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The group of automorphisms of the Lie algebra of derivations of a polynomial algebra

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Abstract

We prove that the group of automorphisms of the Lie algebra $\mathrm{Der}_K(P_n)$ of derivations of a polynomial algebra $P_n = K[x_1, \ldots, x_n]$ over a field of characteristic zero is canonically isomorphic to the the group of automorphisms of the polynomial algebra P_n.

Key Words: Group of automorphisms, monomorphism, Lie algebra, automorphism, locally nilpotent derivation.

Mathematics subject classification 2010: 17B40, 17B20, 17B66, 17B65, 17B30.

1 Introduction

In this paper, module means a left module, K is a field of characteristic zero and K^* is its group of units, and the following notation is fixed:

- $P_n := K[x_1, \ldots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,
- $G_n := \mathrm{Aut}_K(P_n)$ is the group of automorphisms of the polynomial algebra P_n,
- $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n,
- $D_n := \mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of K-derivations of P_n where $[\partial, \delta] := \partial\delta - \delta\partial$,
- $\delta_1 := \mathrm{ad}(\partial_1), \ldots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebra D_n determined by the elements $\partial_1, \ldots, \partial_n$ (where $\mathrm{ad}(a)(b) := [a, b]$),
- $\mathbb{G}_n := \mathrm{Aut}_{\mathrm{Lie}}(D_n)$ is the group of automorphisms of the Lie algebra D_n,
- $\mathcal{D}_n := \bigoplus_{i=1}^n K\partial_i$,
- $\mathcal{H}_n := \bigoplus_{i=1}^n KH_i$ where $H_1 := x_1\partial_1, \ldots, H_n := x_n\partial_n$,
- $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} Kx^\alpha \partial^\beta$ is the n'th *Weyl algebra*,
- for each natural number $n \geq 2$, $\mathfrak{u}_n := K\partial_1 + P_1\partial_2 + \cdots + P_{n-1}\partial_n$ is the *Lie algebra of triangular polynomial derivations* (it is a Lie subalgebra of the Lie algebra D_n) and $\mathrm{Aut}_K(\mathfrak{u}_n)$ is its group of automorphisms.

The aim of the paper is to prove the following theorem.

Theorem 1.1 $\mathbb{G}_n = G_n$.

Structure of the proof. (i) $G_n \subseteq \mathbb{G}_n$ via the group monomorphism (Lemma 2.3.(3))

$$G_n \to \mathbb{G}_n, \ \ \sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

(ii) Let $\sigma \in \mathbb{G}_n$. Then $\partial'_1 := \sigma(\partial_1), \ldots, \partial'_n := \sigma(\partial_n)$ are commuting, locally nilpotent derivations of the polynomial algebra P_n (Lemma 2.6.(1)).

(iii) $\bigcap_{i=1}^n \ker_{P_n}(\partial_i') = K$ (Lemma 2.6.(2)).

(iv)(crux) There exists a polynomial automorphism $\tau \in G_n$ such that $\tau\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$ (Corollary 2.9).

(v) $\mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$ (Proposition 2.5.(3)) where

$$\mathrm{Sh}_n := \{s_\lambda \in G_n \,|\, s_\lambda(x_1) = x_1 + \lambda_1, \ldots, s_\lambda(x_n) = x_n + \lambda_n\}$$

is the *shift group* of automorphisms of the polynomial algebra P_n and $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^n$.

(vi) By (iv) and (v), $\sigma \in G_n$, i.e. $\mathbb{G}_n = G_n$. $\Box$

An analogue of the Jacobian Conjecture is true for D_n. The Jacobian Conjecture claims that *certain* monomorphisms of the polynomial algebra P_n are isomorphisms: *Every algebra endomorphism σ of the polynomial algebra P_n such that $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ is an automorphism.* The condition that $\mathcal{J}(\sigma) \in K^*$ implies that the endomorphism σ is a monomorphism.

Conjecture. *Every homomorphism of the Lie algebra D_n is an automorphism.*

Theorem 1.2 *[4] Every monomorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism.*

Remark. Not every epimorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism. Moreover, there are countably many distinct ideals $\{I_{i\omega^{n-1}} \,|\, i \geq 0\}$ such that

$$I_0 = \{0\} \subset I_{\omega^{n-1}} \subset I_{2\omega^{n-1}} \subset \cdots \subset I_{i\omega^{n-1}} \subset \cdots$$

and the Lie algebras $\mathfrak{u}_n / I_{i\omega^{n-1}}$ and $\mathfrak{u}_n$ are isomorphic (Theorem 5.1.(1), [5]).

Theorems 1.2 and Conjecture have bearing of the Jacobian Conjecture and the Conjecture of Dixmier [8] for the Weyl algebra A_n over a field of characteristic zero that claims: *every homomorphism of the Weyl algebra is an automorphism.* The Weyl algebra A_n is a simple algebra, so every algebra endomorphism of A_n is a monomorphism. This conjecture is open since 1968 for all $n \geq 1$. It is stably equivalent to the Jacobian Conjecture for the polynomial algebras as was shown by Tsuchimoto [9], Belov-Kanel and Kontsevich [7], (see also [2] for a short proof which is based on the author's new inversion formula for polynomial automorphisms [1]).

