Article:

Bavula, V.V. and Lu, T. (2017) The quantum euclidean algebra and its prime spectrum. Israel Journal of Mathematics, 219 (2). pp. 929-958. ISSN 0021-2172

https://doi.org/10.1007/s11856-017-1503-1

The quantum Euclidean algebra and its prime spectrum

V. V. Bavula and T. Lu

Abstract

For the quantum Euclidean algebra, its prime, completely prime and maximal spectra are described (together with inclusions of prime ideals). The centre is generated by two algebraically independent elements (one is quadratic and the other is cubic).

Key Words: Prime ideal, maximal ideal, completely prime ideal, quantum algebra.
Mathematics subject classification 2010: 16D25, 16D70, 16P50, 17B37.

Contents

1 Introduction

Notation. Throughout this paper, $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of non-negative integers. We use the abbreviation $[n] := \frac{q^n-q^{-n}}{q-q^{-1}}$ where $n \in \mathbb{Z}$. In particular, we have $[2] = q + q^{-1}$. Set $\varrho := q^2 - 1$.

The, so-called, '*quantum Euclidean n-space*' appeared over two decades ago. These noncommutative Noetherian algebras are natural quantizations of the coordinate algebra of Euclidean n-space. Much is known about these and related algebras, see the book of Brown and Goodearl [9] for details. Also there are plenty of open questions.

The universal enveloping algebras of semisimple or solvable Lie algebras are relatively well studied comparing to the general/mixed case where a Lie algebra is a skew product of a semisimple Lie algebra and a solvable one. Technique for study such algebras is not yet developed. The same is true for quantizations of universal enveloping algebras and quantum groups. The quantum Euclidean algebra is one of the most basic (but very nontrivial) example of a quantization of the general/mixed case. For almost any noncommutative algebra, a problem of giving explicit description of its prime ideals together with complete picture of inclusions/noninclusions of prime ideals is a challenging one. In the present paper, an answer is given to this problem for the quantum Euclidean algebra (the prime spectrum has complex structure) and a technique is developed that might be useful in studying similar algebras of 'small' Gelfand-Kirillov dimensions.

Fix a field $\mathbb{K}$ of characteristic zero, unless specified otherwise; and an element $q \in \mathbb{K}^*$ which we assume is not a root of unity. The *quantized enveloping algebra* of $\mathfrak{sl}_2$ is the $\mathbb{K}$-algebra $U_q(\mathfrak{sl}_2)$ with generators E, F, K, K^{-1} subject to the defining relations:

$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

There is a Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ defined by

$$\begin{aligned}
&\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(E) = E \otimes 1 + K \otimes E, && \varepsilon(E) = 0, && S(E) = -K^{-1}E,\\
&\Delta(F) = F \otimes K^{-1} + 1 \otimes F, && \varepsilon(F) = 0, && S(F) = -FK.
\end{aligned}$$

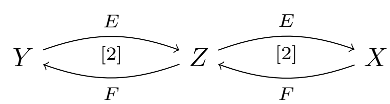
The 3-dimensional *quantum Euclidean space* $\mathbb{K}_q^3$ has been studied in several physical literature, see e.g. [10, 13]. We use different notation here. Recall that the $\mathbb{K}$-algebra $\mathbb{K}_q^3$ is generated by X, Z and Y subject to the following defining relations:

$$XZ = q^2 ZX, \quad YZ = q^{-2} ZY, \quad XY = YX + (q^2 - 1)Z^2.$$

We can make the quantum Euclidean space $\mathbb{K}_q^3$ a $U_q(\mathfrak{sl}_2)$-module algebra by defining

$$\begin{array}{lll} K \cdot X = q^2 X, & K \cdot Z = Z, & K \cdot Y = q^{-2} Y, \\ F \cdot X = Z, & F \cdot Z = Y, & F \cdot Y = 0, \\ E \cdot X = 0, & E \cdot Z = [2]X, & E \cdot Y = [2]Z. \end{array} \tag{1}$$

The actions are indicated in the following picture,



It is straightforward to verify that the actions (1) define on $\mathbb{K}_q^3$ a $U_q(\mathfrak{sl}_2)$-module algebra structure. Then one can form the smash product algebra $\mathcal{E} := \mathbb{K}_q^3 \rtimes U_q(\mathfrak{sl}_2)$. We call this algebra the *quantum Euclidean algebra*. The defining relations for this algebra are given below. Our aim is to study the structure of this algebra and its representation theory.

Definition. The *quantum Euclidean algebra* is the algebra $\mathcal{E}$ generated over $\mathbb{K}$ by the elements K, K^{-1}, E, F, X, Z and Y with defining relations

$$\begin{array}{lll} EK = q^{-2} KE, & FK = q^2 KF, & EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}, \\ XK = q^{-2} KX, & ZK = KZ, & YK = q^2 KY, \\ FX = XF + ZK^{-1}, & FZ = ZF + YK^{-1}, & FY = YF, \\ EX = q^2 XE, & EZ = ZE + [2]X, & EY = q^{-2} YE + [2]Z, \\ XZ = q^2 ZX, & YZ = q^{-2} ZY, & XY = YX + (q^2 - 1)Z^2. \end{array}$$

The aim of the paper is to classify the sets of prime, completely prime and maximal ideals of the algebra $\mathcal{E}$ (Theorem 3.9, Corollary 3.12 and Corollary 3.10, respectively). The centre of the algebra $\mathcal{E}$ is a polynomial algebra $\mathbb{K}[C_1, C_2]$ (Proposition 2.4.(1)) where the elements C_1 and C_2 are found explicitly. The algebra $\mathcal{E}$ is a free module over its centre (Proposition 2.7).

Notation. Set $[K; n] := \frac{q^n K - q^{-n} K^{-1}}{q - q^{-1}}$ for all $n \in \mathbb{Z}$. Then $EF - FE = [K; 0]$ and the following identities hold

$$\begin{array}{ll} E[K; n] = [K; n-2]E, & Y[K; n] = [K; n+2]Y, \\ F[K; n] = [K; n+2]F, & X[K; n] = [K; n-2]X. \end{array}$$

For any algebra A, we denote by $Z(A)$ the centre of A.

2 Ring theoretic properties of the algebra $\mathcal{E}$

In this section, it is proved that the centre $Z(\mathcal{E})$ of the algebra $\mathcal{E}$ is equal to $\mathbb{K}[C_1, C_2]$ (Proposition 2.4.(1)) and the algebra $\mathcal{E}$ is a free $Z(\mathcal{E})$-module (Proposition 2.7). Several important subalgebras and localizations of $\mathcal{E}$ are introduced, all of them turned out to be generalized Weyl algebras. They are instrumental in finding the centre of $\mathcal{E}$ and its prime spectrum (Section 3).

Generalized Weyl algebra. *Definition*, [1, 2, 3]. Let D be a ring, σ be an automorphism of D and a is an element of the centre of D. *The generalized Weyl algebra* $A := D(\sigma, a) := D[X, Y; \sigma, a]$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y \text{ for all } \alpha \in D, \quad YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded where $A_n = Dv_n$, $v_n = X^n$ for $n > 0$, $v_n = Y^{-n}$ for $n < 0$ and $v_0 = 1$. It follows from the above relations that $v_n v_m = (n,m)v_{n+m} = v_{n+m}\langle n,m\rangle$ for some $(n,m) \in D$. If $n > 0$ and $m > 0$ then

$$\begin{aligned} n \geq m: &\quad (n,-m) = \sigma^n(a)\cdots\sigma^{n-m+1}(a), \;\; (-n,m) = \sigma^{-n+1}(a)\cdots\sigma^{-n+m}(a),\\ n \leq m: &\quad (n,-m) = \sigma^n(a)\cdots\sigma(a), \qquad\quad (-n,m) = \sigma^{-n+1}(a)\cdots a, \end{aligned}$$

in other cases $(n,m) = 1$. Clearly, $\langle n,m\rangle = \sigma^{-n-m}((n,m))$.

Definition, [5]. Let D be an ring and σ be its automorphism. Suppose that elements b and ρ belong to the centre of the ring D, ρ is invertible and $\sigma(\rho) = \rho$. Then $E := D\langle\sigma; b, \rho\rangle := D\langle X, Y; \sigma, b, \rho\rangle$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \;\text{ and }\; Y\alpha = \sigma^{-1}(\alpha)Y \;\text{ for all }\; \alpha \in D, \;\text{ and }\; XY - \rho YX = b.$$

The origin of this construction stems from the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$. When we rewrite the defining relations of $Usl(2)$ (where $[a,b] = ab-ba$): $[H,X] = X$, $[H,Y] = -Y$ and $[X,Y] = 2H$ in the equivalent form: $XH = (H-1)X$, $YH = (H+1)Y$ and $XY - YX = 2H$ and notice that $XH = \sigma(H)X$ and $YH = \sigma^{-1}(H)Y$ where σ is an automorphism of the polynomial algebra $K[H]$ given by $\sigma(H) = H - 1$ we come to $Usl(2) = K[H]\langle X, Y; \sigma, 2H\rangle$. The next natural step was to replace the polynomial $2H$ by an arbitrary polynomial $a(H) \in K[H]$. This was done independently in [15] and [1]. That is how the, so-called, *algebras similar to* $Usl(2)$ appeared. It is the algebra $K\langle X, Y, H\rangle$ that satisfies the defining relations:

$$XH = (H-1)X, \;\; YH = (H+1)Y \;\text{ and }\; XY - YX = a(H) \;\text{ where }\; a(H) \in K[H].$$

In 90s, there were many examples like this, various 'quantum deformations' of $Usl(2)$, with a ring D which is a 'small' commutative ring.

If D is commutative domain, $\rho = 1$ and $b = u - \sigma(u)$ for some $u \in D$ (resp., if D is a commutative finitely generated domain over a field K and $\rho \in K^*$) the algebras E were considered in [11] (resp., [12]).

The ring E is *the iterated skew polynomial ring* $E = D[Y;\sigma^{-1}][X;\sigma,\partial]$ where ∂ is the $\sigma-$derivation of $D[Y;\sigma^{-1}]$ such that $\partial D = 0$ and $\partial Y = b$ (here the automorphism σ is extended from D to $D[Y;\sigma^{-1}]$ by the rule $\sigma(Y) = \rho Y$).

An element d of a ring D is *normal* if $dD = Dd$. The next proposition shows that the rings E are GWAs and under a certain (mild) conditions they have a 'canonical' normal element.

Proposition 2.1. *Let* $E = D[X, Y; \sigma, b, \rho]$. *Then*

1. *[5, Lemma 1.2] The ring* E *is the GWA* $D[H][X, Y; \sigma, H]$ *where* $\sigma(H) = \rho H + b$.
2. *[5, Lemma 1.3] The following statements are equivalent:*
   (a) *[5, Corollary 1.4]* $C = \rho(YX + \alpha) = XY + \sigma(\alpha)$ *is a normal element in* E *for some central element* $\alpha \in D$,
   (b) $\rho\alpha - \sigma(\alpha) = b$ *for some central element* $\alpha \in D$.
3. *[5, Corollary 1.4] If one of the equivalent conditions of statement 2 holds then the ring* $E = D[C][X, Y; \sigma, a = \rho^{-1}C - \alpha]$ *is a GWA where* $\sigma(C) = \rho C$.

The next proposition is a corollary of Proposition 2.1 when $\rho = 1$. The rings E with $\rho = 1$ admit a 'canonical' central element (under a mild condition). This proposition is a key one for this paper and is used on many occasions to produce central elements. In the present paper a full generality of the construction is needed, i.e. when the base ring D is *noncommutative*.

Proposition 2.2. *Let* $E = D[X, Y; \sigma, b, \rho = 1]$. *Then*

1. *[5, Lemma 1.5] The following statements are equivalent:*
   (a) $C = YX + \alpha = XY + \sigma(\alpha)$ *is a central element in* E *for some central element* $\alpha \in D$,
   (b) $\alpha - \sigma(\alpha) = b$ *for some central element* $\alpha \in D$.
2. *[5, Corollary 1.6] If one of the equivalent conditions of statement 2 holds then the ring* $E = D[C][X, Y; \sigma, a = C - \alpha]$ *is a GWA where* $\sigma(C) = C$.

If D is commutative the implication $(b) \Rightarrow (a)$ also appeared in [12].

An involution $*$ of $\mathcal{E}$. Recall that an *involution* $*$ on an algebra A is a $\mathbb{K}$-algebra anti-automorphism ($(ab)^* = b^*a^*$) such that $a^{**} = a$ for all elements $a \in A$. There is an involution $*$ of $\mathcal{E}$ defined by the rule:

$$\begin{aligned} F^* &= E, & K^* &= K, & E^* &= F, \\ Y^* &= -[2]KX, & Z^* &= Z, & X^* &= -\frac{1}{[2]}YK^{-1}. \end{aligned} \tag{2}$$

$\mathbb{K}$-basis of the algebra $\mathcal{E}$. Let $(x_1, \ldots, x_5)$ be any permutation of the elements (E, F, X, Y, Z). Then the set $\{K^i x^\alpha \mid i \in \mathbb{Z}, \alpha \in \mathbb{N}^5\}$ is a $\mathbb{K}$-basis of the algebra $\mathcal{E}$ (this follows from the defining relations of $\mathcal{E}$) where $x^\alpha = x_1^{\alpha_1} \cdots x_5^{\alpha_5}$. These $\mathbb{K}$-bases of $\mathcal{E}$ are called *standard*.

