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Multidimensional Inequality and Human Development*

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Abstract

The measurement of inequality from a human development perspective is fundamental. First, we briefly introduce the human development approach and its conceptual basis: the capability approach. We then present the primary challenges for multidimensional inequality measurement, reflecting two types of distributional changes. One is concerned with the dispersions within distributions and the other is concerned with the association between distributions. We next present a review of the most prominent measures within a unifying framework and review surrounding empirical applications. We observe that multidimensional measures have a great potential, but there are challenges to overcome for fulfilling such potential.

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1 Measurement of Inequality within a Human Development Framework

The human development (HD hereafter) approach became increasingly prevalent in the international development agenda with the first Human Development Report (HDR), published by the United Nations Development Programme (UNDP) in 1990. In fact, the Human Development Index (HDI) has become a metric of important reference. The essence of the HD approach is that development must have human beings at the centre of attention. Therefore, it is imperative to capture not only the level of HD but also the inequality in HD across the population. This chapter is an introduction to the field of inequality measurement within a HD perspective, highlighting its scope and challenges.

The HD approach is linked to many conceptual frameworks that go back to Aristotle, and include the basic needs approach, the Social Doctrine of the Catholic Church, human rights and sustainable livelihoods. But it has been fundamentally strengthened by Amartya Sen's capability approach (Alkire and Deneulin, 2009), in which development is "the process of expanding the real freedoms people enjoy" (Sen, 1999, p.3).[1] A key implication is the space in which development must be evaluated.

Evaluating development in the space of resources, let it be income or Rawlsian primary goods, is problematic because these are mere *means* to ends, not *ends* in themselves. Moreover, people have different abilities to convert each specific resource into a certain achievement.[2] Evaluating development in the space of utilities is also problematic because there are *adaptive preferences* by which people in an objective state of deprivation can show high utility levels. Thus, Sen argues that the space of evaluation of development must be that of capabilities and functionings, which is inherently multidimensional.

Functionings are "the various things a person may value ***doing*** or ***being***", which range from fundamental ones such as being adequately nourished, to more sophisticated ones such as taking part in the life of the community. As long as a person's functionings can be expressed by real numbers, the functionings can be summarised by a *functioning vector*. The set of all functioning vectors available to the person form the person's *capability set* or *capabilities*. One particular functioning vector, or a combination of functionings that the person actually chooses from the set of capabilities, reflects that person's *achievements*. Resembling the concept of budget set in the consumer theory, the capability set is the collection of all available functionings or the ***set of opportunities*** and thus represents the person's ***freedom*** to choose or achieve various functionings (see Sen, 1997, p. 394-95).

Sen favours using the 'capability set' over the chosen or 'achieved functionings' as the space for evaluating development, because achieved functionings are merely an element of the entire capability set. The capability set, in contrast, contains *all* available functioning vectors, even those not chosen. This distinction is relevant because two persons may have been observed to choose the same functioning vector, and yet one may have chosen the functioning vector in the absence of any better available alternative (i.e. lacks freedom to choose from), whereas the other may have chosen the functioning vector despite having better available alternatives (i.e. has freedom to choose from).[3] Thus, using the capability set over achieved functioning captures a person's freedom to choose from various alternative functionings regarded as intrinsically valuable (Sen, 1985).[4]

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[1] "*Enlarging people's choices"* is about expanding *valuable* possible beings and doings (Alkire and Deneulin, 2009, p. 34).

[2] Such abilities are influenced by personal heterogeneities, environmental diversities, variations in social climate, differences in relational perspectives and intra-household distribution (Sen, 1997).

[3] A frequent example offered by Sen is that of two persons with low nutritional status, one is due to the lack of resources and another because of the decision to fast.

[4] For Sen, functionings are things people value *and* "have reason to value", implying that social choices need to be made regarding beings and doings that can be considered valuable (Alkire and Deneulin, 2009, ch. 2).

Fleurbaey (2004) contends that the space of functionings does allow measuring freedoms. One may evaluate freedoms, he suggests, by putting substantially higher relative weight on basic functionings for human flourishing, and by combining this with knowledge of the existing legislation of the place where the individual lives. For example, poor educational achievements, low income, and unsatisfactory social relations inevitably reflect the lack of freedom to choose.

In this way, we are soon into the practical challenges faced when shaping a measure of HD, let it be on the level of HD, inequality or poverty. One first practical challenge relates to a long-standing discussion on whether there should be a list of 'central capabilities', and thus, of implied functionings (as required by Fleurbaey). Some capabilities, according to Sen, may be considered basic. Yet, Sen claims that no particular list should be prescribed, because any list needs to be defined according to the purpose of the evaluation, must emerge from deliberative engagement, and must necessarily be contingent to time and space (Sen, 2004). In contrast, Martha Nussbaum argues that a list of central human capabilities is fundamental to avoid issues of omission and power by which people may learn not to want or value certain functionings (Alkire and Deneulin, 2009).[5] At this point it is worth noting that capabilities refer to different *dimensions* of well-being (also sometimes called domains), and within them, there may be one or more *indicators* that proxy the capabilities. Choosing dimensions and indicators to be considered in a measure is a key step. Interestingly, Alkire (2008) points that in practice one finds a striking degree of commonality between different lists of central human capabilities or dimensions that have been suggested.

A second practical challenge, closely related to the first one, is the selection of relative weights. Non-included dimensions receive a zero weight. In turn, weighting the included dimensions and indicators also has important implications as it determines their trade-offs (Decancq and Lugo 2012a). Sen in fact advices using a range of weights on which there is at least some agreement. It is also considered a good practice to perform robustness analysis to (reasonable) changes in the weighting structure (Alkire et al., 2015). However, explicit weights are not the only determinant of trade-offs across dimensions, so are normalization procedures and the aggregation function across dimensions (Decancq and Lugo, 2012a). All these are non-trivial normative decisions in the evaluation of multidimensional well-being, inequality or poverty, that require a sound justification and transparency. Analysis and discussion on these matters can be found elsewhere.[6]

This chapter is organised as follows. Section 2 sets the basis for unidimensional inequality measurement, building on which Section 3 moves to the associated multidimensional framework and presents related challenges. Section 4 discusses axiomatic properties and provides a succinct review of the most prominent multidimensional indices. Section 5 reviews empirical applications these indices. Section 6 concludes.[7]

2 Inequality within Single Dimensional Framework

For a while, let us suppose that human development can be assessed by only a single dimension, which may be either earned incomes or educational attainments. Suppose, there are n (≥ 2) persons in a hypothetical society. For simplicity of presentation, we assume that each of the n persons has an achievement. We denote the achievement of person i by $x_i \in \mathbb{R}_{++}$ for all $i = 1, \ldots, n$, where $\mathbb{R}_{++}$ is the set of strictly positive real

[5] Nussbaum's (2000) ten central capabilities are: (1) life, (2) bodily health, (3) bodily integrity,(4) sense, imagination and though, (5) emotions, (6) practical reason, (7) affiliation, (8) other species, (9) play, (10) control over one's environment.

[6] Alkire (2008); Alkire et al (2015, ch.6); Decancq and Lugo (2012a).

[7] Chakravarty and Lugo (2016) and Zoli (2009) offer related discussions.

numbers, i.e. strictly positive achievements.[8] The collection of all n persons' achievements in the society can be represented by an achievement vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$. An achievement vector may be referred to as a *distribution of achievements*. A higher value of achievement reflects higher level of well-being. We denote the average of all achievements in distribution $\mathbf{x}$ by $\mu(\mathbf{x}) = (x_1 + \cdots + x_n)/n$.

