This is a repository copy of *Nested Distributed Model Predictive Control*.

Nested distributed MPC [*]

Pablo R. Baldivieso Monasterios [*] **Bernardo Hernandez** [*]
Paul A. Trodden [*]

[*] *Department of Automatic Control & Systems Engineering,*
University of Sheffield, Sheffield S1 3JD, UK
(e-mail {prbaldivieso1, bahernandezvicente1, p.trodden}@sheffield.ac.uk)

Abstract: We propose a distributed model predictive control approach for linear time-invariant systems coupled via dynamics. The proposed approach uses the tube MPC concept for robustness to handle the disturbances induced by mutual interactions between subsystems; however, the main novelty here is to replace the conventional linear disturbance rejection controller with a second MPC controller, as is done in tube-based nonlinear MPC. In the distributed setting, this has the advantages that the disturbance rejection controller is able to consider the plans of neighbours, and the reliance on explicit robust invariant sets is removed.

Keywords: Control of constrained systems; Decentralized and distributed control; Distributed control and estimation; Model predictive and optimization-based control

1. INTRODUCTION

Model Predictive Control (MPC) is a mature and popular control technique (Rawlings and Mayne, 2009; Mayne, 2014) that excels in situations where it is prohibitively difficult to design a control law off-line: for example, in the presence of constraints. MPC is inherently, however, a *centralized* control technique, and so its applicability to large-scale systems is limited by the fact that the controller would have to model, sense and control the whole plant. For this reason, significant attention has been given to non-centralized MPC, including decentralized, distributed and hierarchical forms (Scattolini, 2009). The main challenge is how to coordinate the control actions of independent MPC-based controllers, in order that the overall system is stable and satisfies constraints. Many proposals have been made (see Scattolini (2009); Christofides et al. (2013) for excellent surveys), and broadly differ according to the nature or source of the coupling between subsystems and the algorithmic approach taken to coordinate control actions (Maestre and Negenborn, 2014).

The problem tackled in this paper is the fundamental one of controlling dynamically coupled linear time-invariant systems. The problem is challenging because the states and inputs of one subsystem affect others too; therefore, the straightforward application of MPC, even with terminal conditions (Rawlings and Mayne, 2009), does not guarantee constraint satisfaction and stability. A popular approach is to decompose and distribute the MPC problem (or its dual) among the different controllers, and solve the problem iteratively at each time step—with information exchange between controllers—until feasibility or optimality is obtained (Maestre and Negenborn, 2014); however, the price to pay is large amounts of communication, slow convergence

(of iterates) in large systems, and a long time to solve to MPC problem at each step.

In pursuit of iteration-free methods that still achieve desirable guarantees, a few authors (Farina and Scattolini, 2012; Riverso and Ferrari-Trecate, 2012; Trodden et al., 2016; Hernandez and Trodden, 2016) have exploited ideas from *robust* MPC, and particularly tube-based MPC (Mayne et al., 2005). The basic idea is—considering the mutual interactions as exogenous disturbances—to augment the conventional MPC control law with an ancillary, disturbance rejection term, computed off-line and based on the theory of disturbance-invariant sets. The main drawback is the conservatism induced by taking a robust approach to what is a nominal problem, and research efforts have focused on ways in which to reduce this and improve performance: Farina and Scattolini (2012) employ reference trajectories, and consider the disturbances as deviations from these. Riverso and Ferrari-Trecate (2012) employ the tube concept *twice*, designing two disturbance rejection controllers: the first to minimize deviations between a planned nominal trajectory and planned perturbed trajectory, and the second to minimize deviations between the latter and the true perturbed trajectory. Trodden et al. (2016) propose a more straightforward design, with only one disturbance rejection controller and no reference trajectories, but optimize disturbance-invariant sets on-line in order to reduce conservatism.

In this paper, we offer a new contribution to the family of tube-based distributed MPC (DMPC) approaches. The main development here is to replace the ancillary disturbance rejection controller—which is linear in each of Farina and Scattolini (2012); Riverso and Ferrari-Trecate (2012); Trodden et al. (2016)—with an ancillary MPC controller, which operates in a nested fashion with the main controller. This development is inspired by the approach of Mayne et al. (2011) for tube-based *nonlinear* MPC, which introduced the idea of an ancillary MPC controller; in that approach,

the controller is needed because of the non-linearity of the system. Here we employ the second controller for a different purpose, which also leads to two advantages with respect to existing tube-based DMPC: the ancillary controller is able to consider the plans of neighbouring subsystems when optimizing the disturbance rejection control action; perhaps more significantly, the need to explicitly compute and employ disturbance-invariant sets—which are prohibitively complex objects for high-dimension subsystems—is removed.

The problem statement is defined in Section 2. In Section 3, the nested DMPC approach is developed, including optimal control problems and the distributed algorithm. Recursive feasibility and stability are established in section 4. A comprehensive off-line design method to select controller parameters is given in Section 5, before an illustrative example of the approach is presented in Section 6.

Notation: The sets of non-negative and positive reals are denoted, respectively, $\mathbb{R}_0^+$ and $\mathbb{R}^+$. AX denotes the image of a set $X \subset \mathbb{R}^n$ under the linear mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^p$, and is given by $\{Ax : x \in X\}$. For $X, Y \subset \mathbb{R}^n$, the Minkowski sum is $X \oplus Y \triangleq \{x + y : x \in X, y \in Y\}$; for $Y \subset X$. For $X \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, $X \oplus a$ means $X \oplus \{a\}$. A polyhedron is an intersection of a finite number of half-spaces, and a polytope is a closed and bounded polyhedron. A *C-set* is a compact and convex set that contains the origin; in a *PC-set*, the origin is within the interior. The C-set L is said to be a summand of K if there exists a set M such that $K = L \oplus M$. A sequence is defined as $\mathbf{x} = \{x(0), x(1), \ldots\}$, the cardinality of which will be clear from the context. The notation x_{-i} indicates a sequence without the ith member.

2. PROBLEM STATEMENT

We consider the discrete-time dynamics

$$x^+ = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and control input, and x^+ is the state at the next time instance. This system is partitioned or decomposed into M non-overlapping subsystems, in the sense that the state and input may be written $x = (x_1, \ldots, x_M)$ and $u = (u_1, \ldots, u_M)$, where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and input of subsystem i, $\sum_{i \in \mathcal{M}} n_i = n$ and $\sum_{i \in \mathcal{M}} m_i = m$, and the dynamics of subsystem $i \in \mathcal{M} \triangleq \{1, \ldots, M\}$ may be written as

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + \sum_{j \neq i} A_{ij}x_j + B_{ij}u_j. \tag{2}$$

In this equation, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_{ij} \in \mathbb{R}^{n_i \times m_j}$ are the relevant block elements of A and B. The summation term represents the interaction of the states and inputs of other subsystems ($j \neq i$) on the dynamics of subsystem i; without loss of generality, the summation may be performed over $j \in \mathcal{N}_i$, where $\mathcal{N}_i \triangleq \{j \in \mathcal{M} : [A_{ij} \ B_{ij}] \neq 0, j \neq i\}$ is the set of *neighbours* of i.