The 'central advance' method of finding the centre of $\mathcal{E}$ (and related algebras). When we have a complicated algebra, like $\mathcal{E}$, with many defining relations (15 for $\mathcal{E}$), and we want to find its centre, it is not obvious from where to start and in which direction to move. The philosophy/method we use in the paper in finding the centre of $\mathcal{E}$, the, so-called, *central advance*, can be summarized as follows. The algebra $\mathcal{E}$ is 'covered' by a chain of certain rather large subalgebras, they are GWA and have non-trivial central elements that are found explicitly by applying Proposition 2.2. At each step elements are getting complicated but the relations are getting simpler, they tend to be 'more commutative' (i.e., q-commutative or commutative). At the final stage, we consider the left localization $\mathcal{E}_Z$ of the algebra $\mathcal{E}$ at the powers of the element Z and using new generators of $\mathcal{E}$ that have been found in previous steps we find an additional central element C_2 of $\mathcal{E}_Z$ (using Proposition 2.2) turned out to be an element of $\mathcal{E}$.

The quantum Euclidean space $\mathbb{K}_q^3$ is a GWA. Clearly, $\mathbb{K}_q^3 = \mathbb{K}[Z]\langle X, Y; \sigma, b = (q^2-1)Z^2, \rho = 1\rangle$ where $\sigma(Z) = q^2 Z$. The polynomial $\alpha = -\frac{Z^2}{1+q^2}$ is a solution to the equation $\alpha - \sigma(\alpha) = b$. By Proposition 2.2, the algebra $\mathbb{K}_q^3$ is a GWA $\mathbb{K}_q^3 = \mathbb{K}[C, Z][X, Y; \sigma, a = C + \frac{Z^2}{1+q^2}]$ where the automorphism σ of the polynomial algebra $\mathbb{K}[C, Z]$ is given by the rule $\sigma(Z) = q^2 Z$ and $\sigma(C) = C$. The central element C of the algebra $\mathbb{K}_q^3$ can be written as $C = YX - \frac{Z^2}{1+q^2} = XY - \frac{q^4 Z^2}{1+q^2}$. The element

$$C_1 := -(1+q^2)C = Z^2 - (1+q^2)YX = q^4 Z^2 - (1+q^2)XY \tag{3}$$

is a central element of the the GWA

$$\mathbb{K}_q^3 = \mathbb{K}[C_1, Z][X, Y; \sigma, a = \frac{Z^2 - C_1}{1+q^2}] \tag{4}$$

where $\sigma(C_1) = C_1$ and $\sigma(Z) = q^2 Z$. The element C_1 commutes with E, F and K. Therefore, $C_1 \in Z(\mathcal{E})$. In fact, the centre of the algebra $\mathcal{E}$ is a polynomial algebra $\mathbb{K}[C_1, C_2]$ (Proposition 2.4.(1)).

The algebra $\mathbb{E}$ is a GWA. The algebra $\mathbb{E}$ is a subalgebra of $\mathcal{E}$ that is generated by the elements E, Z and X. Then $\mathbb{E} = \mathbb{K}[X]\langle E, Z; \sigma, b = [2]X, \rho = 1\rangle$ where $\sigma(X) = q^2 X$. The element $\alpha = -\varrho^{-1}[2]X \in \mathbb{K}[X]$ is a solution to the equation $\alpha - \sigma(\alpha) = b$. By Proposition 2.2, the algebra $\mathbb{E}$ is the GWA $\mathbb{E} = \mathbb{K}[C, X][E, Z; \sigma, a = C + \varrho^{-1}[2]X]$ where $\sigma(C) = C$, the element $C = ZE - \varrho^{-1}[2]X = EZ - \varrho^{-1}q^2[2]X \in Z(\mathbb{E})$. Hence,

$$\Omega := \varrho C = \varrho ZE - [2]X = \varrho EZ - q^2[2]X \in Z(\mathbb{E}) \tag{5}$$

is a central element of the GWA

$$\mathbb{E} = \mathbb{K}[\Omega, X][E, Z; \sigma, a = \varrho^{-1}(\Omega + [2]X)] \tag{6}$$

where $\sigma(\Omega) = \Omega$ and $\sigma(X) = q^2 X$. Using the fact that the element Ω is a central element of the algebra $\mathbb{E}$ and some of the defining relations of $\mathcal{E}$, we have the following equalities

$$\begin{aligned} &E\Omega = \Omega E, \quad X\Omega = \Omega X, \quad Z\Omega = \Omega Z, \quad Y\Omega = \Omega Y, \\ &\Omega K = q^{-2}K\Omega, \quad F\Omega = \Omega F + \varrho EYK^{-1} - q(K + q^2K^{-1})Z. \end{aligned} \tag{7}$$

The algebra $\mathbb{F} := \mathbb{E}^*$ is a GWA. Recall that $E^* = F$, $Z^* = Z$ and $X^* = -\frac{1}{[2]}YK^{-1}$. Let

$$\varphi := \Omega^* = \varrho ZF + q^2 YK^{-1} = \varrho FZ + YK^{-1}. \tag{8}$$

By (6), the algebra $\mathbb{F}$ is the GWA

$$\mathbb{F} = \mathbb{K}[\varphi, YK^{-1}][F, Z; \sigma_1, a_1 = \varrho^{-1}(\varphi - q^2 YK^{-1})] \tag{9}$$

where $\sigma_1(\varphi) = \varphi$ and $\sigma_1(YK^{-1}) = q^{-2}YK^{-1}$ and $a_1 := ZF = (EZ)^* = \sigma(a)^* = \varrho^{-1}(\Omega + q^2[2]X)^* = \varrho^{-1}(\varphi - q^2 YK^{-1})$. By applying the involution $*$ to the equalities in (7), we have

$$\begin{aligned} &\varphi F = F\varphi, \quad \varphi Y = q^2 Y\varphi, \quad \varphi Z = Z\varphi, \quad \varphi X = q^{-2}X\varphi, \\ &K\varphi = q^{-2}\varphi K, \quad \varphi E = E\varphi - \varrho[2]FX - (q^{-1}K^{-1} + qK)Z. \end{aligned} \tag{10}$$

The algebras R and R_Z are GWAs. By (4), the algebra $R := \mathbb{K}_q^3[K^{\pm 1}; \tau]$ where $\tau(C_1) = C_1, \tau(Z) = Z, \tau(X) = q^2 X$ and $\tau(Y) = q^{-2}Y$ is the GWA

$$R = \mathbb{K}[C_1, Z, K^{\pm 1}][X, Y; \sigma, a = \frac{Z^2 - C_1}{1 + q^2}] \tag{11}$$

where $\sigma(C_1) = C_1, \sigma(Z) = q^2 Z$ and $\sigma(K) = q^{-2}K$. Let R_Z be the localization of the algebra R at the powers of the element Z. By (11), the algebra R_Z is the GWA

$$R_Z = \mathbb{K}[C_1, Z^{\pm 1}, K^{\pm 1}][X, Y; \sigma, a = \frac{Z^2 - C_1}{1 + q^2}]. \tag{12}$$

The element $\Theta := KZ$ belongs to the centre of the algebra R_Z and

$$R_Z = \mathbb{K}[\Theta^{\pm 1}] \otimes \mathbb{K}[C_1, Z^{\pm 1}][X, Y; \sigma, a = \frac{Z^2 - C_1}{1 + q^2}] \tag{13}$$

is the tensor product of algebras. The centre of the second tensor multiple is $\mathbb{K}[C_1]$. Therefore, the centre of the algebra R_Z is equal to

$$Z(R_Z) = \mathbb{K}[\Theta^{\pm 1}, C_1]. \tag{14}$$

Lemma 2.3.

1. *$R_X = \mathbb{K}[C_1] \otimes \mathbb{K}[Z, K^{\pm 1}][X^{\pm 1}; \sigma]$ and $R_Y = \mathbb{K}[C_1] \otimes \mathbb{K}[Z, K^{\pm 1}][Y^{\pm 1}; \sigma^{-1}]$ where $\sigma(Z) = q^2 Z$ and $\sigma(K) = q^{-2}K$.*
2. *$R_{X,Z} = \mathbb{K}[C_1, \Theta^{\pm 1}] \otimes \mathbb{K}[K^{\pm 1}][X^{\pm 1}; \sigma]$ and $R_{Y,Z} = \mathbb{K}[C_1, \Theta^{\pm 1}] \otimes \mathbb{K}[K^{\pm 1}][Y^{\pm 1}; \sigma^{-1}]$ where $\sigma(K) = q^{-2}K$.*
3. *$Z(R) = Z(R_Y) = \mathbb{K}[C_1, \Theta]$, $Z(R_Z) = Z(R_{Y,Z}) = Z(R_{X,Z}) = \mathbb{K}[C_1, \Theta^{\pm 1}]$ (where $\Theta = KZ$).*

Proof. 1. Statement 1 follows from (11).

2. Statement 2 follows from statement 1.

3. Statement 3 follows from statement 2 and the fact that the centre of the algebras $\mathbb{K}[K^{\pm 1}][X^{\pm 1}; \sigma]$ and $\mathbb{K}[K^{\pm 1}][Y^{\pm 1}; \sigma^{-1}]$ is $\mathbb{K}$. □

The algebra $\mathcal{E}_Z$ is a GWA. Let us show that

$$\Omega\varphi - \varphi\Omega = \varrho(q^{-1}K^{-1}C_1 + qKZ^2). \tag{15}$$

Since $\Omega\varphi = \Omega(\varrho FZ + YK^{-1}) \overset{(7)}{=} \varrho\Big(F\Omega - \varrho EYK^{-1} + q(K + q^2K^{-1})Z\Big)Z + \Omega YK^{-1} = \varphi\Omega + q^{-2}\varrho(\Omega - \varrho EZ)YK^{-1} + \varrho q(K + q^2K^{-1})Z^2 = \varphi\Omega + \varrho\big(q^{-1}K^{-1}C_1 + qKZ^2\big)$, we get the equality (15), as required.

Let $\mathcal{E}_Z$ be the localization of the algebra $\mathcal{E}$ at the powers of the element Z. Let A be the subalgebra of $\mathcal{E}_Z$ generated by the algebra R_Z and the elements Ω and φZ^{-1}. In fact, $A = \mathcal{E}_Z$.

The inclusions $R_Z \subseteq A \subseteq \mathcal{E}_Z$ are obvious. In order to show that $A = \mathcal{E}_Z$, if suffices to show that $F, E \in A$. By (5), $E \in A$. By (8), $F \in A$. Therefore, $A = \mathcal{E}_Z$. Introducing the algebra $A = \mathcal{E}_Z$ is a key moment in finding the cubic central element C_2 of the algebra A, see below.

The element Z commutes with Ω and φ. Multiplying the equality (15) by Z^{-1} on the right we obtain the equality

$$\Omega \cdot \varphi Z^{-1} - \varphi Z^{-1} \cdot \Omega = \varrho(q^{-1}C_1K^{-1}Z^{-1} + qKZ) =: b. \tag{16}$$

The elements C_1 and KZ are central in R, hence $b \in Z(R_Z)$. In view of (16), the algebra $\mathcal{E}_Z$ can be written as

$$\mathcal{E}_Z = R_Z\langle \Omega, \varphi Z^{-1}; \sigma, b, \rho = 1\rangle \tag{17}$$

where $\sigma(K) = q^{-2}K$, $\sigma(X) = X$, $\sigma(Y) = Y$, $\sigma(Z) = Z$ and $\sigma(C_1) = C_1$ (see (7)). The central element $\alpha = -q^{-1}C_1K^{-1}Z^{-1} + q^3KZ$ of the algebra R_Z is a solution to the equation $\alpha - \sigma(\alpha) = b$. By Proposition 2.2, the algebra $\mathcal{E}_Z$ is a GWA $\mathcal{E}_Z = R_Z[C][\Omega, \varphi Z^{-1}; \sigma, a = C - \alpha]$ where $\sigma(C) = C$. The element $C = \varphi Z^{-1}\Omega + \alpha = \Omega\varphi Z^{-1} + \sigma(\alpha)$ is a central element of the algebra $\mathcal{E}_Z$. Hence, so is the element

$$C_2 := \varrho^{-1}C = \varrho FEZ - q^2[2]FX + EYK^{-1} + q^2[K;0]Z \in Z(\mathcal{E}_Z). \tag{18}$$

In more detail,

$$\begin{aligned} C &= \varphi Z^{-1}\Omega - q^{-1}C_1K^{-1}Z^{-1} + q^3KZ \\ &= (\varrho F + YK^{-1}Z^{-1})(\varrho EZ - q^2[2]X) - q^{-1}C_1K^{-1}Z^{-1} + q^3KZ \\ &= \varrho^2 FEZ - \varrho q^2[2]FX + \varrho EYK^{-1} - q^{-1}(q[2]YX + C_1)K^{-1}Z^{-1} + (q^3K - \varrho[2]K^{-1})Z \\ &= \varrho^2 FEZ - \varrho q^2[2]FX + \varrho EYK^{-1} + (q^3K - q^3K^{-1})Z. \end{aligned}$$