Inequality in any single dimension is mainly understood through either Pigou-Dalton-progressive (regressive) transfer. A Pigou-Dalton progressive (regressive) transfer takes place whenever one distribution is obtained from another distribution through a rank preserving transfer of achievement from a person with higher (lower) achievement to a person with lower (higher) achievement, while keeping the mean achievement unchanged. Consider two distributions: $\mathbf{x} = (1,2,8,9)$ and $\mathbf{y} = (2,2,8,8)$. Note that $\mu(\mathbf{x}) = \mu(\mathbf{y}) = 5$. Note that $\mathbf{y}$ can be obtained from $\mathbf{x}$ by transferring achievement of one unit from the person with nine units to the person with one unit. In this case, $\mathbf{y}$ is stated to be obtained from $\mathbf{x}$ by a progressive transfer. Technically, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$, $\mathbf{y}$ is stated to be obtained from $\mathbf{x}$ by a Pigou-Dalton progressive transfer if there are two persons i_1 and i_2 such that $x_{i_1} > x_{i_2}$, $y_{i_1} = x_{i_1} - \delta$ and $y_{i_2} = x_{i_2} + \delta$ for any $\delta > 0$ yet $y_{i_1} > y_{i_2}$, and $y_i = x_i$ for all $i \neq i_1, i_2$. Conversely, distribution $\mathbf{x}$, can be obtained from distribution $\mathbf{y}$ by a regressive transfer. Technically, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$, $\mathbf{x}$ is stated to be obtained from $\mathbf{y}$ by a Pigou-Dalton regressive transfer if there are two persons i_1 and i_2 such that $y_{i_1} > y_{i_2}$, $x_{i_1} = y_{i_1} + \delta$ and $x_{i_2} = y_{i_2} - \delta$ for any $0 < \delta < y_{i_2}$, and $x_i = y_i$ for all $i \neq i_1, i_2$.

Whenever a distribution is obtained from another distribution by a sequence of Pigou-Dalton progressive (regressive) transfers, then inequality in the former distribution is lower (higher) than that in the latter distribution.

Pigou-Dalton progressive transfer(s) can be technically expressed using T-transformation(s). A *T-transformation matrix* ($\mathbf{T}$) is a weighted average of an identify matrix $\mathbf{E}$ and a non-identity permutation matrix $\mathbf{P}$, such that $\mathbf{T} = \lambda\mathbf{E} + (1 - \lambda)\mathbf{P}$ where $0 < \lambda < 1$.[9] A permutation matrix is a non-negative square matrix with each row and each column having exactly one element equal to one and the rest being equal to zero. The following combination of $\mathbf{E}$, $\mathbf{P}$ and λ provides the T-transformation matrix for obtaining $\mathbf{y} = (2,2,8,8)$ from $\mathbf{x} = (1,2,8,9)$:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \lambda = 0.875.$$

Thus,

$$\mathbf{T} = \begin{bmatrix} 0.875 & 0 & 0 & 0.125 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.125 & 0 & 0 & 0.875 \end{bmatrix} \text{ and so } \mathbf{y} = \mathbf{x}\mathbf{T}.$$

Distribution $\mathbf{y}$ in this case is obtained from distribution $\mathbf{x}$ by post-multiplying $\mathbf{x}$ by a T-transformation matrix. Whenever a distribution is obtained from another distribution by a sequence of Pigou-Dalton transfers, then the former distribution can be equivalently obtained from the latter by a *finite number* of T-transformations.

The lowest level of inequality or the situation of perfect equality is accomplished whenever everybody receives the same level of achievement. Technically, the situation of

[8] Empirical applications of certain inequality measures require special treatment of negative or zero achievement values.

[9] We define T-transformation here in a strict sense by restricting λ to lie between 0 and 1. Whenever, $\lambda = 1$, a T-transformation matrix coincides with an identity matrix, resulting in no change in the distribution. Whenever, $\lambda = 0$, a T-transformation matrix coincides with a permutation matrix, where elements within an achievement vector merely swap places.

perfect equality in $\mathbf{x}$ is reached whenever every person receives an achievement equal to $\mu(\mathbf{x})$; we denote the equally distributed distribution corresponding to $\mathbf{x}$ as $\bar{\mathbf{x}}$, where $\bar{x}_i = \mu(\mathbf{x})$ for all $i = 1, \dots, n$. A sequence of T-transformations may lead to the situation of perfect equality. For example, a sequential application of the following two T-transformation matrices leads to $\bar{\mathbf{x}} = (5,5,5,5)$ from $\mathbf{x} = (1,2,8,9)$, i.e., $\bar{\mathbf{x}} = \mathbf{x}\mathbf{T}^1\mathbf{T}^2$:

$$\mathbf{T}^1 = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix} \text{ and } \mathbf{T}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ [10]

We introduce the related concept of bistochastic matrix (denoted by $\mathbf{B}$) that we will use in subsequent sections. A *bistochastic matrix* is a non-negative square matrix whose each row and each column sums to one. T-transformation matrices themselves as well as a product of T-transformation matrices are bistochastic matrices. Permutation matrices (which include identity matrices) are also bistochastic matrices. Post-multiplying a distribution by a non-permutation bistochastic matrix does not change the mean of the distribution, but makes the distribution more equal. However, not all bistochastic matrices, especially those with $n \geq 3$ dimensions, can be expressed as a product of T-transformation matrices. An example of a bistochastic matrix that is neither a T-transformation matrix nor a product of T-transformation matrices is:

$$\mathbf{B} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$ [11]

This difference becomes important in the multidimensional context.

Can the understanding of inequality within the single dimensional context be extended straightforwardly to understanding inequality involving multiple dimensions? Does increase or decrease in inequality within each of the many dimensions leads to increase or decrease in overall inequality? We critically answer these questions in the next section.

3 Inequality Involving Multiple Dimensions

We introduce some additional notation that are specific to the multidimensional framework. Suppose, in addition to n (≥ 2) persons in the society, inequality is assessed by d (≥ 2) dimensions. Similar to the single dimensional framework we denote the achievement of person i in dimension j by $x_{ij} \in \mathbb{R}_{++}$ for all $i = 1, \dots, n$ and $j = 1, \dots, d$, where a higher value of x_{ij} denotes higher achievement within dimension j. The collection of all persons' achievements in a society can be represented by an $n \times d$-dimensional achievement matrix $\mathbf{X}$ as:

Dimensions

$$\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix} \textbf{ People}$$

We denote each row i of $\mathbf{X}$ by a d-dimensional vector $\mathbf{x}_{i\cdot}$ summarising person i's achievements in all d dimensions; whereas we denote each column j of $\mathbf{X}$ by an n-dimensional vector $\mathbf{x}_{\cdot j}$ summarising the achievements for all n persons in dimension j. A column vector of achievements is referred to as a *marginal distribution of achievements* and an achievement matrix, which contains all marginal distributions, is referred to as a *joint distribution of achievements*. For definitional purposes, we will denote the set of all possible matrices of size $n \times d$ by $\mathcal{X}_n \in \mathbb{R}_+^{n \times d}$ and all possible achievement matrices by

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10 These two T-transformation matrices are not the unique set of matrices for obtaining $\bar{\mathbf{x}}$ from $\mathbf{x}$.

11 See Marshall and Olkin (1979), Page 23.

$\mathcal{X} = \bigcup_n \mathcal{X}_n$. Like in the single dimensional framework, we let $\mu_j(\mathbf{X}) = \mu(\mathbf{x}_{\cdot j})$ denote the average of all achievements in dimension j. The average achievements across all d dimensions are summarised by vector $\boldsymbol{\mu}(\mathbf{X}) = (\mu_1(\mathbf{X}), \dots, \mu_d(\mathbf{X}))$. We also define the additional vector notation. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^d$, $\mathbf{a} \geq \mathbf{b}$ implies that $a_j \geq b_j$ for all j and $\mathbf{a} > \mathbf{b}$ implies that $a_j \geq b_j$ for all j and $a_j > b_j$ for some j.

Let us now see if the concept of Pigou-Dalton progressive transfer in the single-dimensional framework can be extended to the multidimensional framework. We have already discussed in Section 3 that Pigou-Dalton progressive transfers can be presented using T-transformations. In the multidimensional context, similarly Pigou-Dalton progressive transfer(s) may take place uniformly across all dimensions. For two joint distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a *uniform Pigou-Dalton progressive transfer* (UPDT) whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by pre-multiplying $\mathbf{X}$ by a T-transformation matrix $\mathbf{T}$, i.e., $\mathbf{Y} = \mathbf{TX}$.[12] The *UPDT majorization* requires inequality to be lower if a distribution is obtained from another distribution by a UPDT or a sequence of UPDTs. In the following example, $\mathbf{Y}^1$ is obtained from $\mathbf{X}^1$ by pre-multiplying $\mathbf{X}^1$ by $\mathbf{T}$.

$$\mathbf{Y}^1 = \begin{bmatrix} 2 & 3 \\ 2 & 3 \\ 8 & 9 \\ 8 & 9 \end{bmatrix}; \mathbf{X}^1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 8 & 9 \\ 9 & 10 \end{bmatrix} \text{ and } \mathbf{T} = \begin{bmatrix} 0.875 & 0 & 0 & 0.125 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.125 & 0 & 0 & 0.875 \end{bmatrix}.$$

Note that the same T-transformation has been applied uniformly to both marginal distributions; i.e., $\mathbf{y}_{\cdot j}^1 = \mathbf{T}\mathbf{x}_{\cdot j}^1$ for $j = 1{,}2$. Clearly, $\mu_j(\mathbf{Y}^1) = \mu_j(\mathbf{X}^1)$ for $j = 1{,}2$.

