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Rank-One Matrix Completion with Automatic Rank Estimation via L1-Norm Regularization

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***Abstract*—Completing a matrix from a small subset of its entries, i.e., matrix completion, is a challenging problem arising from many real-world applications, such as machine learning and computer vision. One popular approach to solving the matrix completion problem is based on low-rank decomposition/factorization. Low-rank matrix decomposition-based methods often require a pre-specified rank, which is difficult to determine in practice. In this paper, we propose a novel low-rank decomposition-based matrix completion method with *automatic rank estimation*. Our method is based on rank-one approximation where a matrix is represented as a weighted summation of a set of rank-one matrices. To automatically determine the rank of an incomplete matrix, we impose L1-norm regularization on the weight vector and simultaneously minimize the reconstruction error. After obtaining the rank, we further remove the L1-norm regularizer and refine recovery results. With a correctly estimated rank, we can obtain the optimal solution under certain conditions. Experimental results on both synthetic and real-world data demonstrate that the proposed method not only has good performance in rank estimation, but also achieves better recovery accuracy than competing methods.**

***Index Terms*—Rank estimation, matrix completion, rank-one approximation, low-rank decomposition.**

I. Introduction

Matrix completion aims to recover a whole matrix from its partial observations. It has witnessed a burst of activities, motivated by many applications such as machine learning [1]–[5], image processing [6]–[8], and computer vision [9]–[11]. Most existing methods assume the target matrix has a low-rank structure since most real-world data (e.g., images) are low-rank or approximately low-rank. Thus, for a target matrix $\mathbf{M} \in \mathbb{R}^{I_1 \times I_2}$ with partial observations in an index set $\mathbf{\Omega}$, the matrix completion problem can be formulated as a rank minimization problem:

$$\min_{\mathbf{X}} \ rank(\mathbf{X}) \ \ \text{s.t.} \ \ \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \tag{1}$$

where $rank(\mathbf{X})$ is the rank of $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2}$, and $\mathbf{\Omega} \in \mathbb{R}^{I_1 \times I_2}$ is the binary index matrix: $\mathbf{\Omega}_{ij} = 1$ if $\mathbf{X}_{ij}$ is observed, and $\mathbf{\Omega}_{ij} = 0$ otherwise. $\mathcal{P}_{\mathbf{\Omega}}$ is the associated sampling operator which acquires only the entries indexed by $\mathbf{\Omega}$. However, the model (1) is NP-hard due to the non-convexity and combinational nature of the $rank$ function.

To address this problem, a popular convex relaxation of rank function is based on minimization of the nuclear norm (a.k.a., trace norm or Schatten p-norm with $p = 1$) [1], [12]–[14]. In this way, the rank minimization model (1) is rewritten as a nuclear norm minimization model:

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_* \ \ \text{s.t.} \ \ \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \tag{2}$$

where the nuclear norm $\|\mathbf{X}\|_*$ is the summation of the singular values of $\mathbf{X}$. Assuming the observed entries are uniformly sampled from the original matrix $\mathbf{M}$, Candès and Recht [1] prove that the missing entries can be exactly recovered if $\mathbf{M}$ (with rank R) satisfies certain incoherence conditions and observes at least $O(N^{1.2}R\log(N))$ ($N = \max(I_1, I_2)$) entries. This sampling bound is narrowed to $O(NR\log(N))$ in [13]. A number of nuclear norm minimization-based algorithms have been proposed to solve the convex model (2). Singular Value Thresholding (SVT) [15] employs the linearized Bremgan iterations [16] to solve the dual of a regularized approximation of (2). Accelerated Proximal Gradient with Linesearch algorithm (APGL) [17] accelerates the convergence of SVT by a fast iterative shrinkage thresholding algorithm [18]. Fixed Point Continuation with Approximate singular value decomposition (SVD) (FPCA) [19] addresses the same problem as APGL while utilizing a fast Monte Carlo algorithm for SVD calculations. Soft-Impute [20] exploits a "sparse plus low-rank" structure to allow efficient SVD in each iteration, with accelerated version (AIS-Impute) in [21]. Other well-known works include [22]–[26].

Another class of techniques is based on low-rank matrix decomposition/factorization, which is more suitable for large-scale cases. Since any matrix $\mathbf{Z} \in \mathbb{R}^{I_1 \times I_2}$ can be modeled in a bilateral factorization form: $\mathbf{U}\mathbf{V}^\top$, where $\mathbf{U} \in \mathbb{R}^{I_1 \times R}$, $\mathbf{V} \in \mathbb{R}^{I_2 \times R}$, the low-rank decomposition-based matrix completion model is formulated as:

$$\min_{\mathbf{Z},\mathbf{U},\mathbf{V}} \ \|\mathbf{Z} - \mathbf{U}\mathbf{V}^\top\|_F^2 \ \ \text{s.t.} \ \ \mathcal{P}_{\mathbf{\Omega}}(\mathbf{Z}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \tag{3}$$

where the integer R ($R < \min(I_1, I_2)$) is the rank of matrix $\mathbf{M}$. Gradient-based optimization algorithms such as alternating minimization methods [27]–[30] are widely used to solve the model (3). Although (3) is non-convex, many works demonstrate that low-rank decomposition-based methods can perform more efficiently and are empirically as reliable as the convex methods [31]–[38]. Besides, there have been some works [27], [38], [39] that provide theoretical guarantee for their performance. For example, Jain *et al.* [27] theoretically prove that the alternating minimization also can exactly recover the matrix under certain conditions similar to the conditions given in [1] (decomposition-based methods may require more observations than nuclear norm minimization-based methods ($O(NR\log(N))$) [27]).

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Many matrix completion methods especially low-rank matrix decomposition-based methods often require a pre-specified rank. Determining the rank of an incomplete matrix is a challenging task, with several existing studies [35], [40]–[44]. Based on the model (3), Wen *et al.* [35] propose a low-rank matrix fitting algorithm (LMaFit) that estimates the rank by two heuristic strategies (decreasing rank strategy and increasing rank strategy) and solve it by a nonlinear successive over-relaxation method [45]. Keshavan *et al.* [39], [40], [46] reformulate the LMaFit model (3) into an SVD form and propose a gradient descent algorithm on Grassmann manifold (OptSpace), which integrates the spectral techniques with manifold optimization and determines the rank by computing the SVD of the trimmed observations [40]. Recently, MaCBetH [43] is proposed to improve OptSpace by a different spectral procedure that detects the rank (by estimating the negative eigenvalues of a Bethe Hessian matrix) and a better initialization for the approximation minimization. These three methods have achieved good performance of rank estimation on synthetic matrices while they do not work well on real-world images, at least in our preliminary studies. On the other hand, for a fixed-rank smooth Riemannian manifold algorithm named LRGeomCG [47], Uschmajew and Vandereycken propose GeomPursuit [44] that adds a greedy outer iteration to LRGeomCG to increase the rank with a step-size l for better recovery performance. Based on our empirical studies, however, GeomPursuit does not obtain exact true ranks and becomes much slower for larger matrices.

Rank-one approximation is a specific low-rank matrix decomposition popularly used in matrix completion [48]–[53]. Here, any matrix $\mathbf{Z}$ is represented as the weighted summation of R factorized rank-one matrices: $\mathbf{Z} = \sum_{r=1}^{R} w_r \mathbf{u}_r \mathbf{v}_r^\top = \mathbf{U}\text{diag}(\mathbf{w})\mathbf{V}^\top$, where the weight vector $\mathbf{w} = [w_1, \cdots, w_r, \cdots, w_R]^\top, \mathbf{U} \in \mathbb{R}^{I_1 \times R} = \{\mathbf{u}_r\}_{r=1}^R, \mathbf{V} \in \mathbb{R}^{I_2 \times R} = \{\mathbf{v}_r\}_{r=1}^R$. Actually, SVD is a special rank-one approximation whose factors $\{\mathbf{u}_r\}_{r=1}^R$ and $\{\mathbf{v}_r\}_{r=1}^R$ are orthogonal, and it is used in OptSpace. Wang *et al.* [50], [51] recently propose an efficient rank-one matrix pursuit method (R1MP) by extending orthogonal matching pursuit to the matrix case. R1MP usually achieves better results given a rank higher than the true rank. In other words, R1MP cannot estimate the rank and does not pursue a low-rank approximation.

In this paper, we propose a novel rank-one matrix completion method with *automatic rank estimation*. Under the low-rank assumption, we aim to automatically determine the rank of an incomplete matrix and recover the matrix. When a rank is given, we can minimize the reconstruction error of the rank-one approximation via least squares to predict the missing entries. We present it as **R**ank-**One M**atrix **C**ompletion (**R1MC**). With a correctly estimated rank, R1MC likes other fixed-rank methods such as [32], [47], [54] can achieve the optimal solution for matrix completion under certain conditions [1], [27], according to the Eckart–Young–Mirsky theorem [55], [56]. However, the rank estimation is a difficult task for incomplete matrices. By solving this problem, the main contributions of this paper are:

- We address the rank estimation problem by imposing an L1-norm regularization on the weight vector (analogous to the vector of singular values) while minimizing the reconstruction error. We call this ***L1**-norm regularized rank-one **M**atrix **C**ompletion method with automatic rank estimation* as **L1MC**.
- We further develop *L1MC with refinement* (**L1MC-RF**) by proposing a **refinement strategy**: once the rank is automatically determined by L1MC, we remove the L1-norm regularization, and further refine the recovery results by directly minimizing the reconstruction errors via R1MC. Essentially, L1MC-RF integrates L1MC and R1MC, while R1MC can be replaced by other fixed-rank completion methods such as [32].

Thus, L1MC-RF can automatically estimate the true rank and exactly predict the missing entries under certain conditions [1], [32], [57]. We solve the optimization problem by the block coordinate descent approach (a.k.a., alternating minimization method or nonlinear (block) Gauss-Seidel scheme), where each variable is iteratively updated with all the others fixed.

In the next section, we review necessary preliminaries and related works. We present the proposed methods in Sec. III and then evaluate them in Sec. IV. Finally, a conclusion is drawn in Sec.V.

II. Preliminaries and Related Works

A. Notations

In this paper, a vector is denoted by a bold lower-case letter $\mathbf{x} \in \mathbb{R}^I$ and a matrix is denoted by a bold capital letter $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2}$. The ith entry of a vector $\mathbf{a} \in \mathbb{R}^I$ is denoted by a_i, and the (i,j)th entry of a matrix $\mathbf{X}$ is denoted by X_{ij}. The Frobenius norm of a matrix $\mathbf{X}$ is defined by $\|\mathbf{X}\|_F = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle}$. $\mathbf{\Omega} \in \mathbb{R}^{I_1 \times I_2}$ is a binary index matrix: $\mathbf{\Omega}_{ij} = 1$ if $\mathbf{X}_{ij}$ is observed, and $\mathbf{\Omega}_{ij} = 0$ otherwise. $\mathcal{P}_{\mathbf{\Omega}}$ is the associated sampling operator which acquires only the entries indexed by $\mathbf{\Omega}$, defined as:

$$(\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}))_{ij} = \begin{cases} X_{ij}, & \text{if}(i,j) \in \mathbf{\Omega} \\ 0, & \text{if}(i,j) \in \mathbf{\Omega^c} \end{cases}, \tag{4}$$

where $\mathbf{\Omega}^c$ is the complement of $\mathbf{\Omega}$. We have $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) + \mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X}) = \mathbf{X}$.

B. The Eckart–Young–Mirsky Theorem

Given a matrix $\mathbf{M} \in \mathbb{R}^{I_1 \times I_2}$ with rank R (with singular values of $\mathbf{M}$ $\{\sigma_1 \geq \cdots \sigma_p \geq \sigma_{p+1} \geq \cdots \geq \sigma_R > 0\}$), the optimal low-rank approximation is given by a truncated SVD of $\mathbf{M}$ according to the classical Eckart–Young–Mirsky theorem [55], [56]. That is, if $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$, then $\mathbf{M}_p = \sum_{i=1}^{p} \sigma_i \ \mathbf{u}_i \mathbf{v}_i^\top$ is the unique optimal rank-p approximation ($p < R$) of $\mathbf{M}$. We present the Eckart–Young–Mirsky theorem under Frobenius norm in **Theorem** 1 following [58]. This theorem is employed for matrix/tensor decompositions in [12], [58], [59].