Assumption 1. For each $i \in \mathcal{M}$, (A_{ii}, B_{ii}) is controllable.

Each subsystem $i \in \mathcal{M}$ is subject to local constraints on its states and inputs

$$x_i \in \mathbb{X}_i, \qquad u_i \in \mathbb{U}_i. \tag{3}$$

Assumption 2. For each $i \in \mathcal{M}$, $\mathbb{X}_i$ and $\mathbb{U}_i$ are PC-sets.

The control objective is to steer the states of all subsystems to the origin while satisfying the constraints and minimizing the infinite-horizon cost

$$\sum_{k=0}^{\infty} \sum_{i \in \mathcal{M}} \ell_i(x_i(k), u_i(k)) \tag{4}$$

where $\ell_i(x_i, u_i) \triangleq (1/2)(x_i^\top Q_i x_i + u_i^\top R_i u_i)$ and Q_i, R_i are positive definite for all $i \in \mathcal{M}$.

3. NESTED DISTRIBUTED MPC

The main challenge with respect to controlling the system (1) with independent, decentralized controllers is how to deal with the interactions, for the states and inputs of one subsystem are affected by, and affect, others in the system. The most direct approach ignores these interactions, and employs the nominal prediction model

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{u}_i \tag{5}$$

within an MPC optimization to provide the receding horizon control law $u_i = \bar{\kappa}_i(x_i)$, obtained by applying the first control $\bar{u}_i^0(0; x_i)$ in the optimized sequence. Ignoring interactions in this way, however, can lead to constraint violations and even instability, unless further actions or design steps are taken to coordinate the actions of controllers (Scattolini, 2009).

An alternative approach is to treat all interactions as disturbances to be rejected. The dynamic coupling between subsystems—arising from the decomposition of the large-scale system—induces mutual disturbances upon each subsystem; in fact, we may re-write (2) as the uncertain dynamics

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \tag{6}$$

where $w_i \triangleq \sum_{j \in \mathcal{N}_i}(A_{ij}x_j + B_{ij}u_j)$. This disturbance is, in view of the constraints on each x_j and u_j, contained within the set

$$\mathbb{W}_i \triangleq \bigoplus_{j \in \mathcal{N}_i} A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j, \tag{7}$$

which, because of Assumption 2, is bounded and at least a C-set. The local control problem is then to regulate the uncertain, constrained LTI system (6) which is subject to bounded additive disturbances, and the direct application of a robust MPC technique will (under suitable further assumptions) lead to guaranteed feasibility and stability. For example, one could employ the tube-based approach to robust MPC (Mayne et al., 2005), which retains the nominal model for predictions within an MPC problem with restricted constraints (see the problem $\mathbb{P}_i(\bar{x}_i)$ in the next subsection), but augments the implicit control law with a linear, disturbance rejection control law:

$$u_i = \bar{\kappa}_i(\bar{x}_i) + K_i(x_i - \bar{x}_i).$$

The latter term corrects for the errors introduced by neglecting the disturbance (the interactions) in the predictions. The price to pay is conservatism, for the controllers are designed to be robust to the *whole* space of possible states and inputs of other subsystems: neither the nominal MPC control law nor the linear disturbance rejection controller take into account the planned states and/or inputs of interacting subsystems. Hence, approaches that utilize tubes (Farina and Scattolini, 2012; Riverso and Ferrari-Trecate, 2012; Trodden et al., 2016) have focused on ways in which the conservatism can be reduced.

In this paper, we present a third way to this problem, with the aim of retaining the desirable guarantees that a robust approach brings, but lessening the conservatism and other drawbacks associated with this. In particular, we propose a control law of the form

$$u_i = \kappa_i(x_i) = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(x_i - \bar{x}_i; \bar{\mathbf{x}}_{-i}, \bar{\mathbf{u}}_{-i}), \quad (8)$$

which, inspired by Mayne et al. (2011), replaces the linear disturbance rejection control law of tube MPC with a *second* predictive control law. The second term still acts on the error $x_i - \bar{x}_i$ between the true (perturbed) state and the predicted (nominal) state, but takes into account information shared by other subsystems about their predicted states and inputs. These shared predictions are the outputs of the *first* predictive controller; hence, the controllers for a subsystem work in a nested fashion.

The remainder of this section presents the approach, including the optimal control problems and the algorithm. First, we require the following assumption about the disturbance set, which is common in tube-based MPC, but here effectively limits the strength of coupling between subsystems:

Assumption 3. For each $i \in \mathcal{M}$, $\mathbb{W}_i \subset \text{interior}(\mathbb{X}_i)$.

3.1 Main optimal control problem

The main optimal control problem for subsystem i employs the nominal model (5) to determine, in the presence of tightened constraints, a nominal optimal control sequence and associated nominal state predictions. Formally, this problem is $\mathbb{P}_i(\bar{x}_i)$, defined as

$$\bar{V}_i^0(\bar{x}_i) = \min_{\bar{\mathbf{u}}_i} \sum_{k=0}^{N-1} \ell_i(\bar{x}_i(k), \bar{u}_i(k)) \quad (9)$$

subject to

$$\bar{x}_i(0) = \bar{x}_i, \quad (10a)$$
$$\bar{x}_i(k+1) = A_{ii}\bar{x}_i(k) + B_{ii}\bar{u}_i(k), k = 0, \dots, N-1, \quad (10b)$$
$$\bar{x}_i(k) \in \alpha_i^x \mathbb{X}_i, k = 1, \dots, N-1, \quad (10c)$$
$$\bar{u}_i(k) \in \alpha_i^u \mathbb{U}_i, k = 1, \dots, N-1, \quad (10d)$$
$$\bar{x}_i(N) = 0. \quad (10e)$$

In this problem, the decision variable $\mathbf{u}_i$ is the sequence of (nominal) controls

$$\bar{\mathbf{u}}_i = \{\bar{u}_i(0), \bar{u}_i(1), \dots, \bar{u}_i(N-1)\}.$$

The original state and input constraint sets, $\mathbb{X}_i$ and $\mathbb{U}_i$, are scaled by factors $\alpha_i^x \in (0, 1]$ and $\alpha_i^u \in (0, 1]$ respectively, in order to preserve constraint satisfaction despite the neglecting of the disturbance (interaction) in the predictions. A detailed and comprehensive design procedure for these scalars is given in Section 5.

Remark 4. For simplicity, we use the origin as terminal set; less restrictive conditions are subject to current research.