An analogue of the Conjecture of Dixmier is true for the algebra $\mathbb{I}_1 := K\langle x, \frac{d}{dx}, \int \rangle$ of polynomial integro-differential operators.

Theorem 1.3 (Theorem 1.1, [3]) *Each algebra endomorphism of $\mathbb{I}_1$ is an automorphism.*

In contrast to the Weyl algebra $A_1 = K\langle x, \frac{d}{dx}\rangle$, the algebra of polynomial differential operators, the algebra $\mathbb{I}_1$ is neither a left/right Noetherian algebra nor a simple algebra. The left localizations, $A_{1,\partial}$ and $\mathbb{I}_{1,\partial}$, of the algebras A_1 and $\mathbb{I}_1$ at the powers of the element $\partial = \frac{d}{dx}$ are isomorphic. For the simple algebra $A_{1,\partial} \simeq \mathbb{I}_{1,\partial}$, there are algebra endomorphisms that are not automorphisms [3].

The group of automorphisms of the Lie algebra $\mathfrak{u}_n$. In [6], the group of automorphisms $\mathrm{Aut}_K(\mathfrak{u}_n)$ of the Lie algebra $\mathfrak{u}_n$ of triangular polynomial derivations is found ($n \geq 2$), it is isomorphic to an iterated semi-direct product (Theorem 5.3, [6]),

$$\mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n))$$

where $\mathbb{T}^n$ is an algebraic n-dimensional torus, $\mathrm{UAut}_K(P_n)_n$ is an explicit factor group of the group $\mathrm{UAut}_K(P_n)$ of unitriangular polynomial automorphisms, $\mathbb{F}_n'$ and $\mathbb{E}_n$ are explicit groups that are isomorphic respectively to the groups $\mathbb{I}$ and $\mathbb{J}^{n-2}$ where $\mathbb{I} := (1 + t^2K[[t]], \cdot) \simeq K^{\mathbb{N}}$

and $\mathbb{J} := (tK[[t]], +) \simeq K^{\mathbb{N}}$. Comparing the groups G_n and $\mathrm{Aut}_K(\mathfrak{u}_n)$ we see that the group $(\mathrm{UAut}_K(P_n)_n$ of polynomial automorphisms is a *tiny* part of the group $\mathrm{Aut}_K(\mathfrak{u}_n)$ but in contrast $\mathbb{G}_n = \mathrm{Aut}_K(P_n)$. It is shown that the *adjoint group* of automorphisms $\mathcal{A}(\mathfrak{u}_n)$ of the Lie algebra $\mathfrak{u}_n$ is equal to the group $\mathrm{UAut}_K(P_n)_n$ (Theorem 7.1, [6]). Recall that the *adjoint group* $\mathcal{A}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ is generated by the elements $e^{\mathrm{ad}(g)} := \sum_{i\geq 0} \frac{\mathrm{ad}(g)^i}{i!} \in \mathrm{Aut}_K(\mathcal{G})$ where g runs through all the locally nilpotent elements of the Lie algebra $\mathcal{G}$ (an element g is a *locally nilpotent element* if the inner derivation $\mathrm{ad}(g) := [g, \cdot]$ of the Lie algebra $\mathcal{G}$ is a locally nilpotent derivation).

2 Proof of Theorem 1.1

This section can be seen as a proof of Theorem 1.1. The proof is split into several statements that reflect 'Structure of the proof of Theorem 1.1' given in the Introduction.

The Lie algebra D_n is $\mathbb{Z}^n$-graded. The Lie algebra

$$D_n = \bigoplus_{\alpha\in\mathbb{N}^n} \bigoplus_{i=1}^n Kx^\alpha\partial_i \tag{1}$$

is a $\mathbb{Z}^n$-graded Lie algebra

$$D_n = \bigoplus_{\beta\in\mathbb{Z}^n} D_{n,\beta} \;\; \text{where} \;\; D_{n,\beta} = \bigoplus_{\alpha-e_i=\beta} Kx^\alpha\partial_i,$$

i.e. $[D_{n,\alpha}, D_{n,\beta}] \subseteq D_{n,\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{N}^n$ where $e_1 := (1,0,\dots,0), \dots, e_n := (0,\dots,0,1)$ is the canonical free basis for the free abelian group $\mathbb{Z}^n$. This follows from the commutation relations

$$[x^\alpha\partial_i, x^\beta\partial_j] = \beta_i x^{\alpha+\beta-e_i}\partial_j - \alpha_j x^{\alpha+\beta-e_j}\partial_i. \tag{2}$$

Clearly, for all $i, j = 1, \dots, n$ and $\alpha \in \mathbb{N}^n$,

$$[H_j, x^\alpha\partial_i] = \begin{cases} \alpha_j x^\alpha\partial_i & \text{if } j \neq i, \\ (\alpha_i - 1)x^\alpha\partial_i & \text{if } j = i, \end{cases} \tag{3}$$

$$[\partial_j, x^\alpha\partial_i] = \alpha_j x^{\alpha-e_j}\partial_i. \tag{4}$$