Hence, we get the expression (18) for $C_2 = \varrho^{-1}C$, as required. Since $C_2 \in \mathcal{E}$, it is automatically a central element in $\mathcal{E}$. So, the algebra $\mathcal{E}_Z$ is the GWA

$$\mathcal{E}_Z = R_Z[C_2][\Omega, \varphi Z^{-1}; \sigma, a = \varrho C_2 - \alpha] \tag{19}$$

where $\sigma(C_2) = C_2$. By (18), the central element C_2 can be rewritten in the following two forms

$$C_2 = F\Omega + EYK^{-1} + q^2[K;0]Z, \quad \text{and} \tag{20}$$

$$C_2 = E\varphi - q^2[2]FX + [K;0]Z. \tag{21}$$

Gelfand-Kirillov Conjecture and the centre of $\mathcal{E}$. By (19), the localization $\mathcal{E}_{Z,\Omega}$ of the GWA $\mathcal{E}_Z$ at the powers of the element Ω is the tensor product of algebras

$$\mathcal{E}_{Z,\Omega} = \mathbb{K}[C_2] \otimes R_Z[\Omega^{\pm 1}; \sigma] \tag{22}$$

where $\sigma(K) = q^{-2}K, \sigma(X) = X, \sigma(Y) = Y, \sigma(Z) = Z$ and $\sigma(C_1) = C_1$. By Lemma 2.3.(2),

$$R_{Z,Y} = \mathbb{K}[C_1, \Theta^{\pm 1}] \otimes \mathbb{Y} \quad \text{where } \mathbb{Y} := \mathbb{K}[Z^{\pm 1}][Y^{\pm 1}; \tau],\ \tau(Z) = q^{-2}Z. \tag{23}$$

Then by (19), the localization $\mathcal{E}_{Z,Y}$ of the GWA $\mathcal{E}_Z$ at the powers of the element Y is the tensor product of algebras

$$\mathcal{E}_{Z,Y} = \mathbb{Y} \otimes A \tag{24}$$

where $A := \mathbb{K}[C_1, C_2, \Theta^{\pm 1}][\Omega, \varphi Z^{-1}; \sigma, a = \varrho C_2 + q^{-1}C_1\Theta^{-1} - q^3\Theta]$ is a GWA where $\sigma(C_1) = C_1, \sigma(C_2) = C_2$ and $\sigma(\Theta) = q^{-2}\Theta$. By the very definition, $A \subseteq \mathcal{E}_Z$.

Combining these two results, the localization $\mathcal{E}_{Z,Y,\Omega}$ of the algebra $\mathcal{E}_{Z,Y}$ at the powers of the element Ω is the tensor product of algebras

$$\mathcal{E}_{Z,Y,\Omega} = \mathbb{K}[C_1, C_2] \otimes \mathbb{Y} \otimes \mathbb{T},\ \text{where } \mathbb{T} := \mathbb{K}[\Theta^{\pm 1}][\Omega^{\pm 1}; \sigma'],\ \sigma'(\Theta) = q^{-2}\Theta. \tag{25}$$

We have the following natural inclusion of algebras

$$
\begin{array}{ccccc}
 & & \mathcal{E}_{Z,Y,\Omega} & & \\
 & \nearrow & & \nwarrow & \\
\mathcal{E}_{Z,Y} & & & & \mathcal{E}_{Z,\Omega} \\
 & \nwarrow & & \nearrow & \\
 & & \mathcal{E}_Z & & \\
 & & \uparrow & & \\
 & & \mathcal{E} & &
\end{array}
\qquad (26)
$$

Recall that a $\mathbb{K}$-algebra A admitting a skew field of fractions $\mathrm{Frac}(A)$ is said to satisfy the *quantum Gelfand-Kirillov conjecture* if $\mathrm{Frac}(A)$ is isomorphic to a quantum Weyl field over a purely transcendental field extension of $\mathbb{K}$; see [9, II.10, p. 230]. By (25), the algebra $\mathcal{E}$ satisfies the Gelfand-Kirillov conjecture. We say that two elements $x, y \in A$ *q-commute* if there exists an integer $i \in \mathbb{Z}$ such that $xy = q^i yx$.

Proposition 2.4.

1. *$Z(\mathcal{E}) = Z(\mathcal{E}_{Z,Y}) = Z(\mathcal{E}_{Z,Y,\Omega}) = \mathbb{K}[C_1, C_2]$ is a polynomial ring.*
2. *The involution $*$ fixes $Z(\mathcal{E})$, i.e., $C_1^* = C_1$ and $C_2^* = C_2$.*

Proof. 1. Notice that both the algebras $\mathbb{Y}$ and $\mathbb{T}$ are central, simple, quantum torus. Then by (25), $Z(\mathcal{E}_{Z,Y,\Omega}) = Z(\mathbb{K}[C_1, C_2]) \otimes Z(\mathbb{Y}) \otimes Z(\mathbb{T}) = \mathbb{K}[C_1, C_2]$. Since $\mathbb{K}[C_1, C_2] \subseteq Z(\mathcal{E}_{Z,Y}) \subseteq Z(\mathcal{E}_{Z,\Omega,Y}) \cap \mathcal{E} = \mathbb{K}[C_1, C_2]$, and so $Z(\mathcal{E}_{Z,Y}) = \mathbb{K}[C_1, C_2]$. Similarly, since $\mathbb{K}[C_1, C_2] \subseteq Z(\mathcal{E}) \subseteq Z(\mathcal{E}_{Z,\Omega,Y}) \cap \mathcal{E} = \mathbb{K}[C_1, C_2]$, we have $Z(\mathcal{E}) = \mathbb{K}[C_1, C_2]$.

2. Clearly, $C_1^* = C_1$. By (19) and the fact that $\varphi = \Omega^*$, we have $C_2 = \varrho^{-1}(\Omega^* Z^{-1}\Omega + \alpha)$ where $\alpha = -q^{-1}C_1 K^{-1} Z^{-1} + q^3 KZ = \alpha^*$. Therefore, $C_2^* = C_2$ since $Z^* = Z$. □

Let D be a ring and σ be its automorphism. An ideal I of the ring D is called *σ-stable* if $\sigma(I) = I$. The ring D is called a *σ-simple* ring iff 0 and D are the only σ-stable ideals of D. An automorphism σ of D is called an *inner* automorphism if $\sigma(d) = udu^{-1}$ for all $d \in D$ and some unit u of D.

Theorem 2.5. *[6, Theorem 4.2] Let $A = D(\sigma, a)$ be a GWA. Then A is simple iff*

1. *a is a regular element in D (i.e., a is not a zero divisor),*
2. *D is a σ-simple ring,*
3. *no power of σ is an inner automorphism of D, and*
4. *$Da + D\sigma^i(a) = D$ for all $i \geqslant 1$.*

Lemma 2.6.

1. *The algebra $\mathcal{E}_{Z,Y} = \mathbb{Y} \otimes A$ is a tensor product of algebras where $\mathbb{Y} = \mathbb{K}[Z^{\pm 1}][Y^{\pm 1}; \tau]$ is a central simple algebra and $\tau(Z) = q^{-2}Z$, and $A = \mathbb{K}[C_1, C_2, \Theta^{\pm 1}][\Omega, \varphi Z^{-1}; \sigma, a = \varrho C_2 + q^{-1}C_1\Theta^{-1} - q^3\Theta]$ is a GWA and $\sigma(C_1) = C_1, \sigma(C_2) = C_2$ and $\sigma(\Theta) = q^{-2}\Theta$.*
2. *$Z(A) = \mathcal{Z} := \mathbb{K}[C_1, C_2]$.*
3. *Let $\mathcal{Z}_0 := \mathcal{Z} \setminus \{0\}$. Then the algebra $B := \mathcal{Z}_0^{-1}A = Q[\Theta^{\pm 1}][\Omega, \varphi Z^{-1}; \sigma, a]$ is a simple GWA where $Q := \mathbb{K}(C_1, C_2)$ is the field of rational functions in C_1 and C_2.*

Proof. 1. By (24), $\mathcal{E}_{Z,Y} = \mathbb{Y} \otimes A$ is the tensor product of algebras $\mathbb{Y}$ and A. The algebra $\mathbb{Y}$ is simple and central (since q is not a root of 1).

2. By Proposition 2.4.(1), $Z(\mathcal{E}_{Z,Y}) = \mathcal{Z}$. By statement 1, the algebra $\mathbb{Y}$ is central. So, $Z(\mathcal{E}_{Z,Y}) = Z(\mathbb{Y}) \otimes Z(A) = Z(A)$, i.e., $Z(A) = \mathcal{Z}$.

3. To prove simplicity of the algebra B we use Theorem 2.5. Conditions 1 and 3 of Theorem 2.5 are obvious. Since q is not a root of unity and $\sigma(\Theta) = q^{-2}\Theta$, the algebra $Q[\Theta^{\pm 1}]$ is σ-simple. The algebra $Q[\Theta^{\pm 1}]$ is a localization of the polynomial algebra $\mathbb{K}[C_1, C_2, \Theta]$ and the polynomial $\Theta a = -q^3\Theta^2 + \varrho C_2\Theta + q^{-1}C_1$ is irreducible in $\mathbb{K}[C_1, C_2, \Theta]$. Hence, a is an irreducible polynomial of the Laurent polynomial ring $Q[\Theta^{\pm 1}]$. In particular, the ideal (a) is a maximal ideal of $Q[\Theta^{\pm 1}]$. Clearly, the maximal ideals $\big(\sigma^i(a)\big) = \big(\varrho C_2 + q^{2i-1}C_1\Theta^{-1} - q^{-2i+3}\Theta\big)$, $i \geqslant 0$ are distinct. Therefore, condition 4 of Theorem 2.5 holds, and so A is a simple algebra. □

By (22), the algebra $\mathcal{E}_{Z,\Omega}$ is an iterated Ore extension that can be presented as follows

$$\mathcal{E}_{Z,\Omega} = \mathbb{K}[C_2] \otimes \mathbb{K}[K^{\pm 1}, Z^{\pm 1}][\Omega^{\pm 1}; \tau_1][Y; \tau_2][X; \tau_3, \delta'] \tag{27}$$

where $\tau_1(K) = q^{-2}K, \tau_1(Z) = Z$; $\tau_2(K) = q^2K, \tau_2(Z) = q^{-2}Z, \tau_2(\Omega) = \Omega$; $\tau_3(K) = q^{-2}K, \tau_3(Z) = q^2Z, \tau_3(\Omega) = \Omega, \tau_3(Y) = Y$; $\delta'(K) = \delta'(Z) = \delta'(\Omega) = 0$ and $\delta'(Y) = (q^2-1)Z^2$.

Proposition 2.7. *The algebra $\mathcal{E}$ is a free module over its centre and ${}_{Z(\mathcal{E})}\mathcal{E} = Z(\mathcal{E}) \oplus M$ for some left free $Z(\mathcal{E})$-module M.*

Proof. Let $\mathcal{F}$ be the degree filtration on $\mathcal{E}$ where $\deg(K^{\pm 1}) = 0$ and $\deg(E) = \deg(F) = \deg(X) = \deg(Y) = \deg(Z) = 1$. Then the associated graded algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ is generated by elements $E', F', K^{\pm 1}, X', Y'$ and Z' that satisfy the quadratic relations as in the definition of the algebra $\mathcal{E}$ but all the terms of $\mathcal{F}$-degree 1 (e.g., $\deg_{\mathcal{F}}(YK^{-1}) = 1$) must be deleted. So, all commutation relations of the algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ is of the type $ab = q^iba$ but the relation $X'Y' = Y'X' + (q^2-1)Z'^2$. Clearly, the images C_1' and C_2' of the elements C_1 and C_2 in $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ are equal to $C_1' = Z'^2 - (1+q^2)Y'X'$ and $C_2' = \varrho F'E'Z'$. The elements C_1' and C_2' belong to the centre of the algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$.

On the algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ consider the degree filtration $\mathcal{G}$ where $\deg(Z') = \deg(K^{\pm 1}) = 0$ and $\deg(E') = \deg(F') = \deg(X') = \deg(Y') = 1$. Then the associated graded algebra $\mathcal{E}' = \mathrm{gr}_{\mathcal{G}}\big(\mathrm{gr}_{\mathcal{F}}(\mathcal{E})\big)$ is generated by the elements $\tilde{E}, \tilde{F}, K^{\pm 1}, \tilde{X}, \tilde{Y}$ and $\tilde{Z}$ that satisfy the same defining relations as the algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ but the ration $X'Y' = Y'X' + (q^2-1)Z'^2$ must be replaced by $\tilde{X}\tilde{Y} = \tilde{Y}\tilde{X}$. So, all the canonical generators of the algebra $\mathcal{E}'$ are 'q-commute'. Notice that $\tilde{C_1} = -(1+q^2)\tilde{Y}\tilde{X}$ and $\tilde{C_2} = \varrho\tilde{F}\tilde{E}\tilde{Z}$ are central monomials of the algebra $\mathcal{E}'$.