Theorem 1. ***(The Eckart–Young–Mirsky Theorem)*** *Let $\mathbf{M} \in \mathbb{R}^{I_1 \times I_2}$ has rank $R < \min(I_1, I_2)$ and its SVD is: $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top = \sum_{i=1}^{R} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$, where $\mathbf{\Sigma} = diag(\sigma_1, \cdots, \sigma_p, \cdots, \sigma_R, 0, \cdots, 0)$ and $\sigma_1 \geq \cdots \sigma_p \geq \sigma_{p+1} \geq \cdots \geq \sigma_R > 0$. Denote $\mathcal{M}$ as the set of $I_1 \times I_2$*

matrices with rank $p < R < \min(I_1, I_2)$. The unique optimal solution of:

$$\min_{\mathbf{A}\in\mathcal{M}} \|\mathbf{M}-\mathbf{A}\|_F^2, \ \textit{s.t.} \ \ rank(\mathbf{A}) = p \tag{5}$$

is given by the rank-p approximation (truncated SVD) of $\mathbf{M}$: $\mathbf{M}_p = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$, and we have

$$\min_{\mathbf{A}\in\mathcal{M}} \|\mathbf{M}-\mathbf{A}\|_F^2 = \|\mathbf{M}-\mathbf{M}_p\|_F^2 = \sum_{j=p+1}^{R} \sigma_j^2 \tag{6}$$

C. Existing Completion Methods with Rank Estimation

Rank estimation is important for matrix completion methods requiring a rank a priori [36], [60]. The state-of-the-art matrix completion methods with automatic rank estimation are LMaFit [35], Optspace [39], [40], [46], MaCBetH [43], and GeomPursuit [44].

LMaFit: Based on the low-rank matrix decomposition model (3): $\min_{\mathbf{Z},\mathbf{U},\mathbf{V}} \|\mathbf{Z}-\mathbf{U}\mathbf{V}^\top\|_F^2$ s.t. $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{Z}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, LMaFit [35] is proposed to heuristically estimate a rank starting from an over-estimated rank or under-estimated rank (initially higher or lower than the true rank) for matrix completion. Moreover, LMaFit only requires solving a linear least squares problem per iteration instead of a SVD and integrates an efficient nonlinear successive over-relaxation scheme to accelerate the convergence.

OptSpace: Keshavan *et al.* [46] propose OptSpace with another matrix decomposition form (SVD form):

$$\min_{\mathbf{U},\mathbf{\Sigma},\mathbf{V}} \ \|\mathbf{X}-\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top\|_F^2 \ \ \text{s.t.} \ \ \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \tag{7}$$

where the factor matrices $\mathbf{U}$ and $\mathbf{V}$ have orthogonal columns and $\mathbf{\Sigma}$ is a diagonal matrix. OptSpace consists of three steps [40], [46]. First, it trims the observed matrix $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$ by setting to zero all rows (resp. columns) with more observed entries than twice the average number of observed entries per row (resp. per column). Second, it computes the best rank-R approximation of the trimmed matrix via sparse SVD, where the rank R is estimated as the singular value index if the ratio between two consecutive singular values is minimum [40]. Third, it minimizes the reconstruction error via a special gradient descent method over the Grassmann manifold. Besides, the authors further provide the performance guarantee for OptSpace under moderate incoherence condition [39].

MaCBetH: Recently, Alaa *et al.* [43] propose MaCBetH to improve OptSpace by replacing the first two steps with a different spectral procedure that detects a rank and provides a better initialization for the approximation minimization. In MaCBetH, the rank is estimated as the number of negative eigenvalues of the Bethe Hessian matrix, and the corresponding eigenvectors are used as initial conditions for minimizing the difference between the predicted matrix and the observed entries [43].

GeomPursuit: Another state-of-the-art algorithm, GeomPursuit [44], combines *a greedy outer iteration* that increases the rank with a step-size l *with* a smooth Riemannian algorithm *LRGeomCG* [47] that optimizes the cost function on a fixed-rank manifold. In other words, LRGeomCG needs a fixed rank as input while GeomPursuit can estimate the rank via Greedy rank updates. Based on the empirical studies, however, we found that GeomPursuit cannot obtain exact true ranks though it improves the recovery performance of LRGeomCG. Moreover, it is sensitive to the step-size l and becomes much slower for larger matrices.

On the other hand, **FBCP** [61] is one of recent tensor completion methods which can automatically determine the rank of an incomplete tensor (a matrix is a second-order tensor), where the authors formulate CANDECOMP/PARAFAC (CP) decomposition [62], [63] using a hierarchical probabilistic model and employ a fully Bayesian treatment for automatic rank estimation. Our rank-one approximation model (10) (to be presented in Section III) can be considered as the matrix case of CP decomposition. Here we degenerate FBCP to matrix case to compare with ours and other existing methods.

D. Existing Rank-One Matrix Completion Methods

Given a matrix $\mathbf{Z} \in \mathbb{R}^{I_1\times I_2}$, it can be written as a linear combination of rank-one matrices by extending the atom decomposition [64] to matrix case [48], [50], [52]:

$$\mathbf{Z} = \mathbf{Y}(\boldsymbol{\theta}) = \sum_{i\in\mathcal{I}} \theta_i \mathbf{Y}_i, \tag{8}$$

where $\{\mathbf{Y}_i, i \in \mathcal{I}\}$ is the set of rank-one matrices with $\|\mathbf{Y}_i\|_F = 1$, and $\boldsymbol{\theta}$ is the weight vector: $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_i, \cdots, \theta_{|\mathcal{I}|}]^\top$. Here, the weight vector $\boldsymbol{\theta}$ includes infinite number of weights [50].

Based on the model (8), Wang *et al.* [50], [51] reformulate the matrix completion problem as (9), and propose rank-one matrix pursuit (R1MP):

$$\min_{\theta} \ \|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{Y}(\boldsymbol{\theta}) - \mathbf{M})\|_F^2 \ \ \text{s.t.} \ \ \|\boldsymbol{\theta}\|_0 \le c, \tag{9}$$

where c is an integer and $\|\boldsymbol{\theta}\|_0$ denotes the cardinality of the number of nonzero elements of $\boldsymbol{\theta}$. R1MP alternatively constructs rank-one basis matrices and learns weights of the bases by orthogonal matching pursuit method. R1MP can efficiently obtain better results given a rank higher than the true rank of the original (complete) matrix. In other words, R1MP cannot automatically estimate the rank of an incomplete matrix and does not pursue a low-rank approximation.

III. Proposed Methods

We can represent any matrix $\mathbf{Z} \in \mathbb{R}^{I_1\times I_2}$ as the weighted summation of R factorized rank-one matrices:

$$\begin{aligned} \mathbf{Z} \ &= \ \sum_{r=1}^{R} w_r \mathbf{u}_r \mathbf{v}_r^\top \ = \ \mathbf{U}\text{diag}(\mathbf{w})\mathbf{V}^\top \\ &\text{s.t. } \|\mathbf{u}_r\|_2 = \|\mathbf{v}_r\|_2 = 1, \ \text{for } r = 1, \cdots, R, \end{aligned} \tag{10}$$

where the weight vector $\mathbf{w} = [w_1, \cdots, w_r, \cdots, w_R]^\top, \mathbf{U} \in \mathbb{R}^{I_1\times R} = \{\mathbf{u}_r\}_{r=1}^R, \mathbf{V} \in \mathbb{R}^{I_2\times R} = \{\mathbf{v}_r\}_{r=1}^R$, and R ($R < \min(I_1, I_2)$) is the rank of $\mathbf{Z}$.

Remark 1: This model (10) is different from the model (8) used in [50], [52]: the number of weights (analogous to singular values) of (10) is finite and should be small (low-rank), and we represent each rank-one matrix in a factorization form. Besides, our model (10) is similar to the SVD form (used in OptSpace [46]) but the columns of the factor matrices $\mathbf{U}$ and

$\mathbf{V}$ of our model are not enforced to be orthogonal. In addition, our rank-one matrix decomposition can also be considered as the matrix case of CP decomposition.

Next, we present R1MC and develop our methods L1MC and L1MC-RF progressively.

A. Rank-One Matrix Completion (R1MC) Given True Rank

Based on the rank-one approximation model (10), given a low-rank matrix $\mathbf{M} \in \mathbb{R}^{I_1 \times I_2}$ with partially observed entries in $\mathbf{\Omega}$, i.e., $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, we reformulate the matrix completion problem as:

$$\begin{aligned} &\min_{\mathbf{X},\mathbf{Z}} \ \frac{1}{2}\|\mathbf{X}-\mathbf{Z}\|_F^2 \\ \text{s.t. } &\mathbf{Z}=\sum_{r=1}^{R} w_r \mathbf{u}_r \mathbf{v}_r^\top,\ \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \\ &\|\mathbf{u}_r\|_2=\|\mathbf{v}_r\|_2=1,\ \text{for } r=1,\cdots,R, \end{aligned} \tag{11}$$

where $\mathbf{Z}$ is the summation of R rank-one matrices. Here, we assume the true rank R is known, and we can minimize the reconstruction error via least squares to predict the missing entries. We summarize this **R**ank-**One** **M**atrix **C**ompletion method (**R1MC**) in **Algorithm** 1.

Algorithm 1 **R**ank-**One** **M**atrix **C**ompletion (**R1MC**)

1: **Input:** Incomplete matrix $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, index matrix $\mathbf{\Omega}$, given rank R, maximum iterations K, and stopping tolerance *tol*.
2: **Initialization:** $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, $\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X})=\mathbf{0}$, $\mathbf{Z}=$ zeros(I_1, I_2).
3: **for** $k=1,...,K$ **do**
4: Compute the rank-R approximation of $\mathbf{X}$: $[\mathbf{U}_0\ \mathbf{\Sigma}_0\ \mathbf{V}_0]=$ svd$(\mathbf{X})$, $\mathbf{U}_0=\{\mathbf{u}_r\}_{r=1}^R \in \mathbb{R}^{I_1\times R}$, $\mathbf{\Sigma}_0 \in \mathbb{R}^{R\times R}$, $\mathbf{V}_0=\{\mathbf{v}_r\}_{r=1}^R \in \mathbb{R}^{I_2\times R}$.
5: Set $\mathbf{Z}=\mathbf{U}_0\mathbf{\Sigma}_0\mathbf{V}_0^\top$.
6: Update the missing entries by: $\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{Z})$.
7: If $\|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}-\mathbf{Z})\|_F/\|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})\|_F < tol$ or $\|\mathbf{X}^{k+1}-\mathbf{X}^k\|_F/\|\mathbf{X}^{k+1}\|_F < tol$, break; otherwise, continue.
8: **end for**
9: **output:** $\mathbf{Z}$.

Remark 2: R1MC shares the same spirit as Iterative Hard Thresholding (IHT) [54], [65] and Singular Value Projection (SVP) [32], where the SVD of the target matrix is truncated by keeping the top R singular values and associated singular vectors. On the other hand, R1MC requires less parameter tuning, e.g., no step-size parameter required in [32], [54], [65], so it is simpler to implement and use. *Different* from the convex completion methods such as [1] which relax the rank function via nuclear norm using soft singular value thresholding, R1MC is a non-convex method and obtains a low-rank solution using hard singular value thresholding. *Unlike* the methods in [27] which optimize the underlying matrix in a bilateral factorization form, the matrix is represented as a set of rank-one matrices in R1MC.