The *support* $\mathrm{Supp}(D_n) := \{\beta \in \mathbb{Z}^n \,|\, D_{n,\beta} \neq 0\}$ is a submonoid of $\mathbb{Z}^n$. Let us find the support $\mathrm{Supp}(D_n)$, the graded components $D_{n,\beta}$ and their dimensions $\dim_K D_{n,\beta}$. For each $i = 1, \dots, n$, let $\mathbb{N}^{n,i} := \{\alpha \in \mathbb{N}^n \,|\, \alpha_i = 0\}$ and $P_n^{\partial_i} := \ker_{P_n}(\partial_i)$. It follows from the decompositions $P_n = P_n^{\partial_i} \oplus P_n x_i$ for $i = 1, \dots, n$ that

$$D_n = \bigoplus_{i=1}^n (P_n^{\partial_i} \oplus P_n x_i)\partial_i = \bigoplus_{i=1}^n P_n^{\partial_i}\partial_i \oplus \bigoplus_{i=1}^n P_n H_i,$$

$$D_n = \bigoplus_{i=1}^n P_n^{\partial_i}\partial_i \oplus \bigoplus_{\alpha\in\mathbb{N}^n} x^\alpha\mathcal{H}_n. \tag{5}$$

Hence,

$$\mathrm{Supp}(D_n) = \coprod_{i=1}^n (\mathbb{N}^{n,i} - e_i) \coprod \mathbb{N}^n. \tag{6}$$

$$D_{n,\beta} = \begin{cases} x^\alpha\partial_i & \text{if } \beta = \alpha - e_i \in \mathbb{N}^{n,i} - e_i, \\ x^\beta\mathcal{H}_n & \text{if } \beta \in \mathbb{N}^n. \end{cases} \tag{7}$$

$$\dim_K D_{n,\beta} = \begin{cases} 1 & \text{if } \beta = \alpha - e_i \in \mathbb{N}^{n,i} - e_i, \\ n & \text{if } \beta \in \mathbb{N}^n. \end{cases}$$

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ be its Lie subalgebra. The *centralizer* $C_{\mathcal{G}}(\mathcal{H}) := \{x \in \mathcal{G} \,|\, [x, \mathcal{H}] = 0\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$. In particular, $Z(\mathcal{G}) := C_{\mathcal{G}}(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$. The *normalizer* $N_{\mathcal{G}}(\mathcal{H}) := \{x \in \mathcal{G} \,|\, [x, \mathcal{H}] \subseteq \mathcal{H}\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, it is the largest Lie subalgebra of $\mathcal{G}$ that contains $\mathcal{H}$ as an ideal.

Let V be a vector space over K. A K-linear map $\delta : V \to V$ is called a *locally nilpotent map* if $V = \bigcup_{i\geq 1} \ker(\delta^i)$ or, equivalently, for every $v \in V$, $\delta^i(v) = 0$ for all $i \gg 1$. When δ is a locally nilpotent map in V we also say that δ *acts locally nilpotently* on V. Every *nilpotent* linear map δ, that is $\delta^n = 0$ for some $n \geq 1$, is a locally nilpotent map but not vice versa, in general. Let $\mathcal{G}$ be a Lie algebra. Each element $a \in \mathcal{G}$ determines the derivation of the Lie algebra $\mathcal{G}$ by the rule $\mathrm{ad}(a) : \mathcal{G} \to \mathcal{G}$, $b \mapsto [a, b]$, which is called the *inner derivation* associated with a. The set $\mathrm{Inn}(\mathcal{G})$ of all the inner derivations of the Lie algebra $\mathcal{G}$ is a Lie subalgebra of the Lie algebra $(\mathrm{End}_K(\mathcal{G}), [\cdot, \cdot])$ where $[f, g] := fg - gf$. There is the short exact sequence of Lie algebras

$$0 \to Z(\mathcal{G}) \to \mathcal{G} \stackrel{\mathrm{ad}}{\to} \mathrm{Inn}(\mathcal{G}) \to 0,$$

that is $\mathrm{Inn}(\mathcal{G}) \simeq \mathcal{G}/Z(\mathcal{G})$ where $Z(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$ and $\mathrm{ad}([a, b]) = [\mathrm{ad}(a), \mathrm{ad}(b)]$ for all elements $a, b \in \mathcal{G}$. An element $a \in \mathcal{G}$ is called a *locally nilpotent element* (respectively, a *nilpotent element*) if so is the inner derivation $\mathrm{ad}(a)$ of the Lie algebra $\mathcal{G}$.

The Cartan subalgebra $\mathcal{H}_n$ of D_n. A nilpotent Lie subalgebra C of a Lie algebra $\mathcal{G}$ is called a *Cartan subalgebra* of $\mathcal{G}$ if it coincides with its normalizer. We use often the following obvious observation: *An abelian Lie subalgebra that coincides with its centralizer is a maximal abelian Lie subalgebra.*

Lemma 2.1 *1. $\mathcal{H}_n$ is a Cartan subalgebra of D_n.*

2. $\mathcal{H}_n = C_{D_n}(\mathcal{H}_n)$ is a maximal abelian subalgebra of D_n.