The solution of $\mathbb{P}_i(\bar{x}_i)$ at nominal state $\bar{x}_i$ yields the optimal control and state sequences $\bar{\mathbf{u}}_i^0(\bar{x}_i) = \{\bar{u}_i^0(0; \bar{x}_i), \dots, \bar{u}_i^0(N-1; \bar{x}_i)\}$ and $\bar{\mathbf{x}}_i^0(\bar{x}_i) = \{\bar{x}_i^0(0; \bar{x}_i), \dots, \bar{x}_i^0(N; \bar{x}_i)\}$. It also defines the implicit control law

$$\bar{\kappa}_i(\bar{x}_i) = \bar{u}_i^0(0; \bar{x}_i).$$

In the next section, we define the ancillary optimal control problem that yields the second part of the control law (8).

3.2 Ancillary optimal control problem

The aim of the ancillary MPC controller is to reduce the error between true states and predictions. This error is $e_i \triangleq x_i - \bar{x}_i$, and evolves as

$$e_i^+ = A_{ii}e_i + B_{ii}f_i + \sum_{j \in \mathcal{N}_i} A_{ij}x_j + B_{ij}u_j$$

where $f_i \triangleq u_i - \bar{u}_i$. In a conventional single tube MPC controller approach, $f_i = K_i e_i$, but here we wish to replace this simple linear controller with a controller that can account for predictions of neighbouring subsystems. The above error dynamics are, however, not suitable for use as a prediction model because of the dependency on true states and inputs, x_j and u_j, rather than shared predictions.

To this end, therefore, we define a second nominal subsystem model to use for predictions in the ancillary controller:

$$\hat{x}_i^+ = A_{ii}\hat{x}_i + B_{ii}\hat{u}_i + \bar{w}_i. \quad (11)$$

The disturbance term $\bar{w}_i$ is composed of the predictions $(\bar{x}_j, \bar{u}_j)$ gathered from each of the neighbours, $j \in \mathcal{N}_i$, of agent i such that $\bar{w}_i = \sum_{j \in \mathcal{N}_i} A_{ij}\bar{x}_j + B_{ij}\bar{u}_j$ and $\bar{\mathbf{w}}_i \triangleq \{\bar{w}_i(0), \dots, \bar{w}_i(N)\}$. From this model, we define a nominal state error $\bar{e}_i \triangleq \hat{x}_i - \bar{x}_i$, and control error $\bar{f}_i = \hat{u}_i - \bar{u}_i$, whose dynamics evolve as

$$\bar{e}_i^+ = A_{ii}\bar{e}_i + B_{ii}\bar{f}_i + \bar{w}_i$$

It is this model that is employed in the following, ancillary optimal control problem, $\hat{\mathbb{P}}_i(\bar{e}_i; \bar{\mathbf{w}}_i)$:

$$\hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i) = \min_{\bar{\mathbf{f}}_i} \sum_{k=0}^{H-1} \ell_i(\bar{e}_i(k), \bar{f}_i(k)) \quad (12)$$

subject to, for $k = 0, \dots, H-1$,

$$\bar{e}_i(0) = \bar{e}_i, \quad (13a)$$
$$\bar{e}_i(k+1) = A_{ii}\bar{e}_i(k) + B_{ii}\bar{f}_i(k) + \bar{w}_i(k), \quad (13b)$$
$$\bar{e}_i(k) \in \beta_i^x \mathbb{X}_i, k = 0, \dots, H-1 \quad (13c)$$
$$\bar{f}_i(k) \in \beta_i^u \mathbb{U}_i, k = 0, \dots, H-1 \quad (13d)$$
$$\bar{e}_i(H) = 0. \quad (13e)$$

In this problem, the decision variable is the sequence of controls $\bar{\mathbf{f}}_i = \{\bar{f}_i(0), \dots, \bar{f}_i(H-1)\}$; the horizon is H. The cost function is the same as in the main problem. The parameter $\bar{\mathbf{w}}_i$ denotes the collection of disturbance predictions for subsystems $j \in \mathcal{N}_i$. The state and input constraints are, similar to in the main problem, scaled by factors $\beta_i^x \in (0, 1]$ and $\beta_i^u \in (0, 1]$; detailed design steps are given in Section 5.

The solution of $\hat{\mathbb{P}}_i(\bar{e}_i, \bar{\mathbf{w}}_i)$ defines an implicit control law

$$f_i = \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i).$$

However, this alone is not sufficient to guarantee the recursive feasibility and stability properties that we seek. In particular, if $u_i = \bar{\kappa}_i(\bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i)$ then

$$e_i^+ - \bar{e}_i^+ = A_{ii}(e_i - \bar{e}_i) + (w_i - \bar{w}_i),$$

which is unsatisfactory because the error dynamics here depend on only the spectral radius of A_{ii}: if A_{ii} is unstable, the mismatch between true error e_i and planned error $\bar{e}_i$ diverges. The next section describes how this problem is overcome by adding an extra feedback term to the ancillary control law.

3.3 Modified ancillary control law

We define $\hat{e}_i \triangleq e_i - \bar{e}_i$ and $\hat{w}_i \triangleq w_i - \bar{w}_i$; by definition, $x_i = \bar{x}_i + e_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$, and thus we seek to regulate $\bar{x}_i$, $\bar{e}_i$ and $\hat{e}_i$ to zero. To this end, we add another control, $\hat{f}_i$, to the ancillary control law, i.e., $f_i = \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \hat{f}_i$, so that

$$\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i.$$

With an appropriate choice of feedback law $\hat{f}_i = \mu_i(\hat{e}_i)$, then, this error can be regulated and guaranteed to remain within an invariant set around the origin, despite the disturbance $\hat{w}$. Therefore, the approach we take to designing the additional feedback term is based on the concept of *robust control invariant* (RCI) sets (Raković et al., 2007) and their corresponding invariance-inducing control laws.

Definition 5. (RCI set). A set $\mathcal{R}$ is *robust control invariant* (RCI) for a system $x^+ = f(x, u, w)$ and constraint set $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}$ if (i) $\mathcal{R} \subset \mathbb{X}$ and (ii) for all $x \in \mathcal{R}$, there exists a $u \in \mathbb{U}$ such that $x^+ = f(x, u, w) \in \mathcal{R}$, $\forall w \in \mathbb{W}$.