We have already discussed that the T-transformations and the product of T-transformations can be seen as bistochastic transformations, but not all bistochastic matrices can be presented as products of T-transformation matrices. That means that the transformation of some achievements matrices into others can never be obtained by UPD. A concept referred to as uniform majorization has thus been introduced in the literature (Kolm 1977). For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is stated to be obtained from $\mathbf{X}$ by *uniform bistochastic transformation* (UBT) whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by pre-multiplying $\mathbf{X}$ by a bistochastic matrix $\mathbf{B}$; i.e., $\mathbf{Y} = \mathbf{BX}$. *Uniform majorization* requires inequality to be lower if a distribution is obtained from another distribution by a UBT. As all T-transformation matrices are bistochastic matrices, $\mathbf{Y}^1$ can be obtained from $\mathbf{X}^1$ by UM. Again, note that the same transformation has been applied uniformly to both dimensions, i.e., $\mathbf{y}_{\cdot j}^1 = \mathbf{B}\mathbf{x}_{\cdot j}^1$ and also $\mu_j(\mathbf{Y}^1) = \mu_j(\mathbf{X}^1)$ for $j = 1{,}2$.

In the previous example, each marginal distribution in $\mathbf{Y}^1$ has become more equal than the respective distribution in $\mathbf{X}^1$. Within each marginal distribution, the poorest person is better off at the cost of the richest person being worse off, while leaving the mean achievement unchanged. Should we consider distribution $\mathbf{Y}^1$ to be more equal than distribution $\mathbf{X}^1$? The answer should be 'yes' because the poorest person is unambiguously better off ($\mathbf{y}_{1\cdot}^1 > \mathbf{x}_{1\cdot}^1$) and the richest person is unambiguously worse off ($\mathbf{y}_{4\cdot}^1 < \mathbf{x}_{4\cdot}^1$). Inequality is certainly lower in $\mathbf{Y}^1$ than in $\mathbf{X}^1$.

Can we thus state that multidimensional inequality would be lower whenever one joint distribution is obtained from another by UBT (or UPDT)? The answer is not straightforward. Let us consider another example motivated by Dardanoni (1996), where $\mathbf{Y}^2$ is obtained from $\mathbf{X}^2$, such that $\mathbf{Y}^2 = \mathbf{BX}^2$:

$$\mathbf{Y}^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 7 & 7 \\ 7 & 7 \end{bmatrix}; \mathbf{X}^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 5 & 9 \\ 9 & 5 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

[12] In the single-dimensional context, the distribution across persons is a row vector; in the multidimensional context each marginal distribution across persons is a column vector.

In this case, the achievements of the two richest persons were averaged, while the achievements of the two poorest persons remained unchanged. Clearly, $\mu_j(\mathbf{Y}^2) = \mu_j(\mathbf{X}^2)$ for $j = 1,2$. Suppose, each person's human development is obtained by aggregating her achievements using an aggregation function: $f(x_{i1}^2, x_{i2}^2) = (x_{i1}^2 x_{i2}^2)^{0.5}$, which is a standard concave Cobb-Douglas function. The human development levels for the two poorest persons remain unchanged, i.e., $f(y_{i1}^2, y_{i2}^2) = f(x_{i1}^2, x_{i2}^2)$ for $i = 1,2$. However, the human development levels are certainly higher for the two richest persons, i.e., $f(y_{i1}^2, y_{i2}^2) > f(x_{i1}^2, x_{i2}^2)$ for $i = 3,4$. What we see is that a reduction in inequality within both marginal distributions uniformly in $\mathbf{Y}^2$ has made the two richest persons better off but has left the two poorest persons behind.

Can it thus be claimed that inequality is lower in $\mathbf{Y}^2$ than in $\mathbf{X}^2$? There are two major issues with such comparisons. One is that the transfer is restricted to occur uniformly across all dimensions, which may not be reasonable in practice. Second, transfers do not necessarily take place between a richer person and a poorer person, which is the main essence of the Pigou-Dalton transfer in the single-dimensional context (Lasso de la Vega et al, 2010). For UPDT and UBT, transfers may take place between two persons where one has higher achievements in some dimensions while lower achievements in other dimensions than the other person.

Through a novel approach, Bosmans et al. (2015) provide an explanation for the comparison between $\mathbf{Y}^2$ and $\mathbf{X}^2$ by decomposing the overall multidimensional inequality into an *inequity* component and an *inefficiency* component. *Inequity* within a joint distribution exists as long as all persons do not have the mean achievement within each dimension. *Inefficiency* within a joint distribution exists so far as the well-being level of at least one person can be improved through redistribution without worsening the well-being levels of any other person. The redistribution between $\mathbf{Y}^2$ and $\mathbf{X}^2$ have indeed increased inequity in well-being by leaving the two poorest behind (similar point was raised by Duclos at al. 2011, p.229). However, the well-being levels of the two richest persons have increased without worsening anyone else's well-being level, improving efficiency. The improvement in efficiency may have outweighed the deterioration in inequity, leading to a net improvement in inequality. Bosmans et al. (2015) thus conclude that 'uniform majorization is more successful at capturing the efficiency aspect of multidimensional inequality than at capturing the equity aspect' (p. 99).

Fleurbaey and Trannoy (2003) have proposed another extension of the Pigou-Dalton progressive transfer in the multidimensional context referred to as *Pigou-Dalton Bundle Transfer* (PBT). The transfer is rank preserving and takes place only between a rich person and an unambiguously poorer person. A person h is unambiguously richer than another person k whenever $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$. In $\mathbf{X}^2$, for example, the first two persons are unambiguously poorer than the last two persons. The *PBT majorization* requires inequality to be lower if a distribution is obtained from another distribution by a PBT or a sequence of PBTs.

Adapting from Lasso de la Vega et al. (2010), for any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a PBT whenever there are two persons h and k, such that (i) $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$, (ii) $\mathbf{y}_{k\cdot} = \mathbf{x}_{k\cdot} + \delta$ and $\mathbf{y}_{h\cdot} = \mathbf{x}_{h\cdot} - \delta$ for some $\delta = (\delta_1, \dots, \delta_d) > 0$, (iii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iv) $\mathbf{y}_{h\cdot} \geq \mathbf{y}_{k\cdot}$. What do all these conditions mean? Condition (i) requires that person h has higher achievement than person k in at least one dimension and no less achievement in any dimension before transfer. Condition (ii) requires that achievement(s) of positive amount in at least one dimension is transferred from person h to person k. Condition (iii) requires that achievements of all other persons are identical in $\mathbf{Y}$ and $\mathbf{X}$. Finally, condition (iv) requires that the post-transfer achievements of person h is not lower than that of person k in any dimension, ensuring that post-transfer ranks are preserved.

In the following example, $\mathbf{Y}^3$ is obtained from $\mathbf{X}^3$ by a PBT:

$$\mathbf{Y}^3 = \begin{bmatrix} 2 & 4 \\ 3 & 4 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^3 = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{ and } \delta = (1,2).$$

The first person in $\mathbf{X}^3$ is poorer than all others in both dimensions. An achievement of one unit in dimension 1 ($\delta_1 = 1$) and an achievement of two units in dimension 2 ($\delta_2 = 2$) are transferred from person 2 to person 1. Importantly, after the transfer (i.e., in $\mathbf{Y}^3$) person 1's achievements in no dimension is larger than that of person 2, yet $\mu_j(\mathbf{Y}^3) = \mu_j(\mathbf{X}^3)$ for all j. Inequality $\mathbf{Y}^3$ may surely be claimed to be lower than that in $\mathbf{X}^3$ as required by the PBT majorization.[13]

So far, we have presented extensions of Pigou-Dalton transfers. However, in the multidimensional context, there is a second form of inequality that is concerned with association between dimensions. Let us consider the following example involving $\mathbf{Y}^4$ and $\mathbf{X}^4$ to clarify the point:

$$\mathbf{Y}^4 = \begin{bmatrix} 1 & 10 \\ 2 & 9 \\ 8 & 3 \\ 9 & 2 \end{bmatrix}; \mathbf{X}^4 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 8 & 9 \\ 9 & 10 \end{bmatrix}.$$

Note that each marginal distribution in $\mathbf{Y}^4$ is identical to the respective marginal distribution in $\mathbf{X}^4$, i.e., $\mathbf{y}_{\cdot 1}^4 = \mathbf{x}_{\cdot 1}^4 = (1,2,8,9)$ and $\mathbf{y}_{\cdot 2}^4 = \mathbf{y}_{\cdot 2}^4 = (2,3,7,8)$ and certainly $\mu_j(\mathbf{Y}^4) = \mu_j(\mathbf{X}^4)$ for all j. The main difference between the two joint distributions is observed when we look at the marginal distributions together within each joint distribution. In $\mathbf{X}^4$, both marginals are perfectly positively associated with each other; each poorer person is poorer in both dimensions than each richer person. Contrarily, in $\mathbf{Y}^4$, marginals are oppositely ordered or are perfectly negatively associated with each other. Which distribution is more equal: $\mathbf{Y}^4$ or $\mathbf{X}^4$?