Remark 3: If $\mathbf{M}$ (with rank R) obeys the incoherence property and observes enough randomly sampled entries [1], R1MC can exactly recover the missing entries with high probability. The theoretical guarantees of IHT (R1MC) is first conjectured in [32], and recently [57] theoretically improves the sampling bound for IHT (R1MC): R1MC converges to the exact low-rank solution when the number of known entries is more than $O(NR^2\log^2(N)), N=\max(I_1, I_2)$. Furthermore, Wei *et al.* [57] demonstrate that this sampling complexity can achieve the optimal one $O(NR)$ empirically. In other words, for an incomplete matrix $\mathbf{X}$ with enough observed entries from $\mathbf{M}$ (i.e., $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$), the missing entries of $\mathbf{M}$ can be exactly recovered. In R1MC, we predict the missing entries by iteratively updating: $\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{Z})$ and computing the rank-R approximation of $\mathbf{X}$ by the truncated SVD of $\mathbf{X}$, and finally recover the matrix exactly under the above assumptions. On the other hand, the unique optimal rank-R approximation of $\mathbf{M}$ is given by the truncated SVD of $\mathbf{M}$ according to the Eckart–Young–Mirsky theorem (Theorem 1).

In this way, assuming the true rank R is known, R1MC can achieve the optimal solution for matrix completion under the appropriate conditions. If the input rank is higher (over-estimate) or lower (under-estimate) than the true rank, it may result in poor recovery performance. Therefore, it is important to determine a good rank value (true rank) for low-rank matrix decomposition for matrix completion [36].

B. L1-norm Regularized Rank-One Matrix Completion with Automatic Rank Estimation (L1MC)

To address the important rank estimation issue, we impose L1-norm regularization on the weight vector $\mathbf{w}$ and reformulate the R1MC model (11) as follows:

$$\begin{aligned} &\min_{\mathbf{X},\mathbf{w},\{\mathbf{u}_r,\mathbf{v}_r\}_{r=1}^R,R} \ \mu\|\mathbf{w}\|_1+\frac{1}{2}\|\mathbf{X}-\sum_{r=1}^{R} w_r\ \mathbf{u}_r\mathbf{v}_r^\top\|_F^2, \\ &\text{s.t. } \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \\ &\|\mathbf{u}_r\|_2=\|\mathbf{v}_r\|_2=1,\ \text{for } r=1,\cdots,R, \end{aligned} \tag{12}$$

where μ is the regularization parameter and R is the rank to be estimated. By simultaneously minimizing the L1-norm regularization and the reconstruction error, we can automatically determine the rank of an incomplete matrix and simultaneously predict the missing entries. We name this new **L1**-norm regularized rank-one **M**atrix **C**ompletion method with automatic rank estimation as **L1MC**.

Remark 4: Note that the *weights* in model (11) are analogous to the *singular values*. L1-norm regularization makes the weight vector sparse and leads to a low-rank solution.

Derivation of L1MC via BCD: we employ the Block Coordinate Descent (BCD) method [66] for optimization. The BCD method is also known as the alternating minimization method or nonlinear (block) Gauss-Seidel scheme. We divide the target variables into $R+1$ blocks: $\{\{w_1,\mathbf{u}_1,\mathbf{v}_1\},\cdots,\{w_r,\mathbf{u}_r,\mathbf{v}_r\},\cdots,\{w_R,\mathbf{u}_R,\mathbf{v}_R\},\mathbf{X}\}$. We optimize a group (block) of variables while fixing the other groups (blocks), and update one variable while fixing the other variables in each group. After finishing the update of these $R+1$ blocks variables, we finally determine the rank.

The Lagrangian function with respect to the r-th block $\{w_r,\mathbf{u}_r,\mathbf{v}_r\}$ is:

$$\begin{aligned}\mathcal{L}_{w_r,\mathbf{u}_r\mathbf{v}_r} &= \mu|w_r| + \frac{1}{2}\|\mathbf{X}_r - w_r\mathbf{u}_r\mathbf{v}_r^\top\|_F^2, \\ &\text{s.t. } \mathbf{X}_r = \mathbf{X} - \sum_{q=1}^{r-1} w_q\mathbf{u}_q\mathbf{v}_q^\top, \\ &\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \|\mathbf{u}_r\|_2 = \|\mathbf{v}_r\|_2 = 1,\end{aligned} \tag{13}$$

where $\mathbf{X}_r$ is the residual of the approximation.

1) Update $\mathbf{u}_r, \mathbf{v}_r$ *:* The function (13) with respect to $\mathbf{u}_r$ is,

$$\mathcal{L}_{\mathbf{u}_r} = \frac{1}{2}\|\mathbf{X}_r - w_r\mathbf{u}_r\mathbf{v}_r^\top\|_F^2 \tag{14}$$

Then we set the partial derivative of $\mathcal{L}_{\mathbf{u}_r}$ with respect to $\mathbf{u}_r$ to zero, and get:

$$w_r^2\mathbf{u}_r - w_r\mathbf{X}_r\mathbf{v}_r = 0 \Rightarrow {\mathbf{u}_r}^{(k+1)} = \frac{\mathbf{X}_r^k\mathbf{v}_r^k}{w_r^k}. \tag{15}$$

We normalize $\mathbf{u}_r^{(k+1)} = \frac{\mathbf{u}_r^{(k+1)}}{\|\mathbf{u}_r^{(k+1)}\|_2}$. Note that we only update the blocks with non-zero weights (e.g., $w_r^k \neq 0$).

Similarly, we can update ${\mathbf{v}_r}^{(k+1)}$ by,

$${\mathbf{v}_r}^{(k+1)} = \frac{{\mathbf{X}_r^\top}^k{\mathbf{u}_r}^{(k+1)}}{w_r^k}, \tag{16}$$

and normalize $\mathbf{v}_r^{(k+1)} = \frac{\mathbf{v}_r^{(k+1)}}{\|\mathbf{v}_r^{(k+1)}\|_2}$.

2) Update w_r *:* The function (13) with respect to w_r is,

$$\mathcal{L}_{w_r} = \mu|w_r| + \frac{1}{2}\|\mathbf{X}_r - w_r\ \mathbf{u}_r\mathbf{v}_r^\top\|_F^2. \tag{17}$$

Then we set the partial derivative of $\mathcal{L}_{w_r}$ with respect to w_r to zero,

$$\begin{aligned}\frac{\partial\mathcal{L}_{w_r}}{\partial w_r} &= \frac{\mu|w_r|}{\partial w_r} + \big(w_r - \text{trace}(\mathbf{v}_r\mathbf{u}_r^\top\mathbf{X}_r)\big) \\ &= \frac{\mu|w_r|}{\partial w_r} + w_r - \langle\mathbf{X}_r, \mathbf{v}_r\mathbf{u}_r^\top\rangle = 0.\end{aligned} \tag{18}$$

According to Eq. (18), we know $w_r = \langle\mathbf{X}_r, \mathbf{v}_r\mathbf{u}_r^\top\rangle - \mu\frac{|w_r|}{\partial w_r}$. Based on the *soft thresholding* algorithm [67] for L1-norm regularization, we update $w_r^{(k+1)}$ by:

$$w_r^{(k+1)} = \textit{shrink}_\mu(\langle\mathbf{X}_r^k,\ {\mathbf{v}_r}^{(k+1)}{\mathbf{u}_r^\top}^{(k+1)}\rangle), \tag{19}$$

where *shrink* is the soft thresholding operator [18], [67]:

$$\textit{shrink}_\mu(a) = \begin{cases} a-\mu & (a > \mu) \\ 0 & (|a| \leq \mu) \\ a+\mu & (a < -\mu) \end{cases}. \tag{20}$$

3) Update $\mathbf{X}$*:* The function (12) with respect to $\mathbf{X}$ is,

$$\begin{aligned}&\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{X} - \sum_{r=1}^{R} w_r\ \mathbf{u}_r\mathbf{v}_r^\top\|_F^2, \\ &\text{s.t. } \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \quad \|\mathbf{u}_r\|_2 = \|\mathbf{v}_r\|_2 = 1.\end{aligned} \tag{21}$$

By deriving simply the Karush-Kuhn-Tucker (KKT) conditions for Eq. (21) [68], we can update $\mathbf{X}^{(k+1)}$ by $\mathbf{X}^{(k+1)} = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) + \mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{Z}^{(k+1)})$, where $\mathbf{Z}^{(k+1)} = \sum_{r=1}^{R} w_r^{(k+1)}\ \mathbf{u}_r^{(k+1)}{\mathbf{v}_r^\top}^{(k+1)}$.

Algorithm 2 **L1**-norm Regularized Rank-One **M**atrix **C**ompletion with Automatic Rank Estimation (**L1MC**)

Input: Incomplete matrix $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, index matrix $\mathbf{\Omega}$, regularization parameter μ, initial rank $\hat{R}$, maximum iterations K, and stopping tolerance *tol*.
Initialization: Initialize $\{\mathbf{w} = \{w_r\}_{r=1}^{\hat{R}}, \{\mathbf{u}_r \in \mathbb{R}^{I_1}, \mathbf{v}_r \in \mathbb{R}^{I_2}, \|\mathbf{u}_r\|_2 = \|\mathbf{v}_r\|_2 = 1\}_{r=1}^{\hat{R}}\}$ randomly (normal distribution); Set $\mathbf{Z} = \text{zeros}(I_1, I_2)$, $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, $\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X}) = \mathbf{0}$.
for $k = 1, ..., K$ **do**
  $\mathbf{X_r} = \mathbf{X}$.
  for $r = 1, ..., \hat{R}$ **do**
    if $w_r \neq 0$ **then**
      Update $\mathbf{u}_r$ and $\mathbf{v}_r$ by (15) and (16) respectively.
      Update w_r by (19).
      $\mathbf{X}_r = \mathbf{X}_r - w_r\mathbf{u}_r\mathbf{v}_r^\top$.
    end if
  end for
  Update X: Update $\mathbf{Z} = \mathbf{X} - \mathbf{X}_r$ and the missing entries by: $\mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}^c}(\mathbf{Z})$.
  If $\|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X} - \mathbf{Z})\|_F / \|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})\|_F < \textit{tol}$ or $\|\mathbf{X}^{k+1} - \mathbf{X}^k\|_F / \|\mathbf{X}^{k+1}\|_F < \textit{tol}$, break; otherwise, continue.
end for
Rank Estimation: Only keep the w_r if $w_r > (10^{-3} \times$ ***Sampling Ratio*** $\times$ sum$(\mathbf{w}))$, and then $R^* =$ length$(\mathbf{w})$, and keep corresponding $\{\mathbf{u}_r\}_{r=1}^{R^*}$ and $\{\mathbf{v}_r\}_{r=1}^{R^*}$.
output: $R^*, \mathbf{Z}$.

*4) **Estimate the Rank*** R *:* After iteratively updating all the above variables till convergence or reaching the maximum iterations, we finally determine a rank. By checking the weight vector $\mathbf{w}$, we only keep the weights larger than a threshold (we set the threshold at $10^{-3} \times$ *Sampling Ratio* $\times\ TotalWeight$), i.e., removing the zero and small weights which account for a very small proportion of total weights. Finally, the number of the remaining weights in $\mathbf{w}$ is the estimated rank and we keep the corresponding factors.

We summarize this new matrix completion method with automatic rank estimation, **L1MC**, in **Algorithm** 2. In addition, since we need a initial rank for optimizing our L1MC objective function (12), we denote $\hat{R}$ as the *initial rank* for rank estimation.

Remark 5: In L1MC, we set the threshold in rank estimation at $10^{-3} \times$ ***Sampling Ratio*** $\times\ TotalWeight$, i.e., removing small weights (analogous to singular values) less than 0.01% to 0.09% of *total weights* for data with *SR* $= 10\% - 90\%$ observed entries, respectively. This setting follows the similar idea in [69], where the low-rank matrix is truncated by removing small singular values less than 1% of the L2-norm of the vector of singular values. Furthermore, based on empirical studies, this threshold can be loose to be an ideal value 0 on the synthetic matrices (and real data with SR $> 30\%$). By only removing small singular values which account for a very small proportion of total singular values, we keep most information of the target matrix. This threshold for rank estimation can be fixed with no need of tuning. It

works well in all tested synthetic and real data although we do not have theoretical guarantee for it yet.