Given a RCI set $\mathcal{R}$, Definition 5 implies the existence of a control law $\mu \colon \mathbb{R}^m \mapsto \mathbb{R}^n$, such that the set mapping $\mu(\mathcal{R}) \triangleq \{\mu(x) : x \in \mathcal{R}\} = \{u \in \mathbb{U} : x^+ \in \mathcal{R}, \forall w \in \mathbb{W}\}$ is nonempty. Thus, given a RCI set, $\hat{\mathcal{R}}_i$, for the dynamics of the error $\hat{e}_i$, the respective control action is chosen as $\hat{f}_i = \mu_i(\hat{e}_i)$. The existence, design and computation of this invariant set and control law is discussed in detail on Section 5. For now, we note that the modified ancillary control law is

$$\hat{\kappa}_i(\bar{e}_i, \hat{e}_i, \bar{\mathbf{w}}_i) = \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i),$$

comprising the ancillary MPC control law plus the additional feedback term, and the overall control law for subsystem i is

$$u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i, \hat{e}_i; \bar{\mathbf{w}}_i) = \bar{u}_i^0(0; \bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i).$$

The structure of this three-term controller is worth remarking upon: the first term regulates the nominal state $\bar{x}_i$, while the second term regulates the planned error, accounting for planned (nominal) states and inputs of neighbours. The third term regulates the unplanned errors that arise from using nominal, rather than true, dynamics in the optimal control problems.

3.4 Distributed Control Algorithm

The optimization problems $\mathbb{P}_i(\bar{x}_i)$ and $\hat{\mathbb{P}}_i(\bar{e}_i, \bar{\mathbf{w}}_i)$ are used in the following algorithm.

Algorithm 1. (NeDMPC for subsystem i).

Initial data: Sets $\mathbb{X}_i$, $\mathbb{U}_i$; matrices (A_{ij}, B_{ij}) for $j \in \mathcal{N}_i$; constants α_i^x, α_i^u, β_i^x, β_i^u; states $\bar{x}_i(0) = x_i(0)$, $\bar{e}_i = 0$, $\bar{\mathbf{w}}_i = 0$, $\hat{V}_i^* = +\infty$.

Online Routine:

(1) At time k, controller state $\bar{x}_i$, solve $\mathbb{P}_i(\bar{x}_i)$ to obtain $\bar{\mathbf{u}}_i^0$ and $\bar{\mathbf{x}}_i^0$.
(2) Transmit $(\bar{\mathbf{x}}_i^0, \bar{\mathbf{u}}_i^0)$ to controllers $j \in \mathcal{N}_i$.
(3) Compute $\bar{\mathbf{w}}_i^0 = \{\bar{w}_i^0(l)\}_l$ from received $(\bar{\mathbf{x}}_j^0, \bar{\mathbf{u}}_j^0)$, where $\bar{w}_i^0(l) = \sum_{j \in \mathcal{N}_i}(A_{ij}\bar{x}_j^0(l) + B_{ij}\bar{u}_j^0(l))$, $l = 0 \ldots N$.
(4) At controller state $\bar{e}_i$, solve $\hat{\mathbb{P}}_i(\bar{e}_i; \bar{\mathbf{w}}_i^0)$ to obtain $\bar{f}_i^0$: if feasible and $\hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i^0) \leq \hat{V}_i^*$, set $\bar{\mathbf{w}}_i = \bar{\mathbf{w}}_i^0$ and $\hat{V}_i^* = \hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i^0)$; else, solve $\hat{\mathbb{P}}_i(\bar{e}_i; \bar{\mathbf{w}}_i)$ for $\bar{f}_i^0$.

(5) Measure plant state x_i, calculate $\hat{e}_i = x_i - \bar{x}_i - \bar{e}_i$, and apply $u_i = \bar{u}_i^0 + \bar{f}_i^0 + \mu_i(\hat{e}_i)$.
(6) Update controller states as $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{u}_i^0$ and $\bar{e}_i^+ = A_{ii}\bar{e}_i + B_{ii}\bar{f}_i^0 + \bar{w}_i$—where $\bar{w}_i = \bar{w}_i(0)$—$\bar{\mathbf{w}}_i^+ = \{\bar{w}_i(1), \ldots, \bar{w}_i(N), 0\}$, and $V_i^{*+} = V_i^* - \ell_i(\bar{e}_i, \bar{f}_i^0)$.
(7) Set $k = k+1$, $\bar{x}_i = \bar{x}_i^+$, $\bar{e}_i = \bar{e}_i^+$, $\bar{\mathbf{w}}_i = \bar{\mathbf{w}}_i^+$, $V_i^* = V_i^{*+}$, and go to Step 1.

In step 4, the ancillary problem is solved using the new disturbance sequence, $\bar{\mathbf{w}}_i^0$, formed from the state and input sequences of other subsystems just optimized in Step 1. If this problem is infeasible, or the optimal cost does not decrease sufficiently with respect to the previous solution, the problem is re-solved albeit with the previous disturbance sequence, $\bar{\mathbf{w}}_i$; as will be shown, this problem remains feasible even when the new problem is not, and in fact a feasible solution can be generated without solving the problem.

Remark 6. Guaranteeing the recursive feasibility of the ancillary problem is simple when the disturbance sequence is unchanging, but when the latter changes it is a non-trivial challenge. On the other hand, the feasibility of the ancillary problem depends on the horizon H, and—in view of the fact that $\bar{\mathbf{w}}_i$ is a sequence of N disturbances, with $\bar{w}_i(N) = 0$ always—it is suggested that $H \geq N+1$. In that case, $\bar{w}_i(k) = 0$ for prediction step $k \geq N$.

This completes the description of the approach, including control problems and the algorithm. In Section 5, we present a comprehensive approach to designing the invariance-inducing controller μ_i and the set scaling parameters a_i^x, a_i^u, β_i^x and β_i^u. Before that, we establish recursive feasibility and stability of the approach, which points to necessary and sufficient conditions on the scaling parameters that are useful later in developing the controller design process.

4. RECURSIVE FEASIBILITY AND STABILITY

Recursive feasibility is the main challenge for this approach. In contrast to conventional tube MPC, which uses linearity of the error dynamics and robust positive invariant (RPI) sets to allow the exact determination of constraint tightening margins for robustness, here the error dynamics are nonlinear and the constraint tightening is via scaling factors. In this section, we aim to establish conditions under which the proposed control scheme is recursively feasible and stable. Our approach here uses the notion of robust control invariant (RCI) sets (Raković et al., 2007): we show that, by suitable choices of scaling factors α_i^x, α_i^u, β_i^x and β_i^u, the error states of the controlled system evolve within bounded RCI sets, which may be used to guarantee constraint satisfaction and feasibility; however, we do not seek to obtain an explicit representation of the RCI set, but merely rely on its existence—an implicit form of invariance.