In their pioneering paper, Atkinson and Bourguignon (1982) introduced a second form of multidimensional inequality, requiring multidimensional social evaluations to be sensitive to correlation, or more precisely association, between dimensions. In the context of two dimensions, they obtained different conditions on how bivariate joint distributions could be ranked, whenever these distributions have same marginals but different inter-dependence as presented in the example involving $\mathbf{Y}^4$ and $\mathbf{X}^4$. Decancq (2012) has extended the Atkinson-Bourguignon framework to situations involving three or more dimensions.

In the literature of multidimensional measurement, certain approaches have been proposed to capture sensitivity to change in the inter-dependence between multiple dimensions. One prominent approach is the *Correlation Increasing Transfer* coined by Tsui (1999) motivated by Boland and Proschan (1988). The concept is also referred as *Correlation Increasing Switch* (Bourguignon and Chakravarty 2003), and *Association Increasing Transfer* (Seth 2013). We explain the concept using an example with $\mathbf{Z}^5$, $\mathbf{Y}^5$, and $\mathbf{X}^5$ involving three dimensions.

$$\mathbf{Z}^5 = \begin{bmatrix} 1 & \mathbf{3} & 2 \\ 2 & \mathbf{4} & 3 \\ 8 & 8 & 6 \\ 9 & 9 & 9 \end{bmatrix}; \mathbf{Y}^5 = \begin{bmatrix} 1 & \mathbf{4} & 2 \\ 2 & \mathbf{3} & 3 \\ 8 & \mathbf{8} & \mathbf{6} \\ 9 & \mathbf{9} & \mathbf{9} \end{bmatrix}; \mathbf{X}^5 = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 3 \\ 8 & \mathbf{9} & \mathbf{9} \\ 9 & \mathbf{8} & \mathbf{6} \end{bmatrix}$$

[13] We have not taken preferences of persons into consideration. For discussions on how transfers such as PBT and UM may become incompatible when one allows individual preferences to differ, see Fleurbaey and Trannoy (2003).

If we compare the joint distributions, then clearly their marginals are identical: $\mathbf{z}_{\cdot 1}^5 = \mathbf{y}_{\cdot 1}^5 = \mathbf{x}_{\cdot 1}^5 = (1,2,8,9)$, $\mathbf{z}_{\cdot 2}^5 = \mathbf{y}_{\cdot 2}^5 = \mathbf{x}_{\cdot 2}^5 = (3,4,8,9)$, and $\mathbf{z}_{\cdot 3}^5 = \mathbf{y}_{\cdot 3}^5 = \mathbf{y}_{\cdot 3}^5 = (2,3,6,9)$. In $\mathbf{X}^5$, marginals are not perfectly positively associated. For example, the fourth person has higher achievement than the third person in the first dimension but lower achievements in other dimensions. Their achievements are swapped to obtain $\mathbf{Y}^5$ so that the fourth person has higher achievements in all three dimensions than the third person. Clearly, the association between dimensions is higher in $\mathbf{Y}^5$. Next, the achievements between the first two persons in $\mathbf{Y}^5$ are further swapped to obtain $\mathbf{Z}^5$, increasing the association further. All three dimensions in $\mathbf{Z}^5$ are perfectly positively associated.

Formally, for any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an *association increasing transfer* (AIT) if there are two persons h and k, such that (i) $\mathbf{y}_{hj} = \min\{\mathbf{x}_{hj}, \mathbf{x}_{kj}\}$ and $\mathbf{y}_{kj} = \max\{\mathbf{x}_{hj}, \mathbf{x}_{kj}\}$ for all j, (ii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iii) $\mathbf{Y}$ is not a permutation of $\mathbf{X}$.[14] The third condition in the definition is important as it prevents the transfer from taking place in dimensions where both persons have equal achievements. If $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an *association increasing transfer*, then, conversely, $\mathbf{X}$ is obtained from $\mathbf{Y}$ by an *association decreasing transfer* (ADT).

An *association increasing majorization* requires that inequality should increase when one distribution is obtained from another by an AIT or a sequence of AITs. A *converse association increasing majorization* requires that inequality should fall when one distribution is obtained from another by an AIT or a sequence of AITs.

Should multidimensional inequality increase or decrease due to an association increasing transfer? Tsui (1999), among others, requires inequality to increase (or at least not to decrease) whenever there is an AIT. Implicitly, this requirement assumes dimensions to be substitutes (Atkinson and Bourguignon, 1982; Bourguignon and Chakravarty, 2003). If a good health outcome can *substitute* low income or low educational level, it is preferred that high achievements are spread out across the population. However, if attributes are *complements* – say, if a good health outcome is necessary to achieve higher income or better education, an AIT may produce a preferable distribution. This relationship however has recently been questioned by Bosmans et al. (2015), who show that association increasing majorization may be compatible with complementarity. This controversial topic however requires further research.

Another approach to capture sensitivity to change in the inter-dependence between multiple dimensions has been proposed by Dardanoni (1996), which Decancq and Lugo (2012b) refer to as *unfair rearrangement* (UR). For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an UR if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a sequence of AITs such that there is vector dominance between every pair of persons in $\mathbf{Y}$. In words, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by an UR, then one individual has largest achievements in all dimensions, another individual has second largest achievements in all dimensions and so on. In our example, $\mathbf{Z}^5$ is obtained from both $\mathbf{Y}^5$ and $\mathbf{X}^5$ by UR. Note, however, that UR cannot rank $\mathbf{Y}^5$ and $\mathbf{X}^5$. Should inequality increase due to unfair rearrangement? Similar controversy may arise as in case of the association increasing majorization and converse association increasing majorization.

We finally discuss the concept of *Compensating Transfer* (CT), proposed by Lasso de la Vega et al. (2010), which combines the concept of PBT and ADT. For any two distributions $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a CT whenever there are two persons h and k, such that (i) $\mathbf{x}_{h\cdot} > \mathbf{x}_{k\cdot}$, (ii) $\mathbf{y}_{k\cdot} = \mathbf{x}_{k\cdot} + \delta$ and $\mathbf{y}_{h\cdot} = \mathbf{x}_{h\cdot} - \delta$ for some $\delta = (\delta_1, \ldots, \delta_d) > 0$, (iii) $\mathbf{y}_{i\cdot} = \mathbf{x}_{i\cdot}$ for all $i \neq h, k$, and (iv) $\mathbf{y}_{h\cdot} \geq \mathbf{x}_{k\cdot}$. The definition of CT is analogous to the definition of PBT, but with one crucial difference. The fourth condition in the definition of CT allows reversal

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[14] We prefer to use the broader term 'association' rather than 'correlation'.

of ranks by requiring $\mathbf{y}_{h\cdot} \geq \mathbf{x}_{k\cdot}$, unlike $\mathbf{y}_{h\cdot} \geq \mathbf{y}_{k\cdot}$ in case of PBT. A CT is claimed to lower multidimensional inequality.