Let $A = \mathbb{K}[\tilde{Y}, \tilde{X}]$ and $B = \mathbb{K}[\tilde{F}, \tilde{E}, \tilde{Z}]$. The algebra $\mathcal{E}'$ is the tensor product of vector spaces $A \otimes B \otimes \mathbb{K}[K^{\pm 1}]$. Since, up to constant, $\tilde{C_1}$ is a central monomial in A and $\tilde{C_2}$ is a central monomial in B, the modules ${}_{\mathbb{K}[\tilde{C_1}]}A$ and ${}_{\mathbb{K}[\tilde{C_2}]}B$ are free. Hence, the $\mathbb{K}[\tilde{C_1}, \tilde{C_2}]$-module $\mathcal{E}'$ is free and $\mathbb{K}[\tilde{C_1}, \tilde{C_2}]$ is a direct summand of $\mathcal{E}'$. Then by [8, Lemma 4.2], the $\mathbb{K}[C_1', C_2']$-module $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$ is a free module and $\mathbb{K}[C_1', C_2']$ is a direct summand of $\mathrm{gr}_{\mathcal{F}}(\mathcal{E})$. Hence, the $\mathbb{K}[C_1, C_2]$-module $\mathcal{E}$ is free and $\mathbb{K}[C_1, C_2]$ is a direct summand of $\mathcal{E}$. $\square$

3 The prime spectrum of the algebra $\mathcal{E}$

The aim of this section is to find the posets $\mathrm{Spec}\,(\mathcal{E})$ (Theorem 3.9), $\mathrm{Spec}_c(\mathcal{E})$ (Corollary 3.12) and $\mathrm{Max}(\mathcal{E})$ (Corollary 3.10). The key idea in finding the prime spectrum of the algebra $\mathcal{E}$ is to use localizations which essentially reduce the problem of finding $\mathrm{Spec}\,(\mathcal{E})$ to $\mathrm{Spec}\,(A)$ where A is a GWA in Lemma 2.6.(1). The exceptional curves in the centre of $\mathcal{E}$ is a key for finding spectra $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(\mathcal{E})$.

The exceptional curves $\xi_i\,(i \geqslant 1)$ and exceptional maximal ideals $\mathcal{M}^{ex}$ of $\mathcal{Z}$. The elements of $\mathcal{Z} = \mathbb{K}[C_1, C_2]$,

$$\xi_i = C_1 + \nu_i C_2^2, \quad \text{where } i \geqslant 1 \text{ and } \nu_i = \frac{q^{2i-2}\varrho^2}{(1+q^{2i})^2}, \tag{28}$$

are called the *exceptional curves/elements*. The elements $\{\xi_i\}$ are distinct since the elements $\{\nu_i\}$ are distinct: if $\nu_i = \nu_j$ for some $i \neq j$ then either $(q^{i+j}-1)(q^j - q^i) = 0$ or $(q^{i+j}+1)(q^j+q^i) = 0$, i.e., q is a root of 1, a contradiction.

Remark. For $i \geqslant 1$, $\nu_{-i} = \nu_i$, and so $\xi_{-i} = \xi_i$.

For all $i \neq j$, the ideal (ξ_i, ξ_j) in $\mathcal{Z}$ is equal to (C_1, C_2^2). Clearly, $\xi_i \in (C_1, C_2)$ for all $i \geqslant 1$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{Z}$ that contains the element ξ_i. Then $\mathfrak{m} = (\xi_i, \mathfrak{p})$ for a *unique* $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2])$. If $\mathfrak{m} = (C_1, C_2)$ then $\mathfrak{m} = (\xi_i, C_2)$ for all $i \geqslant 1$.

The set $\mathcal{M}^{ex} := \{\mathfrak{m} \in \mathrm{Max}(\mathcal{Z}) \,|\, \xi_i \in \mathfrak{m} \text{ for some } i \geqslant 1\} \setminus \{(C_1, C_2)\}$ is called the *exceptional set* of maximal ideals of $\mathcal{Z}$. Notice that the maximal ideal (C_1, C_2) is not exceptional, by definition. Clearly,

$$\mathcal{M}^{ex} = \big\{(\xi_i, \mathfrak{p}) \,|\, i \geqslant 1, \mathfrak{p} \in \mathrm{Max}\big(\mathbb{K}[C_2]\big) \setminus \{(C_2)\}\big\} \tag{29}$$

and all maximal ideals $(\xi_i, \mathfrak{p})$ are distinct: if $(\xi_i, \mathfrak{p}) = (\xi_j, \mathfrak{q})$ then either $i = j$ or $i \neq j$. In the first case, $\mathfrak{p} = \mathfrak{q}$ (since otherwise $\mathfrak{p}$ and $\mathfrak{q}$ are distinct maximal ideals of $\mathbb{K}[C_2]$, and so $\mathfrak{p} + \mathfrak{q} = (1)$, hence $(\xi_i, \mathfrak{p}) = (\xi_i, \mathfrak{p}, \xi_j, \mathfrak{q}) = (1)$, a contradiction). In the second case, i.e., when $i \neq j$, $(\xi_i, \mathfrak{p}) = (\xi_i, \xi_j, \mathfrak{p}, \mathfrak{q}) \supseteq (\xi_i, \xi_j) = (C_1, C_2^2)$. Hence, $(\xi_i, \mathfrak{p}) = (C_1, C_2)$, a contradiction.

The set $\mathcal{M}^{ex}$ is the disjoint union

$$\mathcal{M}^{ex} = \bigsqcup_{i \geqslant 1} \mathcal{M}_i^{ex} \quad \text{where } \mathcal{M}_i^{ex} := \big\{(\xi_i, \mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Max}\big(\mathbb{K}[C_2]\big) \setminus \{(C_2)\}\big\}. \tag{30}$$

The contraction map

$$\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathcal{Z}), \quad P \mapsto P \cap \mathcal{Z}, \tag{31}$$

is a surjection since, for each prime ideal $\mathfrak{p}$ of $\mathcal{Z}$, the factor algebra $A/\mathfrak{p}A \simeq \mathcal{Z}/\mathfrak{p}[\Theta^{\pm 1}](\sigma, a)$ is a domain, hence $\mathfrak{p}A \in \mathrm{Spec}\,(A)$; and $\mathfrak{p}A \cap \mathcal{Z} = \mathfrak{p}$ since ${}_{\mathcal{Z}}A = \mathcal{Z} \oplus A'$ for some $\mathcal{Z}$-submodule A' of A (see Lemma 2.6.(1)). Therefore,

$$\mathrm{Spec}\,(A) = \bigsqcup_{\mathfrak{p} \in \mathrm{Spec}\,(\mathcal{Z})} \mathrm{Spec}\,(A; \mathfrak{p}) \tag{32}$$

where $\mathrm{Spec}\,(A; \mathfrak{p}) := \{P \in \mathrm{Spec}\,(A) \,|\, P \cap \mathcal{Z} = \mathfrak{p}\}$.

A prime ideal P of an algebra A is called a *completely prime ideal* if the factor algebra A/P is a domain. The set of completely prime ideals of A is denoted by $\mathrm{Spec}_c(A)$. It is a poset with respect to $\subseteq$. By (32),

$$\mathrm{Spec}_c(A) = \bigsqcup_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{Z})} \mathrm{Spec}_c(A; \mathfrak{p}), \quad \mathrm{Spec}_c(A; \mathfrak{p}) := \mathrm{Spec}_c(A) \cap \mathrm{Spec}(A; \mathfrak{p}). \tag{33}$$

The factor algebras $A/\mathfrak{m}A$ where $\mathfrak{m} \in \mathrm{Max}(\mathcal{Z})$ and $\mathrm{Spec}\,(A; \mathfrak{m})$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{Z}$. Then $L_\mathfrak{m} := \mathcal{Z}/\mathfrak{m}$ is a finite field extension of $\mathbb{K}$ and $A(\mathfrak{m}) := A/\mathfrak{m}A = L_\mathfrak{m}[\Theta^{\pm 1}][x, y; \sigma, \bar{a}]$ is a GWA where $x = \Omega + \mathfrak{m}A, y = \varphi Z^{-1} + \mathfrak{m}A$, σ is an $L_\mathfrak{m}$-automorphism of the algebra $L_\mathfrak{m}[\Theta^{\pm 1}]$ such that $\sigma(\Theta) = q^{-2}\Theta$ and $\bar{a} = a + \mathfrak{m}$ where $a = \varrho C_2 + q^{-1}C_1\Theta^{-1} - q^3\Theta$.

Proposition 3.1. *Let* $\mathfrak{m} \in \mathrm{Max}(\mathcal{Z})$.

1. *The algebra* $A(\mathfrak{m})$ *is simple iff* $\mathfrak{m} \in \mathrm{Max}(\mathcal{Z}) \setminus \mathcal{M}^{ex}$.
2. *If* $\mathfrak{m} \in \mathcal{M}^{ex}$*, i.e.,* $\mathfrak{m} = (\xi_i, \mathfrak{p})$ *for some* $i \geqslant 1$ *and* $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$*, then the algebra* $A(\mathfrak{m})$ *contains a unique proper ideal* $\bar{I}(\mathfrak{m}) = \bar{I}_i(\mathfrak{p}) = \bar{I}(\xi_i, \mathfrak{p})$ *which is necessarily the maximal ideal of* $A(\mathfrak{m})$*,* $\bar{I}(\mathfrak{m})^2 = \bar{I}(\mathfrak{m})$ *and* $A(\mathfrak{m})/\bar{I}(\mathfrak{m}) \simeq M_i(L_\mathfrak{m})$ *is the* $i \times i$ *matrix algebra over the field* $L_\mathfrak{m} := \mathcal{Z}/\mathfrak{m} \simeq \mathbb{K}[C_2]/\mathfrak{p}$*. Furthermore,*

$$i = 1: \quad \bar{I}(\mathfrak{m}) = \bigoplus_{j \geqslant 1} Dy^j \oplus D\alpha_0 \oplus \bigoplus_{j \geqslant 1} Dx^j,$$
$$i \geqslant 2: \quad \bar{I}(\mathfrak{m}) = \bigoplus_{j \geqslant i} Dy^j \oplus \bigoplus_{s=-i+1}^{-1} D\alpha_s y^{|s|} \oplus \bigoplus_{s=0}^{i-1} D\alpha_s x^s \oplus \bigoplus_{j \geqslant i} Dx^j,$$

where $D = L_\mathfrak{m}[\Theta^{\pm 1}]$ *and for* $s = 0, \ldots, i-1$,

$$\alpha_s = \prod_{-i+1+s \leqslant j \leqslant 0} \sigma^j(\Theta - q^{2i}\lambda_i), \quad \alpha_{-s} = \prod_{-i+1 \leqslant j \leqslant s} \sigma^j(\Theta - q^{2i}\lambda_i)$$

and $\lambda_i := \big(\varrho^{-1}q^3(1 + q^{2i})\big)^{-1}C_2 + \mathfrak{m} \in L_\mathfrak{m}$.
3.

$$\mathrm{Spec}\,(A; \mathfrak{m}) = \begin{cases} \{\mathfrak{m}A\}, & \text{if } \mathfrak{m} \notin \mathcal{M}^{ex}, \\ \{\mathfrak{m}A,\ I(\mathfrak{m})\}, & \text{if } \mathfrak{m} \in \mathcal{M}^{ex}, \end{cases}$$

$$\mathrm{Spec}_c(A; \mathfrak{m}) = \begin{cases} \{\mathfrak{m}A\}, & \text{if } \mathfrak{m} \notin \mathcal{M}^{ex} \text{ or } \mathfrak{m} = (\xi_i, \mathfrak{p}),\ i \geqslant 2,\ \mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}, \\ \{\mathfrak{m}A,\ I(\mathfrak{m})\}, & \text{if } \mathfrak{m} = (\xi_1, \mathfrak{p}),\ \mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}. \end{cases}$$

where, for $\mathfrak{m} = (\xi, \mathfrak{p}) \in \mathcal{M}^{ex}$*,* $I(\mathfrak{m}) = I(\xi_i, \mathfrak{p}) = I_i(\mathfrak{p}) = f^{-1}(\bar{I}(\mathfrak{m}))$ *where* $f : A \to A(\mathfrak{m}),\ u \mapsto u + \mathfrak{m}$*. Furthermore,*

$$i = 1: \quad I_1(\mathfrak{p}) = \sum_{j \geqslant 1} D(\varphi Z^{-1})^j \oplus D\alpha_0 \oplus \bigoplus_{j \geqslant 1} D\Omega^j + \mathfrak{m}A,$$

$$i \geqslant 2: \quad I_i(\mathfrak{p}) = \sum_{j \geqslant i} D(\varphi Z^{-1})^j + \sum_{s=-i+1}^{-1} D\alpha_s (\varphi Z^{-1})^{|s|} + \sum_{s=0}^{i-1} D\alpha_s \Omega^s + \sum_{j \geqslant i} D\Omega^j + \mathfrak{m}A,$$

where $\alpha_{\pm s}$ *are as in statement 2 and* $\lambda_i = \big(\varrho^{-1} q^3 (1 + q^{2i})\big)^{-1} C_2$*.*

Proof. 1. The algebra $A(\mathfrak{m})$ is the GWA $L_{\mathfrak{m}}[\Theta^{\pm 1}][x, y; \sigma, a]$. To prove statement 1 we use Theorem 2.5. Conditions 1-3 of Theorem 2.5 hold. So, the GWA $A(\mathfrak{m})$ is simple iff $D\bar{a} + D\sigma^i(\bar{a}) = D$ for all $i \geqslant 1$ where $D = L_{\mathfrak{m}}[\Theta^{\pm 1}]$. Let $a_1 = -q^{-3}\Theta^{-1}\bar{a} = \Theta^2 - \varrho q^{-3} C_2 \Theta - q^{-4} C_1$. Since the polynomial $a_1 \in L_{\mathfrak{m}}[\Theta]$ in Θ is quadratic, the equality $Da_1 + D\sigma^i(a_1) = D$ does not hold iff the polynomials a_1 and $\sigma^i(a_1)$ have common linear multiple not of the form $\mathbb{K}^*\Theta$ iff $a_1 = (\Theta - \lambda)(\Theta - \mu)$ for some distinct nonzero $\lambda, \mu \in L_{\mathfrak{m}}$ and the ideal $(\Theta - \mu)$ of D is equal to $\sigma^i\big((\Theta - \lambda)\big) = \big((q^{-2i}\Theta - \lambda)\big) = \big(\Theta - q^{2i}\lambda\big)$ iff $\mu = q^{2i}\lambda$ and $\lambda \neq 0$ iff

$$\begin{cases} -\lambda - \mu = -(1 + q^{2i})\lambda = -\varrho q^{-3} C_2, \\ \lambda\mu = q^{2i}\lambda^2 = -q^{-4} C_1, \end{cases}$$

iff $C_1 = -q^{2i+4}\lambda^2$ and $C_2 = \varrho^{-1} q^3 (1 + q^{2i})\lambda$ for some $\lambda \neq 0$ iff $\xi_i = C_1 + \nu_i C_2^2 = 0$ in $L_{\mathfrak{m}}$ and $\mathfrak{m} \neq (C_1, C_2)$ (since $\lambda \neq 0$) iff $\mathfrak{m} \in \mathcal{M}^{ex}$.

