  $\langle ?C4 \Rightarrow ?C3 \rangle$  (**is**  $\langle ?A ;; ?B \rangle \Rightarrow \langle ?C ;; ?D \rangle$ )  
**proof** –  
**have**  $\langle ?A \Rightarrow ?C \rangle$   
**using**  $P$  **by** (*rel-auto*)  
**thus**  $?thesis$   
**by** (*simp add: impl-seqr-mono*)  
**qed**  
**thus**  $?thesis$   
**by** (*simp add: subsumption2*)  
**qed**  
**finally show**  $?thesis$   
**by** (*simp add: par-by-merge-def JL4*)  
**qed**

**lemma parallel-commitment-lemma-2:**

**assumes**  $P$  is  $RD2$   
**shows**  $cmt_s \dagger (P \parallel_{M_R(M)} Q) =$   
 $((cmt_s \dagger P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t (cmt_s \dagger Q)) \vee pre_s \dagger (P \parallel_{M_R(M)} Q)$   
**by** (*simp add: parallel-commitment-lemma-1 assms parallel-assm-lemma*)

**lemma parallel-commitment-lemma-3:**

$M$  is  $R1m \implies (\$ok' \wedge N_0 M) ;; II_R^t$  is  $R1m$

by (*rel-simp*, *safe*, *metis+*)

**lemma** *parallel-commitment*:

assumes *P* is SRD *Q* is SRD *M* is RDM

shows  $cmt_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R(M)} Q) \Rightarrow_r cmt_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t cmt_R(Q))$

by (*simp add: parallel-commitment-lemma-2 parallel-commitment-lemma-3 Healthy-if assms cmt\_R-def pre\_s-SRD closure rea-impl-def disj-comm unrest*)

**theorem** *parallel-reactive-design*:

assumes *P* is SRD *Q* is SRD *M* is RDM

shows  $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s($

$(\neg_r ((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q)) \wedge$

$\neg_r (cmt_R(P) \parallel_{N_0(M)} ;; R1(true) (\neg_r pre_R(Q)))) \vdash$

$(cmt_R(P) \parallel_{(\$ok' \wedge N_0 M)} ;; II_R^t cmt_R(Q))$  (**is** *?lhs* = *?rhs*)

**proof** –

have  $(P \parallel_{M_R(M)} Q) = \mathbf{R}_s(pre_R(P \parallel_{M_R(M)} Q) \vdash cmt_R(P \parallel_{M_R(M)} Q))$

by (*metis Healthy-def NSRD-is-SRD SRD-as-reactive-design assms(1) assms(2) assms(3) par-rdes-NSRD*)

also have ... = *?rhs*

by (*simp add: parallel-assm parallel-commitment design-export-spec assms, rel-auto*)

finally show *?thesis* .

qed

**lemma** *parallel-pericondition-lemma1*:

$(\$ok' \wedge P) ;; II_R[true,true/\$ok', \$wait'] = (\exists \$st' \cdot P)[true,true/\$ok',\$wait']$

(**is** *?lhs* = *?rhs*)

**proof** –

have  $?lhs = (\$ok' \wedge P) ;; (\exists \$st' \cdot II)[true,true/\$ok', \$wait']$

by (*rel-blast*)

also have ... = *?rhs*

by (*rel-auto*)

finally show *?thesis* .

qed

**lemma** *parallel-pericondition-lemma2*:

assumes *M* is RDM

shows  $(\exists \$st' \cdot N_0(M))[true,true/\$ok', \$wait'] = ((\$0-wait \vee \$1-wait) \wedge (\exists \$st' \cdot M))$

**proof** –

have  $(\exists \$st' \cdot N_0(M))[true,true/\$ok', \$wait'] = (\exists \$st' \cdot (\$0-wait \vee \$1-wait) \wedge \$tr' \geq_u \$tr_< \wedge M)$

by (*simp add: usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)

also have ... =  $(\exists \$st' \cdot (\$0-wait \vee \$1-wait) \wedge M)$

by (*metis (no-types, hide-lams) Healthy-if R1m-def R1m-idem R2m-def RDM-def assms utp-pred-laws.inf-commute*)

also have ... =  $(\$0-wait \vee \$1-wait) \wedge (\exists \$st' \cdot M)$

by (*rel-auto*)

finally show *?thesis* .

qed

**lemma** *parallel-pericondition-lemma3*:

$(\$0-wait \vee \$1-wait) \wedge (\exists \$st' \cdot M) = ((\$0-wait \wedge \$1-wait \wedge (\exists \$st' \cdot M)) \vee (\neg \$0-wait \wedge \$1-wait \wedge (\exists \$st' \cdot M)) \vee (\$0-wait \wedge \neg \$1-wait \wedge (\exists \$st' \cdot M)))$

by (*rel-auto*)

**lemma** *parallel-pericondition [rdes]*:

fixes *M* :: (*'s, 't*::*trace, 'α*) *rsp merge*



**assumes**  $P$  is SRD  $Q$  is SRD  $M$  is RDM

**shows**  $peri_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r peri_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q))$   
 $\vee post_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q)$   
 $\vee peri_R(P) \parallel_{\exists \$st' \cdot M} post_R(Q)$

**proof** –

**have**  $peri_R(P \parallel_{M_R(M)} Q) =$

$(pre_R(P \parallel_{M_R} M Q) \Rightarrow_r cmt_R P \parallel_{(\$ok' \wedge N_0 M)} ;; II_R[true, true/\$ok', \$wait']) cmt_R Q$

**by** (*simp add: peri-cmt-def parallel-commitment SRD-healths assms usubst unrest assms*)

**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r cmt_R P \parallel_{(\exists \$st' \cdot N_0 M)[true, true/\$ok', \$wait']) cmt_R Q$

**by** (*simp add: parallel-pericondition-lemma1*)

**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r cmt_R P \parallel_{(\$0-wait \vee \$1-wait) \wedge (\exists \$st' \cdot M)} cmt_R Q$

**by** (*simp add: parallel-pericondition-lemma2 assms*)

**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r (([cmt_R P]_0 \wedge [cmt_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\$0-wait \wedge$   
 $\$1-wait \wedge (\exists \$st' \cdot M)))$

$\vee ([cmt_R P]_0 \wedge [cmt_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\neg \$0-wait \wedge \$1-wait$   
 $\wedge (\exists \$st' \cdot M)))$

$\vee ([cmt_R P]_0 \wedge [cmt_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\$0-wait \wedge \neg \$1-wait$   
 $\wedge (\exists \$st' \cdot M)))$

**by** (*simp add: par-by-merge-alt-def parallel-pericondition-lemma3 seqr-or-distr*)

**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r (([peri_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\exists \$st' \cdot M)$

$\vee ([post_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\exists \$st' \cdot M)$

$\vee ([peri_R P]_0 \wedge [post_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\exists \$st' \cdot M)))$

**by** (*simp add: seqr-right-one-point-true seqr-right-one-point-false cmt\_R-def post\_R-def peri\_R-def usubst unrest assms*)

**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r peri_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q)$

$\vee post_R(P) \parallel_{\exists \$st' \cdot M} peri_R(Q)$

$\vee peri_R(P) \parallel_{\exists \$st' \cdot M} post_R(Q))$

**by** (*simp add: par-by-merge-alt-def*)

**finally show** *?thesis* .

**qed**

**lemma** *parallel-postcondition-lemma1:*

$(\$ok' \wedge P) ;; II_R[true, false/\$ok', \$wait'] = P[true, false/\$ok', \$wait']$

(*is ?lhs = ?rhs*)

**proof** –

**have**  $?lhs = (\$ok' \wedge P) ;; II[true, false/\$ok', \$wait']$

**by** (*rel-blast*)

**also have**  $\dots = ?rhs$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *parallel-postcondition-lemma2:*

**assumes**  $M$  is RDM

**shows**  $(N_0(M))[true, false/\$ok', \$wait'] = ((\neg \$0-wait \wedge \neg \$1-wait) \wedge M)$

**proof** –

**have**  $(N_0(M))[true, false/\$ok', \$wait'] = ((\neg \$0-wait \wedge \neg \$1-wait) \wedge \$tr' \geq_u \$tr_{<} \wedge M)$

**by** (*simp add: usubst unrest nmerge-rd0-def ex-unrest Healthy-if R1m-def assms*)

**also have**  $\dots = ((\neg \$0-wait \wedge \neg \$1-wait) \wedge M)$

**by** (*metis Healthy-if R1m-def RDM-R1m assms utp-pred-laws.inf-commute*)

**finally show** *?thesis* .

**qed**

**lemma** *parallel-postcondition [rdes]:*

**fixes**  $M :: ('s, 't :: trace, 'a) rsp merge$

**assumes**  $P$  is SRD  $Q$  is SRD  $M$  is RDM  
**shows**  $post_R(P \parallel_{M_R(M)} Q) = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r post_R(P) \parallel_M post_R(Q))$   
**proof** –  
**have**  $post_R(P \parallel_{M_R(M)} Q) =$   
 $(pre_R(P \parallel_{M_R} M Q) \Rightarrow_r cmt_R P \parallel (\$ok' \wedge N_0 M) ;; II_R[true, false/\$ok', \$wait'] cmt_R Q)$   
**by** (*simp add: post-cmt-def parallel-commitment assms usubst unrest SRD-healths*)  
**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r cmt_R P \parallel (\neg \$0-wait \wedge \neg \$1-wait \wedge M) cmt_R Q)$   
**by** (*simp add: parallel-postcondition-lemma1 parallel-postcondition-lemma2 assms,*  
*simp add: utp-pred-laws.inf-commute utp-pred-laws.inf-left-commute*)  
**also have**  $\dots = (pre_R(P \parallel_{M_R} M Q) \Rightarrow_r post_R P \parallel_M post_R Q)$   
**by** (*simp add: par-by-merge-alt-def seqr-right-one-point-false usubst unrest cmt\_R-def post\_R-def assms*)  
**finally show** *?thesis* .  
**qed**

**lemma** *parallel-precondition-lemma:*

**fixes**  $M :: ('s, 't :: trace, 'a) rsp\ merge$   
**assumes**  $P$  is NSRD  $Q$  is NSRD  $M$  is RDM  
**shows**  $(\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q) =$   
 $((\neg_r pre_R P) \parallel_M ;; R1(true) peri_R Q \vee (\neg_r pre_R P) \parallel_M ;; R1(true) post_R Q)$   
**proof** –  
**have**  $((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) cmt_R(Q)) =$   
 $((\neg_r pre_R(P)) \parallel_{N_0(M)} ;; R1(true) (peri_R(Q) \diamond post_R(Q)))$   
**by** (*simp add: wait'-cond-peri-post-cmt*)  
**also have**  $\dots = (([\neg_r pre_R(P)]_0 \wedge [peri_R(Q) \diamond post_R(Q)]_1 \wedge \$v_{<} =_u \$v) ;; N_0(M) ;; R1(true))$   
**by** (*simp add: par-by-merge-alt-def*)  
**also have**  $\dots = (([\neg_r pre_R(P)]_0 \wedge [peri_R(Q)]_1 \triangleleft \$1-wait' \triangleright [post_R(Q)]_1 \wedge \$v_{<} =_u \$v) ;; N_0(M) ;; R1(true))$   
**by** (*simp add: wait'-cond-def alpha*)  
**also have**  $\dots = ((([\neg_r pre_R(P)]_0 \wedge [peri_R(Q)]_1 \wedge \$v_{<} =_u \$v) \triangleleft \$1-wait' \triangleright ([\neg_r pre_R(P)]_0 \wedge [post_R(Q)]_1 \wedge \$v_{<} =_u \$v)) ;; N_0(M) ;; R1(true))$   
 $(is\ (?P\ ;;\ -) = (?Q\ ;;\ -))$   
**proof** –  
**have**  $?P = ?Q$   
**by** (*rel-auto*)  
**thus** *?thesis* **by** *simp*  
**qed**  
**also have**  $\dots = (([\neg_r pre_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v)[true/\$1-wait'] ;; (N_0 M ;; R1 true)[true/\$1-wait'] \vee$   
 $([\neg_r pre_R P]_0 \wedge [post_R Q]_1 \wedge \$v_{<} =_u \$v)[false/\$1-wait'] ;; (N_0 M ;; R1 true)[false/\$1-wait'])$   
**by** (*simp add: cond-inter-var-split*)  
**also have**  $\dots = (([\neg_r pre_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v) ;; N_0 M[true/\$1-wait'] ;; R1 true \vee$   
 $([\neg_r pre_R P]_0 \wedge [post_R Q]_1 \wedge \$v_{<} =_u \$v) ;; N_0 M[false/\$1-wait'] ;; R1 true)$   
**by** (*simp add: usubst unrest*)  
**also have**  $\dots = (([\neg_r pre_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\$wait' \wedge M) ;; R1 true \vee$   
 $([\neg_r pre_R P]_0 \wedge [post_R Q]_1 \wedge \$v_{<} =_u \$v) ;; (\$wait' =_u \$0-wait \wedge M) ;; R1 true)$   
**proof** –  
**have**  $(\$tr' \geq_u \$tr_{<} \wedge M) = M$   
**using** *RDM-R1m[OF assms(3)]*  
**by** (*simp add: Healthy-def R1m-def conj-comm*)  
**thus** *?thesis*  
**by** (*simp add: nmerge-rd0-def unrest assms closure ex-unrest usubst*)  
**qed**  
**also have**  $\dots = (([\neg_r pre_R P]_0 \wedge [peri_R Q]_1 \wedge \$v_{<} =_u \$v) ;; M ;; R1 true \vee$

( $\text{is } (?P_1 \vee_p ?P_2) = (?Q_1 \vee ?Q_2)$ )  
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \mathbf{\$v}_{<} =_u \mathbf{\$v}) ;; M ;; R1 \text{ true})$

**proof** –

**have**  $?P_1 = ([\neg_r \text{pre}_R P]_0 \wedge [\text{peri}_R Q]_1 \wedge \mathbf{\$v}_{<} =_u \mathbf{\$v}) ;; (M \wedge \mathbf{\$wait}')$  ;;  $R1 \text{ true}$   
**by** (*simp add: conj-comm*)

**hence 1:**  $?P_1 = ?Q_1$

**by** (*simp add: seqr-left-one-point-true seqr-left-one-point-false add: unrest usubst closure assms*)

**have**  $?P_2 = (([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \mathbf{\$v}_{<} =_u \mathbf{\$v}) ;; (M \wedge \mathbf{\$wait}')$  ;;  $R1 \text{ true} \vee$   
 $([\neg_r \text{pre}_R P]_0 \wedge [\text{post}_R Q]_1 \wedge \mathbf{\$v}_{<} =_u \mathbf{\$v}) ;; (M \wedge \neg \mathbf{\$wait}')$  ;;  $R1 \text{ true})$

**by** (*subst seqr-bool-split[of left-wvar wait], simp-all add: usubst unrest assms closure conj-comm*)

**hence 2:**  $?P_2 = ?Q_2$

**by** (*simp add: seqr-left-one-point-true seqr-left-one-point-false unrest usubst closure assms*)

**from 1 2 show**  $?thesis$  **by** *simp*

**qed**

**also have**  $\dots = ((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$

**by** (*simp add: par-by-merge-alt-def*)

**finally show**  $?thesis$  .

**qed**

**lemma** *swap-nmerge-rd0:*

$\text{swap}_m ;; N_0(M) = N_0(\text{swap}_m ;; M)$

**by** (*rel-auto, meson+*)

**lemma** *SymMerge-nmerge-rd0 [closure]:*

$M \text{ is SymMerge} \implies N_0(M) \text{ is SymMerge}$

**by** (*rel-auto, meson+*)

**lemma** *swap-merge-rd':*

$\text{swap}_m ;; N_R(M) = N_R(\text{swap}_m ;; M)$

**by** (*rel-blast*)

**lemma** *swap-merge-rd:*

$\text{swap}_m ;; M_R(M) = M_R(\text{swap}_m ;; M)$

**by** (*simp add: merge-rd-def seqr-assoc[THEN sym] swap-merge-rd'*)

**lemma** *SymMerge-merge-rd [closure]:*

$M \text{ is SymMerge} \implies M_R(M) \text{ is SymMerge}$

**by** (*simp add: Healthy-def swap-merge-rd*)

**lemma** *nmerge-rd1-merge3:*

**assumes**  $M \text{ is RDM}$

**shows**  $\mathbf{M3}(N_1(M)) = (\mathbf{\$ok}' =_u (\mathbf{\$0-ok} \wedge \mathbf{\$1-0-ok} \wedge \mathbf{\$1-1-ok}) \wedge$   
 $\mathbf{\$wait}' =_u (\mathbf{\$0-wait} \vee \mathbf{\$1-0-wait} \vee \mathbf{\$1-1-wait}) \wedge$   
 $\mathbf{M3}(M))$