In the following example $\mathbf{Y}^6$ is obtained from $\mathbf{X}^6$ by a CT.

$$\mathbf{Y}^6 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^6 = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{and } \delta = (3,1).$$

The first person in $\mathbf{X}^6$ is poorer than all others. An achievement of three units in dimension 1 ($\delta_1 = 3$) and an achievement of one unit in dimension 2 ($\delta_2 = 1$) are transferred from person 2 to person 1. After the transfer (i.e., in $\mathbf{Y}^6$), person 1's achievement in the first dimension is higher than that of person 2, but person 1's achievement in the second dimension remains lower. In this case, thus, there is no vector dominance between persons 1 and 2. The CT from $\mathbf{X}^6$ to $\mathbf{Y}^6$ can be broken down into a PBT (from $\mathbf{X}^6$ to $\mathbf{Z}^6$ by δ') and an ADT (from $\mathbf{Z}^6$ to $\mathbf{Y}^6$) as:

$$\mathbf{Y}^6 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{Z}^6 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 5 \end{bmatrix}; \mathbf{X}^6 = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 6 & 7 \\ 9 & 5 \end{bmatrix} \text{and } \delta' = (1,1).$$

Our detailed discussions in this section has exposed the difficulties that one may face while assessing inequality within a multidimensional framework.

4 Multidimensional Inequality Measures in Normative Framework

In this section, we discuss various normative multidimensional inequality measures that have been proposed in the literature. Inequality within a single dimension $\mathbf{x} \in \mathbb{R}^n_{++}$ (using notation from Section 0) is assessed by defining a unidimensional inequality measure, which is a function $I(\mathbf{x})$ that maps the achievements in $\mathbf{x}$ in a real number $\mathbb{R}$. Technically, $I: \mathbb{R}^n_{++} \rightarrow \mathbb{R}$. In the unidimensional context, inequality is exclusively about *inequality across people* within a certain dimension, say income. Similarly, in the multidimensional context, an inequality measure maps from the achievements in $\mathbf{X}$ to a real number $\mathbb{R}$, i.e., $I: \mathcal{X} \rightarrow \mathbb{R}$.

In Section 3, we introduced a set of distributional properties, namely, those related to transfer and those related to association across dimensions. In the next subsection, we introduce some of the important non-distributional properties.

4.1 Non-Distributional Properties

Aside from the distributional properties discussed in Section 3, inequality measures are required to satisfy certain additional properties; some of which are basic whereas others are more controversial.

One basic property, *symmetry* (also called *anonymity*), requires that an inequality measure should be invariant to *who* has each achievement vector. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by simply permuting the achievement vectors in $\mathbf{X}$ (i.e., $\mathbf{Y} = \mathbf{PX}$ where $\mathbf{P}$ is a permutation matrix), then $I(\mathbf{X}) = I(\mathbf{Y})$. A second basic property, *replication invariance* (also called *population principle*), requires that an inequality measure should be invariant to replications of the population. Technically, if $\mathbf{Y} \in \mathcal{X}_{\gamma n}$ is obtained from $\mathbf{X} \in \mathcal{X}_n$ by simply replicating or cloning each person's achievement vector in $\mathbf{X}$ by γ times (where γ is a positive integer and $\gamma > 1$), then $I(\mathbf{X}) = I(\mathbf{Y})$. This property allows comparing inequality across societies with different population sizes. A third basic property, *normalisation*, requires that whenever every person has the same achievement vector the inequality measure should be equal to zero. Technically, for any $\mathbf{X} \in \mathcal{X}_n$, if $\mathbf{x}_{i\cdot} = \mathbf{x}_{k\cdot}$ for all $i \neq k$, then $I(\mathbf{X}) = 0$.

The next set of normative properties are slightly controversial and not all are necessarily compatible with each other. The *scale invariance* (also called zero-degree homogeneity) property requires that an inequality measure should be invariant to proportional changes of all achievements. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $\mathbf{Y} = \delta\mathbf{X}$ for any $\delta > 0$, then $I(\mathbf{X}) = I(\mathbf{Y})$. A more intuitive but related property is *ratio scale invariance*, which requires that if each column vector is multiplied by a factor, then this should not alter the level of inequality. Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $\mathbf{y}_{\cdot j} = \delta_j \mathbf{x}_{\cdot j}$ for any $\delta_j > 0$ for all j, then $I(\mathbf{X}) = I(\mathbf{Y})$. The ratio scale invariance property allows comparing the level of inequality when achievements are presented in alternative units (e.g., income may be assessed by US dollars or British Pounds, education may be assessed in years or months). A third related property but with a weaker requirement than ratio scale invariance property is *unit consistency*, which merely requires that the ordering of distributions by an inequality measure should not alter whenever units of measurement change (Zheng 2007). Suppose for any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_n$, $I(\mathbf{X}) > I(\mathbf{Y})$. If $\mathbf{Y}' \in \mathcal{X}_n$ is obtained from $\mathbf{Y}$ and $\mathbf{X}' \in \mathcal{X}_n$ is obtained from $\mathbf{X}$ so that $\mathbf{y}'_{\cdot j} = \delta_j \mathbf{y}_{\cdot j}$ and $\mathbf{x}'_{\cdot j} = \delta_j \mathbf{x}_{\cdot j}$ for any $\delta_j > 0$ for all j, then $I(\mathbf{X}') > I(\mathbf{Y}')$. Finally, the *translation invariance* property, requires that an inequality measure should be invariant when all achievements within each distribution are changed by certain amounts (Kolm 1976). Technically, for any $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n_{++}$, if $\mathbf{Y}$ is obtained from $\mathbf{X}$ such that $y_{ij} = x_{ij} + \delta_j$ for any $\delta_j \in \mathbb{R}$ for all j and for all i, then $I(\mathbf{X}) = I(\mathbf{Y})$.

Along with other properties, an inequality measure that satisfies scale invariance or ratio scale invariance or unit consistency is referred to as a *relative inequality measure*. An inequality measures that satisfies translation invariance, along with other properties, is referred to as an absolute inequality measure. It should be noted that no inequality measure can simultaneously be relative and absolute. Kolm (1976) refers the relative viewpoint as *rightist*; the absolute viewpoint as the *leftist*. In this chapter, we focus on relative multidimensional inequality measures.

Finally, in the assessment of inequality, it is often required understanding the link between the overall inequality and the inequality of different population subgroups whenever the entire population is divided into a collection of mutually exclusive and collectively exhaustive subgroups. For example, the entire population of a country may be subgrouped across states, provinces, ethnic or religious groups. The *subgroup consistency* property requires that an increase in inequality in one subgroup should lead to an increase in the overall inequality if inequality in other subgroups remains unchanged. The *subgroup decomposability* property requires that the overall inequality can be expressed in terms of the inequality levels of subgroups, their vector of average achievements in different dimensions and their population sizes.

4.2 Structure

We now turn to the structure of inequality measures. Foster (2008) eloquently showed that – except for some limiting cases – all unidimensional inequality measures can be presented as a function of two achievement standards, where an *achievement standard* is a measure of the size of a distribution of achievement.[15] Some of the achievement standards may be viewed as social welfare functions (W). In those cases, an inequality index can be presented as a function of two social evaluation functions: $W(\mathbf{x})$ and $W(\bar{\mathbf{x}})$, where $\bar{\mathbf{x}}$ is obtained from $\mathbf{x}$ such that $\bar{\mathbf{x}} = (\mu(\mathbf{x}), \ldots, \mu(\mathbf{x}))$ or each person in $\bar{\mathbf{x}}$ receives the average achievement $\mu(\mathbf{x})$. An inequality measure is typically presented either as $I(\mathbf{x}) =$

[15] The limiting cases are one of Theil's measures and the variance of logarithms. Foster (2006) uses the term *income standard* to refer the size of any unidimensional income distribution as opposed to the term *achievement standard* that we use here.

$[W(\bar{\mathbf{x}}) - W(\mathbf{x})]/W(\bar{\mathbf{x}})$ whenever $W(\bar{\mathbf{x}}) > W(\mathbf{x})$ or $I(\mathbf{x}) = [W(\mathbf{x}) - W(\bar{\mathbf{x}})]/W(\mathbf{x})$ whenever $W(\bar{\mathbf{x}}) < W(\mathbf{x})$.