Proof. Statements 1 and 2 follows from (6) and (7). □

P_n is a D_n-module. The polynomial algebra P_n is a (left) D_n-module: $D_n \times P_n \to P_n$, $(\partial, p) \mapsto \partial * p$. In more detail, if $\partial = \sum_{i=1}^n a_i \partial_i$ where $a_i \in P_n$ then

$$\partial * p = \sum_{i=1}^n a_i \frac{\partial p}{\partial x_i}.$$

The field K is a D_n-submodule of P_n and

$$\bigcap_{i=1}^n \ker_{P_n}(\partial_i) = K. \qquad (8)$$

Lemma 2.2 *The D_n-module P_n/K is simple with* $\mathrm{End}_{D_n}(P_n/K) = K\mathrm{id}$ *where* id *is the identity map.*

Proof. Let M be a nonzero submodule of P_n/K and $0 \neq p \in M$. Using the actions of $\partial_1, \ldots, \partial_n$ on p we obtain an element of M of the form λx_i for some $\lambda \in K^*$. Hence, $x_i \in M$ and $x^\alpha = x^\alpha \partial_i * x_i \in M$ for all $0 \neq \alpha \in \mathbb{N}^n$. Therefore, $M = P_n/K$. Let $f \in \mathrm{End}_{D_n}(P_n/K)$. Then applying f to the equalities $\partial_i * (x_1 + K) = \delta_{i1}$ for $i = 1, \ldots, n$, we obtain the equalities

$$\partial_i * f(x_1 + K) = \delta_{i1} \;\text{ for }\; i = 1, \ldots, n.$$

Hence, $f(x_1 + K) \in \bigcap_{i=2}^n \ker_{P_n/K}(\partial_i) \cap \ker_{P_n/K}(\partial_i^2) = (K[x_1]/K) \cap \ker_{P_n/K}(\partial_i^2) = K(x_1 + K)$. So, $f(x_1 + K) = \lambda(x_1 + K)$ and so $f = \lambda\mathrm{id}$, by the simplicity of the D_n-module P_n/K.
□

The G_n-module D_n. The Lie algebra D_n is a G_n-module,

$$G_n \times D_n \to D_n, \;\; (\sigma, \partial) \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

Every automorphism $\sigma \in G_n$ is uniquely determined by the elements

$$x_1' := \sigma(x_1), \; \ldots, \; x_n' := \sigma(x_n).$$

Let $M_n(P_n)$ be the algebra of $n \times n$ matrices over P_n. The matrix $J(\sigma) := (J(\sigma)_{ij}) \in M_n(P_n)$, where $J(\sigma)_{ij} = \frac{\partial x_j'}{\partial x_i}$, is called the *Jacobian matrix* of the automorphism (endomorphism) σ and its determinant $\mathcal{J}(\sigma) := \det J(\sigma)$ is called the *Jacobian* of σ. So, the j'th column of $J(\sigma)$ is the *gradient* $\mathrm{grad}\, x_j' := (\frac{\partial x_j'}{\partial x_1}, \ldots, \frac{\partial x_j'}{\partial x_n})^T$ of the polynomial x_j'. Then the derivations

$$\partial_1' := \sigma\partial_1\sigma^{-1}, \; \ldots, \; \partial_n' := \sigma\partial_n\sigma^{-1}$$

are the partial derivatives of P_n with respect to the variables $x_1', \ldots, x_n'$,

$$\partial_1' = \frac{\partial}{\partial x_1'}, \; \ldots, \; \partial_n' = \frac{\partial}{\partial x_n'}. \tag{9}$$

Every derivation $\partial \in D_n$ is a unique sum $\partial = \sum_{i=1}^n a_i\partial_i$ where $a_i = \partial * x_i \in P_n$. Let $\partial := (\partial_1, \ldots, \partial_n)^T$ and $\partial' := (\partial_1', \ldots, \partial_n')^T$ where T stands for the transposition. Then

$$\partial' = J(\sigma)^{-1}\partial, \;\; \text{i.e. } \partial_i' = \sum_{j=1}^n (J(\sigma)^{-1})_{ij}\partial_j \;\; \text{for } \; i = 1, \ldots, n. \tag{10}$$

In more detail, if $\partial' = A\partial$ where $A = (a_{ij}) \in M_n(P_n)$, i.e. $\partial_i = \sum_{j=1}^n a_{ij}\partial_j$. Then for all $i, j = 1, \ldots, n$,

$$\delta_{ij} = \partial_i' * x_j' = \sum_{k=1}^n a_{ik}\frac{\partial x_j'}{\partial x_k}$$

where δ_{ij} is the Kronecker delta function. The equalities above can be written in the matrix form as $AJ(\sigma) = 1$ where 1 is the identity matrix. Therefore, $A = J(\sigma)^{-1}$.