**proof** –

**have**  $\mathbf{M3}(N_1(M)) = \mathbf{M3}(\mathbf{\$ok}' =_u (\mathbf{\$0-ok} \wedge \mathbf{\$1-ok}) \wedge$

$\mathbf{\$wait}' =_u (\mathbf{\$0-wait} \vee \mathbf{\$1-wait}) \wedge$

$\mathbf{\$tr}_{<} \leq_u \mathbf{\$tr}' \wedge$

$(\exists \{\mathbf{\$0-ok}, \mathbf{\$1-ok}, \mathbf{\$ok}_{<}, \mathbf{\$ok}', \mathbf{\$0-wait}, \mathbf{\$1-wait}, \mathbf{\$wait}_{<}, \mathbf{\$wait}'\} \cdot \text{RDM}(M))$ )

**by** (*simp add: nmerge-rd1-def nmerge-rd0-def assms Healthy-if*)

**also have**  $\dots = \mathbf{M3}(\mathbf{\$ok}' =_u (\mathbf{\$0-ok} \wedge \mathbf{\$1-ok}) \wedge \mathbf{\$wait}' =_u (\mathbf{\$0-wait} \vee \mathbf{\$1-wait}) \wedge \text{RDM}(M))$

**by** (*rel-blast*)

**also have**  $\dots = (\mathbf{\$ok}' =_u (\mathbf{\$0-ok} \wedge \mathbf{\$1-0-ok} \wedge \mathbf{\$1-1-ok}) \wedge \mathbf{\$wait}' =_u (\mathbf{\$0-wait} \vee \mathbf{\$1-0-wait}$   
 $\vee \mathbf{\$1-1-wait}) \wedge \mathbf{M3}(\text{RDM}(M)))$

**by** (*rel-blast*)

also have ... = ( $\$ok' =_u (\$0-ok \wedge \$1-0-ok \wedge \$1-1-ok) \wedge \$wait' =_u (\$0-wait \vee \$1-0-wait \vee \$1-1-wait) \wedge \mathbf{M}\mathcal{B}(M)$ )  
 by (*simp add: assms Healthy-if*)  
 finally show *?thesis* .  
 qed

**lemma** *nmerge-rd-merge3*:

$\mathbf{M}\mathcal{B}(N_R(M)) = (\exists \$st_{<} \cdot \$v' =_u \$v_{<}) \triangleleft \$wait_{<} \triangleright \mathbf{M}\mathcal{B}(N_1 M) \triangleleft \$ok_{<} \triangleright (\$tr_{<} \leq_u \$tr')$   
 by (*rel-blast*)

**lemma** *swap-merge-RDM-closed* [*closure*]:

assumes *M is RDM*  
 shows *swap<sub>m</sub> ;; M is RDM*

**proof** –

have  $RDM(\text{swap}_m ;; RDM(M)) = (\text{swap}_m ;; RDM(M))$   
 by (*rel-auto*)  
 thus *?thesis*  
 by (*metis Healthy-def' assms*)

qed

**lemma** *parallel-precondition*:

fixes  $M :: ('s, 't::\text{trace}, 'a) \text{rsp merge}$   
 assumes *P is NSRD Q is NSRD M is RDM*  
 shows  $\text{pre}_R(P \parallel_{M_R(M)} Q) =$

$(\neg_r ((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{peri}_R P) \wedge$   
 $\neg_r ((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{post}_R P))$

**proof** –

have  $a: (\neg_r \text{pre}_R(P)) \parallel_{N_0(M)} ;; R1(\text{true}) \text{cmt}_R(Q) =$   
 $((\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{peri}_R Q \vee (\neg_r \text{pre}_R P) \parallel_M ;; R1(\text{true}) \text{post}_R Q)$   
 by (*simp add: parallel-precondition-lemma assms*)

have  $b: (\neg_r \text{cmt}_R P \parallel_{N_0 M} ;; R1 \text{true} (\neg_r \text{pre}_R Q)) =$   
 $(\neg_r (\neg_r \text{pre}_R(Q)) \parallel_{N_0(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{cmt}_R(P))$

by (*simp add: swap-nmerge-rd0[THEN sym] seqr-assoc[THEN sym] par-by-merge-def par-sep-swap*)

have  $c: (\neg_r \text{pre}_R(Q)) \parallel_{N_0(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{cmt}_R(P) =$   
 $((\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{peri}_R P \vee (\neg_r \text{pre}_R Q) \parallel_{(\text{swap}_m ;; M)} ;; R1(\text{true}) \text{post}_R$   
 $P)$

by (*simp add: parallel-precondition-lemma closure assms*)

show *?thesis*

by (*simp add: parallel-assm closure assms a b c, rel-auto*)

qed

Weakest Parallel Precondition

**definition** *wrR* ::

$('t::\text{trace}, 'a) \text{hrel-rp} \Rightarrow$   
 $('t :: \text{trace}, 'a) \text{rp merge} \Rightarrow$   
 $('t, 'a) \text{hrel-rp} \Rightarrow$   
 $('t, 'a) \text{hrel-rp} (- \text{wr}_R'(-) - [60,0,61] 61)$

where [*upred-defs*]:  $Q \text{wr}_R(M) P = (\neg_r ((\neg_r P) \parallel_M ;; R1(\text{true}) Q))$

**lemma** *wrR-R1* [*closure*]:

$M$  is  $R1m \implies Q$  wr<sub>R</sub>( $M$ )  $P$  is  $R1$   
 by (simp add: wrR-def closure)

**lemma** *R2-rea-not*:  $R2(\neg_r P) = (\neg_r R2(P))$   
 by (rel-auto)

**lemma** *wrR-R2-lemma*:  
 assumes  $P$  is  $R2$   $Q$  is  $R2$   $M$  is  $R2m$   
 shows  $((\neg_r P) \parallel_M Q) ;; R1(true_h)$  is  $R2$

**proof** –  
 have  $(\neg_r P) \parallel_M Q$  is  $R2$   
 by (simp add: closure assms)  
 thus ?thesis  
 by (simp add: closure)

qed

**lemma** *wrR-R2 [closure]*:  
 assumes  $P$  is  $R2$   $Q$  is  $R2$   $M$  is  $R2m$   
 shows  $Q$  wr<sub>R</sub>( $M$ )  $P$  is  $R2$

**proof** –  
 have  $((\neg_r P) \parallel_M Q) ;; R1(true_h)$  is  $R2$   
 by (simp add: wrR-R2-lemma assms)  
 thus ?thesis  
 by (simp add: wrR-def wrR-R2-lemma par-by-merge-seq-add closure)

qed

**lemma** *wrR-RR [closure]*:  
 assumes  $P$  is  $RR$   $Q$  is  $RR$   $M$  is  $RDM$   
 shows  $Q$  wr<sub>R</sub>( $M$ )  $P$  is  $RR$   
 apply (rule RR-intro)  
 apply (simp-all add: unrest assms closure wrR-def rpred)  
 apply (metis (no-types, lifting) Healthy-def' R1-R2c-commute R1-R2c-is-R2 R1-rea-not RDM-R2m  
 RR-implies-R2 assms(1) assms(2) assms(3) par-by-merge-seq-add rea-not-R2-closed  
 wrR-R2-lemma)

done

**lemma** *wrR-RC [closure]*:  
 assumes  $P$  is  $RR$   $Q$  is  $RR$   $M$  is  $RDM$   
 shows  $(Q$  wr<sub>R</sub>( $M$ )  $P)$  is  $RC$   
 apply (rule RC-intro)  
 apply (simp add: closure assms)  
 apply (simp add: wrR-def rpred closure assms )  
 apply (simp add: par-by-merge-def seqr-assoc)

done

**lemma** *wppR-choice [wp]*:  $(P \vee Q)$  wr<sub>R</sub>( $M$ )  $R = (P$  wr<sub>R</sub>( $M$ )  $R \wedge Q$  wr<sub>R</sub>( $M$ )  $R)$

**proof** –  
 have  $(P \vee Q)$  wr<sub>R</sub>( $M$ )  $R =$   
 $(\neg_r ((\neg_r R) ;; U0 \wedge (P ;; U1 \vee Q ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r)$   
 by (simp add: wrR-def par-by-merge-def seqr-or-distl)  
 also have ... =  $(\neg_r ((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v} \vee (\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$\mathbf{v}_{<}' =_u$   
 $\$ \mathbf{v}) ;; M ;; true_r)$   
 by (simp add: conj-disj-distr utp-pred-laws.inf-sup-distrib2)  
 also have ... =  $(\neg_r (((\neg_r R) ;; U0 \wedge P ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r \vee$   
 $((\neg_r R) ;; U0 \wedge Q ;; U1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M ;; true_r))$

by (*simp add: segr-or-distl*)  
 also have ... =  $(P \text{ wr}_R(M) R \wedge Q \text{ wr}_R(M) R)$   
 by (*simp add: wrR-def par-by-merge-def*)  
 finally show *?thesis* .  
 qed

**lemma** *wppR-miracle* [*wp*]:  $\text{false wr}_R(M) P = \text{true}_r$   
 by (*simp add: wrR-def*)

**lemma** *wppR-true* [*wp*]:  $P \text{ wr}_R(M) \text{true}_r = \text{true}_r$   
 by (*simp add: wrR-def*)

**lemma** *parallel-precondition-wr* [*rdes*]:  
 assumes *P is NSRD Q is NSRD M is RDM*  
 shows  $\text{pre}_R(P \parallel_{M_R(M)} Q) = (\text{peri}_R(Q) \text{ wr}_R(M) \text{pre}_R(P) \wedge \text{post}_R(Q) \text{ wr}_R(M) \text{pre}_R(P) \wedge$   
 $\text{peri}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{pre}_R(Q) \wedge \text{post}_R(P) \text{ wr}_R(\text{swap}_m ;; M) \text{pre}_R(Q))$   
 by (*simp add: assms parallel-precondition wrR-def*)

**lemma** *parallel-rdes-def* [*rdes-def*]:  
 assumes *P<sub>1</sub> is RC P<sub>2</sub> is RR P<sub>3</sub> is RR Q<sub>1</sub> is RC Q<sub>2</sub> is RR Q<sub>3</sub> is RR*  
 $\$st' \# P_2 \$st' \# Q_2$   
*M is RDM*  
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \parallel_{M_R(M)} \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s(((Q_1 \Rightarrow_r Q_2) \text{ wr}_R(M) P_1 \wedge (Q_1 \Rightarrow_r Q_3) \text{ wr}_R(M) P_1 \wedge$   
 $(P_1 \Rightarrow_r P_2) \text{ wr}_R(\text{swap}_m ;; M) Q_1 \wedge (P_1 \Rightarrow_r P_3) \text{ wr}_R(\text{swap}_m ;; M) Q_1) \vdash$   
 $((P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee$   
 $(P_1 \Rightarrow_r P_3) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_2) \vee (P_1 \Rightarrow_r P_2) \parallel_{\exists \$st' . M} (Q_1 \Rightarrow_r Q_3)) \diamond$   
 $((P_1 \Rightarrow_r P_3) \parallel_M (Q_1 \Rightarrow_r Q_3)))$  (*is ?lhs = ?rhs*)

**proof** –  
 have *?lhs = R<sub>s</sub> (pre<sub>R</sub> ?lhs ⊢ peri<sub>R</sub> ?lhs ◊ post<sub>R</sub> ?lhs)*  
 by (*simp add: SRD-reactive-tri-design assms closure*)  
 also have ... = *?rhs*  
 by (*simp add: rdes closure unrest assms, rel-auto*)  
 finally show *?thesis* .  
 qed

**lemma** *Miracle-parallel-left-zero*:  
 assumes *P is SRD M is RDM*  
 shows *Miracle*  $\parallel_{RM} P = \text{Miracle}$

**proof** –  
 have  $\text{pre}_R(\text{Miracle} \parallel_{RM} P) = \text{true}_r$   
 by (*simp add: parallel-assm wait'-cond-idem rdes closure assms*)  
 moreover hence  $\text{cmt}_R(\text{Miracle} \parallel_{RM} P) = \text{false}$   
 by (*simp add: rdes closure wait'-cond-idem SRD-healths assms*)  
 ultimately have  $\text{Miracle} \parallel_{RM} P = \mathbf{R}_s(\text{true}_r \vdash \text{false})$   
 by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed*)  
 thus *?thesis*  
 by (*simp add: Miracle-def R1-design-R1-pre*)  
 qed

**lemma** *Miracle-parallel-right-zero*:  
 assumes *P is SRD M is RDM*  
 shows  $P \parallel_{RM} \text{Miracle} = \text{Miracle}$

**proof** –  
 have  $\text{pre}_R(P \parallel_{RM} \text{Miracle}) = \text{true}_r$

by (*simp add: wait'-cond-idem parallel-assm rdes closure assms*)  
**moreover hence**  $cmt_R(P \parallel_{RM} \text{Miracle}) = \text{false}$   
 by (*simp add: wait'-cond-idem rdes closure SRD-healths assms*)  
**ultimately have**  $P \parallel_{RM} \text{Miracle} = \mathbf{R}_s(\text{true}_r \vdash \text{false})$   
 by (*metis NSRD-iff SRD-reactive-design-alt assms par-rdes-NSRD srdes-theory-continuous.weak.top-closed*)  
**thus** *?thesis*  
 by (*simp add: Miracle-def R1-design-R1-pre*)  
**qed**

## 8.1 Example basic merge

**definition** *BasicMerge* ::  $((s, t::\text{trace}, \text{unit}) \text{ rsp}) \text{ merge } (N_B)$  **where**  
*[upred-defs]: BasicMerge* =  $(\text{\$tr}_{<} \leq_u \text{\$tr}' \wedge \text{\$tr}' - \text{\$tr}_{<} =_u \text{\$0-tr} - \text{\$tr}_{<} \wedge \text{\$tr}' - \text{\$tr}_{<} =_u \text{\$1-tr} - \text{\$tr}_{<} \wedge \text{\$st}' =_u \text{\$st}_{<})$

**abbreviation** *rbasic-par* ( $- \parallel_B - [85,86] 85$ ) **where**  
 $P \parallel_B Q \equiv P \parallel_{M_R(N_B)} Q$

**lemma** *BasicMerge-RDM* [*closure*]:  $N_B$  *is RDM*  
 by (*rule RDM-intro, (rel-auto)+*)

**lemma** *BasicMerge-SymMerge* [*closure*]:  
 $N_B$  *is SymMerge*  
 by (*rel-auto*)

**lemma** *BasicMerge'-calc*:  
**assumes**  $\text{\$ok}' \# P \text{\$wait}' \# P \text{\$ok}' \# Q \text{\$wait}' \# Q$  *P is R2 Q is R2*  
**shows**  $P \parallel_{N_B} Q = ((\exists \text{\$st}' \cdot P) \wedge (\exists \text{\$st}' \cdot Q) \wedge \text{\$st}' =_u \text{\$st})$   
**using** *assms*

**proof** –  
**have**  $P: (\exists \{\text{\$ok}', \text{\$wait}'\} \cdot R2(P)) = P$  (**is** *?P' = -*)  
 by (*simp add: ex-unrest ex-plus Healthy-if assms*)  
**have**  $Q: (\exists \{\text{\$ok}', \text{\$wait}'\} \cdot R2(Q)) = Q$  (**is** *?Q' = -*)  
 by (*simp add: ex-unrest ex-plus Healthy-if assms*)  
**have**  $?P' \parallel_{N_B} ?Q' = ((\exists \text{\$st}' \cdot ?P') \wedge (\exists \text{\$st}' \cdot ?Q') \wedge \text{\$st}' =_u \text{\$st})$   
 by (*simp add: par-by-merge-alt-def, rel-auto, blast+*)  
**thus** *?thesis*  
 by (*simp add: P Q*)

**qed**

## 8.2 Simple parallel composition

**definition** *rea-design-par* ::  
 $(s, t::\text{trace}, \alpha) \text{ hrel-rsp} \Rightarrow (s, t, \alpha) \text{ hrel-rsp} \Rightarrow (s, t, \alpha) \text{ hrel-rsp}$  (**infixr**  $\parallel_R 85$ )  
**where** [*upred-defs*]:  $P \parallel_R Q = \mathbf{R}_s((\text{pre}_R(P) \wedge \text{pre}_R(Q)) \vdash (\text{cmt}_R(P) \wedge \text{cmt}_R(Q)))$

**lemma** *RHS-design-par*:  
**assumes**  
 $\text{\$ok}' \# P_1 \text{\$ok}' \# P_2$   
**shows**  $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$

**proof** –  
**have**  $\mathbf{R}_s(P_1 \vdash Q_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2) =$   
 $\mathbf{R}_s(P_1 \llbracket \text{true}, \text{false} / \text{\$ok}, \text{\$wait} \rrbracket \vdash Q_1 \llbracket \text{true}, \text{false} / \text{\$ok}, \text{\$wait} \rrbracket) \parallel_R \mathbf{R}_s(P_2 \llbracket \text{true}, \text{false} / \text{\$ok}, \text{\$wait} \rrbracket \vdash$   
 $Q_2 \llbracket \text{true}, \text{false} / \text{\$ok}, \text{\$wait} \rrbracket)$   
 by (*simp add: RHS-design-ok-wait*)