In the multidimensional context – analogously – almost all multidimensional inequality measures are functions of two social evaluation functions: $W(\mathbf{X})$ and $W(\bar{\mathbf{X}})$, where $\bar{\mathbf{X}}$ is obtained from $\mathbf{X}$ such that $\bar{\mathbf{x}}_{i\cdot} = \boldsymbol{\mu}(\mathbf{X})$ for all $i = 1, \ldots, n$ or each person in $\bar{\mathbf{X}}$ receives the mean achievement in all d dimensions. Similarly, the typical approach to present an inequality measure is:

$$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}. \tag{1}$$

Unlike in the unidimensional framework, multidimensional social welfare functions are constructed using a two-step aggregation approach.[16] Some of them pursue a *row-first* aggregation approach; whereas others pursue a *column-first* aggregation approach. A *row-first* aggregation approach uses an aggregation function in the first step to aggregate the achievements of each person (row of $\mathbf{X}$) in all d dimensions to obtain an *aggregate individual-achievement*; then all n aggregate individual-achievements are aggregated using another aggregation function to obtain the overall *social evaluation*. A *column-first* aggregation approach, on the other hand, uses an aggregation function in the first step to aggregate the achievements in each dimension (column of $\mathbf{X}$) of all n persons to obtain an *aggregate dimensional-achievement*; then all d aggregate dimensional-achievements are aggregated using another aggregation function to obtain the overall *social evaluation*. Given that the column-first aggregation function first aggregates achievements across each dimension, it is not possible to capture the second form of multidimensional inequality for reflecting association between dimensions.[17]

4.3 Measures

In Table 1, we summarise the social evaluation functions of various multidimensional inequality measures proposed in the literature. Interestingly, all the social evaluation functions in Table 1 use either a generalised mean evaluation function or a generalized Gini evaluation function in either step. For $m \geq 2$, for any $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$ and for any $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m_{+}$ such that $\mathbf{w} \geq 0$ and $\sum_{k=1}^{m} w_k = 1$, the generalised mean of order $\beta \in \mathbb{R}$ is defined as:

$$GM(\mathbf{a}; \mathbf{w}, \beta) = \begin{cases} \left(\sum_{k=1}^{m} w_k a_k^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{k=1}^{m} a_k^{w_k} & \text{for } \beta = 0 \end{cases}.^{18} \tag{2}$$

The expression for general mean is also equivalent to the expression of the *constant elasticity of substitution function*. The generalised mean evaluation function for $\beta < 1$ is used to construct the well-known unidimensional Atkinson's inequality measure (Atkinson 1970).

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[16] See Bosmans et al. (2015) for an axiomatic justification of the structure in (1) as well as the two-step aggregation approach.

[17] For a class of standard of living measures that are invariant to the order of aggregation, see Dutta et al. (2003).

[18] The generalised mean takes different forms, such as arithmetic mean ($\beta = 1$), geometric mean ($\beta = 0$), harmonic mean ($\beta = -1$) and Euclidean mean ($\beta = 2$). When $\beta > 1$ ($\beta < 1$), higher (lower) weight is placed on larger elements and $GM(\mathbf{a}; \mathbf{w}, \beta)$ approaches the maximum (minimum) element as $\beta \to \infty$ ($\beta \to -\infty$).

For $m \geq 2$ and for any $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$ the generalized Gini evaluation function is defined as:

$$GG(\mathbf{a};\delta) = \sum_{k=1}^{m} \left[\left(\frac{r_k}{m}\right)^{\delta} - \left(\frac{r_k - 1}{m}\right)^{\delta} \right] a_k; \tag{3}$$

where r_k is the rank of the k^{th} element in $\mathbf{a}$ when all elements are ranked in descending order. Note that $GG(\mathbf{a};\delta)$ is also a type of average, where the k^{th} element is assigned a weight of $w'_k = (r_k/m)^{\delta} - ([r_k - 1]/m)^{\delta}$. It is straightforward to verify that $\sum_{k=1}^{m} w'_k = 1$. In this evaluation function, smaller elements receive larger weight.[19] Setting $\delta = 2$ in equation (3), we obtain the Gini social evaluation function:

$$GG(\mathbf{a};2) = \sum_{k=1}^{m} \left[\frac{(2i-1)}{m^2} \right] a_k. \tag{4}$$

The Gini social evaluation function is used to compute the well-known Gini coefficient.

We should point out that a pioneering multidimensional inequality measure proposed by Maasoumi (1986) differs from equation (1) not only in the general structure, but also in the distribution that is considered as the most equal. Maasoumi (1986) used a row-first aggregation approach, but a key difference from other measures is that the most equal distribution is in which the aggregate individual-achievements are equal and not necessarily when everyone has the equal achievement vector. Consider the following achievement matrices:

$$\mathbf{Y}' = \begin{bmatrix} 3 & 7 \\ 3 & 7 \\ 7 & 3 \\ 7 & 3 \end{bmatrix} \text{ and } \mathbf{X}' = \begin{bmatrix} 5 & 5 \\ 5 & 5 \\ 5 & 5 \\ 5 & 5 \end{bmatrix}.$$

Suppose, the level of human development for each individual is assessed by $U(x'_{i1}, x'_{i2}) = (x'_{i1}x'_{i2})^{0.5}$. Massoumi (1986) would consider both $\mathbf{Y}'$ and $\mathbf{X}'$ to be the most egalitarian; whereas other normative inequality measures would consider only $\mathbf{X}'$ to be the most egalitarian.

It is worth noting that all the inequality measures detailed in Table 1 are sensitive to distribution. In other words, they satisfy UPD majorization or uniform majorization. Additionally, the measures proposed by Bourguignon (1999), Tsui (1995, 1999), Decancq and Lugo (2012b), Seth (2013) and Diez et al. (2007) are sensitive to association between dimensions under appropriately selected parameter restrictions. However, Gajdos and Weymark (2005) and Foster et al. (2005), as they use column-first aggregation, are not sensitive to association between dimensions. In fact, measures proposed by Foster et al. (2005) yield the same level of social evaluation whether a row-first or a column-first aggregation is used. In this case, the social evaluation function may be referred as *path independent*. All measures presented in Table 1 require variables under consideration to be cardinal.

[19] The weight assigned to the largest element is $(1/m)^{\delta} - (0/m)^{\delta}$ or $1/m^{\delta}$; whereas the weight assigned to the smallest element is $(m/m)^{\delta} - ([m-1]/m)^{\delta} = 1 - ([m-1]/m)^{\delta}$.

Table 1: Multidimensional Inequality Measures, Relevant Social Evaluation Functions and the Order of Aggregation

	Order of aggregation	First stage aggregation function	Second stage aggregation function	Inequality measure
1. Tsui (1995)	Row-first	$h_i = \prod_{j=1}^{d} x_{ij}^{\alpha_j}$	$W(\mathbf{X}) = \begin{cases} \left[\frac{1}{n}\sum_{i=1}^{n} h_i\right]^{\frac{1}{\sum_{j=1}^{d}\alpha_j}} \\ \left[\prod_{i=1}^{n} h_i^{\frac{1}{\sum_{j=1}^{d}\alpha_j}}\right]^{\frac{1}{n}} \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
2. Bourguignon (1999)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta < 1 \ \& \ \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i^{\alpha}\, ; 0 < \alpha < 1$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
3. Foster, Lopez-Calva and Szekely (2005)*	Column-first	$h_j = \begin{cases} \left(\sum_{i=1}^{n} x_{ij}^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha < 1 \ \& \ \alpha \neq 0 \\ \left(\prod_{i=1}^{n} x_{ij}\right)^{\frac{1}{n}} & \text{for } \alpha = 0 \end{cases}$	$W(\mathbf{X}) = \begin{cases} \left(\sum_{j=1}^{d} w_j h_j^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha < 1 \ \& \ \alpha \neq 0 \\ \prod_{j=1}^{d} h_j^{w_j} & \text{for } \alpha = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
4. Gajdos and Weymark (2005)**	Column-first	$h_j = \sum_{i=1}^{n}\left[\left(\frac{r_i^j}{n}\right)^{\delta} - \left(\frac{r_i^j - 1}{n}\right)^{\delta}\right] x_{ij};$ $\delta > 0$ & r_i^j is the rank of person i in dimension j	$W(\mathbf{X}) = \begin{cases} \left(\sum_{j=1}^{d} w_j h_j^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{j=1}^{d} h_j^{w_j} & \text{for } \beta = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$