Suppose that a group G acts on a set S. For a nonempty subset T of S, $\mathrm{St}_G(T) := \{g \in G \,|\, gT = T\}$ is the *stabilizer* of the set T in G and $\mathrm{Fix}_G(T) := \{g \in G \,|\, gt = t \text{ for all } t \in T\}$ is the *fixator* of the set T in G. Clearly, $\mathrm{Fix}_G(T)$ is a *normal* subgroup of $\mathrm{St}_G(T)$.

The maximal abelian Lie subalgebra $\mathcal{D}_n$ of D_n.

Lemma 2.3 *1. $C_{D_n}(\mathcal{D}_n) = \mathcal{D}_n$ and so $\mathcal{D}_n$ is a maximal abelian Lie subalgebra of D_n.*

2. $\mathrm{Fix}_{G_n}(\mathcal{D}_n) = \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$.

3. D_n is a faithful G_n-module, i.e. the group homomorphism $G_n \to \mathbb{G}_n$, $\sigma \mapsto \sigma : \partial \mapsto \sigma\partial\sigma^{-1}$, is a monomorphism.

4. $\mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{e\}$.

Proof. 1. Statement 1 follows from (2).

2. Let $\sigma \in \mathrm{Fix}_{G_n}(D_n)$ and $J(\sigma) = (J_{ij})$. By (10), $\partial = J(\sigma)\partial$, and so, for all $i, j = 1, \ldots, n$, $\delta_{ij} = \partial_i * x_j = J_{ij}$, i.e. $J(\sigma) = 1$, or equivalently, by (8),

$$x_1' = x_1 + \lambda_1, \ldots, x_n' = x_n + \lambda_n$$

for some scalars $\lambda_i \in K$, and so $\sigma \in \mathrm{Sh}_n$.

3 and 4. Let $\sigma \in \mathrm{Fix}_{G_n} = (\partial_1, \ldots, \partial_n, H_1, \ldots, H_n)$. Then $\sigma \in \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$, by statement 2. So, $\sigma(x_1) = x_1 + \lambda_1, \ldots, \sigma(x_n) = x_n + \lambda_n$ where $\lambda_i \in K$. Then $x_i\partial_i = \sigma(x_i\partial_i) = (x_i + \lambda_i)\partial_i$ for $i = 1, \ldots, n$, and so $\lambda_1 = \cdots = \lambda_n = 0$. This means that $\sigma = e$. So, $\mathrm{Fix}_{G_n} = (\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{e\}$ and D_n is a faithful G_n-module. $\square$

By Lemma 2.3.(3), we identify the group G_n with its image in $\mathbb{G}_n$.

Lemma 2.4 *1. D_n is a simple Lie algebra.*

2. $Z(D_n) = \{0\}$.

3. $[D_n, D_n] = D_n$.

Proof. 1. Let $0 \neq a \in D_n$ and $\mathfrak{a} = (a)$ be the ideal of the Lie algebra D_n generated by the element a. We have to show that $\mathfrak{a} = D_n$. Using the inner derivations $\delta_1, \ldots, \delta_n$ we see that $\partial_i \in \mathfrak{a}$ for some i. Then $\mathfrak{a} = D_n$ since

$$x^\alpha \partial_j = (\alpha_i + 1)^{-1}[\partial_i, x^{\alpha+e_i}\partial_j] \in \mathfrak{a}$$

for all α and j.

2 and 3. Statements 2 and 3 follow from statement 1. $\square$

Proposition 2.5 *1.* $\mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{e\}$.

2. Let $\sigma, \tau \in \mathbb{G}_n$. Then $\sigma = \tau$ iff $\sigma(\partial_i) = \tau(\partial_i)$ and $\sigma(H_i) = \tau(H_i)$ for $i = 1, \ldots, n$.

3. $\mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$.

Proof. 1. Let $\sigma \in F := \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n)$. We have to show that $\sigma = e$. Since $\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(H_1, \ldots, H_n)$, the automorphism σ respects the weight decomposition of D_n. By (7), $\sigma(x^\alpha \partial_i) = \lambda_{\alpha,i} x^\alpha \partial_i$ for all $\alpha \in \mathbb{N}^{n,i}$ and $i = 1, \ldots, n$ where $\lambda_{\alpha,i} \in K$. Clearly, $\lambda_{0,i} = 1$ for $i = 1, \ldots, n$. Since $\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$, by applying σ to the relations $\alpha_j x^{\alpha - e_j}\partial_i = [\partial_j, x^\alpha \partial_i]$, we get the relations

$$\alpha_j \lambda_{\alpha - e_j, i} x^{\alpha - e_j} \partial_i = [\partial_j, \lambda_{\alpha,i} x^\alpha \partial_i] = \alpha_j \lambda_{\alpha,i} x^{\alpha - e_j} \partial_i.$$