2. Suppose that $\mathfrak{m} = (\xi_i, \mathfrak{p}) \in \mathcal{M}^{ex}$. Then $\xi_i = C_1 + \nu_i C_2^2 \in \mathfrak{m}$. Then $D\bar{a} = D(\Theta - \lambda_i)\sigma^i(\Theta - \lambda_i) = D(\Theta - \lambda_i)(\Theta - q^{2i}\lambda_i)$, see the proof of statement 1. Now, statement 2 follows from [4, Lemma 1].

3. Statement 3 follows from statements 1 and 2. □

The factor algebras $A(\mathfrak{q})$ where $\mathrm{ht}(\mathfrak{q}) = 1$ **and** $\mathrm{Spec}\,(A; \mathfrak{q})$. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{Z} = \mathbb{K}[C_1, C_2]$ of height $\mathrm{ht}(\mathfrak{q}) = 1$. Then $\mathfrak{q} = \mathcal{Z}q$ for some irreducible polynomial $q \in \mathcal{Z}$ which is unique up to multiplication by a non-zero constant. The set $\mathcal{P}^{ex} := \{(\xi_i) := \mathcal{Z}\xi_i \,|\, i \geqslant 1\}$ is called *the set of exceptional height* 1 *prime ideals* of $\mathcal{Z}$.

The algebra $\mathcal{Z}(\mathfrak{q}) := \mathcal{Z}/\mathfrak{q}$ is a domain. The algebra $A(\mathfrak{q}) := A/\mathfrak{q}A = \mathcal{Z}(\mathfrak{q})[\Theta^{\pm 1}][x, y; \sigma, \bar{a}]$ is a GWA where $x = \Omega + \mathfrak{q}A, y = \varphi Z^{-1} + \mathfrak{q}A$, σ is an $\mathcal{Z}(\mathfrak{q})$-automorphism of the algebra $\mathcal{Z}(\mathfrak{q})[\Theta^{\pm 1}]$ where $\sigma(\Theta) = q^{-2}\Theta$ and $\bar{a} = a + \mathfrak{q}$ where $a = \varrho C_2 + q^{-1} C_1 \Theta^{-1} - q^3 \Theta$. The algebra $A(\mathfrak{q})$ is a domain since $\mathcal{Z}(\mathfrak{q})[\Theta^{\pm 1}]$ is so and $\bar{a} \neq 0$. So, $\mathfrak{q}A \in \mathrm{Spec}_c(A)$. Let $L_{\mathfrak{q}}$ be the field of fractions of the domain $\mathcal{Z}(\mathfrak{q})$. The algebra $A(\mathfrak{q})$ is a subalgebra of the GWA $B(\mathfrak{q}) := L_{\mathfrak{q}}[\Theta^{\pm 1}][x, y; \sigma, \bar{a}]$ which is the localization of $A(\mathfrak{q})$ at the central Ore set $\mathcal{Z}(\mathfrak{q}) \setminus \{0\}$. If $\mathfrak{q} = (\xi_i)$, for some $i \geqslant 1$, then $\mathcal{Z}(\xi_i) \simeq \mathbb{K}[C_2]$ and $L_{(\xi_i)} = \mathbb{K}(C_2)$.

Proposition 3.2. *Let* $\mathfrak{q} \in \mathrm{Spec}\,(\mathcal{Z})$ *with* $\mathrm{ht}(\mathfrak{q}) = 1$*.*

1. *The algebra* $B(\mathfrak{q})$ *is simple iff* $\mathfrak{q} \notin \mathcal{P}^{ex}$*.*
2. *If* $\mathfrak{q} \in \mathcal{P}^{ex}$*, i.e.,* $\mathfrak{q} = (\xi_i)$ *for some* $i \geqslant 1$*, then the algebra* $B(\mathfrak{q})$ *contains a unique proper ideal* J_i *which is necessarily the maximal ideal of* $B(\mathfrak{q})$*,* $J_i^2 = J_i$ *and* $B(\mathfrak{q})/J_i \simeq M_i(L_{\mathfrak{q}}) \simeq M_i\big(\mathbb{K}(C_2)\big)$ *is the* $i \times i$ *matrix algebra over the field* $L_{\mathfrak{q}}$*. Furthermore,*

$$i = 1: \quad J_1 = \bigoplus_{j \geqslant 1} Dy^j \oplus D\alpha_0 \oplus \bigoplus_{j \geqslant 1} Dx^j,$$

$$i \geqslant 2: \quad J_i = \bigoplus_{j \geqslant i} Dy^j \oplus \bigoplus_{s=-i+1}^{-1} D\alpha_s y^{|s|} \oplus \bigoplus_{s=0}^{i-1} D\alpha_s x^s \oplus \bigoplus_{j \geqslant i} Dx^j,$$

where $D = L_{\mathfrak{q}}[\Theta^{\pm 1}]$ *and for* $s = 0, \ldots, i-1$*,*

$$\alpha_s = \prod_{-i+1+s \leqslant j \leqslant 0} \sigma^j(\Theta - q^{2i}\lambda_i), \quad \alpha_{-s} = \prod_{-i+1 \leqslant j \leqslant s} \sigma^j(\Theta - q^{2i}\lambda_i)$$

and $\lambda_i = \big(\varrho^{-1} q^3 (1 + q^{2i})\big)^{-1} C_2 + \mathfrak{q} \in L_{\mathfrak{q}}$*.*

3.

$$\mathrm{Spec}\,(A;\mathfrak{q}) = \begin{cases} \{\mathfrak{q}A\}, & \textit{if } \mathfrak{q}\not\in \mathcal{P}^{ex},\\ \{\mathfrak{q}A,\ I_i\}, & \textit{if } \mathfrak{q}=(\xi_i),\ i=1,2,\ldots, \end{cases}$$

$$\mathrm{Spec}_c(A;\mathfrak{q}) = \begin{cases} \{\mathfrak{q}A\}, & \textit{if } \mathfrak{q}\not\in \mathcal{P}^{ex} \textit{ or } \mathfrak{q}=(\xi_i),\ i\geqslant 2,\\ \{\mathfrak{q}A,\ I_1\}, & \textit{if } \mathfrak{q}=(\xi_1), \end{cases}$$

where $I_i := f^{-1}(J_i)$ *and* $f: A\to A/\mathfrak{q}A \to B(\mathfrak{q}),\ u\mapsto u+\mathfrak{q}\mapsto \frac{u+\mathfrak{q}A}{1}$. *Furthermore,*

$$i=1:\quad I_1=\sum_{j\geqslant 1}\mathcal{D}(\varphi Z^{-1})^j + D\alpha_0 + \sum_{j\geqslant 1}\mathcal{D}\Omega^j + \xi_1 A,$$

$$i\geqslant 2:\quad I_i=\sum_{j\geqslant i}\mathcal{D}(\varphi Z^{-1})^j + \sum_{s=-i+1}^{-1}\mathcal{D}\alpha_s(\varphi Z^{-1})^{|s|} + \sum_{s=0}^{i-1}\mathcal{D}\alpha_s\Omega^s + \sum_{j\geqslant i}\mathcal{D}\Omega^j + \xi_i A,$$

where $\mathcal{D}=\mathbb{K}[C_1,C_2,\Theta^{\pm 1}]$, $\alpha_{\pm s}$ *are as above and* $\lambda_i = \big(\varrho^{-1}q^3(1+q^{2i})\big)^{-1}C_2$.

Proof. 1. The algebra $B(\mathfrak{q}) = L_{\mathfrak{q}}[\Theta^{\pm 1}][x,y;\sigma,\bar{a}]$ is a GWA. To show that statement 1 holds we use Theorem 2.5. Conditions 1-3 of Theorem 2.5 hold. So, the algebra $B(\mathfrak{q})$ is not simple iff $D\bar{a}+D\sigma^i(\bar{a})=D$ for some $i\geqslant 1$ iff $Da_1+D\sigma^i(a_1)=D$ for some $i\geqslant 1$ where $a_1 := -q^{-3}\Theta\bar{a} = \Theta^2-\varrho q^{-3}C_2\Theta - q^{-4}C_1$. Then repeating the proof of statement 1 of Proposition 3.1.(1) (simply by replacing the field $L_{\mathfrak{m}}$ by $L_{\mathfrak{q}}$) we see that $Da_1+D\sigma^i(a_1)=D$ iff $\xi_i = C_1+\nu_iC_2^2=0$ in $L_{\mathfrak{q}}$ iff $\xi_i\in\mathfrak{q}$ iff $\mathfrak{q}=(\xi_i)$ (since $\mathrm{ht}\big((\xi_i)\big)=1=\mathrm{ht}(\mathfrak{q})$).

2. Suppose that $\mathfrak{q}=(\xi_i)$ for some $i\geqslant 1$. Then $D\bar{a}=D(\Theta-\lambda_i)\sigma^i(\Theta-\lambda_i)=D(\Theta-\lambda_i)(\Theta-q^{2i}\lambda_i)$, see the proof of statement 2 of Proposition 3.1. Now, statement 2 follows from [4, Lemma 1].

3. Let $P\in\mathrm{Spec}\,(A;\mathfrak{q})$. Then $\bar{P}:=P/\mathfrak{q}A\in\mathrm{Spec}\,(A(\mathfrak{q});0)$, i.e., $\bar{P}\cap\mathcal{Z}(\mathfrak{q})=0$. So, the map

$$\mathrm{Spec}\,\big(A(\mathfrak{q});0\big)\to\mathrm{Spec}\,\big(B(\mathfrak{q})\big),\quad \bar{P}\mapsto B(\mathfrak{q})\bar{P},$$

is a bijection with inverse $Q\to A(\mathfrak{q})\cap Q$. If $\mathfrak{q}\not\in\mathcal{P}^{ex}$ then $B(\mathfrak{q})$ is a simple algebra, by statement 1, hence $\mathrm{Spec}\,\big(A(\mathfrak{q});0\big)=\{0\}$ and then $\mathrm{Spec}\,(A;\mathfrak{q})=\{\mathfrak{q}A\}$. If $\mathfrak{q}\in\mathcal{P}^{ex}$, i.e., $\mathfrak{q}=(\xi_i)$ for some $i\geqslant 1$, then $\mathrm{Spec}\,\big(B(\mathfrak{q})\big)=\{0,J_i\}$, by statement 2. Hence, $\mathrm{Spec}\,(A;\mathfrak{q})=\{\mathfrak{q}A,I_i\}$. Now, the result about $\mathrm{Spec}_c(A;\mathfrak{q})$ follows from statement 2. □

The set $\mathrm{Spec}\,(A;0)$. Let $Q(\mathcal{Z})=\mathbb{K}(C_1,C_2)$ be the field of fractions of $\mathcal{Z}=\mathbb{K}[C_1,C_2]$. The GWA A is a subalgebra of the GWA $B=Q(\mathcal{Z})[\Theta^{\pm 1}][\Omega,\varphi Z^{\pm 1};\sigma,a]$ which is the localization of A at $\mathbb{K}[C_1,C_2]\setminus\{0\}$ (where σ and a are as in Lemma 2.6.(1)). By Lemma 2.6.(3), the algebra B is simple. The next proposition shows that every nonzero prime ideal of A meets the centre of A.

Proposition 3.3.

1. $\mathrm{Spec}\,(A;0)=\mathrm{Spec}_c(A;0)=\{0\}$.

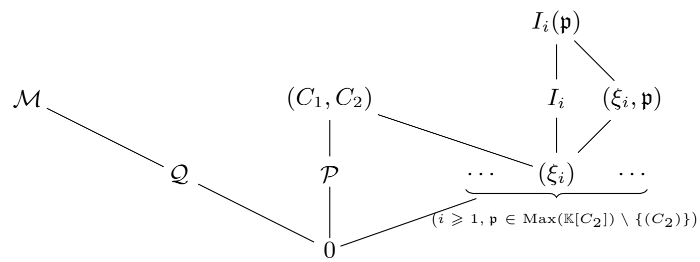
2. $\mathfrak{p}\cap\mathcal{Z}\neq 0$ *for all non-zero prime ideals* $\mathfrak{p}$ *of* A.