Algorithm 3 L1MC with **Ref**inement (**L1MC-RF**)

1: **Input:** Incomplete matrix $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$, index matrix $\mathbf{\Omega}$, regularization parameter μ, initial rank $\hat{R}$, maximum iterations K, and stopping tolerance *tol*.
2: **Step 1:** Obtain R^* by **Algorithm** 2 *L1MC*.
3: **Step 2:** Feed the estimated rank R^* into **Algorithm** 1 *R1MC* to further optimize factors and weights.
4: **output:** $R^*, \mathbf{Z}$.

C. L1MC with Refinement (L1MC-RF)

L1MC can automatically estimate the rank and simultaneously predict the missing entries. However, the L1-norm regularization of model (12) restricts L1MC to directly optimize the factors and weights of rank-one approximation. To improve the recovery performance, we propose a **refinement strategy**. We refine the recovery results by directly minimizing the reconstruction error **without** the L1-norm regularization after rank estimation, i.e., we firstly determine the rank of an incomplete matrix by L1MC, and then we **remove** the L1-norm regularizer and further improve the recovery accuracy. Thus, after the rank estimation step, we reformulate the L1MC model (12) as:

$$\begin{aligned} \min_{\mathbf{X},\{\mathbf{u}_r,w_r,\mathbf{v}_r\}_{r=1}^{R^*}} & \frac{1}{2}\|\mathbf{X}-\sum_{r=1}^{R^*} w_r\ \mathbf{u}_r\mathbf{v}_r^\top\|_F^2, \\ \text{s.t. } & \mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})=\mathcal{P}_{\mathbf{\Omega}}(\mathbf{M}), \\ & \|\mathbf{u}_r\|_2=\|\mathbf{v}_r\|_2=1,\ \text{for } r=1,\cdots,R^*. \end{aligned} \tag{22}$$

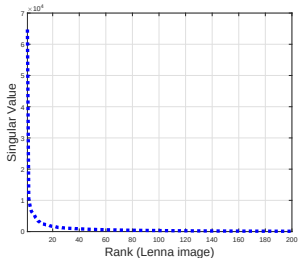
The formulation (22) is equivalent to the R1MC model (11). Therefore, we can directly optimize the factors and weights by R1MC to further refine the recovery results. Note that we also can further refine the recovery results of L1MC by other fixed-rank completion methods such as SVP [32], IHT [54], [65], LRGeomCG [47] and so forth, while R1MC is simpler to implement and use. We denote this integrated-solution as **L1MC with Refinement,** i.e., **L1MC-RF**, summarized in **Algorithm** 3.

In the following section, we evaluate the rank estimation and recovery accuracy of the proposed methods on the synthetic matrices and real-world images.

IV. Experimental Results

We evaluate the performance of the proposed methods from three aspects: i) parameter sensitivity and convergence; ii) importance of estimating the true rank; iii) accuracy of recovery and rank estimation over various sampling ratios given incomplete matrices. We sample $10\%-90\%$ entries from each matrix uniformly at random for training and use **"SR"** for this **S**ampling **R**atio (training ratio). We implemented our methods in MATLAB and all experiments were performed on a PC (Intel Xeon(R) 3.40GHz, 64GB memory).

(a) Original Lenna image.



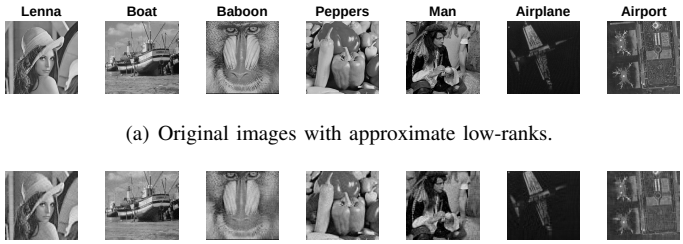
(b) First 200 singular values of Lenna.

Figure 1. An example of low-rank image.

A. Experimental Settings

1) Data: Following [17], [19], [35], [43], we generate the **synthetic matrices**: $\mathbf{M} \in \mathbb{R}^{I_1\times I_2}$ with rank R from two random matrices $\mathbf{M}_1 \in \mathbb{R}^{I_1\times R}$ and $\mathbf{M}_2 \in \mathbb{R}^{I_2\times R}$ with i.i.d. standard Gaussian entries, i.e., $\mathbf{M}=\mathbf{M}_1\mathbf{M}_2^\top$. In this paper, we report the results of *five* synthetic matrices: $\{500\times500\ (R=5), 1000\times1000\ (R=25), 1000\times1000\ (R=50), 2000\times2000\ (R=50), 2000\times2000\ (R=100)\}$.

Moreover, the minimum sampling ratios for guaranteeing the exact recovery of these five matrices are $(\frac{O(NR\log(N))}{(I_1\times I_2)})$: $\{O(6.21\%), O(17.27\%), O(34.54\%), O(19\%), O(38\%)\}$ respectively using nuclear norm minimization-based methods according to [13] (decomposition-based methods may need more observations [27]).



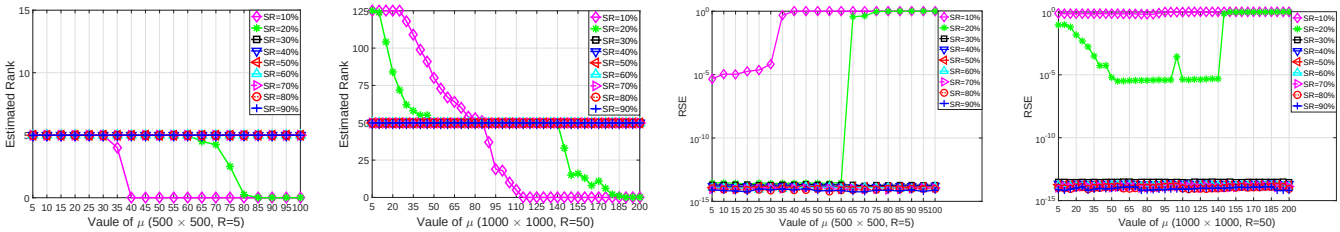
(a) Original images with approximate low-ranks.

(b) Truncated images with exact low-ranks.

Figure 2. Seven real images used for experiments.

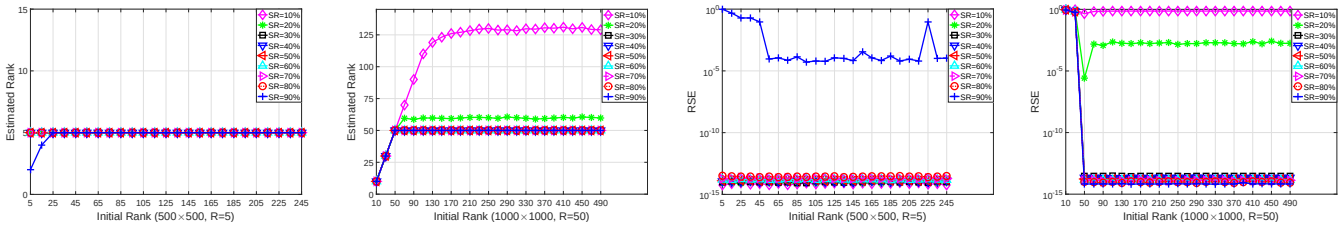
Real data[1]**:** We also evaluate our methods on *seven* real-world images: {Lenna (512×512), Boat (512×512), Baboon (512×512), Peppers (512×512), Man (1024×1024), Airplane (1024×1024), Airport (1024×1024)}. These natural images are approximately low-rank by observing their singular values, as shown in Fig. 1. Following [35] where the authors truncated the SVD of the Boat image to obtain the rank-40 image, we examined the singular value of these images and truncated their SVD to get the images with exact low-ranks: {29 (Lenna), 40 (Boat), 24 (Baboon), 30 (Peppers), 27 (Man), 23 (Airplane), 22 (Airport)}, as shown in Fig. 2. Similarly, the minimum sampling ratios for guaranteeing the exact recovery of these low-rank images are: $\{O(35.33\%), O(48.74\%), O(29.24\%), O(36.55\%), O(18.28\%), O(15.57\%), O(14.89\%)\}$ respectively using nuclear norm minimization-based methods.

[1]Boat image is from http://lmafit.blogs.rice.edu/ and other images are available at http://sipi.usc.edu/database/database.php?volume=misc&image.



(a) 500×500 $(R = 5)$ (b) 1000×1000 $(R = 50)$ (c) 500×500 $(R = 5)$ (d) 1000×1000 $(R = 50)$

Figure 3. *Estimated ranks* and *RSE* of recovering two synthetic matrices via L1MC-RF with (a) (c): $\mu \in [5 : 5 : 100]$ and (b) (d): $\mu \in [5 : 5 : 200]$, respectively.



(a) 500×500 $(R = 5)$ (b) 1000×1000 $(R = 50)$ (c) 500×500 $(R = 5)$ (d) 1000×1000 $(R = 50)$

Figure 4. *Estimated ranks* and *RSE* of recovering two synthetic matrices via L1MC-RF with **initial rank** (a) (c): $\hat{R} \in [5 : 10 : 245]$ and (b) (d): $\hat{R} \in [10 : 20 : 490]$, respectively.

2) Compared Methods: We compare the proposed methods against ***ten*** state-of-the-art matrix completion methods:

- *Four* nuclear norm minimization-based methods: **SVT**[2] [15], **APGL**[3] [17], **FPCA**[4] [19], and **AIS-Impute**[5] [21].
- *Three* low-rank matrix decomposition-based methods with automatic rank estimation: **LMaFit**[6] [35], **OptSpace**[7] [39], and **MaCBetH**[8] [43]
- *Two* Riemannian descent methods: **GeomPursuit** [44] and **LRGeomCG**[9] [47]. GeomPursuit combines LRGeomCG with a rank-adaptive strategy.
- *One* tensor completion method with automatic rank estimation: **FBCP**[10] [61] degenerated to 2D.

We also tested R1MP [50], [51]. In this set of experiments, however, R1MP performs poorly compared with these methods, so its results are not reported here.

3) Evaluation Metrics: Given an incomplete matrix $\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X}) = \mathcal{P}_{\mathbf{\Omega}}(\mathbf{M})$ (input), and the recovered matrix $\mathbf{Z}$ (output), we measure the recovery performance with ground truth $\mathbf{M}$ (with rank R) using following metrics:

- Relative Square Error (RSE) [70]: $\|\mathbf{M} - \mathbf{Z}\|_F / \|\mathbf{M}\|_F$, which refers to the reconstruction error. We use RSE to measure the recovery accuracy and consider the matrix $\mathbf{M}$ successfully recovered if RSE $< 10^{-3}$ [1], [12], [19].
- Relative Square Error on Training (RSE$_{train}$) [15], [35]: $\|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X} - \mathbf{Z})\|_F / \|\mathcal{P}_{\mathbf{\Omega}}(\mathbf{X})\|_F$, which is used for convergence study and stopping criterion.
- Relative error of weight vector (Err$_{\mathbf{w}}$): $\|\mathbf{s} - \mathbf{w}\|_2 / \|\mathbf{s}\|_2$, where $\mathbf{s}$ is the vector that consists of all singular values of the ground truth $\mathbf{M}$.
- Estimated Rank (Est.R).
- Time cost.