Hence $\lambda_{\alpha,i} = \lambda_{\alpha - e_j, i}$ provided $\alpha_j \neq 0$. We conclude that all the coefficients $\lambda_{\alpha,i}$ are equal to one of the coefficients $\lambda_{e_i,j}$ where $i, j = 1, \ldots, n$ and $i \neq j$. The relations $\partial_j = [\partial_i, x_i \partial_j]$ implies the relations $\partial_j = [\partial_i, \lambda_{e_i,j} x_i \partial_j] = \lambda_{e_i,j}\partial_j$, hence all the coefficients $\lambda_{e_i,j}$ are equal to 1. So, $\oplus_{i=1}^n P_n^{\partial_i} \partial_i \subseteq \mathcal{F} := \mathrm{Fix}_{D_n}(\sigma) := \{\partial \in D_n \mid \sigma(\partial) = \partial\}$. To finish the proof of statement 1 it suffices to show that $x^\alpha H_i \in \mathcal{F}$ for all $\alpha \in \mathbb{N}^n$ and $i = 1, \ldots, n$, see (5) and (6). We use induction on $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $|\alpha| = 0$ the statement is obvious as $\sigma \in F$. Suppose that $|\alpha| > 0$. Using the commutation relations

$$[\partial_j, x^\alpha H_i] = \begin{cases} \alpha_j x^{\alpha - e_j} H_i & \text{if } j \neq i, \\ (\alpha_i + 1) x^\alpha \partial_i & \text{if } j = i, \end{cases} \tag{11}$$

the induction and the previous case, we see that

$$[\partial_j, \sigma(x^\alpha H_i) - x^\alpha H_i] = 0 \ \text{ for } \ i = 1, \ldots, n.$$

Therefore, $\sigma(x^\alpha H_i) - x^\alpha H_i \in C_{D_n}(\mathcal{D}_n) = \mathcal{D}_n$. Since the automorphism σ respects the weight decomposition of D_n, we must have $\sigma(x^\alpha H_i) - x^\alpha H_i \in x^\alpha \mathcal{H}_n \cap \mathcal{D}_n = \{0\}$. Hence, $x^\alpha H_i \in \mathcal{F}$, as required.

2. Statement 2 follows from statement 1.

3. Clearly, $\mathrm{Sh}_n \subseteq F = \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$. Let $\sigma \in F$ and $H_i' := \sigma(H_i), \ldots, H_n' := \sigma(H_n)$. Applying the automorphism σ to the commutation relations $[\partial_i, H_j] = \delta_{ij}\partial_i$ gives the relations $[\partial_i, H_j'] = \delta_{ij}\partial_i$. By taking the difference, we see that $[\partial_i, H_j' - H_j] = 0$ for all i and j. Therefore, $H_i' = H_i + d_i$ for some elements $d_i \in C_{D_n}(\mathcal{D}_n) = \mathcal{D}_n$ (Lemma 2.3.(1)), and so $d_i = \sum_{j=1}^n \lambda_{ij}\partial_j$ for some elements $\lambda_{ij} \in K$. The elements $H_1', \ldots, H_n'$ commute, hence

$$[H_j, \partial_i] = [H_i, \partial_j] \ \text{ for all } \ i, j,$$

or equivalently,

$$\lambda_{ij}\partial_j = \lambda_{ji}\partial_i \text{ for all } i, j.$$

This means that $\lambda_{ij} = 0$ for all $i \neq j$, i.e.

$$H'_i = H_i + \lambda_{ii}\partial_i = (x_i + \lambda_{ii})\partial_i = s_\lambda(H_i)$$

where $s_\lambda \in \mathrm{Sh}_n$, $s_\lambda(x_i) = x_i + \lambda_{ii}$ for all i. Then $s_\lambda^{-1}\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \dots, \partial_n, H_1, \dots, H_n) = \{e\}$ (statement 2), and so $\sigma = s_\lambda \in \mathrm{Sh}_n$. $\square$

Lemma 2.6 *Let $\sigma \in \mathbb{G}_n$ and $\partial'_1 := \sigma(\partial_1), \dots, \partial'_n := \sigma(\partial_n)$. Then*

1. *$\partial'_1, \dots, \partial'_n$ are commuting, locally nilpotent derivations of P_n.*
2. *$\bigcap_{i=1}^n \ker_{D_n}(\partial'_i) = K$.*

Proof. 1. The derivations $\partial'_1, \dots, \partial'_n$ commute since $\partial_1, \dots, \partial_n$ are commute. The inner derivations $\delta_1, \dots, \delta_n$ of the Lie algebra D_n are commuting and locally nilpotent. Hence, inner derivations

$$\delta'_1 := \mathrm{ad}(\partial'_1), \dots, \delta'_n := \mathrm{ad}(\partial'_n)$$

of the Lie algebra D_n are commuting and locally nilpotent. The vector space $P_n\partial'_i$ is closed under the derivations δ'_j since

$$\delta'_j(P_n\partial'_i) = [\partial'_j, P_n\partial'_i] = (\partial'_j * P_n) \cdot \partial'_i \subseteq P_n\partial'_i.$$

Therefore, $\partial'_1, \dots, \partial'_n$ are locally nilpotent derivations of the polynomial algebra P_n.