Proof. 1. Let $P\in\mathrm{Spec}\,(A;0)$. Then $P' := S^{-1}P$ (where $S=\mathcal{Z}\setminus\{0\}$) is an ideal of the simple algebra B. Then necessarily $P=0$ since otherwise $P'=B$ and so $P\cap A\neq 0$, a contradiction. Therefore, $\mathrm{Spec}\,(A;0)=\{0\}$. Hence, $\mathrm{Spec}_c(A;0)=\{0\}$.

2. Statement 2 follows from Proposition 3.1, Proposition 3.2 and statement 1. □

The next theorem describes the prime spectrum of the algebra A and the inclusions between prime ideals.

Theorem 3.4. *The prime spectrum* Spec (A) *of the algebra* A *is given below*



(34)

where $\mathcal{M} := \mathrm{Max}(\mathbb{K}[C_1, C_2]) \setminus \big(\mathcal{M}^{ex} \cup \{(C_1, C_2)\}\big)$, $\mathcal{C} := \mathrm{Spec}\,(\mathbb{K}[C_1, C_2])$, $\mathcal{P} := \{\mathfrak{p} \in \mathcal{C} \,|\, \mathfrak{p} \subseteq (C_1, C_2),\ \mathrm{ht}(\mathfrak{p}) = 1\} \setminus \mathcal{P}^{ex}$, $\mathcal{Q} := \{\mathfrak{q} \in \mathcal{C} \,|\, \mathfrak{q} \nsubseteq (C_1, C_2),\ \mathrm{ht}(\mathfrak{q}) = 1\}$. *In more detail, let* $\mathcal{S} = \mathcal{P}, \mathcal{P}^{ex}, \mathcal{Q}, \mathcal{M}, \mathcal{M}^{ex}$. *The set* $\mathcal{S}$, *as a subset of* Spec (A), *is equal to* $\{(\mathfrak{q}) = A\mathfrak{q} \,|\, \mathfrak{q} \in \mathcal{S}\}$.

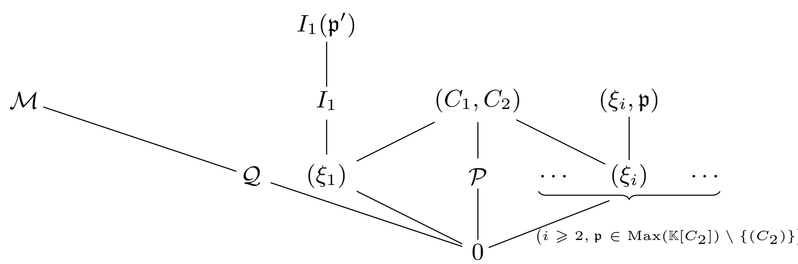
Remark. The inclusion '$\mathcal{Q}$ — $\mathcal{M}$' means a bunch of inclusions '$\mathfrak{q}$ — $\mathfrak{m}$' where $\mathfrak{q} \in \mathcal{Q}$, $\mathfrak{m} \in \mathcal{M}$ and $\mathfrak{q} \subseteq \mathfrak{m}$.

Proof. The theorem follows from (32), Proposition 3.1, Proposition 3.2 and Proposition 3.3. □

By (34), we have the following descriptions of maximal ideals of the algebra A and of the completely prime spectrum $\mathrm{Spec}_c(A)$ of A.

Corollary 3.5.

1. $\mathrm{Max}(A) = \{\mathfrak{q}A \,|\, \mathfrak{q} \in \mathcal{M}\} \sqcup \{A(C_1, C_2)\} \sqcup \{I_i(\mathfrak{p}) \,|\, i \geqslant 1, \mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$.
2. $\mathrm{Spec}_c(A)$:



where $\mathfrak{p}' \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$.

Proof. 1. Statement 1 follows from (34).

2. Statement 2 follows from (34), Proposition 3.1, Proposition 3.2 and Proposition 3.3. □

Prime ideals of the algebra $\mathcal{E}$. For an algebra A, Spec (A) is the set of its prime ideals. The set (Spec $(A), \subseteq)$ is a partially ordered set (poset) with respect to inclusion. Let $f : A \to B$ be an algebra epimorphism. Then Spec (B) can be seen as a subset of Spec (A) via the injection Spec $(B) \to$ Spec (A), $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$. So, Spec $(B) = \{\mathfrak{q} \in \mathrm{Spec}\,(A) \,|\, \ker(f) \subseteq \mathfrak{q}\}$. Given a left denominator set S of the algebra A. Then $\sigma : A \to S^{-1}A$, $a \mapsto s^{-1}a$, is an algebra homomorphism. If the algebra A is a Noetherian algebra then Spec $(S^{-1}A)$ can be seen as a subset of Spec (A) via the injection Spec $(S^{-1}A) \to$ Spec (A), $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$.

Let R be a ring. Then each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$ where $x \in R$. An element $s \in R$ is a *normal* element if $sR = Rs$.

The next two lemmas are used in the proof of Theorem 3.9. Let $\mathbb{V}$ be the ideal of $\mathcal{E}$ generated by the elements X, Y and Z. Since $\mathcal{E}/\mathbb{V} \simeq U_q(\mathfrak{sl}_2)$ is a domain, the ideal $\mathbb{V}$ is a completely prime ideal.

Lemma 3.6. *For each* $n \geqslant 1$, *let* $V_n = \{X^iY^jZ^k \,|\, i, j, k \in \mathbb{N},\ i + j + k = n\}$. *Then*

1. $\mathbb{V}^n = (Z^n) = (Z)^n$.
2. $\mathbb{V}^n = \sum_{v \in V_n} \mathcal{E}v = \sum_{v \in V_n} v\mathcal{E}$.

3. $\bigcap_{n\geqslant 1} \mathbb{V}^n = 0$.

Proof. 2. Let us fix $n \geqslant 1$. Let $L_n = \sum_{v\in V_n} \mathcal{E}v$ and $R_n = \sum_{v\in V_n} v\mathcal{E}$. Then using the commutation relations of the elements in V_1 with the elements $\{E, F, K\}$, we see that $L_n \subseteq R_n$ and $R_n \subseteq L_n$, hence $L_n = R_n$. Similarly, $\mathbb{V}^n \subseteq L_n$. But the inclusion $L_n \subseteq \mathbb{V}^n$ is obvious. Hence, $\mathbb{V}^n = L_n$.

1. To prove statement 1 we use induction on n. Let $n = 1$, using the commutation relations of the element Z with F and E we have the inclusions $X, Y \in (Z)$. Hence, $(Z) = \mathbb{V}$. So, the case $n = 1$ is true.

Suppose that $n > 1$ and the result is true for all $n' < n$. In particular, $\mathbb{V}^{n-1} = (Z^{n-1}) = (Z)^{n-1}$. Let us show, say, that $\mathbb{V}^n = (Z^n)$:

$$\begin{aligned}\mathbb{V}^n &= \mathbb{V}^{n-1}\mathbb{V} = (Z^{n-1})\mathbb{V} = \mathcal{E}Z^{n-1}\mathcal{E}\mathbb{V} = \mathcal{E}Z^{n-1}\mathbb{V}\\ &= \mathcal{E}Z^{n-1}(Z\mathcal{E} + X\mathcal{E} + Y\mathcal{E}) = \mathcal{E}Z^n\mathcal{E} + \mathcal{E}Z^{n-1}X\mathcal{E} + \mathcal{E}Z^{n-1}Y\mathcal{E}.\end{aligned}$$

By Lemma 4.1.(4), $Z^{n-1}X \in (Z^n)$. By Lemma 4.1.(2) and the relation $YZ = q^{-2}ZY$, $Z^{n-1}Y \in (Z^n)$. Therefore, $\mathbb{V}^n = (Z^n)$. Hence, $(Z)^n = (Z^n)$.

3. Statement 3 follows from statement 2. □

Lemma 3.7. *For all $n \geqslant 1$, $(Y^n) = (Y)^n = \mathcal{E}_Z$ where $(Y^n) = \mathcal{E}_Z Y^n \mathcal{E}_Z$ and $(Y) = \mathcal{E}_Z Y \mathcal{E}_Z$.*

Proof. The algebra R_Z is a GWA, see (12), where the defining element a is equal to $\frac{Z^2-C_1}{1+q^2}$. For all $n \geqslant 1$, $(Y^n) \ni (1+q^2)(XY^n - Y^nX) = (1+q^2)Y^{n-1}(\sigma^n(a) - a) = Y^{n-1}(q^{2n}-1)Z^2$. Hence, $(Y^n) = (Y^{n-1}) = \cdots = (Y) = (1)$, as required. □

The following proposition is used in the proof of Theorem 3.9. For an algebra, it identifies the spectra of certain localizations and factor algebras of the algebra with parts of the prime spectrum of the algebra.

Proposition 3.8. *[7] Let R be a Noetherian ring and s be an element of R such that $S_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R\cdot)$). Then $\mathrm{Spec}\,(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$ where $\mathrm{Spec}(R, s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) := \{\mathfrak{q} \in \mathrm{Spec}\,(R) \mid s \notin \mathfrak{q}\}$ and*

(a) the map $\mathrm{Spec}(R, s) \mapsto \mathrm{Spec}\,(R/(s)), \mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ where $\pi : R \to R/(s), r \mapsto r + (s)$,

(b) the map $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto S_s^{-1}\mathfrak{p}$, is a bijection with inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$, where $\sigma : R \to R_s := S_s^{-1}R$, $r \mapsto \frac{r}{1}$.

(c) For all $\mathfrak{p} \in \mathrm{Spec}\,(R, s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R), \mathfrak{p} \not\subseteq \mathfrak{q}$.

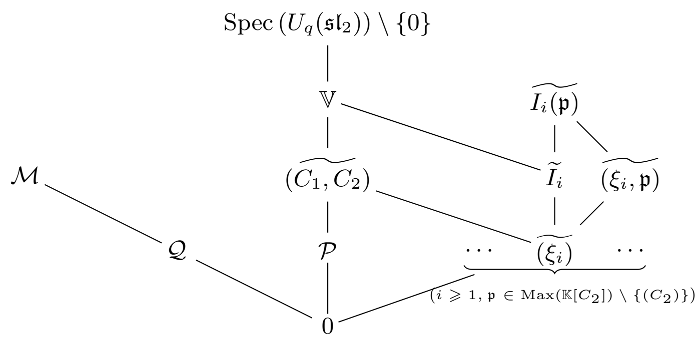
The following diagram reveals the logic behind the proof of Theorem 3.9.

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & \mathcal{E}_Z & \longrightarrow & \mathcal{E}_{Z,Y} \\ \downarrow & & & & \\ \mathcal{E}/(Z) \simeq U_q(\mathfrak{sl}_2) & & & & \end{array} \qquad (35)$$

Theorem 3.9. *The prime spectrum $\mathrm{Spec}\,(\mathcal{E})$ of the algebra $\mathcal{E}$ is the disjoint union*

$$\mathrm{Spec}\,(\mathcal{E}) = \mathrm{Spec}\,(\mathcal{E}/\mathbb{V}) \ \sqcup\ \mathrm{Spec}\,(\mathcal{E}_{Z,Y}) = \mathrm{Spec}\,(\mathcal{E}/\mathbb{V}) \ \sqcup\ \mathrm{Spec}\,(A). \qquad (36)$$

The map $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathcal{E}_{Z,Y})$, $I \mapsto \mathbb{Y} \otimes I$, is a bijection with the inverse $J \mapsto J \cap A$. The map $t : \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathcal{E})$, $I \mapsto \tilde{I} := \mathcal{E} \cap \mathbb{Y} \otimes I_{Z,Y}$ is an injection. The prime spectrum of $\mathcal{E}$ is given *below:*



(37)

where $\mathcal{M} := \mathrm{Max}(\mathbb{K}[C_1, C_2]) \setminus \big(\mathcal{M}^{ex} \cup \{(C_1, C_2)\}\big)$, $\mathcal{C} := \mathrm{Spec}\,(\mathbb{K}[C_1, C_2])$, $\mathcal{P} := \{\mathfrak{p} \in \mathcal{C} \,|\, \mathfrak{p} \subseteq (C_1, C_2),\ \mathrm{ht}(\mathfrak{p}) = 1\} \setminus \mathcal{P}^{ex}$, $\mathcal{Q} := \{\mathfrak{q} \in \mathcal{C} \,|\, \mathfrak{q} \nsubseteq (C_1, C_2),\ \mathrm{ht}(\mathfrak{q}) = 1\}$. *In more detail,*

(a) *the set* $\mathrm{Spec}\,(\mathcal{E}/\mathbb{V})$, *as a subset of* $\mathrm{Spec}\,(\mathcal{E})$, *is equal to* $\big\{(\mathbb{V}, \mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}\,(U_q(\mathfrak{sl}_2)) \setminus \{0\}\big\}$ *(recall that* $\mathcal{E}/\mathbb{V} \simeq U_q(\mathfrak{sl}_2)$*).*

(b) *Let* $\mathcal{S} = \mathcal{P}, \mathcal{P}^{ex}, \mathcal{Q}, \mathcal{M}, \mathcal{M}^{ex}, (C_1, C_2)$. *The set* $\mathcal{S}$, *as a subset of* $\mathrm{Spec}\,(\mathcal{E})$, *is equal to* $\{\widetilde{(\mathfrak{q})} = \mathcal{E} \cap \mathbb{Y} \otimes \mathfrak{q}A \,|\, \mathfrak{q} \in \mathcal{S}\}$.