**also from** *assms*  
**have** ... =  
 $\mathbf{R}_s((R1 (R2c (P_1)) \wedge R1 (R2c (P_2))) \llbracket true, false / \$ok, \$wait \rrbracket) \vdash$   
 $(R1 (R2c (P_1 \Rightarrow Q_1)) \wedge R1 (R2c (P_2 \Rightarrow Q_2))) \llbracket true, false / \$ok, \$wait \rrbracket)$   
**apply** (*simp add: rea-design-par-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest assms*)  
**apply** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp*)  
**using** *assms apply (rel-auto)*  
**done**  
**also have** ... =  
 $\mathbf{R}_s((R2c(P_1) \wedge R2c(P_2)) \vdash$   
 $(R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$   
**by** (*metis (no-types, hide-lams) R1-R2s-R2c R1-conj R1-design-R1-pre RHS-design-ok-wait*)  
**also have** ... =  
 $\mathbf{R}_s((P_1 \wedge P_2) \vdash (R1 (R2s (P_1 \Rightarrow Q_1)) \wedge R1 (R2s (P_2 \Rightarrow Q_2))))$   
**by** (*simp add: R2c-R3h-commute R2c-and R2c-design R2c-idem R2c-not RHS-def*)  
**also have** ... =  $\mathbf{R}_s((P_1 \wedge P_2) \vdash ((P_1 \Rightarrow Q_1) \wedge (P_2 \Rightarrow Q_2)))$   
**by** (*metis (no-types, lifting) R1-conj R2s-conj RHS-design-export-R1 RHS-design-export-R2s*)  
**also have** ... =  $\mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**finally show** *?thesis* .  
**qed**

**lemma** *RHS-tri-design-par*:  
**assumes**  $\$ok' \# P_1 \$ok' \# P_2$   
**shows**  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$   
**by** (*simp add: RHS-design-par assms unrest wait'-cond-conj-exchange*)

**lemma** *RHS-tri-design-par-RR* [*rdes-def*]:  
**assumes**  $P_1$  *is RR*  $P_2$  *is RR*  
**shows**  $\mathbf{R}_s(P_1 \vdash Q_1 \diamond R_1) \parallel_R \mathbf{R}_s(P_2 \vdash Q_2 \diamond R_2) = \mathbf{R}_s((P_1 \wedge P_2) \vdash (Q_1 \wedge Q_2) \diamond (R_1 \wedge R_2))$   
**by** (*simp add: RHS-tri-design-par unrest assms*)

**lemma** *RHS-comp-assoc*:  
**assumes**  $P$  *is NSRD*  $Q$  *is NSRD*  $R$  *is NSRD*  
**shows**  $(P \parallel_R Q) \parallel_R R = P \parallel_R Q \parallel_R R$   
**by** (*rdes-eq cls: assms*)

**end**

## 9 Productive Reactive Designs

**theory** *utp-rdes-productive*  
**imports** *utp-rdes-parallel*  
**begin**

### 9.1 Healthiness condition

A reactive design is productive if it strictly increases the trace, whenever it terminates. If it does not terminate, it is also classed as productive.

**definition** *Productive* ::  $(s, t::trace, \alpha)$  *hrel-rsp*  $\Rightarrow (s, t, \alpha)$  *hrel-rsp* **where**  
[*upred-defs*]:  $Productive(P) = P \parallel_R \mathbf{R}_s(true \vdash true \diamond (\$tr <_u \$tr'))$

**lemma** *Productive-RHS-design-form*:  
**assumes**  $\$ok' \# P \$ok' \# Q \$ok' \# R$



**shows**  $Productive(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash Q \diamond (R \wedge \$tr <_u \$tr'))$   
**using** *assms* **by** (*simp add: Productive-def RHS-tri-design-par unrest*)

**lemma** *Productive-form*:

$Productive(SRD(P)) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

**proof** –

**have**  $Productive(SRD(P)) = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) \parallel_R \mathbf{R}_s(true \vdash true \diamond (\$tr <_u \$tr'))$

**by** (*simp add: Productive-def SRD-as-reactive-tri-design*)

**also have**  $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

**by** (*simp add: RHS-tri-design-par unrest*)

**finally show** *?thesis* .

**qed**

A reactive design is productive provided that the postcondition, under the precondition, strictly increases the trace.

**lemma** *Productive-intro*:

**assumes**  $P$  is SRD  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P)) \$wait' \# pre_R(P)$

**shows**  $P$  is Productive

**proof** –

**have**  $P : \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) = P$

**proof** –

**have**  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge post_R(P)))$

**by** (*metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)

**also have**  $\dots = \mathbf{R}_s(pre_R(P) \vdash (pre_R(P) \wedge peri_R(P)) \diamond (pre_R(P) \wedge (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*metis assms(2) utp-pred-laws.inf.absorb1 utp-pred-laws.inf.assoc*)

**also have**  $\dots = \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr'))$

**by** (*metis (no-types, hide-lams) design-export-pre wait'-cond-conj-exchange wait'-cond-idem*)

**finally show** *?thesis*

**by** (*simp add: SRD-reactive-tri-design assms(1)*)

**qed**

**thus** *?thesis*

**by** (*metis Healthy-def RHS-tri-design-par Productive-def ok'-pre-unrest unrest-true utp-pred-laws.inf-right-idem utp-pred-laws.inf-top-right*)

**qed**

**lemma** *Productive-post-refines-tr-increase*:

**assumes**  $P$  is SRD  $P$  is Productive  $\$wait' \# pre_R(P)$

**shows**  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge post_R(P))$

**proof** –

**have**  $post_R(P) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*metis Healthy-def Productive-form assms(1) assms(2)*)

**also have**  $\dots = R1(R2c(pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*simp add: rea-post-RHS-design unrest usubst assms, rel-auto*)

**also have**  $\dots = R1((pre_R(P) \Rightarrow (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*simp add: R2c-impl R2c-preR R2c-postR R2c-and R2c-tr-less-tr' assms*)

**also have**  $(\$tr <_u \$tr') \sqsubseteq (pre_R(P) \wedge \dots)$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *Continuous-Productive [closure]*: Continuous Productive

**by** (*simp add: Continuous-def Productive-def, rel-auto*)

## 9.2 Reactive design calculations

**lemma** *preR-Productive* [rdes]:

**assumes** *P is SRD*

**shows**  $pre_R(Productive(P)) = pre_R(P)$

**proof** –

**have**  $pre_R(Productive(P)) = pre_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*metis Healthy-def Productive-form assms*)

**thus** *?thesis*

**by** (*simp add: rea-pre-RHS-design usubst unrest R2c-not R2c-preR R1-preR Healthy-if assms*)

**qed**

**lemma** *periR-Productive* [rdes]:

**assumes** *P is NSRD*

**shows**  $peri_R(Productive(P)) = peri_R(P)$

**proof** –

**have**  $peri_R(Productive(P)) = peri_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*metis Healthy-def NSRD-is-SRD Productive-form assms*)

**also have**  $\dots = R1 (R2c (pre_R P \Rightarrow_r peri_R P))$

**by** (*simp add: rea-peri-RHS-design usubst unrest R2c-not assms closure*)

**also have**  $\dots = (pre_R P \Rightarrow_r peri_R P)$

**by** (*simp add: R1-rea-impl R2c-rea-impl R2c-preR R2c-peri-SRD*

*R1-peri-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)

**finally show** *?thesis*

**by** (*simp add: SRD-peri-under-pre assms unrest closure*)

**qed**

**lemma** *postR-Productive* [rdes]:

**assumes** *P is NSRD*

**shows**  $post_R(Productive(P)) = (pre_R(P) \Rightarrow_r post_R(P) \wedge \$tr <_u \$tr')$

**proof** –

**have**  $post_R(Productive(P)) = post_R(\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')))$

**by** (*metis Healthy-def NSRD-is-SRD Productive-form assms*)

**also have**  $\dots = R1 (R2c (pre_R P \Rightarrow_r post_R P \wedge \$tr' >_u \$tr))$

**by** (*simp add: rea-post-RHS-design usubst unrest assms closure*)

**also have**  $\dots = (pre_R P \Rightarrow_r post_R P \wedge \$tr' >_u \$tr)$

**by** (*simp add: R1-rea-impl R2c-rea-impl R2c-preR R2c-and R1-extend-conj' R2c-post-SRD*

*R1-post-SRD assms closure R1-tr-less-tr' R2c-tr-less-tr'*)

**finally show** *?thesis* .

**qed**

**lemma** *preR-frame-seq-export*:

**assumes** *P is NSRD P is Productive Q is NSRD*

**shows**  $(pre_R P \wedge (pre_R P \wedge post_R P) ;; Q) = (pre_R P \wedge (post_R P ;; Q))$

**proof** –

**have**  $(pre_R P \wedge (post_R P ;; Q)) = (pre_R P \wedge ((pre_R P \Rightarrow_r post_R P) ;; Q))$

**by** (*simp add: SRD-post-under-pre assms closure unrest*)

**also have**  $\dots = (pre_R P \wedge (((\neg_r pre_R P) ;; Q) \vee (pre_R P \Rightarrow_r R1(post_R P)) ;; Q))$

**by** (*simp add: NSRD-is-SRD R1-post-SRD assms(1) rea-impl-def seqr-or-distl R1-preR Healthy-if*)

**also have**  $\dots = (pre_R P \wedge (((\neg_r pre_R P) ;; Q) \vee (pre_R P \wedge post_R P) ;; Q))$

**proof** –

**have**  $(pre_R P \vee \neg_r pre_R P) = R1 true$

**by** (*simp add: R1-preR rea-not-or*)

**then show** *?thesis*

**by** (*metis (no-types, lifting) R1-def conj-comm disj-comm disj-conj-distr rea-impl-def seqr-or-distl utp-pred-laws.inf-top-left utp-pred-laws.sup.left-idem*)

qed  
**also have** ... =  $(pre_R P \wedge (((\neg_r pre_R P) \vee (pre_R P \wedge post_R P) ;; Q)))$   
**by** (*simp add: NSRD-neg-pre-left-zero assms closure SRD-healths*)  
**also have** ... =  $(pre_R P \wedge (pre_R P \wedge post_R P) ;; Q)$   
**by** (*rel-blast*)  
**finally show** *?thesis ..*  
qed

### 9.3 Closure laws

**lemma** *Productive-rdes-intro*:

**assumes**  $(\$tr <_u \$tr') \sqsubseteq R \$ok' \# P \$ok' \# Q \$ok' \# R \$wait \# P \$wait' \# P$   
**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))$  *is Productive*

**proof** (*rule Productive-intro*)

**show**  $\mathbf{R}_s(P \vdash Q \diamond R)$  *is SRD*

**by** (*simp add: RHS-tri-design-is-SRD assms*)

**from** *assms(1)* **show**  $(\$tr' >_u \$tr) \sqsubseteq (pre_R (\mathbf{R}_s(P \vdash Q \diamond R)) \wedge post_R (\mathbf{R}_s(P \vdash Q \diamond R)))$

**apply** (*simp add: rea-pre-RHS-design rea-post-RHS-design usubst assms unrest*)

**using** *assms(1)* **apply** (*rel-auto*)

**apply** *fastforce*

**done**

**show**  $\$wait' \# pre_R (\mathbf{R}_s(P \vdash Q \diamond R))$

**by** (*simp add: rea-pre-RHS-design rea-post-RHS-design usubst R1-def R2c-def R2s-def assms unrest*)

qed

We use the  $R4$  healthiness condition to characterise that the postcondition must extend the trace for a reactive design to be productive.

**lemma** *Productive-rdes-RR-intro*:

**assumes**  $P$  *is RR*  $Q$  *is RR*  $R$  *is RR*  $R$  *is R4*

**shows**  $(\mathbf{R}_s(P \vdash Q \diamond R))$  *is Productive*

**using** *assms* **by** (*simp add: Productive-rdes-intro R4-iff-refine unrest*)

**lemma** *Productive-Miracle [closure]: Miracle is Productive*

**unfolding** *Miracle-tri-def Healthy-def*

**by** (*subst Productive-RHS-design-form, simp-all add: unrest*)

**lemma** *Productive-Chaos [closure]: Chaos is Productive*

**unfolding** *Chaos-tri-def Healthy-def*

**by** (*subst Productive-RHS-design-form, simp-all add: unrest*)

**lemma** *Productive-intChoice [closure]*:

**assumes**  $P$  *is SRD*  $P$  *is Productive*  $Q$  *is SRD*  $Q$  *is Productive*

**shows**  $P \sqcap Q$  *is Productive*

**proof** –

**have**  $P \sqcap Q =$

$\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond (post_R(P) \wedge \$tr <_u \$tr')) \sqcap \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond (post_R(Q) \wedge \$tr <_u \$tr'))$

**by** (*metis Healthy-if Productive-form assms*)

**also have** ... =  $\mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond ((post_R P \wedge \$tr' >_u \$tr) \vee (post_R Q \wedge \$tr' >_u \$tr)))$

**by** (*simp add: RHS-tri-design-choice*)

**also have** ... =  $\mathbf{R}_s((pre_R P \wedge pre_R Q) \vdash (peri_R P \vee peri_R Q) \diamond (((post_R P) \vee (post_R Q)) \wedge \$tr' >_u \$tr))$

by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 also have ... is Productive  
 by (simp add: Healthy-def Productive-RHS-design-form unrest)  
 finally show ?thesis .  
 qed

**lemma** *Productive-cond-rea* [closure]:  
 assumes  $P$  is SRD  $P$  is Productive  $Q$  is SRD  $Q$  is Productive  
 shows  $P \triangleleft b \triangleright_R Q$  is Productive

**proof** –  
 have  $P \triangleleft b \triangleright_R Q =$   
 $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) \triangleleft b \triangleright_R \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond (\text{post}_R(Q)$   
 $\wedge \$tr <_u \$tr'))$   
 by (metis Healthy-if Productive-form assms)  
 also have ... =  $\mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond ((\text{post}_R P \wedge \$tr' >_u$   
 $\$tr) \triangleleft b \triangleright_R (\text{post}_R Q \wedge \$tr' >_u \$tr)))$   
 by (simp add: cond-srea-form)  
 also have ... =  $\mathbf{R}_s((\text{pre}_R P \triangleleft b \triangleright_R \text{pre}_R Q) \vdash (\text{peri}_R P \triangleleft b \triangleright_R \text{peri}_R Q) \diamond (((\text{post}_R P) \triangleleft b \triangleright_R (\text{post}_R$   
 $Q)) \wedge \$tr' >_u \$tr))$   
 by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 also have ... is Productive  
 by (simp add: Healthy-def Productive-RHS-design-form unrest)  
 finally show ?thesis .  
 qed

**lemma** *Productive-seq-1* [closure]:  
 assumes  $P$  is NSRD  $P$  is Productive  $Q$  is NSRD  
 shows  $P ;; Q$  is Productive

**proof** –  
 have  $P ;; Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) ;; \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond$   
 $(\text{post}_R(Q)))$   
 by (metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms(1) assms(2)  
 assms(3))  
 also have ... =  $\mathbf{R}_s((\text{pre}_R P \wedge (\text{post}_R P \wedge \$tr' >_u \$tr) \text{wp}_r \text{pre}_R Q) \vdash$   
 $(\text{peri}_R P \vee ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{peri}_R Q)) \diamond ((\text{post}_R P \wedge \$tr' >_u \$tr) ;;$   
 $\text{post}_R Q))$   
 by (simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero  
 SRD-healths ex-unrest wp-rea-def disj-upred-def)  
 also have ... =  $\mathbf{R}_s((\text{pre}_R P \wedge (\text{post}_R P \wedge \$tr' >_u \$tr) \text{wp}_r \text{pre}_R Q) \vdash$   
 $(\text{peri}_R P \vee ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; \text{peri}_R Q)) \diamond ((\text{post}_R P \wedge \$tr' >_u \$tr) ;;$   
 $\text{post}_R Q \wedge \$tr' >_u \$tr))$   
**proof** –  
 have  $((\text{post}_R P \wedge \$tr' >_u \$tr) ;; R1(\text{post}_R Q)) = ((\text{post}_R P \wedge \$tr' >_u \$tr) ;; R1(\text{post}_R Q) \wedge \$tr'$   
 $>_u \$tr)$   
 by (rel-auto)  
 thus ?thesis  
 by (simp add: NSRD-is-SRD R1-post-SRD assms)  
 qed  
 also have ... is Productive  
 by (rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def)  
 finally show ?thesis .  
 qed

**lemma** *Productive-seq-2* [closure]:  
 assumes  $P$  is NSRD  $Q$  is NSRD  $Q$  is Productive

shows  $P ;; Q$  is Productive

**proof** –

have  $P ;; Q = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P))) ;; \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{peri}_R(Q) \diamond (\text{post}_R(Q) \wedge \$tr <_u \$tr'))$

by (*metis Healthy-def NSRD-is-SRD SRD-reactive-tri-design Productive-form assms*)

also have  $\dots = \mathbf{R}_s((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{ pre}_R Q) \vdash (\text{peri}_R P \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; (\text{post}_R Q \wedge \$tr' >_u \$tr)))$

by (*simp add: RHS-tri-design-composition-wp rpred unrest closure assms wp NSRD-neg-pre-left-zero SRD-healths ex-unrest wp-rea-def disj-upred-def*)

also have  $\dots = \mathbf{R}_s((\text{pre}_R P \wedge \text{post}_R P \text{ wp}_r \text{ pre}_R Q) \vdash (\text{peri}_R P \vee (\text{post}_R P ;; \text{peri}_R Q)) \diamond (\text{post}_R P ;; (\text{post}_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr))$

**proof** –

have  $(R1(\text{post}_R P) ;; (\text{post}_R Q \wedge \$tr' >_u \$tr) \wedge \$tr' >_u \$tr) = (R1(\text{post}_R P) ;; (\text{post}_R Q \wedge \$tr' >_u \$tr))$

by (*rel-auto*)

thus *?thesis*

by (*simp add: NSRD-is-SRD R1-post-SRD assms*)

qed

also have  $\dots$  is Productive

by (*rule Productive-rdes-intro, simp-all add: unrest assms closure wp-rea-def*)

finally show *?thesis* .

qed

end

## 10 Guarded Recursion

**theory** *utp-rdes-guarded*

**imports** *utp-rdes-productive*

**begin**

### 10.1 Traces with a size measure

Guarded recursion relies on our ability to measure the trace's size, in order to see if it is decreasing on each iteration. Thus, we here equip the trace algebra with the *ucard* function that provides this.