2. Let $\lambda \in \bigcap_{i=1}^n \ker_{P_n}(\partial'_i)$. Then

$$\lambda\partial'_1 \in C_{D_n}(\partial'_1, \dots, \partial'_n) = \sigma(C_{D_n}(\partial_1, \dots, \partial_n)) = \sigma(C_{D_n}(\mathcal{D}_n)) = \sigma(\mathcal{D}_n) = \sigma(\bigoplus_{i=1}^n K\partial_i) = \bigoplus_{i=1}^n K\partial'_i,$$

since $C_{D_n}(\mathcal{D}_n) = \mathcal{D}_n$, Lemma 2.3.(1). Then $\lambda \in K$ since otherwise the infinite dimensional space $\bigoplus_{i\geq 0} K\lambda^i\partial'_1$ would be a subspace of a finite dimensional space $\sigma(\mathcal{D}_n)$. $\square$

The following lemma is well-known and it is easy to prove.

Lemma 2.7 *Let ∂ be a locally nilpotent derivation of a commutative K-algebra A such that $\partial(x) = 1$ for some element $x \in A$. Then $A = A^\partial[x]$ is a polynomial algebra over the ring $A^\partial := \ker(\partial)$ of constants of the derivation ∂ in the variable x.*

The next theorem is the most important point in the proof of Theorem 1.1 and, roughly speaking, the main reason why Theorem 1.1 holds.

Theorem 2.8 *Let $\partial'_1, \dots, \partial'_n$ be commuting, locally nilpotent derivations of the polynomial algebra P_n such that $\bigcap_{i=1}^n \ker_{P_n}(\partial'_i) = K$. Then there exist polynomials $x'_1, \dots, x'_n \in P_n$ such that*

$$\partial'_i * x'_j = \delta_{ij}. \quad (12)$$

Moreover, the algebra homomorphism

$$\sigma : P_n \to P_n, \quad x_1 \mapsto x'_1, \dots, x_n \mapsto x'_n$$

is an automorphism such that $\partial'_i = \sigma\partial_i\sigma^{-1} = \frac{\partial}{\partial x'_i}$ for $i = 1, \dots, n$.

Proof. Case $n = 1$: By Lemma 2.6, the derivation ∂'_1 of the polynomial algebra P_1 is a locally nilpotent derivation with $K'_1 := \ker_{P_1}(\partial'_1) = K$. Hence, $\partial'_1 * x'_1 = 1$ for some polynomial $x'_1 \in P_1$. By Lemma 2.7, $K[x_1] = K'_1[x'_1] = K[x'_1]$, and so $\sigma : K[x_1] \to K[x_1]$, $x \mapsto x'_1$, is an automorphism such that $\partial'_1 = \frac{d}{dx'_1} = \sigma \frac{d}{dx_1} \sigma^{-1}$.

Case $n \geq 2$. Let $K'_i := \ker_{P_n}(\partial'_i)$ for $i = 1, \ldots, n$. Clearly, $K \subseteq K'_i$.

(i) $K'_i \neq K$ *for* $i = 1, \ldots, n$: If $K'_i = K$ for some i then by the same argument as in the case $n = 1$ there exists a polynomial $x'_i \in P_n$ such that $\partial'_i * x'_i = 1$, and so $P_n = K'_i[x'_i] = K[x_i]$, a contradiction.

(ii) Let m be the maximum of card(I) such $\emptyset \neq I \subseteq \{1, \ldots, n-1\}$ and $\bigcap_{i \in I} K'_i \neq K$. By (i), $2 \leq m \leq n-1$. Changing (if necessary) the order of the derivations $\partial'_1, \ldots, \partial'_n$ we may assume that $A := \bigcap_{i=1}^m K'_i \neq K$. Then the algebra A is infinite dimensional (since $K \neq A \subseteq P_n$) and invariant under the action of the derivations ∂'_j for $j = m+1, \ldots, n$. By the choice of m,

$$A^{\partial'_j} = K'_j \cap \bigcap_{i=1}^m K'_i = K \ \text{ for } \ j = m+1, \ldots, n$$

and the derivations ∂'_j acts locally nilpotently on the algebra $A^{\partial'_j}$. Therefore, for each index $j = m+1, \ldots, n$, there exists an element $x'_j \in A$ such that $\partial'_j * x'_j = 1$, and so (Lemma 2.7)

$$A = A^{\partial'_j}[x'_j] = K[x'_j] \ \text{ for } \ j = m+1, \ldots, n. \tag{13}$$

(ii)(a) Suppose that $m = n-1$, i.e. $\partial'_i * x'_n = \delta_{in}$ for all $i = 1, \ldots, n$. By Lemma 2.7, $P_n = K'_n[x'_n]$. The algebra K'_n admits the set of commuting, locally nilpotent derivations

$$\partial''_1 := \partial'_1|_{K'_n}, \ldots, \partial''_{n-1} := \partial'_{n-1}|_{K'_n}$$

with $\bigcap_{i=1}^{n-1} \ker_{K'_n}(\partial''_i) = K'_n \cap \bigcap_{i=1}^{n-1} K'_i = K$.