Remark. The inclusion '$\mathcal{Q}$ — $\mathcal{M}$' means a bunch of inclusions '$\mathfrak{q}$ — $\mathfrak{m}$' where $\mathfrak{q} \in \mathcal{Q}$, $\mathfrak{m} \in \mathcal{M}$ and $\mathfrak{q} \subseteq \mathfrak{m}$.

Proof. Recall that $\mathbb{V} = (Z)$, by Lemma 3.6.(1). The algebra $\mathcal{E}$ is a Noetherian domain, hence so is the algebra $\mathcal{E}_Z$. By Lemma 3.6.(1) and Proposition 3.8 (where $s = Z$),

$$\mathrm{Spec}\,(\mathcal{E}) = \mathrm{Spec}\,(\mathcal{E}/\mathbb{V}) \ \sqcup\ \mathrm{Spec}\,(\mathcal{E}_Z). \tag{38}$$

By Lemma 3.7, every ideal of the algebra $\mathcal{E}_Z$ that contains an element Y^i ($i \geqslant 1$) is equal to $\mathcal{E}_Z$. Therefore,

$$\mathrm{Spec}\,(\mathcal{E}_Z) = \mathrm{Spec}\,(\mathcal{E}_{Z,Y}). \tag{39}$$

By Lemma 2.6.(1), the algebra $\mathcal{E}_{Z,Y} = \mathbb{Y} \otimes A$ is a tensor product of algebras where $\mathbb{Y} = \mathbb{K}[Z^{\pm 1}][Y^{\pm 1}; \tau]$ is a central simple algebra. Therefore, the map

$$\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathcal{E}_{Z,Y}), \quad I \mapsto \mathbb{Y} \otimes I, \tag{40}$$

is a bijection with the inverse $J \mapsto J \cap A$, and so (36) follows. Furthermore, the map

$$\mathrm{Spec}_c(A) \to \mathrm{Spec}\,(\mathcal{E}_{Z,Y}), \quad I \mapsto \mathbb{Y} \otimes I, \tag{41}$$

is a bijection with the inverse $J \mapsto J \cap A$. Now, it is obvious that the map t is an injection. By Proposition 3.8.(c), none of the prime ideals in $\mathrm{Spec}\,(\mathcal{E}/\mathbb{V})$ is contained in a prime ideal of $\mathrm{Spec}\,(A)$. By the very definition of the ideals $\widetilde{(\xi_i)}$ and $\widetilde{(I_i)}$ we have inclusions as in (37). Since $(C_1, C_2, \mathfrak{p}) = (1)$ for all $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$, these are the only inclusions in (37). $\square$

By (37), we have the following description of the set $\mathrm{Max}(\mathcal{E})$ of maximal ideals of $\mathcal{E}$.

Corollary 3.10.

1. $\mathrm{Max}(\mathcal{E}) = \mathrm{Max}(U) \bigsqcup \mathcal{M} \bigsqcup \big\{\widetilde{I_i(\mathfrak{p})} \,|\, i \geqslant 1, \mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}\big\}$.
2. *For all nonzero* $\mathfrak{p} \in \mathrm{Spec}\,(\mathcal{E})$, $\mathfrak{p} \cap Z(\mathcal{E}) \neq 0$.

The next corollary is a description of the GK-dimensions of simple factor algebras of $\mathcal{E}$.

Corollary 3.11. *If* $M \in \mathrm{Max}(\mathcal{E})$ *then* $\mathrm{GK}\,(\mathcal{E}/M) \in \{0, 1, 2, 4\}$. *In more detail,*

1. $\mathrm{GK}\,(\mathcal{E}/M) = 0$ *iff* M *is the annihilator of a simple finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*
2. $\mathrm{GK}\,(\mathcal{E}/M) = 1$ *iff* $M \in \mathrm{Max}\big(U_q(\mathfrak{sl}_2)\big) \setminus \mathcal{F}$ *where* $\mathcal{F}$ *is the set of annihilators of simple finite dimensional* $U_q(\mathfrak{sl}_2)$*-modules.*

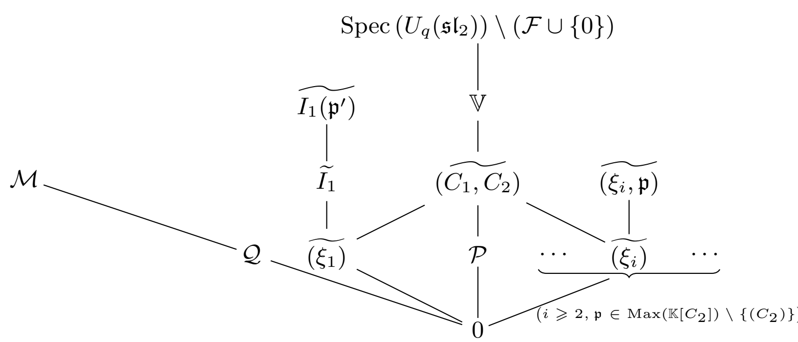
3. $\mathrm{GK}\,(M) = 2$ *iff* $M = \widetilde{I_i(\mathfrak{p})}$ *where* $i \geqslant 1$ *and* $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$.
4. $\mathrm{GK}\,(M) = 4$ *iff* $M = \widetilde{(\mathfrak{q})}$ *where* $\mathfrak{q} \in \mathcal{M}$.

Proof. The corollary follows from an explicit description of $\mathrm{Max}(\mathcal{E})$ (Corollary 3.10.(1)). If $M \in \mathcal{F}$ (resp. $M \in \mathrm{Max}(U_q(\mathfrak{sl}_2)) \setminus \mathcal{F}$) then $\mathrm{GK}\,(\mathcal{E}/M) = 0$ (resp. $\mathrm{GK}\,(\mathcal{E}/M) = 1$).

If $M = \widetilde{I_i(\mathfrak{p})}$ where $i \geqslant 1$ and $\mathfrak{p} \in \mathrm{Max}\big(\mathbb{K}[C_2]\big) \setminus \{(C_2)\}$ then $\mathrm{GK}\,\big(\mathcal{E}/\widetilde{I_i(\mathfrak{p})}\big) = \mathrm{GK}\,\Big(\mathcal{E}_{Z,Y}/\big(\mathbb{Y} \otimes I_i(\mathfrak{p})\big)\Big) = \mathrm{GK}\,\Big(\mathbb{Y} \otimes \big(A/I_i(\mathfrak{p})\big)\Big) = \mathrm{GK}\,(\mathbb{Y}) = 2$ since the algebra $A/I_i(\mathfrak{p})$ is finite dimensional. If $M = \mathcal{E}\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of $\mathbb{K}[C_1, C_2]$ then $\mathrm{GK}\,(\mathcal{E}/M) = \mathrm{GK}\,(\mathcal{E}_{Z,Y}/M_{Z,Y}) = \mathrm{GK}\,\big(\mathbb{Y} \otimes (A/A\mathfrak{m})\big) = 4$ by using the standard filtrations on the algebras $\mathbb{Y}$ and $A/A\mathfrak{m} \simeq L_{\mathfrak{m}}[\Theta^{\pm 1}][x, y; \sigma, a = \varrho C_2 + q^{-1}C_1\Theta^{-1} - q^3\Theta]$ (see (24)) where $L_{\mathfrak{m}} = \mathbb{K}[C_1, C_2]/\mathfrak{m}$ is a finite field extension of $\mathbb{K}$. $\square$

The next corollary is a description of the set $\mathrm{Spec}_c(\mathcal{E})$ of completely prime ideals of $\mathcal{E}$.

Corollary 3.12. $\mathrm{Spec}_c(\mathcal{E})$:



where $\mathfrak{p}' \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$ *and* $\mathcal{F}$ *is the set of annihilators of simple finite dimensional* $U_q(\mathfrak{sl}_2)$*-modules of dimension* $\geqslant 2$.

Proof. Let Λ be an algebra. Notice that Λ is a domain iff $\mathbb{Y} \otimes \Lambda$ is a domain. Now, the corollary follows from Corollary 3.5 and Theorem 3.9. $\square$

The factor algebra $\mathcal{E}_Z/(\Omega)$. Recall that the algebra $\mathcal{E}_Z$ can be presented as a GWA, see (19). Let (Ω) be the two sided ideal of $\mathcal{E}_Z$ generated by the element Ω. The proposition below shows that the factor algebra $\mathcal{E}_Z/(\Omega)$ is a GWA over a Laurent polynomial ring in two variables.

Proposition 3.13.

1. *The algebra* $\mathcal{E}_Z/(\Omega) \simeq \mathbb{K}[\Theta^{\pm 1}, Z^{\pm 1}][X, Y; \sigma, a = \frac{Z^2 + q^2\Theta^2}{1+q^2}]$ *is a GWA where* $\sigma(\Theta) = \Theta$, $\sigma(Z) = q^2 Z$ *(recall that* $\Theta = KZ$*) and* $Z\big(\mathcal{E}_Z/(\Omega)\big) = \mathbb{K}[\Theta, \Theta^{-1}]$.
2. $Z\big(\frac{\mathcal{E}}{(\Omega)\cap\mathcal{E}}\big) = \mathbb{K}[\Theta]$.
3. *The ideal* (Ω) *of* $\mathcal{E}_Z$ *can also be written as* $(\Omega) = (\Omega, \Omega^*) = (\Omega, \varphi) = (\varphi) = (\Omega^*)$.
4. $\widetilde{I}_1 = (\Omega) \cap \mathcal{E}$.

Proof. 1. Let $I = (\Omega)$ and the automorphism σ be as in (19). We show first that $\varphi \in I$. By (15), the element $e = C_1 + q^2K^2Z^2 \in I$. Then the element $[F, e] = q^2(q^2+1)(\varrho K^2 Z^2 F + KZY) \in I$. Since the elements K and Z are invertible in $\mathcal{E}_Z$, we deduce that $\varphi = \varrho ZF + q^2 Y K^{-1} \in I$.

Now, let us prove that $\mathcal{E}_Z/I \simeq \mathcal{D}/(a, \sigma(a))$ where $\mathcal{D} := R_Z[C_2]$, $a = \varrho C_2 + q^{-1}C_1K^{-1}Z^{-1} - q^3KZ$ and $(a, \sigma(a))$ is the ideal of $\mathcal{D}$ generated by the elements a and $\sigma(a)$: Since the element Ω is a homogeneous element of the GWA $\mathcal{E}_Z = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}_{Z,i}$, i.e., $I = \bigoplus_{i\in\mathbb{Z}} I_i$ where $I_i = I \cap \mathcal{E}_{Z,i}$. We have shown that $\Omega, \varphi \in I$, then it is clear that $\bigoplus_{i\in\mathbb{Z}\setminus\{0\}} \mathcal{E}_{Z,i} \in I$. So, $\mathcal{E}_Z/I \simeq \mathcal{D}/\mathcal{D} \cap I$. But

$$\mathcal{D} \cap I = \mathcal{D} \cap (\Omega\mathcal{E}_Z + \mathcal{E}_Z\Omega) = \Omega\mathcal{E}_{Z,-1} + \mathcal{E}_{Z,-1}\Omega = \Omega\varphi Z^{-1}\mathcal{D} + \mathcal{D}\varphi Z^{-1}\Omega = \sigma(a)\mathcal{D} + \mathcal{D}a = (a, \sigma(a)).$$

Now, $a = \varrho C_2 - \alpha$ and $\sigma(a) = \varrho C_2 - \sigma(a)$ where $\alpha = -q^{-1}C_1\Theta^{-1} + q^3K\Theta$, see (19). The element $\sigma(a) - a = \alpha - \sigma(\alpha) = b = \varrho(q^{-1}C_1\Theta^{-1} + q\Theta)$, see (16), belongs to the ideal $(a, \sigma(a))$ of $\mathcal{D}$ and $(a, \sigma(a)) = (a, b)$.

By (13), $\Theta \in Z(\mathcal{D})$. Since $\mathcal{D} = R_Z \otimes \mathbb{K}[C_2]$, we deduce using (14) that $Z(\mathcal{D}) = \mathbb{K}[C_1, C_2, \Theta^{\pm 1}]$. Notice that $a, b \in Z(\mathcal{D})$ and $Z(\mathcal{D})/(a,b) \simeq \mathbb{K}[\Theta^{\pm 1}]$, by using the explicit expressions for a and b. Now, using the fact that $\mathcal{D} = R_Z \otimes \mathbb{K}[C_2] = \mathbb{K}[C_1, C_2, \Theta^{\pm 1}, Z^{\pm 1}][X, Y; \sigma, \frac{Z^2 - C_1}{1+q^2}]$ and $a, b \in \mathbb{K}[C_1, C_2, \Theta^{\pm 1}]$, we have

$$\mathcal{D}/(a,b) \simeq \mathbb{K}[\Theta^{\pm 1}, Z^{\pm 1}][X, Y; \sigma, \frac{Z^2 + q^2\Theta^2}{1+q^2}]$$

where $\sigma(\Theta) = \Theta$ and $\sigma(Z) = q^2 Z$ (we use the fact that the image of b in $\mathcal{D}/(a,b)$ is zero, and so $C_1 = -q^2\Theta^2$ in $\mathcal{D}/(a,b)$). The localization A_X of the algebra $A := \mathcal{E}_Z/(\Omega)$ at the powers of the element X is the skew Laurent polynomial algebra $\mathbb{K}[\Theta^{\pm 1}, Z^{\pm 1}][X^{\pm 1}; \sigma]$ which is a subalgebra of the simple algebra $B := \mathbb{K}(\Theta)[Z^{\pm 1}][X^{\pm 1}; \sigma]$ with $Z(B) = \mathbb{K}(\Theta)$. Hence, $Z(A) = A \cap Z(B) = \mathbb{K}[\Theta, \Theta^{-1}]$.