4) Parameter Settings: In this paper, we set the maximum iterations $K = 500$ for all methods based on our preliminary studies, and set the regularization parameter $\mu = 50$ and the initial rank $\hat{R} = \text{round}(1/8 \times \min(I_1, I_2))$ for L1MC-RF by default (to be studied in Sec. IV. B). We use two stopping criteria: RSE$_{train}$ and $\|\mathbf{X}^{k+1} - \mathbf{X}^k\|_F / \|\mathbf{X}^{k+1}\|_F$ [19], [71], and terminate the proposed methods if one of stopping criteria is met. Since we found that $tol = 1e-14$ is small enough to obtain very good recoverability and rank estimation, we set the stopping tolerance $tol = 1e-14$ for all methods. Other parameters of the compared methods have followed the original papers. We repeat the runs 10 times and report the average results.

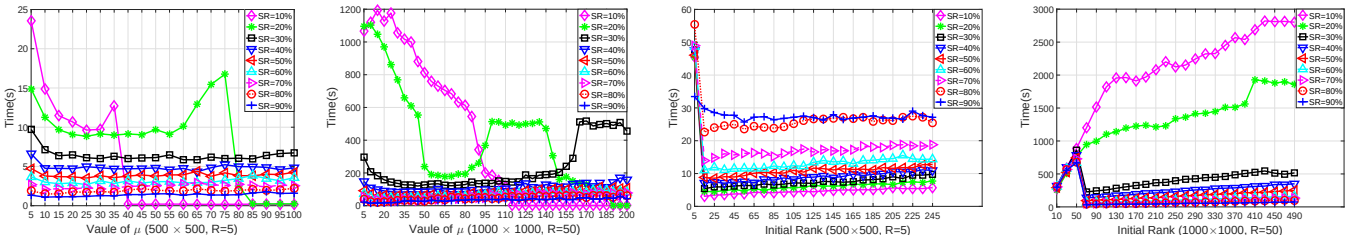
B. Parameter Sensitivity

Firstly, we examine the parameter sensitivity of our methods, including the regularization parameter μ and the initial rank $\hat{R}$ used for rank estimation.

1) Sensitivity of Regularization Parameter μ: We evaluate L1MC-RF with parameter $\mu \in [5 : 5 : 100]$ and $\mu \in [5 : 5 : 200]$ on two synthetic matrices: 500×500 $(R = 5)$ and 1000×1000 $(R = 50)$, respectively. Here we set the initial rank $\hat{R} = \{50, 100\}$ for these two matrices, respectively.

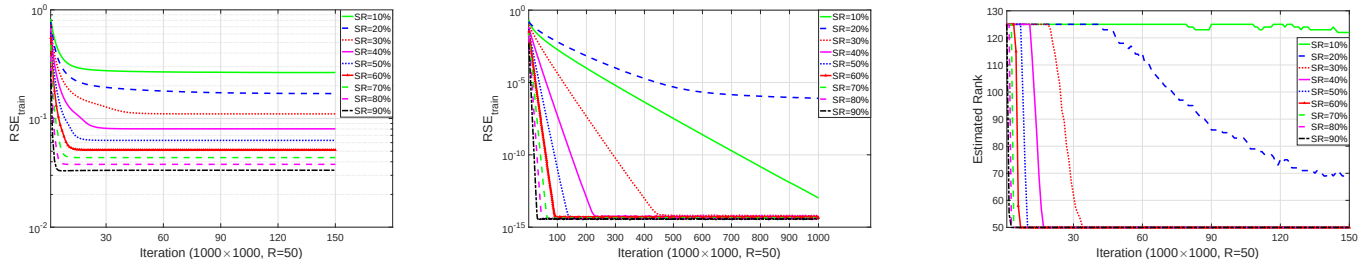
As seen from Fig. 3, it is clear that L1MC-RF is not sensitive to the values of parameter μ: with different values of μ, L1MC-RF performs well on both rank estimation and matrix completion on the whole. Specifically, there are two special scenarios: 1) If we only observe very few entries (e.g.,

[2]http://www.math.ust.hk/~jfcai/.
[3]http://www.math.nus.edu.sg/~mattohkc/NNLS.html.
[4]http://www1.se.cuhk.edu.hk/~sqma/softwares.html.
[5]https://github.com/quanmingyao/AIS-impute.
[6]http://lmafit.blogs.rice.edu/.
[7]http://web.engr.illinois.edu/~swoh/software/optspace/.
[8]https://github.com/alaa-saade/macbeth_matlab.
[9]http://www.unige.ch/math/vandereycken/matrix_completion.html.
[10]http://www.bsp.brain.riken.jp/~qibin/homepage/Software.html.



(a) About $\mu : 500 \times 500$ $(R = 5)$ (b) About $\mu : 1000 \times 1000$ $(R = 50)$ (c) About $\hat{R} : 500 \times 500$ $(R = 5)$ (d) About $\hat{R} : 1000 \times 1000$ $(R = 50)$

Figure 5. *Time costs* of recovering two synthetic matrices via L1MC-RF with (a): $\mu \in [5 : 5 : 100]$ and (b): $\mu \in [5 : 5 : 200]$, respectively; via L1MC-RF with **initial rank** (c): $\hat{R} \in [5 : 10 : 245]$ and (d): $\hat{R} \in [10 : 20 : 490]$, respectively.



(a) RSE$_{train}$ for L1MC (Step 1 of L1MC-RF) (b) RSE$_{train}$ for R1MC (Step 2 of L1MC-RF) (c) Rank estimation by L1MC

Figure 6. *Convergence curves* of recovering the synthetic matrix 1000×1000 $(R = 50)$ with $10\% - 90\%$ observations via L1MC-RF.

SR $= 10\%$) from a smaller matrix with lower rank (e.g., $500 \times 500, R = 5$), a larger μ (e.g., $\mu = 80$) makes the L1-norm regularization dominate the whole objective function (12) and results in zero rank and failure of recovery, as observed from Figs. 3(a) and 3(c); 2) Figs. 3(b) and 3(d) show that we may need to choose a good μ for L1MC-RF to recover a larger matrix with higher rank (e.g., $1000 \times 1000, R = 50$) *only if* the observations are much less than the sampling bound (e.g., SR $< 30\%$), where it is difficult to recover the matrix exactly. On the other hand, a smaller μ (e.g., $\mu = 5$) costs L1MC-RF more time as shown in Figs. 5(a) and 5(b).

In short, we do not need to tune the parameter μ to estimate a good rank and achieve a good recovery result. For simplicity, we fix $\mu = 50$ for the proposed methods by default.

2) Sensitivity of Parameter $\hat{R}$ (Initial Rank): We test on two synthetic matrices 500×500 $(R = 5)$ and 1000×1000 $(R = 50)$ via L1MC-RF with *initial rank* $\hat{R} \in [5 : 10 : 245]$ and $\hat{R} \in [10 : 20 : 490]$, respectively.

As observed from Fig. 4, it is obvious that L1MC-RF is also not sensitive to the values of the initial rank $\hat{R}$: with different values of $\hat{R}$, L1MC-RF has good stable performance in rank estimation and matrix completion almost in all cases. Besides, a higher initial rank $\hat{R}$ increases computational cost, as shown in Figs. 5(c) and 5(d). We set the initial rank $\hat{R} = \text{round}(1/8 \times \min(I_1, I_2))$ by default under the low-rank assumption.

C. Convergence Study

We demonstrate the convergence of our methods in Fig. 6 for recovering the synthetic matrix 1000×1000 $(R = 50)$. Here, we set $tol = eps$ (machine precision) to allow L1MC-RF to pursue the best result until reaching the maximum iterations. Since L1MC-RF consists of L1MC (Step 1 of L1MC-RF) and R1MC (Step 2 of L1MC-RF), we study their convergence in terms of training error as shown in Figs. 6(a) and 6(b) respectively.

L1MC converges within 50 iterations as observed from Fig. 6(a). Fig. 6(b) shows that R1MC converges within 200 iterations for the easy problems (e.g., SR $> 30\%$), while it needs more iterations to achieve convergence if the problem is harder (e.g., SR $= 30\%$). For the two cases of SR $= \{10\%, 20\%\}$, since the sampling ratios are much less than the sampling bound for this synthetic matrix, L1MC-RF (R1MC) fails to find the solution within 1000 iterations.

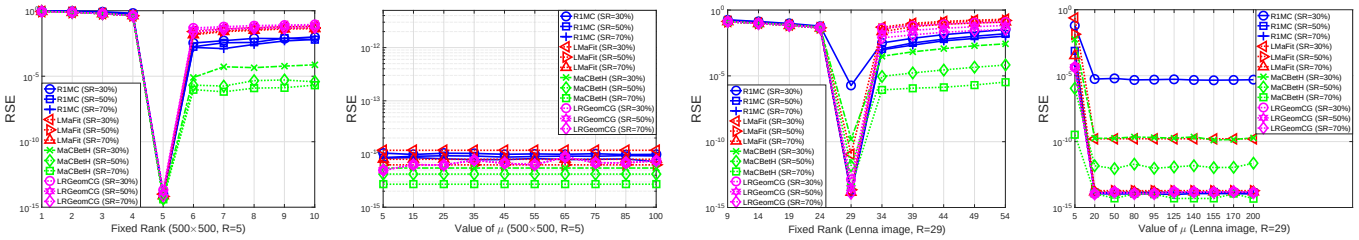
Besides, Fig. 6(c) shows that L1MC successfully determines the true rank within 50 iterations when observing enough entries (SR $\geq 30\%$). In short, L1MC converges faster than R1MC and we set the maximum iterations $K = 500$ for the proposed methods by default.

D. Effects of Rank Value on Matrix Completion Performance

Here, we present studies that investigate the effects of rank estimation accuracy on matrix completion performance of four methods: R1MC, LMaFit, MaCBetH and LRGeomCG. Besides, we also studied OptSpace: it can achieve good recovery results given true or higher-than-true ranks on synthetic matrices while it fails to recover real-world image even given true ranks. Here we dot not report its results for simplicity.

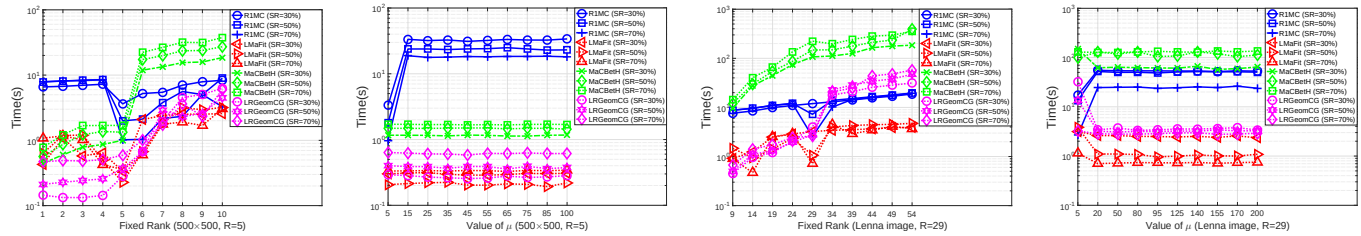
We compare their matrix completion performance with two ways of rank determination: (i) setting the rank manually; (ii) setting μ in L1MC to estimate the rank. We show the results of recovering both synthetic and real matrices with SR $= \{30\%, 50\%, 70\%\}$ in Figs. 7 and 8.

- As seen from Figs. 7(a) and 7(c), the recovery performance (in RSE) of all four methods is highly sensitive to the manually set rank value. Even a slight error in the rank value can lead to serious performance degradation. Only given the true ranks, all the four methods can achieve their best completion results in all cases.
- In contrast, Figs. 7(b) and 7(d) show the corresponding results with L1MC rank estimation by setting μ to a range



(a) Using manually fixed ranks (b) Using estimated ranks via μs (c) Using manually fixed ranks (d) Using estimated ranks via μs

Figure 7. *RSE* of recovering 500×500 $(R = 5)$ and Lenna image $(R = 29)$ via completion methods as given **manually fixed rank** in (a) and (c), and **estimated rank by L1MC with the different values of** μ in (b) and (d).