(ii)(b) Suppose that $m < n-1$. By (13),

$$K^* x'_{m+1} + K = K^* x'_{m+2} + K = \cdots = K^* x'_n + K,$$

and so $\lambda_j := \partial'_j * x'_n \in K$ for $j = m+1, \ldots, n-1$. Hence, $(\partial'_j - \lambda_j \partial'_n) * x'_n = 0$ for $j = m+1, \ldots, n-1$. A linear combination of commuting, locally nilpotent derivations is a locally nilpotent derivation (the proof boils down to the case $\partial + \delta$ of two commuting, locally nilpotent derivations, then the result follows from $(\partial + \delta)^m = \sum_{i=0}^m \binom{m}{i} \partial^i \delta^{m-i}$ and $\partial^i \delta^{m-i} = \delta^{m-i} \partial^i$). Using the set of commuting, locally nilpotent derivations $\partial'_1, \ldots, \partial'_n$ that satisfy (12) we obtain the set of commuting, locally nilpotent derivations

$$\delta'_1 := \partial'_1, \ \ldots, \ \delta'_m := \partial'_m, \ \delta'_{m+1} := \partial'_{m+1} - \lambda_{m+1} \partial'_n, \ \ldots, \ \delta'_{n-1} := \partial'_{n-1} - \lambda_{n-1} \partial'_n, \ \delta'_n := \partial_n$$

that satisfy (12) with

$$\delta'_i * x'_n = \delta_{in} \ \text{ for } \ i = 1, \ldots, n.$$

Then repeating the arguments of (ii)(a), we see that $P_n = K'_n[x'_n]$. The algebra K'_n admits the set of commuting, locally nilpotent derivations

$$\partial''_1 := \delta'_1|_{K'_n}, \ \ldots, \ \partial''_{n-1} := \delta'_{n-1}|_{K'_n}$$

with

$$\bigcap_{i=1}^{n-1} \ker_{K'_n}(\partial''_i) = K'_n \cap \bigcap_{i=1}^{n-1} \ker_{P_n}(\delta'_i) = K'_n \cap \bigcap_{i=1}^{n-1} \ker_{P_n}(\partial'_i) = \bigcap_{i=1}^{n} K'_i = K.$$

(iii) Using the cases (ii)(a) and (ii)(b) $n-1$ more times we find polynomials $x'_1, \ldots, x'_n$ and commuting set of locally nilpotent derivations of P_n, say, $\Delta_1, \ldots, \Delta_n$ that satisfy (12) and such that

(α) $\Delta_i * x'_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$;

(β) the n-tuple of derivations $\Delta = (\Delta_1, \ldots, \Delta_n)^T$ is obtained from the n-tuple of derivations $\partial' = (\partial'_1, \ldots, \partial'_n)^T$ by unitriangular (hence invertible) scalar matrix $\Lambda = (\lambda_{ij}) \in M_n(K)$ such that $\Delta = \Lambda\partial'$; and

(γ) (where $K''_1 := \ker_{P_n}(\Delta_1), \ldots, K''_n := \ker_{P_n}(\Delta_n)$)

$$\begin{aligned} P_n &= K''_n[x'_n] = (K''_{n-1} \cap K''_n)[x'_{n-1}, x'_n] = \cdots = (\bigcap_{i=s}^{n} K''_i)[x'_s, \ldots, x'_n] = \cdots \\ &= (\bigcap_{i=1}^{n} K''_i)[x'_1, \ldots, x'_n] = K[x'_1, \ldots, x'_n]. \end{aligned}$$

(iv) Replacing the row $x' = (x'_1, \ldots, x'_n)$ by the row $x'\Lambda$ gives the required elements of the theorem. Indeed, by (α), $\Lambda \cdot (\partial'_i * x'_j) = 1$, the identity $n \times n$ matrix. Hence, $(\partial'_i * x'_j) \cdot \Lambda = 1$, as required.

(v) Let $x'_1, \ldots, x'_n$ be the set of polynomials as in the theorem. Then σ is an algebra automorphism (see (γ) and (iv)) such that $\partial'_i = \sigma\partial_i\sigma^{-1} = \frac{\partial}{\partial x'_i}$ for $i = 1, \ldots, n$. $\Box$

Corollary 2.9 *Let $\sigma \in \mathbb{G}_n$. Then $\tau\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$ for some $\tau \in G_n$.*

Proof. By Lemma 2.6, the elements $\partial'_1 := \sigma(\partial_1), \ldots, \partial'_n := \sigma(\partial_n)$ satisfy the assumptions of Theorem 2.8. By Theorem 2.8, $\partial'_1 := \tau^{-1}(\partial_1), \ldots, \partial'_n := \tau^{-1}(\partial_n)$ for some $\tau \in G_n$. Therefore, $\tau\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$. $\Box$

Proof of Theorem 1.1. Let $\sigma \in \mathbb{G}_n$. By Corollary 2.9, $\tau\sigma \in \mathrm{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$ (Proposition 2.5.(3)). Therefore, $\sigma \in G_n$, i.e. $\mathbb{G}_n = G_n$. $\Box$

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