**class** *size-trace* = *trace* + *size* +

**assumes**

*size-zero*:  $\text{size } 0 = 0$  **and**

*size-nzero*:  $s > 0 \implies \text{size}(s) > 0$  **and**

*size-plus*:  $\text{size}(s + t) = \text{size}(s) + \text{size}(t)$

— These axioms may be stronger than necessary. In particular,  $0 < ?s \implies 0 < \#_u(?s)$  requires that a non-empty trace have a positive size. But this may not be the case with all trace models and is possibly more restrictive than necessary. In future we will explore weakening.

**begin**

**lemma** *size-mono*:  $s \leq t \implies \text{size}(s) \leq \text{size}(t)$

by (*metis le-add1 local.diff-add-cancel-left' local.size-plus*)

**lemma** *size-strict-mono*:  $s < t \implies \text{size}(s) < \text{size}(t)$

by (*metis cancel-ab-semigroup-add-class.add-diff-cancel-left' local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero local.size-plus zero-less-diff*)

**lemma** *trace-strict-prefixE*:  $xs < ys \implies (\bigwedge zs. \llbracket ys = xs + zs; \text{size}(zs) > 0 \rrbracket \implies \text{thesis}) \implies \text{thesis}$

by (metis local.diff-add-cancel-left' local.less-iff local.minus-gr-zero-iff local.size-nzero)

**lemma** size-minus-trace:  $y \leq x \implies \text{size}(x - y) = \text{size}(x) - \text{size}(y)$

by (metis diff-add-inverse local.diff-add-cancel-left' local.size-plus)

end

Both natural numbers and lists are measurable trace algebras.

**instance** nat :: size-trace

by (intro-classes, simp-all)

**instance** list :: (type) size-trace

by (intro-classes, simp-all add: zero-list-def less-list-def' plus-list-def prefix-length-less)

**syntax**

-size :: logic  $\Rightarrow$  logic (size<sub>u</sub>'(-))

**translations**

size<sub>u</sub>(t) == CONST uop CONST size t

## 10.2 Guardedness

**definition** gvirt :: (('t::size-trace,'α) rp × ('t,'α) rp) chain **where**

[upred-defs]:  $\text{gvirt}(n) \equiv (\$tr \leq_u \$tr' \wedge \text{size}_u(\&tt) <_u \ll n \gg)$

**lemma** gvirt-chain: chain gvirt

apply (simp add: chain-def, safe)

apply (rel-simp)

apply (rel-simp)+

done

**lemma** gvirt-limit:  $\sqcap (\text{range gvirt}) = (\$tr \leq_u \$tr')$

by (rel-auto)

**definition** Guarded :: (('t::size-trace,'α) hrel-rp  $\Rightarrow$  ('t,'α) hrel-rp)  $\Rightarrow$  bool **where**

[upred-defs]:  $\text{Guarded}(F) = (\forall X n. (F(X) \wedge \text{gvirt}(n+1)) = (F(X \wedge \text{gvirt}(n)) \wedge \text{gvirt}(n+1)))$

**lemma** GuardedI:  $\llbracket \bigwedge X n. (F(X) \wedge \text{gvirt}(n+1)) = (F(X \wedge \text{gvirt}(n)) \wedge \text{gvirt}(n+1)) \rrbracket \implies \text{Guarded } F$

by (simp add: Guarded-def)

Guarded reactive designs yield unique fixed-points.

**theorem** guarded-fp-uniq:

assumes mono F F ∈  $\llbracket id \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$  Guarded F

shows  $\mu F = \nu F$

**proof** –

have constr F gvirt

using assms

by (auto simp add: constr-def gvirt-chain Guarded-def tcontr-alt-def')

hence  $(\$tr \leq_u \$tr' \wedge \mu F) = (\$tr \leq_u \$tr' \wedge \nu F)$

apply (rule constr-fp-uniq)

apply (simp add: assms)

using gvirt-limit apply blast

done

moreover have  $(\$tr \leq_u \$tr' \wedge \mu F) = \mu F$

**proof** –

```

have  $\mu F$  is R1
  by (rule SRD-healths(1), rule Healthy-mu, simp-all add: assms)
thus ?thesis
  by (metis Healthy-def R1-def conj-comm)
qed
moreover have  $(\$tr \leq_u \$tr' \wedge \nu F) = \nu F$ 
proof -
  have  $\nu F$  is R1
  by (rule SRD-healths(1), rule Healthy-nu, simp-all add: assms)
  thus ?thesis
  by (metis Healthy-def R1-def conj-comm)
qed
ultimately show ?thesis
  by (simp)
qed

lemma Guarded-const [closure]: Guarded  $(\lambda X. P)$ 
  by (simp add: Guarded-def)

lemma UINF-Guarded [closure]:
  assumes  $\bigwedge P. P \in A \implies \text{Guarded } P$ 
  shows Guarded  $(\lambda X. \bigcap P \in A \cdot P(X))$ 
proof (rule GuardedI)
  fix  $X n$ 
  have  $\bigwedge Y. ((\bigcap P \in A \cdot P Y) \wedge \text{gvrt}(n+1)) = ((\bigcap P \in A \cdot (P Y \wedge \text{gvrt}(n+1))) \wedge \text{gvrt}(n+1))$ 
  proof -
    fix  $Y$ 
    let ?lhs =  $((\bigcap P \in A \cdot P Y) \wedge \text{gvrt}(n+1))$  and ?rhs =  $((\bigcap P \in A \cdot (P Y \wedge \text{gvrt}(n+1))) \wedge \text{gvrt}(n+1))$ 
    have  $a: ?lhs \llbracket \text{false} / \$ok \rrbracket = ?rhs \llbracket \text{false} / \$ok \rrbracket$ 
      by (rel-auto)
    have  $b: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{true} / \$wait \rrbracket$ 
      by (rel-auto)
    have  $c: ?lhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket = ?rhs \llbracket \text{true} / \$ok \rrbracket \llbracket \text{false} / \$wait \rrbracket$ 
      by (rel-auto)
    show ?lhs = ?rhs
      using a b c
      by (rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all)
  qed
  moreover have  $((\bigcap P \in A \cdot (P X \wedge \text{gvrt}(n+1))) \wedge \text{gvrt}(n+1)) = ((\bigcap P \in A \cdot (P (X \wedge \text{gvrt}(n))) \wedge \text{gvrt}(n+1))) \wedge \text{gvrt}(n+1))$ 
  proof -
    have  $(\bigcap P \in A \cdot (P X \wedge \text{gvrt}(n+1))) = (\bigcap P \in A \cdot (P (X \wedge \text{gvrt}(n))) \wedge \text{gvrt}(n+1))$ 
    proof (rule UINF-cong)
      fix  $P$  assume  $P \in A$ 
      thus  $(P X \wedge \text{gvrt}(n+1)) = (P (X \wedge \text{gvrt}(n)) \wedge \text{gvrt}(n+1))$ 
        using Guarded-def assms by blast
    qed
    thus ?thesis by simp
  qed
  ultimately show  $((\bigcap P \in A \cdot P X) \wedge \text{gvrt}(n+1)) = ((\bigcap P \in A \cdot (P (X \wedge \text{gvrt}(n)))) \wedge \text{gvrt}(n+1))$ 
    by simp
qed

```

```

lemma intChoice-Guarded [closure]:
  assumes Guarded  $P$  Guarded  $Q$ 

```

**shows** *Guarded*  $(\lambda X. P(X) \sqcap Q(X))$   
**proof** –  
**have** *Guarded*  $(\lambda X. \sqcap F \in \{P, Q\} \cdot F(X))$   
**by** (*rule UINF-Guarded, auto simp add: assms*)  
**thus** *?thesis*  
**by** (*simp*)  
**qed**

**lemma** *cond-srea-Guarded* [*closure*]:  
**assumes** *Guarded P Guarded Q*  
**shows** *Guarded*  $(\lambda X. P(X) \triangleleft b \triangleright_R Q(X))$   
**using** *assms* **by** (*rel-auto*)

A tail recursive reactive design with a productive body is guarded.

**lemma** *Guarded-if-Productive* [*closure*]:  
**fixes**  $P :: ('s, 't :: \text{size-trace}, 'a) \text{hrel-rsp}$   
**assumes** *P is NSRD P is Productive*  
**shows** *Guarded*  $(\lambda X. P ;; \text{SRD}(X))$   
**proof** (*clarsimp simp add: Guarded-def*)

— We split the proof into three cases corresponding to valuations for ok, wait, and wait' respectively.

**fix**  $X n$   
**have**  $a: (P ;; \text{SRD}(X) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket =$   
 $(P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{false}/\$ok \rrbracket$   
**by** (*simp add: usubst closure SRD-left-zero-1 assms*)  
**have**  $b: ((P ;; \text{SRD}(X) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket =$   
 $((P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{true}/\$wait \rrbracket$   
**by** (*simp add: usubst closure SRD-left-zero-2 assms*)  
**have**  $c: ((P ;; \text{SRD}(X) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{false}/\$wait \rrbracket =$   
 $((P ;; \text{SRD}(X \wedge \text{gvert } n) \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}/\$ok \rrbracket) \llbracket \text{false}/\$wait \rrbracket$   
**proof** –  
**have**  $1: (P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $(P \llbracket \text{true}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvert } n)) \llbracket \text{true}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*metis (no-types, lifting) Healthy-def R3h-wait-true SRD-healths(3) SRD-idem*)  
**have**  $2: (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } X) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } (X \wedge \text{gvert } n)) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**proof** –  
**have**  $\text{exp}: \bigwedge Y :: ('s, 't, 'a) \text{hrel-rsp}. (P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
 $=$   
 $((\neg_r \text{pre}_R P) ;; (\text{SRD}(Y)) \llbracket \text{false}/\$wait \rrbracket \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (\text{SRD}$   
 $Y) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket)$   
 $\wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$

**proof** –  
**fix**  $Y :: ('s, 't, 'a) \text{hrel-rsp}$   
**have**  $(P \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket \wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket =$   
 $((\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
 $\wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*metis (no-types) Healthy-def Productive-form assms(1) assms(2) NSRD-is-SRD*)  
**also have**  $\dots =$   
 $((R1(R2c(\text{pre}_R(P) \Rightarrow (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')))) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$   
 $\wedge \text{gvert} (\text{Suc } n)) \llbracket \text{true}, \text{false}/\$ok, \$wait \rrbracket$   
**by** (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def RD1-def RD2-def usubst unrest assms closure design-def*)  
**also have**  $\dots =$   
 $((\neg_r \text{pre}_R(P) \vee (\$ok' \wedge \text{post}_R(P) \wedge \$tr <_u \$tr')) \llbracket \text{false}/\$wait' \rrbracket ;; (\text{SRD } Y) \llbracket \text{false}/\$wait \rrbracket$



$\wedge$  *gvert* (*Suc n*) $[[true,false/\$ok,\$wait]]$   
**by** (*simp add: impl-alt-def R2c-disj R1-disj R2c-not assms closure R2c-and R2c-preR rea-not-def R1-extend-conj' R2c-ok' R2c-post-SRD R1-tr-less-tr' R2c-tr-less-tr'*)  
**also have ... =**  
 $(((\neg_r \text{pre}_R P) ;; (SRD(Y))[[false/\$wait]] \vee (\$ok' \wedge \text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD Y))[[false/\$wait]]) \wedge$  *gvert* (*Suc n*) $[[true,false/\$ok,\$wait]]$   
**by** (*simp add: usubst unrest assms closure seqr-or-distl NSRD-neg-pre-left-zero SRD-healths*)  
**also have ... =**  
 $(((\neg_r \text{pre}_R P) ;; (SRD(Y))[[false/\$wait]] \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD Y))[[true,false/\$ok,\$wait]])$   
 $\wedge$  *gvert* (*Suc n*) $[[true,false/\$ok,\$wait]]$   
**proof –**  
**have**  $(\$ok' \wedge \text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD Y))[[false/\$wait]] =$   
 $((\text{post}_R P \wedge \$tr' >_u \$tr) \wedge \$ok' =_u true) ;; (SRD Y))[[false/\$wait]]$   
**by** (*rel-blast*)  
**also have ... =**  $(\text{post}_R P \wedge \$tr' >_u \$tr)[[true/\$ok']] ;; (SRD Y)[[false/\$wait]][[true/\$ok]]$   
**using** *seqr-left-one-point*[*of ok (post<sub>R</sub> P  $\wedge$  \$tr' ><sub>u</sub> \$tr) True (SRD Y)[[false/\\$wait]]*]  
**by** (*simp add: true-alt-def [THEN sym]*)  
**finally show ?thesis by** (*simp add: usubst unrest*)  
**qed**  
**finally**  
**show**  $(P[[false/\$wait']] ;; (SRD Y)[[false/\$wait]] \wedge \text{gvert} (Suc n))[[true,false/\$ok,\$wait]] =$   
 $(((\neg_r \text{pre}_R P) ;; (SRD(Y))[[false/\$wait]] \vee (\text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD Y))[[true,false/\$ok,\$wait]])$   
 $\wedge \text{gvert} (Suc n)[[true,false/\$ok,\$wait]] .$   
**qed**  
**have 1:** $((\text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD X))[[true,false/\$ok,\$wait]] \wedge \text{gvert} (Suc n) =$   
 $((\text{post}_R P \wedge \$tr' >_u \$tr) ;; (SRD (X \wedge \text{gvert} n)))[[true,false/\$ok,\$wait]] \wedge \text{gvert} (Suc n)$   
**apply** (*rel-auto*)  
**apply** (*rename-tac tr st more ok wait tr' st' more' tr<sub>0</sub> st<sub>0</sub> more<sub>0</sub> ok'*)  
**apply** (*rule-tac x=tr<sub>0</sub> in exI, rule-tac x=st<sub>0</sub> in exI, rule-tac x=more<sub>0</sub> in exI*)  
**apply** (*simp*)  
**apply** (*erule trace-strict-prefixE*)  
**apply** (*rename-tac tr st ref ok wait tr' st' ref' tr<sub>0</sub> st<sub>0</sub> ref<sub>0</sub> ok' zs*)  
**apply** (*rule-tac x=False in exI*)  
**apply** (*simp add: size-minus-trace*)  
**apply** (*subgoal-tac size(tr) < size(tr<sub>0</sub>)*)  
**apply** (*simp add: less-diff-conv2 size-mono*)  
**using** *size-strict-mono* **apply** *blast*  
**apply** (*rename-tac tr st more ok wait tr' st' more' tr<sub>0</sub> st<sub>0</sub> more<sub>0</sub> ok'*)  
**apply** (*rule-tac x=tr<sub>0</sub> in exI, rule-tac x=st<sub>0</sub> in exI, rule-tac x=more<sub>0</sub> in exI*)  
**apply** (*simp*)  
**apply** (*erule trace-strict-prefixE*)  
**apply** (*rename-tac tr st more ok wait tr' st' more' tr<sub>0</sub> st<sub>0</sub> more<sub>0</sub> ok' zs*)  
**apply** (*auto simp add: size-minus-trace*)  
**apply** (*subgoal-tac size(tr) < size(tr<sub>0</sub>)*)  
**apply** (*simp add: less-diff-conv2 size-mono*)  
**using** *size-strict-mono* **apply** *blast*  
**done**  
**have 2:** $(\neg_r \text{pre}_R P) ;; (SRD X)[[false/\$wait]] = (\neg_r \text{pre}_R P) ;; (SRD(X \wedge \text{gvert} n))[[false/\$wait]]$   
**by** (*simp add: NSRD-neg-pre-left-zero closure assms SRD-healths*)  
**show ?thesis**  
**by** (*simp add: exp 1 2 utp-pred-laws.inf-sup-distrib2*)  
**qed**