(a) Using manually fixed ranks (b) Using estimated ranks via μs (c) Using manually fixed ranks (d) Using estimated ranks via μs

Figure 8. *Time costs* of recovering 500×500 $(R = 5)$ and Lenna image $(R = 29)$ via completion methods as given **manually fixed rank** in (a) and (c), and **estimated rank by L1MC with the different values of** μ in (b) and (d).

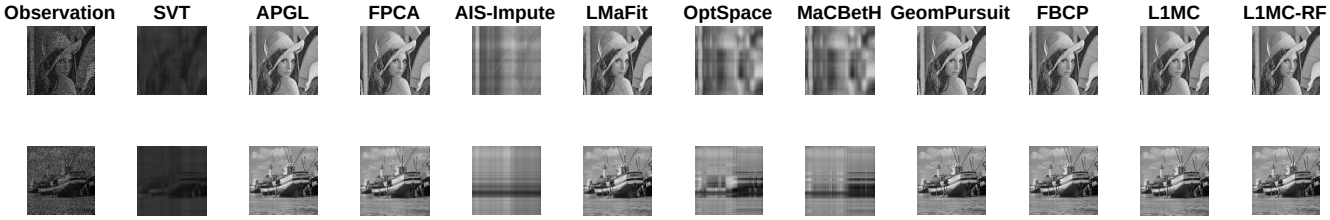


Figure 9. *Recovery results* of the proposed L1MC, L1MC-RF and existing *nine* methods on Lenna $(R = 29)$ and Boat $(R = 40)$ image with 50% observations (best viewed on screen).

of values. We can see a wide range of μ values lead to their best performance of all methods. Such range of μ is wider for data with a larger rank (or dimension) as seen from Fig. 7(d) (also refers to Figs. 3(b) and 3(d)).

- Figs. 8(a) and 8(c) shows that these four methods cost less time the given true ranks in most cases, compared to the cases of given lower/higher-than-true-ranks. With estimated ranks by L1MC with different values of μ, the computational costs are stable with respect to different μ on the whole, as observed from Figs. 8(b) and 8(d).

This study shows the advantage of L1MC in automatic rank estimation, compared to manually fixing the rank. L1MC greatly simplifies parameter tuning where a simple setting of μ from a wide range of feasible values works for a wide range of methods and data. This not only improves the recovery performance but also reduces the time cost in parameter tuning.

Moreover, these results demonstrate the importance of estimating the true rank for matrix completion methods requiring a rank a priori. In the following, we will compare the recovery performance and rank estimation of our methods against the competing algorithms in detail.

E. Completion Performance and Rank Estimation Comparison

We compare recovery accuracy (RSE), time cost (seconds), and rank estimation of the proposed methods against the *night* existing competing algorithms on the *five* synthetic matrices and *seven* real-world images. We tested all the methods on these matrices with $10\% - 90\%$ observed entries, and report here the results of SR $= \{30\%, 50\%, 70\%\}$ (total 36 cases) in Table I and II for simplicity. We use "**" and "–" to indicate that the method diverges (i.e., SVT) and does not terminate in 48 hours (i.e., FBCP) in some cases, respectively.

1) Recovery Accuracy: We report the recovery accuracy (RSE) and time cost in Table I, where we highlight the **best** results (smallest RSE) in **bold** fonts and the second best (second smallest RSE) results in underline in each row for easy comparison. From Table I, we have the following observations:

In terms of **recovery accuracy** on the **synthetic matrices** (total 15 cases), L1MC-RF *consistently* recovers these matrices successfully (RSE $< 10^{-3}$) and obtains very small reconstruction errors of order 10^{-14} in all 15 cases. In fact, L1MC-RF can achieve better results with smaller reconstruction errors of order 10^{-15} if we relax the $tol = 1e - 15$. In addition, OptSpace and MaCBetH are among the top two recovery

Table I
RECOVERY ACCURACY (RSE) AND TIME COST (SECONDS) OF DIFFERENT COMPLETION METHODS ON SYNTHETIC AND REAL DATA (**SR** = **S**AMPLE **R**ATIO = 30%, 50%, 70%). WE HIGHLIGHT THE **BEST** RESULTS IN **BOLD** FONTS AND SECOND BEST RESULTS IN UNDERLINE.

Problem		**SVT** [15]		**APGL** [17]		**FPCA** [19]		**AIS-Impute** [21]		**LMaFit** [35]		**Optspace** [39]		**MaCBetH** [43]		**GeomPursuit** [44]		**FBCP** [61]		**L1MC**		**L1MC-RF**	
Data	SR (%)	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time	RSE	Time
Synthetic	30	1.59E-13	12.2	1.24E-04	3.0	8.00E-07	46.6	7.52E-07	0.77	1.06E-14	0.2	4.57E-15	6.0	**4.31E-15**	0.8	3.20E-06	20.9	7.21E-13	90.7	2.18E-01	1.5	1.84E-14	5.1
500×500	50	1.31E-13	13.8	1.15E-04	2.6	4.27E-08	49.9	4.24E-07	1.15	1.03E-14	0.4	**3.35E-15**	4.6	4.85E-15	1.1	1.67E-06	34.1	2.31E-13	100.1	1.26E-01	1.1	1.23E-14	3.0
R = 5	70	1.07E-13	15.8	1.11E-04	2.4	5.39E-06	1.0	2.95E-07	1.70	8.93E-15	0.3	**2.89E-15**	5.6	3.56E-15	1.6	2.88E-07	80.7	1.07E-13	83.5	8.83E-02	1.0	1.02E-14	1.8
Synthetic	30	3.55E-13	96.0	4.12E-02	27.2	4.73E-05	4.5	4.12E-07	12.79	1.39E-14	1.8	**2.39E-15**	802.1	4.41E-15	13.6	1.30E-06	1185.6	5.86E-13	1423.6	1.19E-01	31.8	2.21E-14	52.6
1000×1000	50	4.31E-13	93.5	1.45E-03	11.9	2.34E-06	4.6	2.17E-07	13.64	8.31E-15	1.4	**1.90E-15**	776.9	3.07E-15	16.9	3.20E-07	2278.6	1.79E-13	1408.3	6.45E-02	19.4	1.45E-14	29.3
R = 25	70	6.28E-13	124.0	1.26E-04	11.6	8.11E-06	5.4	1.47E-07	18.57	9.59E-15	1.9	1.85E-15	722.9	**1.34E-15**	19.4	9.93E-08	1755.5	8.56E-14	1356.7	4.40E-02	12.7	1.13E-14	17.3
Synthetic	30	3.91E-09	468.8	3.18E-02	149.1	1.00E-04	8.1	5.36E-07	74.75	2.00E-14	4.1	**5.98E-15**	32028.2	6.00E-15	102.0	9.11E-07	1796.4	1.80E-12	6851.8	1.45E-01	66.3	2.91E-14	119.9
1000×1000	50	5.14E-13	221.0	1.62E-04	49.0	9.57E-06	8.6	2.40E-07	53.94	1.37E-14	2.5	3.37E-15	46723.0	**2.61E-15**	112.7	3.68E-07	2250.5	4.43E-13	1539.7	7.06E-02	29.0	1.71E-14	46.3
R = 50	70	7.69E-13	175.2	3.76E-03	30.6	2.17E-05	8.7	1.54E-07	70.85	1.10E-14	2.9	2.63E-15	58715.9	**2.13E-15**	117.0	1.15E-07	3079.3	1.96E-13	1460.0	4.59E-02	19.3	1.30E-14	26.4
Synthetic	30	1.66E-04	482.9	2.91E-02	60.0	4.65E-02	225.3	4.64E-02	13.07	3.95E-02	50.3	4.64E-02	1046.6	4.64E-02	30.8	8.15E-09	17226.9	4.21E-10	67412.9	2.91E-02	380.5	**2.09E-14**	460.0
2000×2000	50	3.19E-05	671.0	1.52E-02	50.9	4.64E-02	222.1	4.64E-02	19.11	3.91E-02	61.1	4.64E-02	1218.6	4.64E-02	45.1	2.63E-09	26333.8	–	–	1.75E-02	164.4	**1.41E-14**	207.4
R = 50	70	8.89E-06	852.2	7.31E-04	62.9	4.15E-02	229.3	4.64E-02	26.50	3.90E-02	71.7	4.64E-02	1470.8	4.64E-02	57.2	6.20E-10	35139.0	–	–	1.23E-02	88.6	**1.03E-14**	106.6
Synthetic	30	7.15E-04	1800.7	3.21E-02	83.7	3.31E-02	200.5	3.30E-02	9.61	5.25E-02	42.3	3.30E-02	993.1	3.30E-02	33.0	7.07E-09	33362.0	–	–	2.12E-02	750.3	**2.93E-14**	999.8
2000×2000	50	1.20E-04	1542.7	3.20E-02	109.7	3.31E-02	246.2	3.30E-02	24.17	3.02E-02	74.1	3.30E-02	1499.7	3.30E-02	43.7	1.58E-10	45820.5	–	–	1.29E-02	267.0	**1.69E-14**	335.6
R = 100	70	2.80E-05	1367.1	3.19E-02	92.4	3.31E-02	218.0	3.30E-02	25.94	3.01E-02	64.6	3.30E-02	1452.6	3.30E-02	51.7	1.34E-10	58936.6	–	–	8.90E-03	126.3	**1.10E-14**	155.5
Lenna	30	**	**	1.11E-03	78.7	7.73E-02	240.1	2.70E-01	10.96	6.61E-02	3.7	2.46E-01	47.9	2.57E-01	1.5	1.74E-06	154.5	2.59E-02	918.9	1.68E-02	45.1	**8.33E-07**	62.9
(512×512)	50	6.63E-01	8102.3	6.74E-04	29.8	3.94E-02	249.2	2.69E-01	1.83	4.40E-02	4.8	2.28E-01	62.6	2.25E-01	2.4	5.92E-07	186.7	2.97E-06	1274.7	6.21E-03	11.2	**2.83E-14**	19.5
R = 40	70	9.27E-02	487.8	5.71E-04	9.0	2.77E-02	15.7	2.99E-01	1.27	3.91E-02	5.8	2.19E-01	80.1	2.25E-01	3.2	1.31E-07	237.0	8.67E-10	1709.0	3.78E-03	4.8	**1.87E-14**	7.9
Boat	30	**	**	2.82E-03	143.2	9.74E-02	223.7	1.78E-01	35.32	6.31E-02	4.3	2.42E-01	59.4	2.36E-01	1.1	**3.20E-06**	294.2	9.43E-02	544.0	3.44E-02	103.2	1.47E-02	136.4
(512×512)	50	6.78E-01	8021.5	1.01E-03	47.9	6.71E-02	249.4	2.47E-01	1.12	4.39E-02	5.9	1.94E-01	64.9	2.17E-01	2.1	1.07E-06	211.6	5.47E-06	1881.7	8.82E-03	14.3	**1.99E-09**	36.1
R = 29	70	1.13E-01	491.3	7.36E-04	14.4	7.80E-02	15.0	2.47E-01	1.34	3.76E-02	7.1	1.93E-01	71.8	1.93E-01	3.1	4.03E-07	240.1	4.41E-06	2337.3	4.65E-03	5.3	**3.05E-14**	12.8
Baboon	30	**	**	8.52E-04	40.2	7.63E-02	162.0	1.57E-01	14.17	5.30E-02	3.7	1.72E-01	41.5	1.77E-01	0.8	7.45E-07	131.6	7.53E-07	675.4	1.25E-02	28.2	**3.92E-11**	41.2
(512×512)	50	6.39E-01	6332.8	5.92E-04	13.7	2.27E-07	83.0	2.24E-01	1.46	4.41E-02	5.3	1.51E-01	62.3	1.76E-01	1.1	4.38E-07	166.1	2.94E-07	789.0	5.27E-03	7.8	**2.23E-14**	12.8
R = 24	70	7.86E-02	408.0	5.21E-04	6.0	4.52E-05	9.8	2.24E-01	1.73	3.59E-02	6.6	1.33E-01	93.0	1.76E-01	1.3	1.17E-07	209.3	1.34E-07	1412.6	3.33E-03	3.6	**1.64E-14**	5.5
Peppers	30	**	**	1.14E-03	109.2	7.95E-02	310.8	7.47E-02	106.76	7.41E-02	3.6	2.89E-01	62.6	2.45E-01	2.2	1.31E-06	113.8	1.10E-02	1135.7	1.82E-02	61.6	**4.82E-07**	29.0
(512×512)	50	6.65E-01	9098.9	6.75E-04	37.0	3.73E-02	297.6	1.84E-01	5.99	5.48E-02	5.1	1.96E-01	134.0	2.13E-01	3.4	4.76E-07	125.0	5.05E-06	1938.7	6.45E-03	15.7	**3.06E-14**	28.2
R = 30	70	9.31E-02	657.9	5.67E-04	12.9	3.17E-02	20.7	3.67E-01	2.06	4.46E-02	6.6	1.90E-01	178.4	2.04E-01	5.1	2.53E-07	155.8	3.64E-07	2151.7	3.89E-03	6.6	**1.96E-14**	11.0
Man	30	**	**	6.01E-04	26.7	1.84E-04	48.0	2.27E-01	12.71	7.77E-02	13.0	2.74E-01	327.5	2.29E-01	15.4	6.70E-07	981.8	2.59E-06	4458.9	6.45E-03	271.3	**3.06E-14**	304.5
(1024×1024)	50	1.36E-01	103689.7	5.14E-04	19.1	3.97E-05	56.0	4.08E-01	7.14	7.25E-02	19.0	2.76E-01	472.5	1.86E-01	34.1	2.51E-07	1563.1	1.20E-12	9937.5	3.25E-03	84.3	**1.73E-14**	96.9
R = 27	70	1.45E-12	456.4	4.84E-04	17.3	4.19E-05	56.7	4.53E-01	5.63	6.71E-02	23.3	2.55E-01	765.8	1.85E-01	42.0	8.23E-08	2128.4	1.96E-13	8360.5	2.17E-03	34.2	**1.10E-14**	39.3
Airplane	30	4.16E-01	13056.8	2.46E-03	53.6	3.17E-04	38.1	5.33E-08	18.96	6.10E-02	10.3	2.74E-01	233.5	2.34E-01	13.4	2.22E-06	1282.3	1.77E-05	10311.3	1.57E-02	238.3	**1.06E-10**	283.4
(1024×1024)	50	1.03E-11	270.7	2.83E-03	24.6	4.51E-05	37.1	2.31E-01	13.45	1.04E-02	11.7	2.48E-01	386.9	2.23E-01	21.2	5.69E-07	2425.0	6.03E-05	5855.4	7.83E-03	107.0	**2.00E-14**	122.4
R = 23	70	2.36E-12	112.9	4.47E-04	12.1	4.59E-05	44.8	3.24E-01	14.03	**1.01E-14**	9.7	2.47E-01	431.5	2.12E-01	33.2	1.42E-07	2904.4	3.56E-07	3957.7	5.17E-03	54.9	1.50E-14	62.1
Airport	30	**	**	5.69E-04	24.2	1.86E-04	38.7	2.75E-01	2.37	2.82E-02	33.6	2.14E-01	243.4	2.24E-01	7.9	6.81E-07	745.9	4.16E-12	2996.8	6.61E-03	215.7	**2.59E-14**	245.5
(1024×1024)	50	5.65E-04	27994.4	4.95E-04	13.0	3.05E-05	38.9	2.74E-01	4.29	2.71E-02	42.0	1.96E-01	381.6	2.04E-01	14.3	3.49E-07	1152.2	6.46E-13	7525.8	3.44E-03	78.8	**1.56E-14**	90.5
R = 22	70	6.86E-13	148.6	4.69E-04	13.1	4.02E-05	47.3	2.74E-01	6.26	3.32E-03	38.8	1.77E-01	601.2	1.77E-01	25.0	1.64E-07	1571.7	1.53E-13	6303.0	2.32E-03	37.9	**1.14E-14**	42.9