In order to establish robust constraint satisfaction, it is sufficient to show that the state x_i of subsystem i is contained within a set, say $\mathcal{X}_i$, that is robust positively invariant for the dynamics $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i$ and constraint sets $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$ under the control law $u_i = \kappa_i(x_i)$: that is, given $x_i \in \mathcal{X}_i \subseteq \mathbb{X}_i$, $A_{ii}x_i + B_{ii}\kappa_i(x_i) + w_i \in \mathcal{X}_i$ and $\kappa_i(x_i) \in \mathbb{U}_i$. In our approach, however, the true state satisfies $x_i = \bar{x}_i + e_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$, about which

the following is known: the nominal state $\bar{x}_i$ resides within $\bar{\mathcal{X}}_i^N$, defined as the feasibility region of $\mathbb{P}_i(\bar{x}_i)$:

$$\bar{\mathcal{X}}_i^N \triangleq \{\bar{x}_i : \bar{\mathcal{U}}_i^N(\bar{x}_i) \neq \emptyset\}, \qquad (14)$$

where $\bar{\mathcal{U}}_i^N(\bar{x}_i) \triangleq \{\bar{\mathbf{u}}_i : (10a)–(10e) \text{ are satisfied}\}$; the planned error $\bar{e}_i$, given $\bar{\mathbf{w}}_i$, resides within $\bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i)$, the feasibility region of $\hat{\mathbb{P}}_i(\bar{e}_i; \bar{\mathbf{w}}_i)$:

$$\bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i) \triangleq \{\bar{e}_i \in \mathbb{X}_i : \bar{\mathcal{F}}_i^N(\bar{e}_i; \bar{\mathbf{w}}_i) \neq \emptyset\}, \qquad (15)$$

where $\bar{\mathcal{F}}_i^N(\bar{e}_i; \bar{\mathbf{w}}_i) \triangleq \{\mathbf{f}_i : (13b)–(13e) \text{ are satisfied}\}$; finally, we suppose that the unplanned error $\hat{e}_i$ resides within some set $\hat{\mathcal{R}}_i$. Then our task is to develop conditions under which $x_i \in \bar{\mathcal{X}}_i^N \oplus \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathcal{R}}_i$ implies (i) $x_i^+ \in \bar{\mathcal{X}}_i^N \oplus \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathcal{R}}_i$, (ii) all constraints are satisfied, and (iii) all MPC problems remain feasible (i.e., $\bar{x}_i^+ \in \bar{\mathcal{X}}_i^N$ and $\bar{e}_i^+ \in \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i^+)$). To this end, noting that $\bar{\mathcal{X}}_i^N \subseteq \alpha_i^x \mathbb{X}_i$ by construction, we make the following assumptions, which may also be interpreted as design conditions that guide Section 5:

Assumption 7. The set $\hat{\mathcal{R}}_i$ is RCI for the system $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i$ and constraint set $(\xi_i^x \mathbb{X}_i, \xi_i^u \mathbb{U}_i, \hat{\mathbb{W}}_i)$, for some $\xi_i^x \in [0,1)$ and $\xi_i^u \in [0,1)$, and where $\hat{\mathbb{W}}_i \triangleq \bigoplus_{j \in \mathcal{N}_i}(1 - \alpha_j^x)A_{ij}\mathbb{X}_j \oplus (1 - \alpha_j^u)B_{ij}\mathbb{U}_j$. An invariance inducing control law for $\hat{\mathcal{R}}_i$ is $\hat{f}_i = \mu_i(\hat{e}_i)$.

Assumption 8. The constants $(\alpha_i^x, \beta_i^x, \xi_i^x)$ and $(\alpha_i^u, \beta_i^u, \xi_i^u)$ are chosen such $\alpha_i^x + \beta_i^x + \xi_i^x \leq 1$ and $\alpha_i^u + \beta_i^u + \xi_i^u \leq 1$.

The following result establishes recursive feasibility and constraint satisfaction under these assumptions. To aid the statement of the result, we first make the following definitions: $\bar{\mathbb{W}}_i = \bigoplus_{j \in \mathcal{N}_i}(\alpha_j^x A_{ij}\mathbb{X}_j \oplus \alpha_j^u B_{ij}\mathbb{U}_j)$ is the set of admissible disturbances arising from the solutions of the main optimal control problems for $j \in \mathcal{N}_i$; $\bar{\mathcal{W}}_i^N \triangleq \bar{\mathbb{W}}_i \times \bar{\mathbb{W}}_i \times \cdots \times \bar{\mathbb{W}}_i \times \{0\}$ is the sequence of such sets. Given a disturbance sequence $\bar{\mathbf{w}}_i = \{\bar{w}_i(0), \ldots, \bar{w}_i(N-1), 0\} \in \bar{\mathcal{W}}_i^N$, $\bar{\mathbf{w}}_i^+ = \{\bar{w}_i(1), \ldots, \bar{w}_i(N-1), 0, 0\}$ is the tail of that sequence.

Proposition 9. (Recursive feasibility). Suppose that Assumptions 7–8 hold. Then, for each subsystem $i \in \mathcal{M}$,

(i) If $\bar{x}_i \in \bar{\mathcal{X}}_i^N$ then $\bar{x}_i^+ \in \bar{\mathcal{X}}_i^N$.

(ii) If $\bar{e}_i \in \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i)$, for some $\bar{\mathbf{w}}_i \in \bar{\mathcal{W}}_i^N$, then $\bar{e}_i^+ \in \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i^+)$.

(iii) Given $\bar{x}_i(0) = x_i(0) \in \bar{\mathcal{X}}_i^N$, the subsystem $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i$ under the control law $u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i, \hat{e}_i; \bar{\mathbf{w}}_i) = \bar{u}_i^0(0; \bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i)$ satisfies $x_i \in \mathbb{X}_i$ and $u \in \mathbb{U}_i$ for all time.

Proof. For part (i), because the nominal model is linear, $\alpha_i^x \mathbb{X}_i$ and $\alpha_i^u \mathbb{U}_i$ are PC-sets, and the terminal constraint is control invariant, the set $\bar{\mathcal{X}}_i^N$ is compact, contains the origin and satisfies $\bar{\mathcal{X}}_i^N \supseteq \bar{\mathcal{X}}_i^{N-1} \supseteq \cdots \supseteq \bar{\mathcal{X}}_i^0 = \{0\}$. Moreover, $\bar{\mathcal{X}}_i^N$ is positively invariant for $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$, which is sufficient to prove the claim. (For a detailed proof, see Rawlings and Mayne (2009, Proposition 2.11).) The same arguments applied to $\bar{\mathcal{E}}_i(\bar{\mathbf{w}})$ establish part (ii).