show ?thesis

proof –

have  $(P ;; (SRD X) \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket =$   
     $((P \llbracket true / \$wait ' \rrbracket ;; (SRD X) \llbracket true / \$wait \rrbracket \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket \vee$   
     $(P \llbracket false / \$wait ' \rrbracket ;; (SRD X) \llbracket false / \$wait \rrbracket \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket)$   
by (subst seqr-bool-split[of wait], simp-all add: usubst utp-pred-laws.distrib(4))

also

have ... =  $((P \llbracket true / \$wait ' \rrbracket ;; (SRD (X \wedge gvirt n)) \llbracket true / \$wait \rrbracket \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket$   
 $\vee$   
     $(P \llbracket false / \$wait ' \rrbracket ;; (SRD (X \wedge gvirt n)) \llbracket false / \$wait \rrbracket \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket)$   
by (simp add: 1 2)

also

have ... =  $((P \llbracket true / \$wait ' \rrbracket ;; (SRD (X \wedge gvirt n)) \llbracket true / \$wait \rrbracket \vee$   
     $P \llbracket false / \$wait ' \rrbracket ;; (SRD (X \wedge gvirt n)) \llbracket false / \$wait \rrbracket) \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket$   
by (simp add: usubst utp-pred-laws.distrib(4))

also have ... =  $(P ;; (SRD (X \wedge gvirt n)) \wedge gvirt (n+1)) \llbracket true, false / \$ok, \$wait \rrbracket$   
by (subst seqr-bool-split[of wait], simp-all add: usubst)

finally show ?thesis by (simp add: usubst)

qed

qed

show  $(P ;; SRD(X) \wedge gvirt (Suc n)) = (P ;; SRD(X \wedge gvirt n) \wedge gvirt (Suc n))$

apply (rule-tac bool-eq-splitI[of in-var ok])

apply (simp-all add: a)

apply (rule-tac bool-eq-splitI[of in-var wait])

apply (simp-all add: b c)

done

qed

### 10.3 Tail recursive fixed-point calculations

declare upred-semiring.power-Suc [simp]

lemma mu-csp-form-1 [rdes]:

fixes  $P :: ('s, 't::size-trace, 'α) hrel-rsp$

assumes  $P$  is NSRD  $P$  is Productive

shows  $(\mu X \cdot P ;; SRD(X)) = (\prod i \cdot P \wedge (i+1)) ;; Miracle$

proof –

have 1: Continuous  $(\lambda X. P ;; SRD X)$

using SRD-Continuous

by (clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac x=A in spec, simp)

have 2:  $(\lambda X. P ;; SRD X) \in \llbracket id \rrbracket_H \rightarrow \llbracket SRD \rrbracket_H$

by (blast intro: funcsetI closure assms)

with 1 2 have  $(\mu X \cdot P ;; SRD(X)) = (\nu X \cdot P ;; SRD(X))$

by (simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure)

also have ... =  $(\prod i. ((\lambda X. P ;; SRD X) \wedge \wedge i) false)$

by (simp add: sup-continuous-lfp 1 sup-continuous-Continuous false-upred-def)

also have ... =  $(\lambda X. P ;; SRD X) \wedge \wedge 0) false \sqcap (\prod i. ((\lambda X. P ;; SRD X) \wedge \wedge (i+1)) false)$

by (subst Sup-power-expand, simp)

also have ... =  $(\prod i. ((\lambda X. P ;; SRD X) \wedge \wedge (i+1)) false)$

by (simp)

also have ... =  $(\prod i. P \wedge (i+1)) ;; Miracle$

proof (rule SUP-cong, simp-all)

```

fix i
show P ;; SRD (((λX. P ;; SRD X) ^ i) false) = (P ;; P ^ i) ;; Miracle
proof (induct i)
  case 0
  then show ?case
    by (simp, metis srdes-hcond-def srdes-theory-continuous.healthy-top)
next
  case (Suc i)
  then show ?case
    by (simp add: Healthy-if NSRD-is-SRD SRD-power-comp SRD-seqr-closure assms(1) seqr-assoc[THEN
sym] srdes-theory-continuous.weak.top-closed)
  qed
  qed
also have ... = (∏ i. P ^ (i+1)) ;; Miracle
  by (simp add: seq-Sup-distr)
finally show ?thesis
  by (simp add: UINF-as-Sup[THEN sym])
qed

```

```

lemma mu-csp-form-NSRD [closure]:
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows (μ X · P ;; SRD(X)) is NSRD
  by (simp add: mu-csp-form-1 assms closure)

```

```

lemma mu-csp-form-1':
  fixes P :: ('s, 't::size-trace, 'α) hrel-rsp
  assumes P is NSRD P is Productive
  shows (μ X · P ;; SRD(X)) = (P ;; P*) ;; Miracle
proof -
  have (μ X · P ;; SRD(X)) = (∏ i ∈ UNIV · P ;; P ^ i) ;; Miracle
    by (simp add: mu-csp-form-1 assms closure ustar-def)
  also have ... = (P ;; P*) ;; Miracle
    by (simp only: seq-UINF-distl[THEN sym], simp add: ustar-def)
  finally show ?thesis .
qed

```

```

declare upred-semiring.power-Suc [simp del]

```

end

## 11 Reactive Design Programs

```

theory utp-rdes-prog
imports
  utp-rdes-normal
  utp-rdes-tactics
  utp-rdes-parallel
  utp-rdes-guarded
  UTP-KAT.utp-kleene
begin

```

### 11.1 State substitution

```

lemma srd-subst-RHS-tri-design [usubst]:

```

$\lceil \sigma \rceil_{S\sigma} \dagger \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s(\lceil \sigma \rceil_{S\sigma} \dagger P) \vdash (\lceil \sigma \rceil_{S\sigma} \dagger Q) \diamond (\lceil \sigma \rceil_{S\sigma} \dagger R)$   
**by** (*rel-auto*)

**lemma** *srd-subst-SRD-closed* [*closure*]:

**assumes** *P is SRD*

**shows**  $\lceil \sigma \rceil_{S\sigma} \dagger P$  *is SRD*

**proof** –

**have**  $SRD(\lceil \sigma \rceil_{S\sigma} \dagger (SRD P)) = \lceil \sigma \rceil_{S\sigma} \dagger (SRD P)$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*metis Healthy-def assms*)

**qed**

**lemma** *preR-srd-subst* [*rdes*]:

$pre_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger pre_R(P)$

**by** (*rel-auto*)

**lemma** *periR-srd-subst* [*rdes*]:

$peri_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger peri_R(P)$

**by** (*rel-auto*)

**lemma** *postR-srd-subst* [*rdes*]:

$post_R(\lceil \sigma \rceil_{S\sigma} \dagger P) = \lceil \sigma \rceil_{S\sigma} \dagger post_R(P)$

**by** (*rel-auto*)

**lemma** *srd-subst-NSRD-closed* [*closure*]:

**assumes** *P is NSRD*

**shows**  $\lceil \sigma \rceil_{S\sigma} \dagger P$  *is NSRD*

**by** (*rule NSRD-RC-intro, simp-all add: closure rdes assms unrest*)

## 11.2 Assignment

**definition** *assigns-srd* ::  $'s$  *usubst*  $\Rightarrow$   $(s, t::trace, \alpha)$  *hrel-rsp*  $(\langle - \rangle_R)$  **where**

[*upred-defs*]:  $assigns-srd \sigma = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge [\langle \sigma \rangle_a]_S \wedge \$\Sigma_S' =_u \$\Sigma_S))$

**syntax**

*-assign-srd* :: *svids*  $\Rightarrow$  *uexprs*  $\Rightarrow$  *logic*  $(\langle - \rangle :=_R \langle - \rangle)$

*-assign-srd* :: *svids*  $\Rightarrow$  *uexprs*  $\Rightarrow$  *logic* (**infixr** :=<sub>R</sub> 90)

**translations**

*-assign-srd xs vs*  $\Rightarrow$  *CONST assigns-srd* (*-mk-usubst* (*CONST id*) *xs vs*)

*-assign-srd x v*  $\Leftarrow$  *CONST assigns-srd* (*CONST subst-upd* (*CONST id*) *x v*)

*-assign-srd x v*  $\Leftarrow$  *-assign-srd* (*-spvar x*) *v*

$x, y :=_R u, v \Leftarrow$  *CONST assigns-srd* (*CONST subst-upd* (*CONST subst-upd* (*CONST id*) (*CONST svar x*) *u*) (*CONST svar y*) *v*)

**lemma** *assigns-srd-RHS-tri-des* [*rdes-def*]:

$\langle \sigma \rangle_R = \mathbf{R}_s(true_r \vdash false \diamond \langle \sigma \rangle_r)$

**by** (*rel-auto*)

**lemma** *assigns-srd-NSRD-closed* [*closure*]:  $\langle \sigma \rangle_R$  *is NSRD*

**by** (*simp add: rdes-def closure unrest*)

**lemma** *preR-assigns-srd* [*rdes*]:  $pre_R(\langle \sigma \rangle_R) = true_r$

**by** (*simp add: rdes-def rdes closure*)

**lemma** *periR-assigns-srd* [rdes]:  $\text{peri}_R(\langle\sigma\rangle_R) = \text{false}$   
 by (*simp add: rdes-def rdes closure*)

**lemma** *postR-assigns-srd* [rdes]:  $\text{post}_R(\langle\sigma\rangle_R) = \langle\sigma\rangle_r$   
 by (*simp add: rdes-def rdes closure rpred*)

### 11.3 Conditional

**lemma** *preR-cond-srea* [rdes]:  
 $\text{pre}_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge \text{pre}_R(P) \vee [\neg b]_{S<} \wedge \text{pre}_R(Q))$   
 by (*rel-auto*)

**lemma** *periR-cond-srea* [rdes]:  
 assumes *P is SRD Q is SRD*  
 shows  $\text{peri}_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge \text{peri}_R(P) \vee [\neg b]_{S<} \wedge \text{peri}_R(Q))$

**proof** –

have  $\text{peri}_R(P \triangleleft b \triangleright_R Q) = \text{peri}_R(R1(P) \triangleleft b \triangleright_R R1(Q))$   
 by (*simp add: Healthy-if SRD-healths assms*)

thus *?thesis*

by (*rel-auto*)

**qed**

**lemma** *postR-cond-srea* [rdes]:  
 assumes *P is SRD Q is SRD*  
 shows  $\text{post}_R(P \triangleleft b \triangleright_R Q) = ([b]_{S<} \wedge \text{post}_R(P) \vee [\neg b]_{S<} \wedge \text{post}_R(Q))$

**proof** –

have  $\text{post}_R(P \triangleleft b \triangleright_R Q) = \text{post}_R(R1(P) \triangleleft b \triangleright_R R1(Q))$   
 by (*simp add: Healthy-if SRD-healths assms*)

thus *?thesis*

by (*rel-auto*)

**qed**

**lemma** *NSRD-cond-srea* [closure]:  
 assumes *P is NSRD Q is NSRD*  
 shows  $P \triangleleft b \triangleright_R Q \text{ is NSRD}$

**proof** (*rule NSRD-RC-intro*)

show  $P \triangleleft b \triangleright_R Q \text{ is SRD}$

by (*simp add: closure assms*)

show  $\text{pre}_R(P \triangleleft b \triangleright_R Q) \text{ is RC}$

**proof** –

have 1:  $([\neg b]_{S<} \vee \neg_r \text{pre}_R P) ;; R1(\text{true}) = ([\neg b]_{S<} \vee \neg_r \text{pre}_R P)$

by (*metis (no-types, lifting) NSRD-neg-pre-unit aext-not assms(1) seqr-or-distl st-lift-R1-true-right*)

have 2:  $([b]_{S<} \vee \neg_r \text{pre}_R Q) ;; R1(\text{true}) = ([b]_{S<} \vee \neg_r \text{pre}_R Q)$

by (*simp add: NSRD-neg-pre-unit assms seqr-or-distl st-lift-R1-true-right*)

show *?thesis*

by (*simp add: rdes closure assms*)

**qed**

show  $\$st' \# \text{peri}_R(P \triangleleft b \triangleright_R Q)$

by (*simp add: rdes assms closure unrest*)

**qed**

### 11.4 Assumptions

**definition** *AssumeR* :: '*s cond*  $\Rightarrow$  ('*s*, '*t*::*trace*, '*α*) *hrel-rsp* ( $[-]^\top_R$ ) **where**  
 [*upred-defs*]: *AssumeR* *b* =  $II_R \triangleleft b \triangleright_R \text{Miracle}$

**lemma** *AssumeR-rdes-def* [*rdes-def*]:  
 $[b]^\top_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond [b]^\top_r)$   
**unfolding** *AssumeR-def* **by** (*rdes-eq*)

**lemma** *AssumeR-NSRD* [*closure*]:  $[b]^\top_R$  *is NSRD*  
**by** (*simp add: AssumeR-def closure*)

**lemma** *AssumeR-false*:  $[\text{false}]^\top_R = \text{Miracle}$   
**by** (*rel-auto*)

**lemma** *AssumeR-true*:  $[\text{true}]^\top_R = II_R$   
**by** (*rel-auto*)

**lemma** *AssumeR-comp*:  $[b]^\top_R ;; [c]^\top_R = [b \wedge c]^\top_R$   
**by** (*rdes-simp*)

**lemma** *AssumeR-choice*:  $[b]^\top_R \sqcap [c]^\top_R = [b \vee c]^\top_R$   
**by** (*rdes-eq*)

**lemma** *AssumeR-refine-skip*:  $II_R \sqsubseteq [b]^\top_R$   
**by** (*rdes-refine*)

**lemma** *AssumeR-test* [*closure*]:  $\text{test}_R [b]^\top_R$   
**by** (*simp add: AssumeR-refine-skip nsrd-thy.utest-intro*)

**lemma** *Star-AssumeR*:  $[b]^\top_R^{\star R} = II_R$   
**by** (*simp add: AssumeR-NSRD AssumeR-test nsrd-thy.Star-test*)

**lemma** *AssumeR-choice-skip*:  $II_R \sqcap [b]^\top_R = II_R$   
**by** (*rdes-eq*)

**lemma** *cond-srea-AssumeR-form*:  
**assumes**  $P$  *is NSRD*  $Q$  *is NSRD*  
**shows**  $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P \sqcap [\neg b]^\top_R ;; Q)$   
**by** (*rdes-eq cls: assms*)

**lemma** *cond-srea-insert-assume*:  
**assumes**  $P$  *is NSRD*  $Q$  *is NSRD*  
**shows**  $P \triangleleft b \triangleright_R Q = ([b]^\top_R ;; P \triangleleft b \triangleright_R [\neg b]^\top_R ;; Q)$   
**by** (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure RA1 assms cond-srea-AssumeR-form*)

**lemma** *AssumeR-cond-left*:  
**assumes**  $P$  *is NSRD*  $Q$  *is NSRD*  
**shows**  $[b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([b]^\top_R ;; P)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssumeR-cond-right*:  
**assumes**  $P$  *is NSRD*  $Q$  *is NSRD*  
**shows**  $[\neg b]^\top_R ;; (P \triangleleft b \triangleright_R Q) = ([\neg b]^\top_R ;; Q)$   
**by** (*rdes-eq cls: assms*)

## 11.5 Guarded commands

**definition** *GuardedCommR* ::  $'s$  *cond*  $\Rightarrow ('s, 't::\text{trace}, 'a)$  *hrel-rsp*  $\Rightarrow ('s, 't, 'a)$  *hrel-rsp* ( $- \rightarrow_R -$  [85, 86] 85) **where**  
*gcmd-def*[*rdes-def*]:  $\text{GuardedCommR } g \ A = A \triangleleft g \triangleright_R \text{Miracle}$

**lemma** *gcmd-false[simp]*:  $(\text{false} \rightarrow_R A) = \text{Miracle}$   
**unfolding** *gcmd-def* **by** (*pred-auto*)

**lemma** *gcmd-true[simp]*:  $(\text{true} \rightarrow_R A) = A$   
**unfolding** *gcmd-def* **by** (*pred-auto*)

**lemma** *gcmd-SRD*:  
**assumes** *A is SRD*  
**shows**  $(g \rightarrow_R A)$  *is SRD*  
**by** (*simp add: gcmd-def SRD-cond-srea assms sres-theory-continuous.weak.top-closed*)

**lemma** *gcmd-NSRD [closure]*:  
**assumes** *A is NSRD*  
**shows**  $(g \rightarrow_R A)$  *is NSRD*  
**by** (*simp add: gcmd-def NSRD-cond-srea assms NSRD-Miracle*)