	Order of aggregation	First stage aggregation function	Second stage aggregation function	Inequality measure
5. Decancq and Lugo (2012b)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \sum_{i=1}^{n} \left[\left(\frac{r_i}{n}\right)^{\delta} - \left(\frac{r_i - 1}{n}\right)^{\delta}\right] h_i$; $\delta > 0$ & r_i is the rank of person i in dimension j	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
6. Seth (2013)	Row-first	$h_i = \begin{cases} \left(\sum_{j=1}^{d} w_j x_{ij}^{\beta}\right)^{\frac{1}{\beta}} & \text{for } \beta \leq 1 \;\&\; \beta \neq 0 \\ \prod_{j=1}^{d} x_{ij}^{w_j} & \text{for } \beta = 0 \end{cases}$	$W(\mathbf{X}) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} h_i^{\alpha}\right)^{\frac{1}{\alpha}} & \text{for } \alpha \leq 1 \;\&\; \alpha \neq 0 \\ \left(\prod_{i=1}^{n} h_i\right)^{\frac{1}{n}} & \text{for } \alpha = 0 \end{cases}$	$I(\mathbf{X}) = 1 - \frac{W(\mathbf{X})}{W(\bar{\mathbf{X}})}$
7. Tsui (1999) and Diez et al. (2007)	Row-first	$h_i = \prod_{j=1}^{d} x_{ij}^{\alpha_j}$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i$	$I(\mathbf{X}) = \phi\left(\rho \prod_{j=1}^{d} \mu_j^{\tau}\left[\frac{W(X)}{W(\bar{X})} - 1\right]\right)$
8. Tsui (1999)	Row-first	$h_i = \sum_{j=1}^{d} \delta_j \log\left(x_{ij}\right)$	$W(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} h_i$	$I(\mathbf{X}) = \phi\left(\rho\left[\frac{W(X)}{W(\bar{X})} - 1\right]\right)$

* Both stages of aggregation use the same parameter α. This social evaluation function is path independent.

** Here we present only one measure from the class of indices proposed by Gajdos and Wermark (2005) based on Gini social evaluation function. The first stage aggregation function may be a *generalized* Gini social evaluation function $h_j(x_{\cdot j}) = \sum_{i=1}^{n} a_i\, x_{ij}^{Ord}$ such that $0 < a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} a_i = 1$.

5 Empirical Applications

Many of the indices presented in Table 1 of Section 4 have been used to measure inequality in human development.[20] Two early inequality-adjusted indices of human development are the Gender-related Development Index (GDI) (Anand and Sen, 1995), and Inequality-Adjusted Human Development index (Hicks, 1997). Both use the same three dimensions: health, knowledge and living standard, and apply column-first aggregation. The GDI first computes an equally distributed equivalent (EDE) achievement (with $\beta = -1$ in Equation (1)) aggregating the achievements of males and females in each dimension, and then averages these three dimensional EDE achievements. Similarly, Hicks first computes an aggregate achievement for each dimension adjusting for Gini coefficient within the corresponding dimension capturing inequality (similar to Gajdos and Weymark (2005)), and then averages these three inequality-adjusted achievements.[21] Data unavailability however imposes serious constraints for Hicks' index. Inequality could only be captured across income quintiles for living standard, across six ordered categories of education for knowledge, and across mortality statistics by age, gender and area of residence for health.

Building upon Anand and Sen (1995) and Hicks (1997), Foster et al. (2005) propose a family of distribution sensitive HDIs, reported in Table 1. Unlike its two predecessors, these indices are invariant to the order to aggregation and are subgroup consistent. These indices have been used to study the link between the level of and the inequality in human development in Mexico using the 2000 population census data. Inequality in living standard is captured across per capita household incomes and inequality in education by weighted average of literacy and attendance across households. However, health inequality could only be captured across municipalities using infant survival rates. Despite data limitations, when sensitivity to inequality considered, Mexican states' rankings change considerably from when inequality is ignored. A particular index with $\alpha = 0$ from their family of indices has been used to construct UNDP's inequality-adjusted HDI (UNDP, 2010; Alkire and Foster, 2010).

Analysing the cross-country results based on inequality-adjusted human development indices, Seth and Villar (2017b) observe a consistent inverse relationship: lower level of human development is associated with larger loss in human development due to existing inequality.

The three inequality adjusted indices so far represent a very significant progress by incorporating one form of inequality, but they are not sensitive to the joint distribution of achievements. They either use column-first aggregation or are invariant to the order of aggregation, yet there are strong arguments in favour of requiring sensitivity to the joint distribution of achievements (Seth 2009, 2013; Decancq 2017; Seth and Villar 2017a).

Seth (2009, 2013) proposes a family of well-being indices (detailed in Table 1) by row-first aggregation that are sensitive to both forms of multidimensional inequality. These indices, at the first stage, aggregate the achievements of each person across all dimensions using a generalised mean of order β, and then in the second stage aggregate individual achievements of a general mean of order α. The restriction that $\alpha \neq \beta$ makes these indices sensitive to the joint distribution of achievements.[22] Seth (2009) applies these indices to the same Mexican dataset studied by Foster et al. (2005); whereas Seth (2013) uses these indices to study the change in the level of human development in Indonesia between 1997 and 2000. In both cases new insights can be obtained from using indices that are sensitive

[20] For a review of inequality sensitive indices of human development, see Seth and Villar (2017a,b).

[21] The Gini coefficient adjusted achievement of a dimension is computed as $\mu(1 - G)$, where μ is the average dimensional achievement and G is the dimensional Gini coefficient.

[22] The Foster et al (2005) family of indices is a sub-family of the Seth family of indices, when $\alpha = \beta$.

to inequality across dimensions.[23] The 2010 HDR has replaced the GDI by the Gender Inequality Index (GII), which is based on Seth (2009). The GII is constructed in three steps. First, the achievements of each gender group are aggregated using a geometric mean ($\beta = 0$). Second, the aggregate achievements of both genders are aggregated using a harmonic mean ($\alpha = -1$). Third, the normalised short-fall of the overall index, obtained in the second step, from the level of human development with perfect gender equality is the GII.[24]

Decancq and Lugo (2012b) perform an empirical application of two families of multidimensional inequality indices using Russian data. The two families of indices are based on Gini social evaluation functions: one family uses a column-first aggregation (Gajdos and Weymark 2005) and thus is insensitive to association; whereas another family uses a row-first aggregation (Decancq and Lugo, 2012b) and is thus sensitive to association. They consider four dimensions: equivalent real expenditure, health, years of schooling and housing. For this application, however, the authors cardinalise some of the non-cardinal indicators. They find through a simulation exercise that the assessed levels and trends of inequality differ substantially when judged with association-sensitive indices as opposed to when judged with association-insensitive indices.

Using the same family of indices as Seth (2009), Decancq (2017) proposes and estimates a variant of the OECD's 'Better Life Index' (BLI), making it sensitive to inequality. He uses some variant of BLI indicators from micro data using the Gallup World Poll survey. Findings suggest that incorporating inequality does change country rankings, and again, countries with lower BLI tend to have larger loss due to multidimensional inequality.

Reviewing the applications of multidimensional inequality indices so far one may extract a few general observations. First, the field seems to be still very fertile for further empirical investigations. Applications are still relatively few and yet all of them show interesting insights of the effects of incorporating inequality in the measurement of human well-being or development. Second, some of the most prominent applications have used column-first aggregation, despite the recognised importance of considering the joint distribution of achievements. This may be no coincidence. Two factors may be at interplay. First, row-first aggregation type of measures require micro-data on the relevant variables. This on itself may be not such a great limitation, as the availability of household surveys micro data keeps increasing. However, variables to implement measures sensitive to the joint distribution need to be of cardinal nature, or otherwise, require a prior *cardinalisation* procedure (as performed in the work by Decancq and Lugo, 2012b and Decancq, 2017). While life expectancy can be considered a cardinal variable, finding an equivalent meaningful cardinal variable at the individual level seems not so straightforward. Thus, it may be that the combined requirements, namely cardinally meaningful variables at the micro-data level, does represent a practical limitation for a burst in the implementation of multidimensional inequality indices. Note, in fact, that the recent surge in implementations of the sister measures of multidimensional poverty has been with measures that work with dichotomised achievements, facilitating the use of ordinal variables, which are predominant in multidimensional analysis.

[23] For further insight, see our working paper version (Seth and Santos, 2017).

[24] See the technical note at http://hdr.undp.org/en/content/gender-inequality-index-gii.

6 Concluding Remarks

The technology for measuring multidimensional inequality has greatly evolved and has a lot of potential for monitoring human development. Considering what we have presented in this chapter we shall end with two final remarks.

In the first place, while there seems to be an increasing consensus regarding the transfer properties (PBT has overcome controversies posed by UPD and UM), there is still debate over the association sensitive properties, which, in fact, reflects the discussion on whether dimensions of development are substitutes or complements.