For (iii), suppose that at time k, $\bar{x}_i \in \bar{\mathcal{X}}_i^N$, $\bar{e}_i \in \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i)$ with $\bar{\mathbf{w}}_i \in \bar{\mathcal{W}}_i^N$, and $\hat{e}_i \in \hat{\mathcal{R}}_i$. Then $x_i \in \bar{\mathcal{X}}_i^N \oplus \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathcal{R}}_i \subseteq \alpha_i^x \mathbb{X}_i \oplus \beta_i^x \mathbb{X}_i \oplus \xi_i^x \mathbb{X}_i = (\alpha_i^x + \beta_i^x + \xi_i^x)\mathbb{X}_i \subseteq \mathbb{X}_i$. The applied control is $u_i = \bar{u}_i^0(0; \bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i) \in \alpha_i^u \mathbb{U}_i \oplus \beta_i^u \mathbb{U} \oplus \xi_i^u \mathbb{U}_i \subseteq \mathbb{U}_i$. Then, because of parts (i) and (ii), $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \in \bar{\mathcal{X}}_i^N \oplus \bar{\mathcal{E}}_i^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathcal{R}}_i$. To complete

the proof, however, we must consider the possibility that the disturbance sequence at the successor state is $\hat{\mathbf{w}}_i^0 \neq \bar{\mathbf{w}}_i^+$: in that case, if $\hat{\mathbb{P}}_i(\bar{e}_i^+; \hat{\mathbf{w}}_i^0)$ is feasible then $x_i^+ \in \bar{\mathcal{X}}_i^N \oplus \bar{\mathcal{E}}_i^N(\hat{\mathbf{w}}_i^0) \oplus \hat{\mathcal{R}}_i$, which is still within $\mathbb{X}_i$ by construction, and $u_i = \bar{u}_i^0(0; x_i^+) + \bar{f}_i^0(0; \bar{e}_i^+, \hat{\mathbf{w}}_i^0) + \mu_i(\hat{e}_i^+) \subseteq \mathbb{U}_i$. If $\hat{\mathbb{P}}_i(\bar{e}_i^+; \hat{\mathbf{w}}_i^0)$ is not feasible, then $\hat{\mathbb{P}}_i(\bar{e}_i^+; \bar{\mathbf{w}}_i^+)$ *is* feasible (by the tail), and $u_i = \bar{u}_i^0(0; x_i^+) + \bar{f}_i^0(0; \bar{e}_i^+, \bar{\mathbf{w}}_i^+) + \mu_i(\hat{e}_i^+) \subseteq \mathbb{U}_i$. This establishes recursive feasibility of the algorithm.

Finally, if, at time 0, $\bar{x}_i = x_i \in \bar{\mathcal{X}}_i^N$ then $\bar{e}_i = 0$. Moreover, if $\bar{\mathbf{w}}_i = 0$, then—trivially—$\bar{e}_i \in \bar{\mathcal{E}}_i^N(0)$ and both the main and ancillary problems are feasible. By recursion, feasibility is retained at the next step, and the proof is complete. $\square$

Having established recursive feasibility and constraint satisfaction, the main result follows. The following assumption is supposed to hold.

Assumption 10. (Decentralized stabilizability). The RCI control laws $u_i = \mu_i(x_i)$ asymptotically stabilize the system $x^+ = Ax + Bu$.

Theorem 11. (Asymptotic stability). For each $i \in \mathcal{M}$, (i) the origin is asymptotically stable for the composite subsystem

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$$
$$\bar{e}_i^+ = A_{ii}\bar{e}_i + B_{ii}\bar{f}_i(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \bar{w}_i.$$

(ii) The origin is asymptotically stable for $x_i^+ = A_{ii}x_i + B_{ii}\kappa_i(x_i) + w_i$. The region of attraction is $\bar{\mathcal{X}}_i^N \subseteq \alpha_i^x \mathbb{X}_i$.

Proof. For (i), asymptotic stability of 0 for $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$ follows from the following facts: the value function $\bar{V}_i^0(\bar{x}_i)$ satisfies, for all $\bar{x}_i \in \bar{\mathcal{X}}_i^N$,

$$\bar{V}_i^0(\bar{x}_i) \geq \ell_i(\bar{x}_i, \bar{\kappa}_i(\bar{x}_i)),$$
$$\bar{V}_i^0(0) = 0,$$
$$\bar{V}_i^0(\bar{x}_i^+) - \bar{V}_i^0(\bar{x}_i) \leq -\ell_i(\bar{x}_i, \bar{\kappa}_i(\bar{x}_i)).$$

Therefore $\{\bar{V}_i^0(\bar{x}_i)\} \to 0$ and $\bar{x}_i \to 0$, $\bar{u}_i \to 0$. Similar arguments applied to $\hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i)$—together with the fact that because $\bar{w}_i$ is a linear function of $(\bar{x}_j, \bar{u}_j)$ for $j \in \mathcal{N}_i$, then $\bar{w}_i \to 0$ and $\bar{\mathbf{w}}_i \to 0$—establish that $\bar{e}_i \to 0$; the possibility that $\hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i)$ does not attain the necessary decrease between $(\bar{e}_i, \bar{\mathbf{w}}_i)$ and $(\bar{e}_i^+, \hat{\mathbf{w}}_i^0)$ (where $\hat{\mathbf{w}}_i^0 \neq \mathbf{w}_i$) is eliminated by the checking step in the algorithm.

For (ii), because $x_i \in \bar{x}_i + \bar{e}_i + \hat{e}_i$ and $\bar{x}_i, \bar{e}_i \to 0$, then $x_i \to \hat{e}_i$ and $u_i \to \mu_i(x_i)$. Under the decentralized stabilizability assumption, then $x \to 0$ and so each $x_i \to 0$. $\square$

5. SELECTION OF THE SCALING CONSTANTS

In this section, a methodology is given for the design of the scaling constants α_i^x, α_i^u, β_i^x and β_i^u for the main and ancillary problems, and the RCI controller $\mu_i(\cdot)$. The approach we take is to employ the optimized RCI set design proposed by Raković et al. (2007); however, we do not explicitly construct the set $\hat{\mathcal{R}}_i$, but use the optimization to produce the scaling constants and the control law.

5.1 Revision of optimized robust control invariance

In Raković et al. (2007), the problem of computing an RCI set for $x^+ = Ax + Bu + w$ and $(\mathbb{X}, \mathbb{U}, \mathbb{W})$, with $\mathbb{W}$ a C-set,

is posed as linear programming (LP) problem. The set, and corresponding control set, are the polytopes

$$\mathcal{R}_h(\mathbf{M}_h) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\mathbb{W}, \quad \mu(\mathcal{R}_h(\mathbf{M}_h)) = \bigoplus_{l=0}^{h-1} M_l\mathbb{W},$$

where the matrices $D_l(\mathbf{M}_h), l = 0 \ldots h$ are defined as

$$D_0(\mathbf{M}_h) = I, \quad D_l(\mathbf{M}_k) \triangleq A^l + \sum_{j=0}^{l-1} A^{l-1-j} B M_j, l \geq 1$$

with $M_j \in \mathbb{R}^{m \times n}$ and $\mathbf{M}_h \triangleq (M_0, M_1, \ldots, M_{h-1})$, such that $D_h(\mathbf{M}_h) = 0, h \geq n$. The set of matrices that satisfy these conditions is given by $\mathbb{M}_h \triangleq \{\mathbf{M}_h : D_h(\mathbf{M}_h) = 0\}$. Constraint satisfaction is guaranteed if $\mathcal{R}_h(\mathbf{M}_h) \subseteq \eta\mathbb{X}$ and $\mu(\mathcal{R}_h(\mathbf{M}_h)) \subseteq \theta\mathbb{U}$, with $(\eta, \theta) \in [0, 1] \times [0, 1]$.