2. Statement 2 follows from statement 1.

3. Recall that $\varphi = \Omega^*$ and $\varphi \in (\Omega)$, see the proof of statement 1. So, $(\Omega) = (\Omega, \Omega^*)$. Now, $(\Omega, \Omega^*) = (\Omega, \Omega^*)^* = (\Omega)^* = (\Omega^*) = (\varphi)$.

4. By statement 1, the ideal (Ω) is a completely prime ideal of the algebra $\mathcal{E}_Z$. Hence, the ideal $(\Omega) \cap \mathcal{E}$ is a completely prime prime but not maximal ideal of the algebra $\mathcal{E}$ (see statement 2) that obviously contains Ω and is contained in the Spec (A) part of Spec $(\mathcal{E})$, see Theorem 3.9. By Corollary 3.12, $\widetilde{I_1}$ is the only completely prime ideal of $\mathcal{E}$ that is not maximal, contained in the Spec (A) part of Spec $(\mathcal{E})$ and contains Ω. Therefore, $(\Omega) \cap \mathcal{E} = \widetilde{I_1}$. □

Theorem 3.14.

1. *The ideal $\widetilde{I_1}$ of $\mathcal{E}$ is generated by the elements $\Omega, \varphi, \xi_1, EF + q\varrho^{-2}(K + q^2K^{-1})$ and $\Theta - q^2\lambda_1$ where $\lambda_1 = \big(\varrho^{-1}q^3(1+q^2)\big)^{-1}C_2$.*
2. *The ideal $\widetilde{I_1(\mathfrak{p})}$, where $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$, is generated by the elements in statement 1 and $\mathfrak{p}$.*
3. *$\mathcal{E}/\widetilde{I_1} = \mathbb{K}[C_2] \otimes A_1$ is the tensor product of the polynomial algebra $\mathbb{K}[C_2]$ and the central simple GWA $A_1 := \mathbb{K}[K^{\pm 1}][E, F; \sigma, a_1 = -q\varrho^{-2}(q^2K + K^{-1})]$ where $\sigma(K) = q^{-2}K$.*
4. *For each $\mathfrak{p} \in \mathrm{Max}(\mathbb{K}[C_2]) \setminus \{(C_2)\}$, the algebra $\mathcal{E}/\widetilde{I_1(\mathfrak{p})} \simeq \mathbb{K}[C_2]/\mathfrak{p} \otimes A_1$ is a simple algebra with centre $L_\mathfrak{p} := \mathbb{K}[C_2]/\mathfrak{p}$. The algebra $\mathcal{E}/\widetilde{I_1(\mathfrak{p})}$ is the simple GWA $L_\mathfrak{p}[K^{\pm 1}][E, F; \sigma, a_1]$.*

Proof. 1 and 3. By Proposition 3.2.(2), the elements Ω, φ, ξ_1 and $\Theta - q^2\lambda_1$ belong to the ideal $\widetilde{I_1}$. Let $\mathfrak{a}$ be the ideal of $\mathcal{E}$ generated by these elements and $\mathfrak{b}$ be the ideal of $\mathcal{E}$ generated by $\mathfrak{a}$ and the element $b := EF + q\varrho^{-2}(K + q^2K^{-1})$. Clearly, $\mathfrak{a} \subseteq \mathfrak{b}$ and $\mathfrak{a} \subseteq \widetilde{I_1}$. It remains to show that $\mathfrak{b} = \widetilde{I_1}$. The key idea of the proof is the fact that the ideal $\widetilde{I_1}$ is completely prime. The proof consists of two steps:

(i) $b \in \widetilde{I_1}$ (hence $\mathfrak{b} \subseteq \widetilde{I_1}$), and

(ii) $\mathcal{E}/\mathfrak{b} \simeq \mathbb{K}[C_2] \otimes A_1$.

From (ii), it can be easily deduced that $\mathfrak{b} = \widetilde{I_1}$ by using properties of GWAs.

(i) $b \in \widetilde{I_1}$: The equalities $\Omega = \varrho EZ - q^2[2]X$ and $\varphi = \varrho FZ + YK^{-1}$ yields the equivalence relations modulo the ideal $\mathfrak{a}$:

$$X \equiv q^{-2}[2]^{-1}\varrho EZ \mod \mathfrak{a} \quad \text{and} \quad Y \equiv -\varrho FZK \mod \mathfrak{a}. \tag{42}$$

Similarly, by taking the equalities $C_2 = F\Omega + EYK^{-1} + q^2[K;0]Z$ (see (20)) and $\mathfrak{a} \ni \Theta - q^2\lambda_1 = \Theta - q^2\big(\varrho^{-1}q^3(1+q^2)\big)^{-1}C_2$ modulo $\mathfrak{a}$ and using (42), we have the equivalence relations

$$\big(-\varrho EF + q^2[K;0]\big)Z \equiv C_2 \equiv q\varrho^{-1}(1+q^2)KZ \mod \mathfrak{a}. \tag{43}$$

Recall that $\mathfrak{a} \subseteq \widetilde{I_1}$. By taking there equivalence relations modulo $\widetilde{I_1}$ and then deleting Z on both sides (recall that $\mathcal{E}/\widetilde{I_1}$ is a domain, Corollary 3.12) and then dividing by $-\varrho$ and moving all the elements to the left we have the inclusion $EF + \varrho^{-1}\big(-q^2[K;0] + q\varrho^{-1}(1+q^2)K\big) \in \widetilde{I_1}$. By making simplifications using the expression $[K;0] = \frac{K-K^{-1}}{q-q^{-1}}$, the inclusion above can be written as $b \in \widetilde{I}_1$. Hence, $\mathfrak{b} \subseteq \widetilde{I}_1$.

(ii) $\mathcal{E}/\mathfrak{b} \simeq \mathbb{K}[C_2] \otimes A_1$: Let $\bar{\mathcal{E}} = \mathcal{E}/\mathfrak{b}$. Recall that $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \widetilde{I}_1$. By (42) and (43), the algebra $\bar{\mathcal{E}}$ is generated by the (images of the elements) $K^{\pm 1}, C_2, E$ and F. The element C_2 is a central element of $\bar{\mathcal{E}}$. The subalgebra $A_1' = \mathbb{K}\langle K^{\pm 1}, E, F\rangle$ of $\bar{\mathcal{E}}$ satisfies the relation (where $\sigma(K) = q^{-2}K$) $EK = \sigma(K)E, FK = \sigma^{-1}(K)F, EF = -q\varrho^{-2}(K + q^2K^{-1}) = \sigma(a_1)$ (since $b = 0$) and $FE = [F,E] + EF = -\frac{K-K^{-1}}{q-q^{-1}} + \sigma(a_1) = -q\varrho^{-2}(q^2K + K^{-1}) = a_1$. So, the algebra A_1' is an epimorphic image of the simple algebra A_1 (use Theorem 2.5 and the fact that q is not a root of unity), hence $A_1' = A_1$. Since $\mathfrak{b} \subseteq \widetilde{I}_1$, the domain $\mathcal{E}/\widetilde{I}_1$ is an epimorphic image of the tensor product of algebras $\mathbb{K}[C_2] \otimes A_1$ which is a domain. The algebra A_1 is a central simple algebra. So, any ideal of $\mathbb{K}[C_2] \otimes A_1$ is of the form $\mathfrak{p} \otimes A_1$. The GWA $\frac{\mathbb{K}[C_2]\otimes A_1}{\mathfrak{p}\otimes A_1} \simeq L_{\mathfrak{p}}[K^{\pm 1}][E, F; \sigma, a_1]$ is a domain iff $\mathfrak{p} \in \operatorname{Spec}(\mathbb{K}[C_2])$ where $L_{\mathfrak{p}} = \mathbb{K}[C_2]/\mathfrak{p}$. By Theorem 2.5, for any non-zero prime (i.e., maximal) ideal $\mathfrak{p}$ of $\mathbb{K}[C_2]$ the GWA $L_{\mathfrak{p}} \otimes A_1$ is simple. Since the algebra $\mathcal{E}/\widetilde{I}_1$, which is a domain, is not simple we must have $\mathcal{E}/\widetilde{I}_1 \simeq \mathbb{K}[C_2] \otimes A_1$.

2 and 4. Statements 2 and 4 follow from statements 1 and 3. □

4 Appendix

We collect some useful commutation relations in the following lemma.

Lemma 4.1. *The following identities hold in the algebra $\mathcal{E}$.*

1. $FX^i = X^iF + \frac{1-q^{-4i}}{1-q^{-4}}X^{i-1}ZK^{-1}$.
2. $FZ^i = Z^iF + \frac{1-q^{2i}}{1-q^2}YZ^{i-1}K^{-1}$.
3. $EY^i = q^{-2i}Y^iE + \frac{1-q^{-4i}}{1-q^{-4}}[2]ZY^{i-1}$.
4. $EZ^i = Z^iE + \frac{1-q^{2i}}{1-q^2}[2]Z^{i-1}X$.
5. $ZF^i = F^iZ - \frac{1-q^{2i}}{1-q^2}F^{i-1}YK^{-1}$.
6. $ZE^i = E^iZ + \frac{(q^{-2i}-1)(q^2+1)^2}{q^2-1}E^{i-1}X$.
7. $XF^i = F^iX - \frac{q^{2i}-1}{q^2-1}F^{i-1}ZK^{-1} + \frac{(q^{2i}-q^2)(q^{2i}-1)}{(q^4-1)(q^2-1)}F^{i-2}YK^{-2}$.
8. $XY^i = Y^iX + \frac{q^{4i}-1}{q^2+1}Y^{i-1}Z^2$.
9. $EF^i = F^iE + [i]F^{i-1}[K; 1-i]$.

Proof. The equalities are proved by induction on i and using the defining relations of the algebra $\mathcal{E}$. □

References

[1] V. V. Bavula, Finite-dimensionality of Ext^n and Tor_n of simple modules over a class of algebras. *Funct. Anal. Appl.* 25 (1991) no. 3, 229–230.

[2] V. V. Bavula, Simple $D[X, Y; \sigma, a]$-modules. *Ukrainian Math. J.* **44** (1992) no. 12, 1500–1511.

[3] V. V. Bavula, Generalized Weyl algebras and their representations, *St. Petersburg Math. J.* Vol. 4 (1993) no. 1, 71–92.

[4] V. V. Bavula, Description of bilateral ideals in a class of noncommutative rings. I, *Ukrainian Math. J.* **45** (1993) no. 2, 209–220.

[5] V. V. Bavula, Global dimension of generalized Weyl algebras, *Canadian Mathematical Society Conference Proceedings*, Volume **18**, (1996) 81–107.

[6] V. V. Bavula, Filter dimension of algebras and modules, a simplicity criterion of generalized Weyl algebras, *Comm. Algebra* **24** (6), (1996) 1971–1972.

[7] V. V. Bavula and T. Lu, The prime spectrum and simple modules over the quantum spatial ageing algebra, *Algebra Represent. Theory*, (2016). DOI: 10.1007/s10468-016-9613-8. (Published online: 21 April 2016). arXiv:1509.04736.

[8] J. Bernstein, V. Lunts, A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center,*Amer. J. Math.* **118** (1996) no. 5, 979–987.

[9] K. A. Brown, K. R. Goodearl, Lectures on Algebraic Quantum Groups, Advanced Course in Math. CRM Barcelona, vol. 2. Birkhauser, Basel 2002.

[10] B. L. Cerchiai, J. Madore, S. Schraml and J. Wess, Structure of the three-dimensional quantum Euclidean space, *Eur. Phys. J. C Part. Fields* **16** (2000) no. 1, 169–180.

[11] D. A. Jordan, Iterated skew polynomial rings and quantum groups, *J. Algebra* **156** (1993), 194–218.

[12] D. A. Jordan and I. E. Wells, Invariants for automorphisms of certain iterated skew polynomial rings. *Proc. Edinburgh Math. Soc.* (2) **39** (1996) no. 3, 461–472.

[13] A. Lorek, W. Weich and J. Wess, Non-commutative Euclidean and Minkowski structures, *Z. Phys.* **C 76** (1997) 375–386.

[14] K. R. Goodearl and E. S. Letzter, Prime and Primitive Spectra of Multiparameter Quantum Affine Spaces, *Canadian Mathematical Society Conference Proceedings*, **22**, (1998).

[15] S. P. Smith, A class of algebras similar to the enveloping algebra of $sl(2)$. *Trans. Amer. Math. Soc.* **322** (1990) no. 1, 285–314.

V. V. Bavula
Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk

T. Lu
Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: smp12tl@sheffield.ac.uk