Second, the framework of multidimensional inequality measurement is a rigorous technical framework. Yet it is the actual selection of dimensions and indicators of relevant capabilities and functionings – something briefly discussed in the introduction, what can make it operational to the measurement of human development. Real world data frequently pose significant limitations that require careful assessment and assumptions. This is no trivial matter and while it exceeds the coverage of this chapter, it is worth reminding. In connection to this, the fact that many of the typical available variables are of ordinal nature, combined with the requirement of micro-data to consider the joint distribution in the measures, poses an additional challenge, requiring further research.

7 References

Alkire, S. (2008). "Choosing Dimensions: The Capability Approach and Multidimensional Poverty", in Kakwani, N. and Silbre, J. (eds), *The Many Dimensions of Poverty*. New York: Palgrave Macmillan, 89-119.

Alkire, S., and Foster, J. (2010). 'Designing the inequality-adjusted human development index (IHDI)', OPHI Working Paper 37, University of Oxford.

Alkire, S., Foster, J. E., Seth, S. , Santos, M.E., Roche, J.M. and Ballon, P. (2015). *Multidimensional Poverty Measurement and Analysis: A Counting Approach .* Oxford University Press.

Alkire. S. and Deneulin, S. (2009), "The Human Development and Capability Approach", in Deneulin, S. and Shahani, L. *An Introduction to the Human Development and Capability Approach*, chapter 2. Earthscan. London.

Anand, S., and Sen, A. K. (1995). Gender inequality in human development: theories and measurement. Working Paper No. 95.05, Center for Population and Development Studies, Harvard University, Ca

Atkinson, A.B., and Bourguignon, F., (1982). "The comparison of multidimensioned distributions of economic status". *Review of Economic Studies* 49, 183–201.

Boland, P. J. and Proschan, F. (1988). 'Multivariate Arrangement Increasing Functions with Applications in Probability and Statistics'. *Journal of Multivariate Analysis*, 25(2): 286–298.

Bosmans, K., Decancq, K., Ooghe, E. (2015). "What do normative indices of multidimensional inequality really measure?". *Journal of Public Economics* 130: 94-104.

Bourguignon, F., (1999). Comment on 'Multidimensioned approaches to welfare analysis'. In: Maasoumi, E., Silber, J. (Eds.), *Handbook of Income Inequality Measurement.* Kluwer Academic, London.

Bourguignon, F., Chakravarty, S.R., (2003). "The measurement of multidimensional poverty." Journal of Economic Inequality 1, 25–49.

Chakravarty, S. and Lugo, M. A. (2016). "Multidimensional Indicators of Inequality and Poverty", in Alder, M. D. and Fleurbaey, M. (eds.), *The Oxford Handbook of Well-Being and Public Policy*.

Dardanoni, V., (1995). "On multidimensional inequality measurement". In: Dagum, C.,Lemmi, A. (Eds.), *Income Distribution, Social Welfare, Inequality and Poverty* vol 6. JAI Press, Standford, CT: 201-207.

Decacnq, K. (2017). "Measuring Multidimensional Inequality in the OECD Member Countries with a Distribution-Sensitive Better Life Index", *Social Indicators Research* 131:1057–1086.

Decancq, K. (2012). "Elementary multivariate rearrangements and stochastic dominance on a Fréchet class". *Journal of Economic Theory* 147: 1450-1459.

Decancq, K. and Lugo, M. A. (2012). "Inequality of Well-being: A Multidimensional Approach". *Economica* 79: 721-746.

Decancq, K. and M. A. Lugo, (2012a). "Weights in Multidimensional Indices of Wellbeing: An Overview", *Econometric Reviews*, 32(1): 7-34

Decancq, K. and M. A. Lugo, (2012b). "Inequality of Wellbeing: A Multidimensional Approach", *Economica* 79: 721–746

Diez, H., Lasso de la Vega M. C., Urrutia, A. M. (2007). "Unit-Consistent Aggregative Multidimensional Inequality Measures: A Characterization". Working Papers 66, ECINEQ, Society for the Study of Economic Inequality.

Duclos, J. Y., Sahn, D. E., and Younger, S. D. (2011). 'Partial Multidimensional Inequality Orderings'. *Journal of Public Economics*, 95(3–4): 225–238.

Dutta, I., Pattanaik, P. K. and Xu, Y. (2003). "On Measuring Deprivation and the Standard of Living in a Multidimensional Framework on the Basis of Aggregate Data".

Economica, 70: 197–221.
Fleurbaey, M. (2004), "Equality of Functionings". Mimeo.
Fleurbaey, M. and Trannoy, A. (2003), "The Impossibility of a Paretian Egalitarian". *Social Choice and Welfare* 21: 243-263.
Foster, J. E. (2006) 'Inequality Measurement', in D. Clark (ed.) *The Elgar Companion to Development Studies*, Edward Elgar, Cheltenham
Foster, J.E., L. F. Lopez-Calva, and Székely, M. (2005). "Measuring the distribution of human development: Methodology and application to Mexico", *Journal of Human Development* 6 (1): 5–30.
Gajdos, T., and Weymark, J.A., (2005). "Multidimensional generalized Gini indices". *Economic Theory* 26, 471–496.
Hicks, D. A. (1997). 'The inequality-adjusted human development index: a constructive proposal', *World Development*, vol. 25(8), pp. 1283–1298.
Kolm, S-C. (1976). "Unequal Inequalities I", *Journal of Economic Theory* 12: 416-442.
Kolm, S. C. (1977). 'Multidimensional Egalitarianisms'. *The Quarterly Journal of Economics*, 91(1): 1–13.
Lasso de la Vega, C., Urrutia, A., and de Sarachu, A. (2010), "Characterizing multidimensional inequality measures which fulfil the Pigou-Dalton bundle principle". *Social Choice and Welfare* 35: 319-329.
Maasoumi, E., (1986). "The measurement and decomposition of multi-dimensional inequality." *Econometrica* 54, 991–998.
Marshall, A. W. and Olkin, I. (1979). *Theory of Majorization and its Applications*. Academic.
Nussbaum, M. (2003). "Capabilities as Fundamental Entitlements: Sen and Social Justice". *Feminist Economics* 9(2-3): 33-59.
Pattanaik, P. K. and Xu, Y. (2007). "Minimal relativism, dominance, and standard of living comparisons based on functionings", *Oxford Economic Papers* 59: 354–374.
Roemer J. E. (1998), *Equality of opportunity*, Cambridge, Mass.: Harvard University Press.
Sen, A. K. (1985). "Well-Being, Agency and Freedom: The Dewey Lectures 1984. *The Journal of Philosophy*, Vol. 82(4): 169-221
Sen, A. K. (1997). "From Income Inequality to Economic Inequality". *Southern Economic Journal* 64 (2): 383-401.
Sen, A. K. (1999). *Development as Freedom*.New York: Knopf.
Sen, A. K. (2004). "Capabilities, lists and public reason: Continuing the conversation", *Feminist Economics* 10(3): 77–80.
Seth, S. (2009). 'Inequality, interactions, and human development', *Journal of Human Development and Capabilities*, vol. 10(3), pp. 375–396.
Seth, S. and M. E. Santos (2017), Multidimensional Inequality and Human Development, OPHI working paper XX, University of Oxford.
Seth, S. and Villar, A. (2017a). "Measuring Human Development and Human Deprivations". OPHI Working Paper No 110. University of Oxford.
Seth, S. and Villar, A. (2017b). "Human Development, Inequality, and Poverty: Empirical Findings". OPHI Working Paper No 111. University of Oxford.
Seth, S., (2013). "A class of distribution and association sensitive multidimensional welfare indices". *Journal of Economic Inequality* 11, 133–162.
Tsui, K.-Y., (1995). "Multidimensional generalizations of the relative and absolute inequality indices: the Atkinson–Kolm–Sen approach". *Journal of Economic Theory* 67, 251–265.
Tsui, K.-Y., (1999). "Multidimensional inequality and multidimensional generalized entropy
United Nations Development Programme (UNDP) (1990). *Human Development Report*. *Concept and Measurement of Human Development*. New York.
United Nations Development Programme (UNDP) (2010). *Human Development Report*. *The Real Wealth of Nations: Pathways to Human Development.* New York.

Zoli, C. (2009). "From Unidimensional to Multidimensional Inequality, Welfare and Poverty Measurement." University of Verona. Mimeo.