The optimization problem defined to compute these sets is

$$\mathbb{P}_h^{\mathcal{R}} : \min\{\delta : \gamma \in \Gamma\}, \tag{16}$$

where $\gamma = (\mathbf{M}_h, \eta, \theta, \delta)$, and the set $\Gamma = \{\gamma : \mathbf{M}_h \in \mathbb{M}_h, \mathcal{R}_h(\mathbf{M}_h) \subseteq \eta\mathbb{X}, \mu(\mathcal{R}_h(\mathbf{M}_h)) \subseteq \theta\mathbb{U}, (\eta, \theta) \in [0, 1] \times [0, 1], q_\eta \eta + q_\theta \theta \leq \delta\}$; q_η and q_θ are weights to express a preference for the relative contraction of state and input constraint sets. Feasibility of this problem is linked to the existence of an RCI set: if $\mathbb{P}_h^{\mathcal{R}}$ is feasible, then $\mathcal{R}_h(\mathbf{M}_h)$ satisfies the RCI properties (Raković et al., 2007).

Remark 12. (Invariance inducing control law). The computation of the invariance inducing law, $\mu : \mathcal{R}_h(\mathbf{M}_h) \to \mathbb{U}$, does not require an explicit computation of the respective set. The only required information is the structure of the given set, *i.e.*, the solution of (16), from which a suitable selection map is computed, see Raković et al. (2007).

5.2 Design procedure for each subsystem

Recall that in the control algorithm proposed in the previous section, the state error $e_i = x_i - \bar{x}_i$ was decomposed into planned error $\bar{e}_i = \hat{x}_i - \bar{x}_i$ and an unplanned error $\hat{e}_i = x_i - \hat{x}_i$; thus, $e_i = \bar{e}_i + \hat{e}_i$. Our aim is to determine the RCI control law $\hat{f}_i = \mu_i(\hat{e}_i)$ associated with the unplanned error dynamics $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i$. The principal challenge here is that it is not possible, *a priori*, to define the unplanned error set $\hat{\mathbb{W}}_i$. Instead, we consider that an RCI problem $\mathbb{P}_h^{\mathcal{R}}$ is associated with the error dynamics $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and constraint sets $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$, and call this problem $\mathbb{P}_h^{\mathcal{R}_i}$, with the set $\mathcal{R}_{i,h}$ defined by adding appropriate i subscripts to its generating sets and matrices. The rationale for this is as follows.

The disturbance $w_i = \sum_{j \in \mathcal{N}_i} A_{ij}x_j + B_{ij}u_j$ arising from the state and input coupling, is decomposed into two terms: $w_i = \bar{w}_i + \hat{w}_i$. The first term, $\bar{w}_i = \sum_{j \in \mathcal{N}_i} A_{ij}\bar{x}_j + B_{ij}\bar{u}_j$, is the planned disturbance obtained from the predictions, while the second term, $\hat{w}_i = \sum_{j \in \mathcal{N}_i} A_{ij}(x_j - \bar{x}_j) + B_{ij}(u_j - \bar{u}_j)$, is the unplanned disturbance. Since $\bar{x}_j \in \alpha_j^x\mathbb{X}_j$, $\bar{u}_j \in \alpha_j^u\mathbb{U}_j$ then $\bar{\mathbb{W}}_i = \bigoplus_{j \in \mathcal{N}_i}(\alpha_i^x A_{ij}\mathbb{X}_j \oplus \alpha_i^u B_{ij}\mathbb{U}_j)$, In addition, if we bound $e_j \in (1 - \alpha_j^x)\mathbb{X}_j$ and $f_j \in (1 - \alpha_j^u)\mathbb{U}_j$, it is possible to write $\hat{\mathbb{W}}_i = \bigoplus_{j \in \mathcal{N}_i}((1 - \alpha_i^x)A_{ij}\mathbb{X}_j \oplus (1 - \alpha_i^u)B_{ij}\mathbb{U}_j)$ and so, $w_i \in \mathbb{W}_i = \bar{\mathbb{W}}_i \oplus \hat{\mathbb{W}}_i$, *i.e.*, $\bar{\mathbb{W}}_i$ and $\hat{\mathbb{W}}_i$ are summands of the known $\mathbb{W}_i$. The next results follow directly from the definition of RCI sets and the results of Raković et al. (2007):

Proposition 13. Suppose Assumptions 1–3 hold. If $\mathbb{P}_h^{\mathcal{R}_i}$ is feasible for $i \in \mathcal{M}$, then $\mathcal{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$.

Proposition 14. Suppose $\tilde{\mathbb{W}}_i \subset \mathbb{W}_i$ is a PC-set and a summand of $\mathbb{W}_i$. If $\mathcal{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$, then $\tilde{\mathcal{R}}_{i,h}(\mathbf{M}_{i,h}) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\tilde{\mathbb{W}}_i \subset \mathcal{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \tilde{\mathbb{W}}_i)$.

The implication of the second result is that it is possible to first determine an RCI set for the known disturbance set $\mathbb{W}_i$, and then, from that, determine an RCI set (with the same structure) for the set $\hat{\mathbb{W}}_i$, because the latter is a summand. Therefore, the design is summarized as follows:

(1) The problem $\mathbb{P}_h^{\mathcal{R}_i}$ associated with the known $\mathbb{W}_i$ is solved to yield $\gamma_{i,h} = (\mathbf{M}_{i,h}, \eta_i, \theta_i, \delta_i)$, where η_i and θ_i are scalings of $\mathbb{X}_i$ and $\mathbb{U}_i$ such that $\mathcal{R}_{i,h} \subset \eta_i\mathbb{X}_i$ and $\mu(\mathcal{R}_{i,h}) \subset \theta_i\mathbb{U}_i$ respectively.

(2) Given that, under the RCI control law, $e_i \in \mathcal{R}_{i,h} \subset \eta_i\mathbb{X}_i$ and $f_i \in \mu(\mathcal{R}_{i,h}) \subset \theta_i\mathbb{U}_i$, we select

$$\alpha_i^x = 1 - \eta_i$$
$$\alpha_i^u = 1 - \theta_i.$$

Then $x_i = \bar{x}_i + e_i \in \alpha_i^x\mathbb{X}_i \oplus \eta_i\mathbb{X}_i = \mathbb{X}_i$, as required, with a similar expression for u_i.