**lemma** *gcmd-Productive [closure]*:  
**assumes** *A is NSRD A is Productive*  
**shows**  $(g \rightarrow_R A)$  *is Productive*  
**by** (*simp add: gcmd-def closure assms*)

**lemma** *gcmd-seq-distr*:  
**assumes** *B is NSRD*  
**shows**  $(g \rightarrow_R A) ;; B = (g \rightarrow_R (A ;; B))$   
**by** (*simp add: Miracle-left-zero NSRD-is-SRD assms cond-st-distr gcmd-def*)

**lemma** *gcmd-nondet-distr*:  
**assumes** *A is NSRD B is NSRD*  
**shows**  $(g \rightarrow_R (A \sqcap B)) = (g \rightarrow_R A) \sqcap (g \rightarrow_R B)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssumeR-as-gcmd*:  
 $[b]^\top_R = b \rightarrow_R II_R$   
**by** (*rdes-eq*)

## 12 Generalised Alternation

**definition** *AlternateR*

$:: 'a \text{ set} \Rightarrow ('a \Rightarrow 's \text{ upred}) \Rightarrow ('a \Rightarrow ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a)$   
*hrel-rsp* **where**  
 $[upred-defs, rdes-def]: \text{AlternateR } I \ g \ A \ B = (\bigcap i \in I \cdot ((g \ i) \rightarrow_R (A \ i))) \sqcap ((\neg (\bigvee i \in I \cdot g \ i)) \rightarrow_R B)$

**definition** *AlternateR-list*

$:: ('s \text{ upred} \times ('s, 't::\text{trace}, 'a) \text{ hrel-rsp}) \text{ list} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  **where**  
 $[upred-defs, ndes-simp]: \text{AlternateR-list } xs \ P = \text{AlternateR } \{0..<\text{length } xs\} (\lambda i. \text{map fst } xs \ ! \ i) (\lambda i. \text{map snd } xs \ ! \ i) \ P$

**syntax**

*-altindR-els*  $:: p\text{trn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ -\in \ \cdot \ - \rightarrow \ - \ \text{else} \ - \ fi)$   
*-altindR*  $:: p\text{trn} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ -\in \ \cdot \ - \rightarrow \ - \ fi)$   
*-altgcommR-els*  $:: g\text{comms} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (if_R \ - \ \text{else} \ - \ fi)$   
*-altgcommR*  $:: g\text{comms} \Rightarrow \text{logic} \ (if_R \ - \ fi)$

## translations

$if_R i \in I \cdot g \rightarrow A \text{ else } B \text{ fi} \rightarrow \text{CONST AlternateR } I (\lambda i. g) (\lambda i. A) B$   
 $if_R i \in I \cdot g \rightarrow A \text{ fi} \rightarrow \text{CONST AlternateR } I (\lambda i. g) (\lambda i. A) (\text{CONST Chaos})$   
 $if_R i \in I \cdot (g \ i) \rightarrow A \text{ else } B \text{ fi} \leftarrow \text{CONST AlternateR } I g (\lambda i. A) B$   
 $\text{-altgcommR } cs \rightarrow \text{CONST AlternateR-list } cs (\text{CONST Chaos})$   
 $\text{-altgcommR } (-gcomm\text{-show } cs) \leftarrow \text{CONST AlternateR-list } cs (\text{CONST Chaos})$   
 $\text{-altgcommR-els } cs P \rightarrow \text{CONST AlternateR-list } cs P$   
 $\text{-altgcommR-els } (-gcomm\text{-show } cs) P \leftarrow \text{CONST AlternateR-list } cs P$

**lemma** *AlternateR-NSRD-closed* [closure]:

**assumes**  $\bigwedge i. i \in I \implies A \ i \text{ is NSRD } B \text{ is NSRD}$   
**shows**  $(if_R i \in I \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) \text{ is NSRD}$

**proof** (cases  $I = \{\}$ )

**case** *True*

**then show** *?thesis* **by** (simp add: *AlternateR-def assms*)

**next**

**case** *False*

**then show** *?thesis* **by** (simp add: *AlternateR-def closure assms*)

**qed**

**lemma** *AlternateR-empty* [simp]:

$(if_R i \in \{\} \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) = B$   
**by** (*rdes-simp*)

**lemma** *AlternateR-Productive* [closure]:

**assumes**

$\bigwedge i. i \in I \implies A \ i \text{ is NSRD } B \text{ is NSRD}$

$\bigwedge i. i \in I \implies A \ i \text{ is Productive } B \text{ is Productive}$

**shows**  $(if_R i \in I \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) \text{ is Productive}$

**proof** (cases  $I = \{\}$ )

**case** *True*

**then show** *?thesis*

**by** (simp add: *assms(4)*)

**next**

**case** *False*

**then show** *?thesis*

**by** (simp add: *AlternateR-def closure assms*)

**qed**

**lemma** *AlternateR-singleton*:

**assumes**  $A \ k \text{ is NSRD } B \text{ is NSRD}$

**shows**  $(if_R i \in \{k\} \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) = (A(k) \triangleleft g(k) \triangleright_R B)$

**by** (simp add: *AlternateR-def, rdes-eq cls: assms*)

Convert an alternation over disjoint guards into a cascading if-then-else

**lemma** *AlternateR-insert-cascade*:

**assumes**

$\bigwedge i. i \in I \implies A \ i \text{ is NSRD}$

$A \ k \text{ is NSRD } B \text{ is NSRD}$

$(g(k) \wedge (\bigvee i \in I \cdot g(i))) = \text{false}$

**shows**  $(if_R i \in \text{insert } k \ I \cdot g \ i \rightarrow A \ i \text{ else } B \text{ fi}) = (A(k) \triangleleft g(k) \triangleright_R (if_R i \in I \cdot g(i) \rightarrow A(i) \text{ else } B \text{ fi}))$

**proof** (cases  $I = \{\}$ )

**case** *True*

**then show** *?thesis* **by** (simp add: *AlternateR-singleton assms*)



next

case *False*

have 1:  $(\prod i \in I \cdot g i \rightarrow_R A i) = (\prod i \in I \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(A i) \vdash \text{peri}_R(A i) \diamond \text{post}_R(A i)))$

by (*simp add: NSRD-is-SRD SRD-reactive-tri-design assms(1) cong: UINF-cong*)

from *assms(4)* show *?thesis*

by (*simp add: AlternateR-def 1 False*)

(*rdes-eq cls: assms(1-3) False cong: UINF-cong*)

qed

## 12.1 Choose

**definition** *choose-srd* ::  $(\text{'s}, \text{'t}::\text{trace}, \text{'}\alpha) \text{ hrel-rsp } (\text{choose}_R) \text{ where}$   
[*upred-defs, rdes-def*]:  $\text{choose}_R = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \text{true}_r)$

**lemma** *preR-choose* [*rdes*]:  $\text{pre}_R(\text{choose}_R) = \text{true}_r$

by (*rel-auto*)

**lemma** *periR-choose* [*rdes*]:  $\text{peri}_R(\text{choose}_R) = \text{false}$

by (*rel-auto*)

**lemma** *postR-choose* [*rdes*]:  $\text{post}_R(\text{choose}_R) = \text{true}_r$

by (*rel-auto*)

**lemma** *choose-srd-SRD* [*closure*]: *choose<sub>R</sub> is SRD*

by (*simp add: choose-srd-def closure unrest*)

**lemma** *NSRD-choose-srd* [*closure*]: *choose<sub>R</sub> is NSRD*

by (*rule NSRD-intro, simp-all add: closure unrest rdes*)

## 12.2 State Abstraction

**definition** *state-srea* ::

$\text{'s itself} \Rightarrow (\text{'s}, \text{'t}::\text{trace}, \text{'}\alpha, \text{'}\beta) \text{ rel-rsp} \Rightarrow (\text{unit}, \text{'t}, \text{'}\alpha, \text{'}\beta) \text{ rel-rsp}$  **where**  
[*upred-defs*]:  $\text{state-srea } t P = \langle \exists \{ \$st, \$st' \} \cdot P \rangle_S$

**syntax**

*-state-srea* ::  $\text{type} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{state } \cdot \cdot \cdot [0,200] 200)$

**translations**

$\text{state } 'a \cdot P == \text{CONST state-srea TYPE('}a) P$

**lemma** *R1-state-srea*:  $R1(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R1(P))$

by (*rel-auto*)

**lemma** *R2c-state-srea*:  $R2c(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R2c(P))$

by (*rel-auto*)

**lemma** *R3h-state-srea*:  $R3h(\text{state } 'a \cdot P) = (\text{state } 'a \cdot R3h(P))$

by (*rel-auto*)

**lemma** *RD1-state-srea*:  $RD1(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD1(P))$

by (*rel-auto*)

**lemma** *RD2-state-srea*:  $RD2(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD2(P))$

by (*rel-auto*)

**lemma** *RD3-state-srea*:  $RD3(\text{state } 'a \cdot P) = (\text{state } 'a \cdot RD3(P))$   
 by (*rel-auto*, *blast+*)

**lemma** *SRD-state-srea [closure]*:  $P \text{ is SRD} \implies \text{state } 'a \cdot P \text{ is SRD}$   
 by (*simp add: Healthy-def R1-state-srea R2c-state-srea R3h-state-srea RD1-state-srea RD2-state-srea RHS-def SRD-def*)

**lemma** *NSRD-state-srea [closure]*:  $P \text{ is NSRD} \implies \text{state } 'a \cdot P \text{ is NSRD}$   
 by (*metis Healthy-def NSRD-is-RD3 NSRD-is-SRD RD3-state-srea SRD-RD3-implies-NSRD SRD-state-srea*)

**lemma** *preR-state-srea [rdes]*:  $\text{pre}_R(\text{state } 'a \cdot P) = \langle \forall \{ \$st, \$st' \} \cdot \text{pre}_R(P) \rangle_S$   
 by (*simp add: state-srea-def, rel-auto*)

**lemma** *periR-state-srea [rdes]*:  $\text{peri}_R(\text{state } 'a \cdot P) = \text{state } 'a \cdot \text{peri}_R(P)$   
 by (*rel-auto*)

**lemma** *postR-state-srea [rdes]*:  $\text{post}_R(\text{state } 'a \cdot P) = \text{state } 'a \cdot \text{post}_R(P)$   
 by (*rel-auto*)

### 12.3 While Loop

**definition** *WhileR* ::  $'s \text{ upred} \Rightarrow ('s, 't::\text{size-trace}, 'a) \text{ hrel-rsp} \Rightarrow ('s, 't, 'a) \text{ hrel-rsp}$  (*while<sub>R</sub> - do - od*)  
 where  
 $\text{WhileR } b \ P = (\mu_R X \cdot (P ;; X) \triangleleft b \triangleright_R \text{II}_R)$

**lemma** *Sup-power-false*:  
 fixes  $F :: 'a \text{ upred} \Rightarrow 'a \text{ upred}$   
 shows  $(\prod i. (F \hat{\hat{}} i) \text{ false}) = (\prod i. (F \hat{\hat{}} (i+1)) \text{ false})$   
**proof** –  
 have  $(\prod i. (F \hat{\hat{}} i) \text{ false}) = (F \hat{\hat{}} 0) \text{ false} \sqcap (\prod i. (F \hat{\hat{}} (i+1)) \text{ false})$   
 by (*subst Sup-power-expand, simp*)  
 also have  $\dots = (\prod i. (F \hat{\hat{}} (i+1)) \text{ false})$   
 by (*simp*)  
 finally show *?thesis* .  
**qed**

**theorem** *WhileR-iter-expand*:  
 assumes  $P \text{ is NSRD}$   $P \text{ is Productive}$   
 shows  $\text{while}_R \ b \ \text{do } P \ \text{od} = (\prod i. (P \triangleleft b \triangleright_R \text{II}_R) \hat{\hat{}} i ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R \text{II}_R))$  (**is** *?lhs = ?rhs*)  
**proof** –

have  $1:\text{Continuous } (\lambda X. P ;; \text{SRD } X)$   
 using *SRD-Continuous*  
 by (*clarsimp simp add: Continuous-def seq-SUP-distl[THEN sym], drule-tac x=A in spec, simp*)  
 have  $2:\text{Continuous } (\lambda X. P ;; \text{SRD } X \triangleleft b \triangleright_R \text{II}_R)$   
 by (*simp add: 1 closure assms*)  
 have  $?lhs = (\mu_R X \cdot P ;; X \triangleleft b \triangleright_R \text{II}_R)$   
 by (*simp add: WhileR-def*)  
 also have  $\dots = (\mu X \cdot P ;; \text{SRD}(X) \triangleleft b \triangleright_R \text{II}_R)$   
 by (*auto simp add: srd-mu-equiv closure assms*)  
 also have  $\dots = (\nu X \cdot P ;; \text{SRD}(X) \triangleleft b \triangleright_R \text{II}_R)$   
 by (*auto simp add: guarded-fp-uniq Guarded-if-Productive[OF assms] funcsetI closure assms*)  
 also have  $\dots = (\prod i. ((\lambda X. P ;; \text{SRD } X \triangleleft b \triangleright_R \text{II}_R) \hat{\hat{}} i) \text{ false})$   
 by (*simp add: sup-continuous-lfp 2 sup-continuous-Continuous false-upred-def*)  
 also have  $\dots = (\prod i. ((\lambda X. P ;; \text{SRD } X \triangleleft b \triangleright_R \text{II}_R) \hat{\hat{}} (i+1)) \text{ false})$   
 by (*simp add: Sup-power-false*)  
 also have  $\dots = (\prod i. (P \triangleleft b \triangleright_R \text{II}_R) \hat{\hat{}} i ;; (P ;; \text{Miracle} \triangleleft b \triangleright_R \text{II}_R))$

```

proof (rule SUP-cong, simp)
  fix i
  show (( $\lambda X. P ;; SRD X \triangleleft b \triangleright_R II_R$ ) ^^ (i + 1)) false = (P  $\triangleleft b \triangleright_R II_R$ ) ^ i ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ )
 $\triangleright_R II_R$ )
  proof (induct i)
    case 0
    thm if-eq-cancel
    then show ?case
      by (simp, metis srdes-hcond-def srdes-theory-continuous.healthy-top)
  next
  case (Suc i)
  show ?case
  proof -
    have (( $\lambda X. P ;; SRD X \triangleleft b \triangleright_R II_R$ ) ^^ (Suc i + 1)) false =
      P ;; SRD ((( $\lambda X. P ;; SRD X \triangleleft b \triangleright_R II_R$ ) ^^ (i + 1)) false)  $\triangleleft b \triangleright_R II_R$ 
    by simp
    also have ... = P ;; SRD ((P  $\triangleleft b \triangleright_R II_R$ ) ^ i ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ ))  $\triangleleft b \triangleright_R II_R$ 
    using Suc.hyps by auto
    also have ... = P ;; ((P  $\triangleleft b \triangleright_R II_R$ ) ^ i ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ ))  $\triangleleft b \triangleright_R II_R$ 
    by (metis (no-types, lifting) Healthy-if NSRD-cond-srea NSRD-is-SRD NSRD-power-Suc
NSRD-srd-skip SRD-cond-srea SRD-seqr-closure assms(1) power.power-eq-if seqr-left-unit srdes-theory-continuous.top-closed)
    also have ... = (P  $\triangleleft b \triangleright_R II_R$ ) ^ Suc i ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ )
    proof (induct i)
      case 0
      then show ?case
        by (simp add: NSRD-is-SRD SRD-cond-srea SRD-left-unit SRD-seqr-closure SRD-srdes-skip
assms(1) cond-L6 cond-st-distr srdes-theory-continuous.top-closed)
      next
      case (Suc i)
      have 1:  $II_R ;; ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) ^ i) = ((P \triangleleft b \triangleright_R II_R) ;; (P \triangleleft b \triangleright_R II_R) ^ i)$ 
        by (simp add: NSRD-is-SRD RA1 SRD-cond-srea SRD-left-unit SRD-srdes-skip assms(1))
      then show ?case
      proof -
        have  $\bigwedge u. (u ;; (P \triangleleft b \triangleright_R II_R) ^ Suc i) ;; (P ;; (Miracle) \triangleleft b \triangleright_R (II_R)) \triangleleft b \triangleright_R (II_R) =$ 
          ((u  $\triangleleft b \triangleright_R II_R$ ) ;; (P  $\triangleleft b \triangleright_R II_R$ ) ^ Suc i) ;; (P ;; (Miracle)  $\triangleleft b \triangleright_R (II_R)$ )
        by (metis (no-types) Suc.hyps 1 cond-L6 cond-st-distr power.power.power-Suc)
        then show ?thesis
        by (simp add: RA1 upred-semiring.power-Suc)
      qed
    qed
    finally show ?thesis .
  qed
  qed
  qed
  also have ... = ( $\prod i \cdot (P \triangleleft b \triangleright_R II_R) ^ i$ ) ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ )
  by (simp add: UINF-as-Sup-collect')
  finally show ?thesis .
qed