results on three small synthetic matrices (500×500 with $R = 5$, 1000×1000 with $R = 25$, and 1000×1000 with $R = 50$), though L1MC-RF gives results of one order lower only. LMaFit obtains similar results as L1MC-RF on these smaller matrices. However, LMaFit, OptSpace and MaCBetH do not keep their good performance on the larger matrices ($2000 \times 2000, R = \{50, 100\}$), where L1MC-RF is still the winner and outperforms the second best (GeomPursuit and FBCP) by several orders of magnitude. Moreover, on these large matrices, GeomPursuit costs more than 10 hours in a few cases and FBCP fails to recover them within 48 hours in most cases.

On the **real-world images** (total 21 cases), only L1MC-RF consistently achieves the top two results in all cases except one (Boat image with SR $= 30\%$) where GeomPursuit obtains the best result. GeomPursuit and FBCP achieve the second best results following L1MC-RF in 16 out of 21 cases, while they are more time consuming (about 10 and 45 times slower than L1MC-RF on average respectively). Moreover, FBCP needs more memory. OptSpace and MaCBetH fail to recover these real-world images and estimate wrong ranks (as shown in Table II). LMaFit also does not work well in the 21 cases except one (Airplane image with 70% observations) where it achieves the smallest reconstruction error. In addition, SVT, APGL and AIS-Impute take the second place in a few cases while SVT often fails to converge if the observed entries are fewer (e.g., SR$\leq 30\%$).

In a nutshell, L1MC-RF has shown good recoverability: it outperforms the three decomposition-based methods as well as GeomPursuit and FBCP on average, and also achieves smaller reconstruction errors than the four nuclear norm minimization-based methods in all cases. For illustration, we show two examples of recovering the Boat and Lenna images with 50% observations in Fig. 9.

2) Time Cost: In terms of **computational cost**, L1MC-RF is not the fastest while our focus here is accuracy and our implementation is not optimized for efficiency. It is worth noting that AIS-Impute is the fastest algorithm due to its

Table II
ESTIMATED RANK (EST.R) AND RELATIVE ERROR OF SINGULAR VALUES OF DIFFERENT METHODS ON SYNTHETIC AND REAL DATA. WE HIGHLIGHT THE ***Correct Estimated Rank*** IN **BOLD** AND *italic* FONTS, **SMALLEST $\mathbf{ERR_w}$** RESULTS IN **BOLD** FONTS AND SECOND SMALLEST $\mathrm{ERR_w}$ IN UNDERLINE.

Problem		**SVT** [15]		**APGL** [17]		**FPCA** [19]		**AIS-Impute** [21]		**LMaFit** [35]		**Optspace** [39]		**MaCBetH** [43]		**GeomPursuit** [44]		**FBCP** [61]		**L1MC**		**L1MC-RF**	
Data	SR(%)	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$	Est.R	$\mathrm{Err_w}$
Synthetic	30	***5***	2.41E-15	***5***	1.19E-04	***5***	4.62E-07	***5***	7.23E-07	***5***	5.77E-16	***5***	1.65E-16	***5***	**1.03E-16**	7	5.77E-08	***5***	7.01E-13	***5***	2.11E-01	***5***	5.36E-16
500×500	50	***5***	2.53E-15	***5***	1.13E-04	***5***	2.10E-08	***5***	4.17E-07	5	6.80E-16	***5***	1.85E-16	***5***	**1.03E-16**	7	2.78E-08	***5***	2.27E-13	***5***	1.24E-01	***5***	3.71E-16
$R=5$	70	***5***	3.21E-15	***5***	1.10E-04	***5***	8.62E-07	***5***	2.93E-07	***5***	5.36E-16	***5***	1.65E-16	***5***	**1.44E-16**	7	4.88E-09	***5***	1.05E-13	***5***	8.76E-02	***5***	3.50E-16
Synthetic	30	***25***	1.64E-15	95	1.83E-03	***25***	1.91E-05	***25***	3.79E-07	***25***	7.03E-16	***25***	**3.79E-16**	***25***	1.44E-16	27	8.09E-09	***25***	5.46E-13	***25***	1.10E-01	***25***	8.48E-16
1000×1000	50	***25***	1.44E-15	***25***	1.59E-04	***25***	4.91E-07	***25***	2.09E-07	***25***	6.13E-16	***25***	4.87E-16	***25***	**1.98E-16**	27	3.06E-09	***25***	1.72E-13	***25***	6.23E-02	***25***	5.23E-16
$R=25$	70	***25***	1.73E-15	***25***	1.24E-04	***25***	2.45E-06	***25***	1.44E-07	***25***	5.41E-16	***25***	3.43E-16	***25***	**2.34E-16**	27	6.34E-10	***25***	8.29E-14	***25***	4.31E-02	***25***	3.97E-16
Synthetic	30	69	4.44E-10	125	2.98E-03	***50***	2.36E-05	***50***	4.46E-07	***50***	6.41E-16	***50***	3.97E-16	***50***	**7.69E-17**	60	3.55E-09	***50***	1.53E-12	***50***	1.24E-01	***50***	7.18E-16
1000×1000	50	***50***	1.18E-15	***50***	1.49E-04	***50***	1.35E-06	***50***	2.22E-07	***50***	5.13E-16	***50***	3.46E-16	***50***	**8.97E-17**	52	1.85E-09	***50***	4.05E-13	***50***	6.55E-02	***50***	5.00E-16
$R=50$	70	***50***	1.14E-15	51	1.60E-04	***50***	1.76E-06	***50***	1.48E-07	***50***	6.03E-16	***50***	2.95E-16	***50***	**6.41E-17**	52	2.25E-10	***50***	1.84E-13	***50***	4.41E-02	***50***	2.56E-16
Synthetic	30	***50***	1.21E-05	28	7.54E-04	1	1.08E-03	1	1.07E-03	12	7.32E-04	1	1.07E-03	1	1.07E-03	52	2.04E-11	***50***	2.31E-13	***50***	5.07E-03	***50***	**1.40E-15**
2000×2000	50	***50***	1.71E-06	39	4.87E-04	1	1.09E-03	1	1.09E-03	12	7.56E-04	1	1.09E-03	1	1.09E-03	52	9.99E-12	–	–	***50***	3.12E-03	***50***	**1.27E-15**
$R=50$	70	***50***	3.57E-07	***50***	1.35E-04	12	5.37E-04	1	1.07E-03	12	7.49E-04	1	1.07E-03	1	1.07E-03	52	9.68E-13	–	–	***50***	2.21E-03	***50***	**1.30E-15**
Synthetic	30	***100***	8.28E-05	5	6.12E-04	1	5.55E-04	1	5.52E-04	12	1.29E-04	1	5.52E-04	1	5.52E-04	102	1.20E-11	–	–	***100***	2.66E-03	***100***	**2.18E-15**
2000×2000	50	***100***	9.73E-06	5	6.05E-04	1	5.45E-04	1	5.43E-04	12	4.42E-04	1	5.43E-04	1	5.43E-04	102	1.72E-13	–	–	***100***	1.62E-03	***100***	**1.41E-15**
$R=100$	70	***100***	1.66E-06	5	6.06E-04	1	5.41E-04	1	5.41E-04	12	4.46E-04	1	5.41E-04	1	5.41E-04	102	1.77E-13	–	–	***100***	1.14E-03	***100***	**1.49E-15**
Lenna	30	**	**	***29***	3.25E-04	19	2.21E-03	18	4.11E-02	21	9.37E-04	2	3.00E-02	2	3.26E-02	31	1.00E-08	27	5.64E-04	***29***	5.02E-03	***29***	**6.69E-09**
(512×512)	50	142	6.40E-01	***29***	2.47E-04	26	7.71E-04	4	4.09E-02	24	7.82E-04	3	2.60E-02	3	2.52E-02	31	3.82E-09	37	3.29E-09	***29***	2.28E-03	***29***	**1.31E-15**
$R=29$	70	401	4.29E-02	***29***	2.25E-04	28	9.85E-05	1	4.55E-02	25	6.87E-04	3	2.41E-02	3	2.53E-02	31	7.83E-10	***29***	1.17E-12	***29***	1.49E-03	***29***	**5.36E-16**
Boat	30	**	**	***40***	4.87E-04	19	3.25E-03	58	2.20E-02	32	3.61E-04	1	2.92E-02	1	2.78E-02	42	**2.20E-08**	18	4.19E-03	64	6.99E-03	64	1.21E-03
(512×512)	50	145	6.61E-01	***40***	2.76E-04	27	2.22E-03	1	3.08E-02	33	5.19E-04	3	1.85E-02	2	2.36E-02	42	3.88E-09	52	3.30E-09	***40***	2.41E-03	***40***	**1.37E-11**
$R=40$	70	404	5.14E-02	***40***	2.35E-04	31	1.82E-04	1	3.10E-02	34	5.57E-04	3	1.85E-02	3	1.85E-02	42	1.56E-09	54	7.05E-09	***40***	1.48E-03	***40***	**6.83E-16**
Baboon	30	**	**	***24***	2.48E-04	16	2.22E-03	54	1.75E-02	20	5.01E-04	2	1.45E-02	2	1.53E-02	26	5.18E-09	25	1.52E-09	***24***	3.68E-03	***24***	**1.80E-13**
(512×512)	50	142	6.24E-01	***24***	2.04E-04	***24***	1.80E-08	1	2.52E-02	21	7.33E-04	3	1.12E-02	2	1.54E-02	26	3.78E-09	***24***	1.87E-10	***24***	1.82E-03	***24***	**1.01E-15**
$R=24$	70	402	3.71E-02	***24***	1.90E-04	***24***	1.96E-07	1	2.53E-02	22	5.96E-04	4	8.83E-03	2	1.54E-02	26	5.86E-10	***24***	1.07E-10	***24***	1.21E-03	***24***	**9.09E-16**
Peppers	30	**	**	***30***	3.57E-04	21	2.40E-03	140	1.38E-02	21	1.25E-03	3	4.14E-02	4	2.86E-02	32	9.40E-09	29	3.62E-04	***30***	5.82E-03	***30***	**2.02E-07**
(512×512)	50	143	6.43E-01	***30***	2.66E-04	27	7.41E-04	16	3.51E-02	23	1.19E-03	6	1.87E-02	5	2.23E-02	32	3.07E-09	42	3.20E-09	***30***	2.55E-03	***30***	**9.53E-16**
$R=30$	70	402	4.30E-02	***30***	2.41E-04	29	1.91E-04	1	6.94E-02	25	8.44E-04	6	1.79E-02	5	2.06E-02	32	9.19E-10	32	8.70E-10	***30***	1.65E-03	***30***	**1.10E-15**
Man	30	**	**	***27***	2.90E-04	***27***	1.41E-06	14	5.39E-02	22	2.21E-03	5	3.62E-02	8	2.43E-02	29	1.34E-09	30	2.04E-09	***27***	3.10E-03	***27***	**8.96E-16**
(1024×1024)	50	180	1.24E-01	***27***	2.65E-04	***27***	1.27E-07	4	9.73E-02	22	2.35E-03	5	3.82E-02	10	1.67E-02	29	9.21E-10	***27***	1.72E-13	***27***	1.67E-03	***27***	**4.89E-16**
$R=27$	70	***27***	1.91E-15	***27***	2.55E-04	***27***	1.96E-07	1	1.08E-01	23	2.24E-03	6	3.36E-02	10	1.69E-02	29	3.83E-10	***27***	4.80E-14	***27***	1.15E-03	***27***	**7.06E-16**
Airplane	30	28	2.64E-01	34	4.77E-04	***23***	2.75E-06	***23***	2.54E-08	21	1.23E-03	4	3.62E-02	6	2.57E-02	25	8.33E-09	58	3.57E-08	***23***	7.53E-03	***23***	**9.70E-13**
(1024×1024)	50	***23***	8.05E-13	36	3.70E-04	***23***	1.99E-07	12	5.66E-02	***23***	2.50E-04	5	3.05E-02	6	2.44E-02	25	3.63E-09	28	7.56E-08	***23***	4.05E-03	***23***	**1.60E-15**
$R=23$	70	***23***	3.57E-15	***23***	2.39E-04	***23***	2.59E-07	8	7.91E-02	***23***	**9.76E-16**	5	3.05E-02	7	2.23E-02	25	4.77E-10	28	5.50E-10	***23***	2.76E-03	***23***	1.20E-15
Airport	30	**	**	***22***	2.33E-04	***22***	1.68E-06	1	3.82E-02	21	3.91E-04	4	2.27E-02	3	2.47E-02	24	3.22E-09	***22***	8.71E-13	***22***	2.70E-03	***22***	**9.19E-16**
(1024×1024)	50	105	5.00E-05	***22***	2.15E-04	***22***	1.41E-07	1	3.83E-02	21	3.96E-04	5	1.92E-02	4	2.07E-02	24	1.73E-09	***22***	1.35E-13	***22***	1.49E-03	***22***	**8.87E-16**
$R=22$	70	***22***	1.61E-15	***22***	2.08E-04	***22***	2.42E-07	1	3.83E-02	***22***	5.02E-05	6	1.56E-02	6	1.56E-02	24	6.72E-10	***22***	3.26E-14	***22***	1.03E-03	***22***	**7.90E-16**