(3) The selection of suitable ξ_i^x and ξ_i^u is done by finding values such that the sets $\hat{\mathcal{R}}_{i,h}$ and $\mu_i(\mathcal{R}_{i,h})$ corresponding to the unplanned disturbance set $\hat{\mathbb{W}}_i = \bigoplus_{j \in \mathcal{N}_i}((1 - \alpha_j^x)A_{ij}\mathbb{X}_j \oplus (1 - \alpha_j^u)B_{ij}\mathbb{U}_j)$ being contained within $\xi_i^x\mathbb{X}_i$ and $\xi_i^u\mathbb{U}_i$. The set $\hat{\mathbb{W}}_i$ is computed and the RCI problem $\mathbb{P}_h^{\hat{\mathcal{R}}_i}$ is solved for $\tilde{\gamma}_{(i,h)} = (\mathbf{M}_{i,h}, \tilde{\eta}_i, \tilde{\theta}_i, \tilde{\delta}_i)$ to yield the scaling factors

$$\xi_i^x = \tilde{\eta}_i$$
$$\xi_i^u = \tilde{\theta}_i.$$

(4) The selection of the constants β_i^x and β_i^u follows from Assumption 8 in order to satisfy constraint satisfaction

$$\beta_i^x = 1 - \alpha_i^x - \xi_i^i$$
$$\beta_i^u = 1 - \alpha_i^u - \xi_i^u.$$

(5) The control law $\hat{f}_i = \mu_i(\hat{e}_i)$ is computed from the matrices $\mathbf{M}_{i,h}$, using the minimal selection map procedure described in Raković et al. (2007).

6. ILLUSTRATIVE EXAMPLE

To illustrate the feasibility of the design methodology, we consider an example based on the system from Farina and Scattolini (2012), which comprises four trucks, each with dynamics

$$\frac{d}{dt}\begin{bmatrix} r_i \\ v_i \end{bmatrix} = A_{ii}^c \begin{bmatrix} r_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u_i + w_i$$

where r_i is the displacement of truck i from a datum, v_i is its velocity and u_i is the control input (acceleration). The disturbance w_i arises via the coupling between trucks: truck 1 (mass $m_1 = 3$ kg) is coupled to truck 2 (mass $m_2 = 2$ kg) via a spring (stiffness $k_{12} = 0.5$) and damper ($h_{12} = 0.2$). Likewise, truck 2 (mass $m_3 = 3$ kg) is coupled to truck 3 (mass $m_4 = 6$ kg) via $k_{34} = 1$ and $h_{34} = 0.3$. Finally,
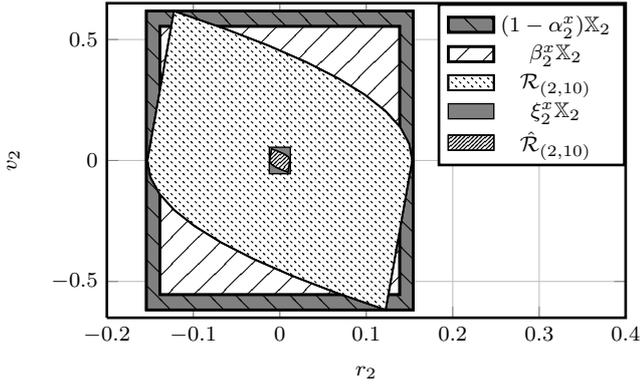
Fig. 1. For truck 2 (and $h = 10$), the different scalings of the state constraint set $\mathbb{X}_2$ and the RCI sets $\mathcal{R}_{2,10}$ and $\hat{\mathcal{R}}_{2,10}$: the main controller, ancillary controller and RCI controller operate within the regions $\alpha_i^x \mathbb{X}_2$, $\beta_i^x \mathbb{X}_2$ and $\eta_i^x \mathbb{X}_2$ respectively; the space $(1 - \alpha_2^x)\mathbb{X}_2$ is divided between the ancillary controller $(\beta_2^x \mathbb{X}_2)$ and the RCI controller $(\xi_2^x \mathbb{X}_2)$ such that $1 - \alpha_2^x = \beta_2^x + \xi_2^x$.

truck 3 is coupled to truck 4 via $k_{23} = 0.75$ and $h_{23} = 0.25$. The initial conditions are $x_1 = (1.8, -2)$, $x_2 = (0.5, 5)$, $x_3 = (-0.9, -5)$, and $x_4 = (-1.8, 2)$. The problem is to steer the trucks to equilibrium while satisfying constraints on displacement ($|r_i| \leq 2$), speed ($|v_i| \leq 8$) and acceleration ($|u_i| \leq 4$ for $i = 1, 2, 3$, and $|u_4| \leq 6$). In each case, the controllers are designed with $Q_i = I$, $R_i = 1$ and horizon $N = 25$. Before applying Algorithm 1 to the system we obtain the scaling constants $(\alpha_i^x, \beta_i^x, \xi_i^x)$ and $(\alpha_i^u, \beta_i^u, \xi_i^u)$ for each truck $i \in \mathcal{M}$—see Table 1 for the values obtained through the procedure detailed in Section 5.

In Figure 1, the different scalings of the state constraint sets are shown for truck 2, and also the corresponding RCI sets. Thus, for truck 2, 92.28% of the state constraint set is allocated to the the main optimal control problem, which is concerned with regulating the nominal subsystem (*i.e.*, neglecting interactions). On the other hand, the ancillary problem—which regulates the planned errors—has 7.36% of the original state constraint sets. The remaining 0.36% of the state constraint set is allocated to the RCI control law to handle unplanned disturbances.

7. CONCLUSIONS

A distributed MPC algorithm for dynamically coupled linear systems was proposed. Subsystem controllers solve (once, at each time step) local optimal control problems to determine control sequences and state trajectories, and exchange information about these. The main feature of the proposed algorithm is the use of a secondary MPC controller for each subsystem, which acts on the shared

Table 1. Designed values of scaling factors.

	Truck 1	Truck 2	Truck 3	Truck 4
α_i^x	0.9784	0.9228	0.9342	0.9816
β_i^x	0.0199	0.0736	0.0628	0.0172
ξ_i^x	0.0017	0.0036	0.0029	0.0012
α_i^u	0.9921	0.9808	0.9759	0.9910
β_i^u	0.0073	0.0183	0.0230	0.0084
ξ_i^u	0.0006	0.0009	0.0011	0.0006

plans of other subsystems and aims to reject the uncertainty caused by neglecting interactions in the main problems. Recursive feasibility and stability are guaranteed under provided assumptions, and a design methodology was given for the off-line selection of controller parameters and illustrated with an example.

A key advantage of the proposed approach, in addition to the guaranteed feasibility and stability and despite this being a tube-based method, is the absence of invariant sets in the optimal control problems. This makes the approach potentially applicable to higher-dimensional subsystems.

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