```

**theorem** WhileR-star-expand:

**assumes** P is NSRD P is Productive

**shows** while<sub>R</sub> b do P od = (P  $\triangleleft b \triangleright_R II_R$ )<sup>\*R</sup> ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ ) (**is** ?lhs = ?rhs)

**proof** -

**have** ?lhs = ( $\prod i \cdot (P \triangleleft b \triangleright_R II_R) ^ i$ ) ;; (P ;; Miracle  $\triangleleft b \triangleright_R II_R$ )

**by** (simp add: WhileR-iter-expand seq-UINF-distr' assms)

**also have** ... =  $(P \triangleleft b \triangleright_R II_R)^* \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$   
**by** (*simp add: ustar-def*)  
**also have** ... =  $((P \triangleleft b \triangleright_R II_R)^* \;; II_R) \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$   
**by** (*simp add: seqr-assoc SRD-left-unit closure assms*)  
**also have** ... =  $(P \triangleleft b \triangleright_R II_R)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$   
**by** (*simp add: nsrd-thy.Star-def*)  
**finally show** ?thesis .  
**qed**

**lemma** *WhileR-NSRD-closed* [closure]:

**assumes** *P is NSRD P is Productive*  
**shows** *while<sub>R</sub> b do P od is NSRD*  
**by** (*simp add: WhileR-star-expand assms closure*)

**theorem** *WhileR-iter-form-lemma*:

**assumes** *P is NSRD*  
**shows**  $(P \triangleleft b \triangleright_R II_R)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R) = ([b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R$

**proof** –

**have**  $(P \triangleleft b \triangleright_R II_R)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R) = ([b]^\top_R \;; P \sqcap [\neg b]^\top_R)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: AssumeR-NSRD NSRD-right-unit NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

**also have** ... =  $([b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-denest assms(1)*)

**also have** ... =  $([b]^\top_R \;; P)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*metis (no-types, hide-lams) RD3-def RD3-idem Star-AssumeR nsrd-thy.Star-def*)

**also have** ... =  $([b]^\top_R \;; P)^{*R} \;; (P \;; \text{Miracle} \triangleleft b \triangleright_R II_R)$

**by** (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-invol assms(1)*)

**also have** ... =  $([b]^\top_R \;; P)^{*R} \;; ([b]^\top_R \;; P \;; \text{Miracle} \sqcap [\neg b]^\top_R)$

**by** (*simp add: AssumeR-NSRD NSRD-Miracle NSRD-right-unit NSRD-seqr-closure NSRD-srd-skip assms(1) cond-srea-AssumeR-form*)

**also have** ... =  $([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; \text{Miracle} \sqcap (([b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R)$

**by** (*simp add: upred-semiring.distrib-left*)

**also have** ... =  $[b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R$

**proof** –

**have**  $([b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R = (II_R \sqcap ([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P) \;; [\neg b]^\top_R$

**by** (*simp add: AssumeR-NSRD NSRD-seqr-closure nsrd-thy.Star-unfoldr-eq assms(1)*)

**also have** ... =  $[\neg b]^\top_R \sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P) \;; [\neg b]^\top_R$

**by** (*metis (no-types, lifting) AssumeR-NSRD AssumeR-as-gcmd NSRD-srd-skip Star-AssumeR nsrd-thy.Star-slide gcmd-seq-distr skip-srea-self-unit urel-dioid.distrib-right'*)

**also have** ... =  $[\neg b]^\top_R \sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; [b \vee \neg b]^\top_R) \;; [\neg b]^\top_R$

**by** (*simp add: AssumeR-true NSRD-right-unit assms(1)*)

**also have** ... =  $[\neg b]^\top_R \sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; [b]^\top_R) \;; [\neg b]^\top_R$

$\sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; [\neg b]^\top_R) \;; [\neg b]^\top_R$

**by** (*metis (no-types, hide-lams) AssumeR-choice upred-semiring.add-assoc upred-semiring.distrib-left upred-semiring.distrib-right*)

**also have** ... =  $[\neg b]^\top_R \sqcap ([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; ([b]^\top_R \;; [\neg b]^\top_R)$

$\sqcap ([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; ([\neg b]^\top_R \;; [\neg b]^\top_R)$

**by** (*simp add: RA1*)

**also have** ... =  $[\neg b]^\top_R \sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; \text{Miracle})$

$\sqcap (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; [\neg b]^\top_R)$

**by** (*simp add: AssumeR-comp AssumeR-false*)

**finally have**  $([b]^\top_R \;; P)^{*R} \;; [\neg b]^\top_R \sqsubseteq (([b]^\top_R \;; P)^{*R} \;; [b]^\top_R \;; P \;; \text{Miracle})$

**by** (*simp add: semilattice-sup-class.le-supI1*)

**thus** ?thesis

**by** (*simp add: semilattice-sup-class.le-iff-sup*)

qed  
 finally show *?thesis* .  
 qed

**theorem** *WhileR-iter-form*:  
 assumes *P is NSRD P is Productive*  
 shows  $\text{while}_R b \text{ do } P \text{ od} = ([b]^\top_R ;; P)^{*R} ;; [\neg b]^\top_R$   
 by (*simp add: WhileR-iter-form-lemma WhileR-star-expand assms*)

**theorem** *WhileR-false*:  
 assumes *P is NSRD*  
 shows  $\text{while}_R \text{false do } P \text{ od} = II_R$   
 by (*simp add: WhileR-def rpred closure sdes-theory-continuous.LFP-const*)

**theorem** *WhileR-true*:  
 assumes *P is NSRD P is Productive*  
 shows  $\text{while}_R \text{true do } P \text{ od} = P^{*R} ;; \text{Miracle}$   
 by (*simp add: WhileR-iter-form AssumeR-true AssumeR-false SRD-left-unit assms closure*)

**lemma** *WhileR-insert-assume*:  
 assumes *P is NSRD P is Productive*  
 shows  $\text{while}_R b \text{ do } ([b]^\top_R ;; P) \text{ od} = \text{while}_R b \text{ do } P \text{ od}$   
 by (*simp add: AssumeR-NSRD AssumeR-comp NSRD-seqr-closure Productive-seq-2 RA1 WhileR-iter-form assms*)

**theorem** *WhileR-rdes-def [rdes-def]*:  
 assumes *P is RC Q is RR R is RR \$st' \# Q R is R4*  
 shows  $\text{while}_R b \text{ do } \mathbf{R}_s(P \vdash Q \diamond R) \text{ od} =$   
 $\mathbf{R}_s([b]^\top_r ;; R)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r P) \vdash ([b]^\top_r ;; R)^{*r} ;; [b]^\top_r ;; Q \diamond ([b]^\top_r ;; R)^{*r} ;; [\neg b]^\top_r$   
 (*is ?lhs = ?rhs*)

**proof** –  
 have  $?lhs = ([b]^\top_R ;; \mathbf{R}_s(P \vdash Q \diamond R))^{*R} ;; [\neg b]^\top_R$   
 by (*simp add: WhileR-iter-form Productive-rdes-RR-intro assms closure*)  
 also have  $\dots = ?rhs$   
 by (*simp add: rdes-def assms closure unrest rpred wp del: rea-star-wp*)  
 finally show *?thesis* .  
 qed

Refinement introduction law for reactive while loops

**theorem** *WhileR-refine-intro*:  
 assumes  
 – Closure conditions  
 $Q_1 \text{ is RC } Q_2 \text{ is RR } Q_3 \text{ is RR } \$st' \# Q_2 \ Q_3 \text{ is R4}$   
 – Refinement conditions  
 $([b]^\top_r ;; Q_3)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$   
 $P_2 \sqsubseteq [b]^\top_r ;; Q_2$   
 $P_2 \sqsubseteq [b]^\top_r ;; Q_3 ;; P_2$   
 $P_3 \sqsubseteq [\neg b]^\top_r$   
 $P_3 \sqsubseteq [b]^\top_r ;; Q_3 ;; P_3$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \sqsubseteq \text{while}_R b \text{ do } \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) \text{ od}$   
**proof** (*simp add: rdes-def assms, rule sdes-tri-refine-intro^*)  
 show  $([b]^\top_r ;; Q_3)^{*r} \text{ wp}_r ([b]_{S<} \Rightarrow_r Q_1) \sqsubseteq P_1$   
 by (*simp add: assms*)  
 show  $P_2 \sqsubseteq (P_1 \wedge ([b]^\top_r ;; Q_3)^{*r} ;; [b]^\top_r ;; Q_2)$   
**proof** –

```

have P2 ⊆ ([b]⊤r ;; Q3)*r ;; [b]⊤r ;; Q2
  by (simp add: assms rea-assume-RR rrel-thy.Star-inductl seq-RR-closed seqr-assoc)
thus ?thesis
  by (simp add: utp-pred-laws.le-infI2)
qed
show P3 ⊆ (P1 ∧ ([b]⊤r ;; Q3)*r ;; [¬ b]⊤r)
proof -
  have P3 ⊆ ([b]⊤r ;; Q3)*r ;; [¬ b]⊤r
  by (simp add: assms rea-assume-RR rrel-thy.Star-inductl seqr-assoc)
  thus ?thesis
  by (simp add: utp-pred-laws.le-infI2)
qed
qed

```

## 12.4 Iteration Construction

**definition** *IterateR*

```

:: 'a set ⇒ ('a ⇒ 's upred) ⇒ ('a ⇒ ('s, 't::size-trace, 'α) hrel-rsp) ⇒ ('s, 't, 'α) hrel-rsp
where IterateR A g P = whileR (∨ i∈A · g(i)) do (ifR i∈A · g(i) → P(i) fi) od

```

**syntax**

```

-iter-srd :: pttrn ⇒ logic ⇒ logic ⇒ logic ⇒ logic (doR -∈- · - → - fi)

```

**translations**

```

-iter-srd x A g P => CONST IterateR A (λ x. g) (λ x. P)
-iter-srd x A g P <= CONST IterateR A (λ x. g) (λ x'. P)

```

**lemma** *IterateR-NSRD-closed* [closure]:

```

assumes
  ∧ i. i ∈ I ⇒ P(i) is NSRD
  ∧ i. i ∈ I ⇒ P(i) is Productive
shows doR i∈I · g(i) → P(i) fi is NSRD
by (simp add: IterateR-def closure assms)

```

**lemma** *IterateR-empty*:

```

doR i∈{} · g(i) → P(i) fi = IIR
by (simp add: IterateR-def srd-mu-equiv closure rpred gfp-const WhileR-false)

```

**lemma** *IterateR-singleton*:

```

assumes P k is NSRD P k is Productive
shows doR i∈{k} · g(i) → P(i) fi = whileR g(k) do P(k) od (is ?lhs = ?rhs)
proof -
  have ?lhs = whileR g k do P k < g k ▷R Chaos od
  by (simp add: IterateR-def AlternateR-singleton assms closure)
  also have ... = whileR g k do [g k]⊤R ;; (P k < g k ▷R Chaos) od
  by (simp add: WhileR-insert-assume closure assms)
  also have ... = whileR g k do P k od
  by (simp add: AssumeR-cond-left NSRD-Chaos WhileR-insert-assume assms)
  finally show ?thesis .
qed

```

## 12.5 Substitution Laws

**lemma** *srd-subst-Chaos* [usubst]:

```

σ †S Chaos = Chaos
by (rdes-simp)

```

**lemma** *srd-subst-Miracle* [*usubst*]:  
 $\sigma \dagger_S \text{Miracle} = \text{Miracle}$   
**by** (*rdes-simp*)

**lemma** *srd-subst-skip* [*usubst*]:  
 $\sigma \dagger_S II_R = \langle \sigma \rangle_R$   
**by** (*rdes-eq*)

**lemma** *srd-subst-assigns* [*usubst*]:  
 $\sigma \dagger_S \langle \varrho \rangle_R = \langle \varrho \circ \sigma \rangle_R$   
**by** (*rdes-eq*)

## 12.6 Algebraic Laws

**theorem** *assigns-srd-id*:  $\langle id \rangle_R = II_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-comp*:  $\langle \sigma \rangle_R ;; \langle \varrho \rangle_R = \langle \varrho \circ \sigma \rangle_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-Miracle*:  $\langle \sigma \rangle_R ;; \text{Miracle} = \text{Miracle}$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-Chaos*:  $\langle \sigma \rangle_R ;; \text{Chaos} = \text{Chaos}$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-cond* :  $\langle \sigma \rangle_R \triangleleft b \triangleright_R \langle \varrho \rangle_R = \langle \sigma \triangleleft b \triangleright_s \varrho \rangle_R$   
**by** (*rdes-eq*)

**theorem** *assigns-srd-left-seq*:  
**assumes** *P is NSRD*  
**shows**  $\langle \sigma \rangle_R ;; P = \sigma \dagger_S P$   
**by** (*rdes-simp cls: assms*)

**lemma** *AlternateR-seq-distr*:  
**assumes**  $\bigwedge i. A\ i \text{ is NSRD } B \text{ is NSRD } C \text{ is NSRD}$   
**shows**  $(if_R\ i \in I \cdot g\ i \rightarrow A\ i \text{ else } B\ fi) ;; C = (if_R\ i \in I \cdot g\ i \rightarrow A\ i ;; C \text{ else } B ;; C\ fi)$   
**proof** (*cases*  $I = \{\}$ )  
**case** *True*  
**then show** *?thesis* **by** (*simp*)  
**next**  
**case** *False*  
**then show** *?thesis*  
**by** (*simp add: AlternateR-def upred-semiring.distrib-right seq-UINF-distr gcnd-seq-distr assms(3)*)  
**qed**

**lemma** *AlternateR-is-cond-srea*:  
**assumes** *A is NSRD B is NSRD*  
**shows**  $(if_R\ i \in \{a\} \cdot g \rightarrow A \text{ else } B\ fi) = (A \triangleleft g \triangleright_R B)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AlternateR-Chaos*:  
 $if_R\ i \in A \cdot g(i) \rightarrow \text{Chaos}\ fi = \text{Chaos}$   
**by** (*cases*  $A = \{\}$ , *simp*, *rdes-eq*)

**lemma** *choose-srd-par*:  
*choose<sub>R</sub>*  $\parallel_R$  *choose<sub>R</sub>* = *choose<sub>R</sub>*  
**by** (*rdes-eq*)

## 12.7 Lifting designs to reactive designs

**definition** *des-rea-lift* :: '*s*, '*t*::*trace*, '*α*) *hrel-rsp* (**R<sub>D</sub>**) **where**  
[*upred-defs*]: **R<sub>D</sub>**(*P*) = **R<sub>s</sub>**([*pre<sub>D</sub>*(*P*)]<sub>*S*</sub> ⊢ (*false* ◊ (*\$tr*' =<sub>*u*</sub> *\$tr* ∧ [*post<sub>D</sub>*(*P*)]<sub>*S*</sub>)))

**definition** *des-rea-drop* :: '*s*, '*t*::*trace*, '*α*) *hrel-rsp* ⇒ '*s* *hrel-des* (**D<sub>R</sub>**) **where**  
[*upred-defs*]: **D<sub>R</sub>**(*P*) = ⊔<sub>*v*</sub> (*pre<sub>R</sub>*(*P*)) [ *\$tr*/*\$tr*' ] ⊔<sub>*v*</sub> *\$st* ]<sub>*S*</sub><  
⊔<sub>*n*</sub> ⊔<sub>*v*</sub> (*post<sub>R</sub>*(*P*)) [ *\$tr*/*\$tr*' ] ⊔<sub>*v*</sub> { *\$st*, *\$st*' } ]<sub>*S*</sub>

**lemma** *ndesign-rea-lift-inverse*: **D<sub>R</sub>**(**R<sub>D</sub>**(*p* ⊔<sub>*n*</sub> *Q*)) = *p* ⊔<sub>*n*</sub> *Q*  
**apply** (*simp add: des-rea-lift-def des-rea-drop-def rea-pre-RHS-design rea-post-RHS-design*)  
**apply** (*simp add: R1-def R2c-def R2s-def usubst unrest*)  
**apply** (*rel-auto*)  
**done**

**lemma** *ndesign-rea-lift-injective*:  
**assumes** *P* is **N** *Q* is **N** **R<sub>D</sub>** *P* = **R<sub>D</sub>** *Q* (is ?*RP*(*P*) = ?*RQ*(*Q*))  
**shows** *P* = *Q*

**proof** –  
**have** ?*RP*([*pre<sub>D</sub>*(*P*)]< ⊔<sub>*n*</sub> *post<sub>D</sub>*(*P*)) = ?*RQ*([*pre<sub>D</sub>*(*Q*)]< ⊔<sub>*n*</sub> *post<sub>D</sub>*(*Q*))  
**by** (*simp add: ndesign-form assms*)  
**hence** [*pre<sub>D</sub>*(*P*)]< ⊔<sub>*n*</sub> *post<sub>D</sub>*(*P*) = [*pre<sub>D</sub>*(*Q*)]< ⊔<sub>*n*</sub> *post<sub>D</sub>*(*Q*)  
**by** (*metis ndesign-rea-lift-inverse*)  
**thus** ?*thesis*  
**by** (*simp add: ndesign-form assms*)  
**qed**