C-mex programming. LMaFit and MaCBetH are faster than L1MC-RF since they use an efficient nonlinear successive over-relaxation scheme and employ the *minFunc* software for acceleration, respectively. On the other hand, FBCP is the slowest among these completion methods. GeomPursuit is much slower than L1MC-RF in each case although it takes the most second best results, i.e., L1MC-RF is more than 89 times and 10 times faster than GeomPursuit on average on the synthetic and real matrices, respectively. Moreover, SVT and OptSpace are also slower than L1MC-RF especially in some cases (e.g., on 1000×1000 with $R=50$), which is probably due to their heavy SVD computation.

3) Rank Estimation: We also report the corresponding estimated rank (Est.R) and relative error of singular values ($\mathrm{Err_w}$) in Table II, where we highlighted the ***correct estimated rank*** in **bold** and *italic* fonts, **smallest $\mathbf{Err_w}$** in **bold** fonts and second smallest $\mathrm{Err_w}$ in underline in each row. For the methods without the rank estimation step, we compute the estimated ranks by SVD of the recovered matrices.

From Table II, we observe that: L1MC-RF (L1MC) successfully determines the true ranks of the given incomplete images in all 36 cases excepting one (Boat image with SR $= 30\%$), where L1MC does not have enough observations. GeomPursuit performs best with the smallest $\mathrm{Err_w}$ in this case, which results in the best recovery result. L1MC can determine the true rank, while its weight vector (singular values) is far from the ground truth ($\mathrm{Err_w} > 10^{-3}$), resulting in poor recovery performance. This demonstrates the significance of our **refinement strategy** in L1MC-RF: *with the refinement strategy, L1MC-RF further refines the factors and weights via R1MC to pursue an optimal solution for matrix completion*. In other words, L1MC-RF not only can automatically estimate the true rank exactly but also obtain the true singular values. On the other hand, though GeomPursuit cannot obtain exact true ranks, it consistently learns the singular values with small errors ($\mathrm{Err_w}$ of order less than 10^{-7}), which leads to good recovery performance. Moreover, FBCP does not always successfully determine the true rank but it also obtains the singular values with very small

errors, which helps it achieve the second best recovery results in 7 out of 36 cases.

4) Limitation on Matrices with Exponentially Decaying Singular Values: Although L1MC-RF can outperform others in results presented so far, it has limitation on matrices with exponentially decaying singular values, which can be found in certain real world applications [72]. Such singular value distribution makes truncation in L1MC-RF more difficult while GeomPursuit performs much better by design. We generated three types of such matrices with *slow, moderate,* and *fast* exponentially decaying singular values following the setting in [44]. The rank was fixed at 20. Table III shows that GeomPursuit gives much better results than L1MC-RF on the moderate exponentially decaying scenario. The experiments on the other two (slow and fast) scenarios show similar results. An interesting future work could be to extend L1MC-RF to handle such cases better, e.g., with more adaptive thresholding or ideas in GeomPursuit.

F. Summary of Experimental Results

- The proposed L1MC-RF is simple to implement and not sensitive to its parameters (the regularization parameter μ and the initial rank $\hat{R}$). It has fast convergence in rank estimation and good efficiency.
- Estimating the true rank is important for matrix completion methods requiring a pre-specified rank to achieve good results. L1MC-RF has good stable performance in both matrix completion and rank estimation. L1MC-RF consistently recovers all the synthetic matrices exactly with very small reconstruction errors and efficiently achieves the top two results almost in all cases on the real-world images.
- The four nuclear norm minimization-based methods (SVT, APGL, FPCA and AIS-Impute) successfully recover the matrices (RSE $< 10^{-3}$) in about half of the total cases but obtain much lower accuracies than L1MC-RF on average.
- The three low-rank matrix decomposition-based methods (LMaFit, OptSpace and MaCBetH) have shown their good recoverability on most synthetic matrices while fail to estimate the true ranks and predict the missing entries on the real-world images overall. Moreover, Optspace is the slowest among the compared decomposition-based methods (including L1MC-RF) due to its SVD computation, which also makes SVT slower than L1MC-RF.
- GeomPursuit consistently recovers the matrices successfully and take the most second places (15 cases). It even achieves the best result in one case where Boat image with 70% missing entries. However, GeomPursuit cannot estimate the exact true ranks. Besides, GeomPursuit is also very time consuming: it is more than 57 times slower than L1MC-RF on the whole.
- FBCP obtains true ranks correctly in half of the total cases and achieves the second best recovery results in 7 cases. However, it is the slowest among the compared methods: it is more than 64 times slower than L1MC-RF on average and even costs more than 48 hours on large matrices (e.g., $2000 \times 2000, R = 100$) in most cases.
- Although our methods can outperform the competing methods on the whole, it cannot obtain good results on the matrices with exponentially decaying singular values. In this special scenario, GeomPursuit works much better.

Table III
COMPARISON RESULTS OF RECOVERING SYNTHETIC MATRIX ($R = 20$) WITH MODERATE EXPONENTIALLY DECAYING SINGULAR VALUES VIA GEOMPURSUIT AND L1MC-RF.

Problem		**GeomPursuit** [44]		**L1MC-RF**	
Data	SR (%)	RSE	Est.R	RSE	Est.R
Synthetic Matrix	30	**6.86E-15**	*20*	3.62E-03	5
With Moderate Exponentially	50	**1.41E-15**	*20*	3.35E-03	5
Decaying Singular Values	70	**1.40E-15**	*20*	8.38E-04	6

V. CONCLUSION

In this paper, we have proposed a novel low-rank matrix completion method with automatic rank estimation, based on rank-one approximation. We have first presented R1MC that minimizes the reconstruction error given a fixed rank to predict the missing entries. Here, if the given rank is the true rank of the target incomplete matrix, R1MC can achieve the optimal solution for the matrix completion problems under moderate conditions. We then solved the challenging rank estimation problem by developing L1MC method that simultaneously minimizes the L1-norm of weight vector and the reconstruction error. Once the rank is automatically estimated by L1MC, we have further proposed a refinement strategy: we remove the L1-norm regularization and then obtain the refined results by directly optimizing the rank-one approximation model (e.g., using R1MC). This whole process is named as L1MC-RF. With the experiments on synthetic and real-world data, we have demonstrated that the proposed L1MC-RF is easy to implement and not sensitive to its parameters. More importantly, L1MC-RF can efficiently estimate the true rank and recover the incomplete matrix exactly under certain conditions, which outperforms the competing methods on the whole. Nonetheless, our methods cannot work well on the special matrices with exponentially decaying singular values, which will be an interesting future work.

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