**lemma** *des-rea-lift-closure* [*closure*]: **R<sub>D</sub>**(*P*) is *SRD*  
**by** (*simp add: des-rea-lift-def RHS-design-is-SRD unrest*)

**lemma** *preR-des-rea-lift* [*rdes*]:  
*pre<sub>R</sub>*(**R<sub>D</sub>**(*P*)) = *R1*([*pre<sub>D</sub>*(*P*)]<sub>*S*</sub>)  
**by** (*rel-auto*)

**lemma** *periR-des-rea-lift* [*rdes*]:  
*peri<sub>R</sub>*(**R<sub>D</sub>**(*P*)) = (*false* ◁ [*pre<sub>D</sub>*(*P*)]<sub>*S*</sub> ▷ (*\$tr* ≤<sub>*u*</sub> *\$tr*'))  
**by** (*rel-auto*)

**lemma** *postR-des-rea-lift* [*rdes*]:  
*post<sub>R</sub>*(**R<sub>D</sub>**(*P*)) = ((*true* ◁ [*pre<sub>D</sub>*(*P*)]<sub>*S*</sub> ▷ (¬ *\$tr* ≤<sub>*u*</sub> *\$tr*')) ⇒ (*\$tr*' =<sub>*u*</sub> *\$tr* ∧ [*post<sub>D</sub>*(*P*)]<sub>*S*</sub>)  
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast*

**lemma** *ndes-rea-lift-closure* [*closure*]:  
**assumes** *P* is **N**  
**shows** **R<sub>D</sub>**(*P*) is *NSRD*

**proof** –  
**obtain** *p* *Q* **where** *P*: *P* = (*p* ⊔<sub>*n*</sub> *Q*)  
**by** (*metis H1-H3-commute H1-H3-is-normal-design H1-idem Healthy-def assms*)  
**show** ?*thesis*  
**apply** (*rule NSRD-intro*)  
**apply** (*simp-all add: closure rdes unrest P*)  
**apply** (*rel-auto*)



done  
qed

lemma *R-D-mono*:

assumes  $P$  is **H**  $Q$  is **H**  $P \sqsubseteq Q$

shows  $\mathbf{R}_D(P) \sqsubseteq \mathbf{R}_D(Q)$

apply (simp add: des-rea-lift-def)

apply (rule srdes-tri-refine-intro')

  apply (auto intro: H1-H2-refines assms aext-mono)

  apply (rel-auto)

apply (metis (no-types, hide-lams) aext-mono assms(3) design-post-choice

  semilattice-sup-class.sup.orderE utp-pred-laws.inf.coboundedI1 utp-pred-laws.inf commute utp-pred-laws.sup.order-iff

done

Homomorphism laws

lemma *R-D-Miracle*:

$\mathbf{R}_D(\top_D) = \text{Miracle}$

by (simp add: Miracle-def, rel-auto)

lemma *R-D-Chaos*:

$\mathbf{R}_D(\perp_D) = \text{Chaos}$

proof –

have  $\mathbf{R}_D(\perp_D) = \mathbf{R}_D(\text{false} \vdash_r \text{true})$

  by (rel-auto)

also have ... =  $\mathbf{R}_s(\text{false} \vdash \text{false} \diamond (\$tr' =_u \$tr))$

  by (simp add: Chaos-def des-rea-lift-def alpha)

also have ... =  $\mathbf{R}_s(\text{true})$

  by (rel-auto)

also have ... = *Chaos*

  by (simp add: Chaos-def design-false-pre)

finally show ?thesis .

qed

lemma *R-D-inf*:

$\mathbf{R}_D(P \sqcap Q) = \mathbf{R}_D(P) \sqcap \mathbf{R}_D(Q)$

by (rule antisym, rel-auto+)

lemma *R-D-cond*:

$\mathbf{R}_D(P \triangleleft [b]_{D<} \triangleright Q) = \mathbf{R}_D(P) \triangleleft b \triangleright_R \mathbf{R}_D(Q)$

by (rule antisym, rel-auto+)

lemma *R-D-seq-ndesign*:

$\mathbf{R}_D(p_1 \vdash_n Q_1) ;; \mathbf{R}_D(p_2 \vdash_n Q_2) = \mathbf{R}_D((p_1 \vdash_n Q_1) ;; (p_2 \vdash_n Q_2))$

apply (rule antisym)

  apply (rule SRD-refine-intro)

    apply (simp-all add: closure rdes ndesign-composition-wp)

  using dual-order.trans apply (rel-blast)

  using dual-order.trans apply (rel-blast)

  apply (rel-auto)

  apply (rule SRD-refine-intro)

    apply (simp-all add: closure rdes ndesign-composition-wp)

    apply (rel-auto)

    apply (rel-auto)

  apply (rel-auto)

done

**lemma** *R-D-seq*:  
**assumes**  $P$  is  $\mathbf{N}$   $Q$  is  $\mathbf{N}$   
**shows**  $\mathbf{R}_D(P) ;; \mathbf{R}_D(Q) = \mathbf{R}_D(P ;; Q)$   
**by** (*metis R-D-seq-ndesign assms ndesign-form*)

These laws are applicable only when there is no further alphabet extension

**lemma** *R-D-skip*:  
 $\mathbf{R}_D(\mathbf{II}_D) = (\mathbf{II}_R :: ('s, 't::trace, unit) hrel-rsp)$   
**apply** (*rel-auto*) **using** *minus-zero-eq* **by** *blast+*

**lemma** *R-D-assigns*:  
 $\mathbf{R}_D(\langle\sigma\rangle_D) = (\langle\sigma\rangle_R :: ('s, 't::trace, unit) hrel-rsp)$   
**by** (*simp add: assigns-d-def des-rea-lift-def alpha assigns-srd-RHS-tri-des, rel-auto*)

**end**

## 13 Instantaneous Reactive Designs

**theory** *utp-rdes-instant*  
**imports** *utp-rdes-prog*  
**begin**

**definition** *ISR D1* ::  $('s, 't::trace, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp$  **where**  
 $[upred-defs]: \mathbf{ISR D1}(P) = P \parallel_R \mathbf{R}_s(\mathit{true}_r \vdash \mathit{false} \diamond (\$tr' =_u \$tr))$

**definition** *ISR D* ::  $('s, 't::trace, 'a) hrel-rsp \Rightarrow ('s, 't, 'a) hrel-rsp$  **where**  
 $[upred-defs]: \mathbf{ISR D} = \mathbf{ISR D1} \circ \mathbf{NSRD}$

**lemma** *ISR D1-idem*:  $\mathbf{ISR D1}(\mathbf{ISR D1}(P)) = \mathbf{ISR D1}(P)$   
**by** (*rel-auto*)

**lemma** *ISR D1-monotonic*:  $P \sqsubseteq Q \Longrightarrow \mathbf{ISR D1}(P) \sqsubseteq \mathbf{ISR D1}(Q)$   
**by** (*rel-auto*)

**lemma** *ISR D1-RHS-design-form*:  
**assumes**  $\$ok' \# P$   $\$ok' \# Q$   $\$ok' \# R$   
**shows**  $\mathbf{ISR D1}(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \mathit{false} \diamond (R \wedge \$tr' =_u \$tr))$   
**using** *assms* **by** (*simp add: ISR D1-def choose-srd-def RHS-tri-design-par unrest, rel-auto*)

**lemma** *ISR D1-form*:  
 $\mathbf{ISR D1}(\mathbf{SRD}(P)) = \mathbf{R}_s(\mathit{pre}_R(P) \vdash \mathit{false} \diamond (\mathit{post}_R(P) \wedge \$tr' =_u \$tr))$   
**by** (*simp add: ISR D1-RHS-design-form SRD-as-reactive-tri-design unrest*)

**lemma** *ISR D1-rdes-def* [*rdes-def*]:  
 $\llbracket P \text{ is } \mathbf{RR}; R \text{ is } \mathbf{RR} \rrbracket \Longrightarrow \mathbf{ISR D1}(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s(P \vdash \mathit{false} \diamond (R \wedge \$tr' =_u \$tr))$   
**by** (*simp add: ISR D1-def rdes-def closure rpred*)

**lemma** *ISR D-intro*:  
**assumes**  $P$  is  $\mathbf{NSRD}$   $\mathit{peri}_R(P) = (\neg_r \mathit{pre}_R(P)) (\$tr' =_u \$tr) \sqsubseteq \mathit{post}_R(P)$   
**shows**  $P$  is  $\mathbf{ISR D}$

**proof** –  
**have**  $\mathbf{R}_s(\mathit{pre}_R(P) \vdash \mathit{peri}_R(P) \diamond \mathit{post}_R(P))$  is  $\mathbf{ISR D1}$   
**apply** (*simp add: Healthy-def rdes-def closure assms(1-2)*)  
**using** *assms(3)* **least-zero** **apply** (*rel-blast*)

**done**  
**hence**  $P$  is ISRD1  
 by (simp add: SRD-reactive-tri-design closure assms(1))  
**thus** ?thesis  
 by (simp add: ISRD-def Healthy-comp assms(1))  
**qed**

**lemma** ISRD1-rdes-intro:  
**assumes**  $P$  is RR  $Q$  is RR ( $\$tr' =_u \$tr$ )  $\sqsubseteq Q$   
**shows**  $\mathbf{R}_s(P \vdash false \diamond Q)$  is ISRD1  
**unfolding** Healthy-def  
**by** (simp add: ISRD1-rdes-def assms closure unrest utp-pred-laws.inf.absorb1)

**lemma** ISRD-rdes-intro [closure]:  
**assumes**  $P$  is RC  $Q$  is RR ( $\$tr' =_u \$tr$ )  $\sqsubseteq Q$   
**shows**  $\mathbf{R}_s(P \vdash false \diamond Q)$  is ISRD  
**unfolding** Healthy-def  
**by** (simp add: ISRD-def closure Healthy-if ISRD1-rdes-def assms unrest utp-pred-laws.inf.absorb1)

**lemma** ISRD-implies-ISRD1:  
**assumes**  $P$  is ISRD  
**shows**  $P$  is ISRD1  
**proof** –  
**have** ISRD( $P$ ) is ISRD1  
 by (simp add: ISRD-def Healthy-def ISRD1-idem)  
**thus** ?thesis  
 by (simp add: assms Healthy-if)  
**qed**

**lemma** ISRD-implies-SRD:  
**assumes**  $P$  is ISRD  
**shows**  $P$  is SRD  
**proof** –  
**have**  $1: ISRD(P) = \mathbf{R}_s((\neg_r (\neg_r pre_R P) ;; R1 true \wedge R1 true) \vdash false \diamond (post_R P \wedge \$tr' =_u \$tr))$   
 by (simp add: NSRD-form ISRD1-def ISRD-def RHS-tri-design-par rdes-def unrest closure)  
**moreover have** ... is SRD  
 by (simp add: closure unrest)  
**ultimately have** ISRD( $P$ ) is SRD  
 by (simp)  
**with** assms **show** ?thesis  
 by (simp add: Healthy-def)  
**qed**

**lemma** ISRD-implies-NSRD [closure]:  
**assumes**  $P$  is ISRD  
**shows**  $P$  is NSRD  
**proof** –  
**have**  $1: ISRD(P) = ISRD1(RD3(SRD(P)))$   
 by (simp add: ISRD-def NSRD-def SRD-def, metis RD1-RD3-commute RD3-left-subsumes-RD2)  
**also have** ... = ISRD1(RD3( $P$ ))  
 by (simp add: assms ISRD-implies-SRD Healthy-if)  
**also have** ... = ISRD1 ( $\mathbf{R}_s((\neg_r pre_R P) wp_r false_h \vdash (\exists \$st' \cdot peri_R P) \diamond post_R P)$ )  
 by (simp add: RD3-def, subst SRD-right-unit-tri-lemma, simp-all add: assms ISRD-implies-SRD)  
**also have** ... =  $\mathbf{R}_s((\neg_r pre_R P) wp_r false_h \vdash false \diamond (post_R P \wedge \$tr' =_u \$tr))$   
 by (simp add: RHS-tri-design-par ISRD1-def unrest choose-srd-def rpred closure ISRD-implies-SRD)

*assms*)  
**also have** ... = (... ;;  $II_R$ )  
**by** (*rdes-simp*, *simp add: RHS-tri-normal-design-composition' closure assms unrest ISRD-implies-SRD*  
*wp rpred wp-rea-false-RC*)  
**also have** ... *is RD3*  
**by** (*simp add: Healthy-def RD3-def seqr-assoc*)  
**finally show** *?thesis*  
**by** (*simp add: SRD-RD3-implies-NSRD Healthy-if assms ISRD-implies-SRD*)  
**qed**

**lemma** *ISRD-form*:  
**assumes** *P is ISRD*  
**shows**  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr)) = P$

**proof** –  
**have**  $P = \text{ISRD1}(P)$   
**by** (*simp add: ISRD-implies-ISRD1 assms Healthy-if*)  
**also have** ... =  $\text{ISRD1}(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond \text{post}_R(P)))$   
**by** (*simp add: SRD-reactive-tri-design ISRD-implies-SRD assms*)  
**also have** ... =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr))$   
**by** (*simp add: ISRD1-rdes-def closure assms*)  
**finally show** *?thesis ..*  
**qed**

**lemma** *ISRD-elim* [*RD-elim*]:  
 $\llbracket P \text{ is ISRD}; Q(\mathbf{R}_s(\text{pre}_R(P) \vdash \text{false} \diamond (\text{post}_R(P) \wedge \$tr' =_u \$tr)) \rrbracket \implies Q(P)$   
**by** (*simp add: ISRD-form*)

**lemma** *skip-srd-ISRD* [*closure*]:  $II_R$  *is ISRD*  
**by** (*rule ISRD-intro*, *simp-all add: rdes closure*)

**lemma** *assigns-srd-ISRD* [*closure*]:  $\langle \sigma \rangle_R$  *is ISRD*  
**by** (*rule ISRD-intro*, *simp-all add: rdes closure, rel-auto*)

**lemma** *seq-ISRD-closed*:  
**assumes** *P is ISRD Q is ISRD*  
**shows**  $P ;; Q \text{ is ISRD}$   
**apply** (*insert assms*)  
**apply** (*erule ISRD-elim*)  
**apply** (*simp add: rdes-def closure assms unrest*)  
**apply** (*rule ISRD-rdes-intro*)  
**apply** (*simp-all add: rdes-def closure assms unrest*)  
**apply** (*rel-auto*)  
**done**

**lemma** *ISRD-Miracle-right-zero*:  
**assumes** *P is ISRD*  $\text{pre}_R(P) = \text{true}_r$   
**shows**  $P ;; \text{Miracle} = \text{Miracle}$   
**by** (*rdes-simp cls: assms*)

A recursion whose body does not extend the trace results in divergence

**lemma** *ISRD-recurse-Chaos*:  
**assumes** *P is ISRD*  $\text{post}_R P ;; \text{true}_r = \text{true}_r$   
**shows**  $(\mu_R X \cdot P ;; X) = \text{Chaos}$   
**proof** –  
**have**  $1: (\mu_R X \cdot P ;; X) = (\mu X \cdot P ;; \text{SRD}(X))$

```

  by (auto simp add: srdes-theory-continuous.utp-lfp-def closure assms)
have ( $\mu X \cdot P$  ;; SRD(X))  $\sqsubseteq$  Chaos
proof (rule gfp-upperbound)
  have  $P$  ;; Chaos  $\sqsubseteq$  Chaos
  apply (rdes-refine-split cls: assms)
  using assms(2) apply (rel-auto, metis (no-types, lifting) dual-order.antisym order-refl)
  apply (rel-auto)+
  done
  thus  $P$  ;; SRD Chaos  $\sqsubseteq$  Chaos
  by (simp add: Healthy-if srdes-theory-continuous.bottom-closed)
qed
thus ?thesis
  by (metis 1 dual-order.antisym srdes-theory-continuous.LFP-closed srdes-theory-continuous.bottom-lower)
qed

lemma recursive-assign-Chaos:
  ( $\mu_R X \cdot \langle \sigma \rangle_R$  ;; X) = Chaos
  by (rule ISRD-recurse-Chaos, simp-all add: closure rdes, rel-auto)

end

```

## 14 Meta-theory for Reactive Designs

```
theory utp-rea-designs
```

```
imports
```

```

  utp-rdes-healths
  utp-rdes-designs
  utp-rdes-triples
  utp-rdes-normal
  utp-rdes-contracts
  utp-rdes-tactics
  utp-rdes-parallel
  utp-rdes-prog
  utp-rdes-instant
  utp-rdes-guarded

```

```
begin end
```

## References

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- [3] M. V. M. Oliveira. *Formal Derivation of State-Rich Reactive Programs using Circus*. PhD thesis, Department of Computer Science - University of York, UK, 2006. YCST-2006-02.