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Infinite-Dimensional Lie Algebras Determined by the Space of Symmetric Squares of Hyperelliptic Curves*

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Abstract

We construct Lie algebras of vector fields on universal bundles of symmetric squares of hyperelliptic curves of genus $g = 1, 2, \ldots$. For each of these Lie algebras, the Lie subalgebra of vertical fields has commuting generators, while the generators of the Lie subalgebra of projectable fields determines the canonical representation of the Lie subalgebra with generators L_{2q}, $q = -1, 0, 1, 2, \ldots$, of the Witt algebra. As an application, we obtain integrable polynomial dynamical systems.

Introduction

The connection of the theory of infinite-dimensional Lie algebras with the classical theory of symmetric polynomials [1] and the modern theory of integrable systems is widely known and fruitful (see [2]). In this article we obtain a description of the Lie algebras $\mathcal{G}(\mathcal{E}^2_{N,0})$ of vector fields on the spaces of the universal bundles $\mathcal{E}^2_{N,0}$ (see Definition 2) of the *symmetric squares of the hyperelliptic curves*

$$V_{\mathbf{x}} = \left\{ (X,Y) \in \mathbb{C}^2 : Y^2 = \prod_{i=1}^{N} (X - x_i) \right\}, \qquad \mathbf{x} = (x_1, \ldots, x_N).$$

The Lie algebra $\mathcal{G}(\mathcal{E}^2_{N,0})$ contains the Lie subalgebra of fields lifted from the base $\mathrm{Sym}^N(\mathbb{C})$ (such fields are said to be *base*), i.e., *horizontal* and *projectable* fields (see [3] and [4, p. 337]) over the polynomial Lie algebra $\mathcal{G}(\mathrm{Sym}^N(\mathbb{C}))$ (see [5], [6]) of derivations of the ring of symmetric polynomials in $x_1, \ldots, x_N$. Lie algebras with such a structure form an important class of Lie algebroids; see [7].

The Lie algebra $\mathcal{G}(\mathrm{Sym}^N(\mathbb{C}))$ naturally arises and plays an important role in various areas of mathematics and mathematical physics, including the isospectral deformation method and the classical method of separation of variables. In fundamental works (see, e.g., [5] and [8]) as coordinates on $\mathrm{Sym}^N(\mathbb{C})$ the elementary symmetric functions $e_1, \ldots, e_N$ were chosen. In [5], in terms of the action of the permutation group S_N on $\mathbb{C}^N$, the operation of convolution of invariants was introduced and basis vector fields on $\mathrm{Sym}^N(\mathbb{C})$ were defined, which are independent at any point of the variety of regular orbits. At each point of the variety of irregular orbits these

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fields generate the tangent space to the stratum of the discriminant hypersurface containing the given point. Zakalyukin's well-known construction yields basis vector fields $V_i = \sum V_{i,j}(\mathbf{e})\frac{\partial}{\partial e_j}$, $\mathbf{e} = (e_1, \ldots, e_N)$, with *symmetric* matrix $V_{i,j}$, which are tangent to the discriminant.

In this article we show that the use of the Newton polynomials $p_1, \ldots, p_N$ makes it possible to substantially simplify the formulas and, most importantly, employ the remarkable infinite-dimensional Lie subalgebra W_{-1} of the Witt algebra W in computations. The generators of the Lie algebra W_{-1} are $L_{-2}, L_0, L_2, \ldots$, and the commutation relations have the form $[L_{2q_1}, L_{2q_2}] = 2(q_2 - q_1)L_{2(q_1+q_2)}$.

For any N, the Lie algebra W_{-1} has a faithful canonical representation in the Lie algebra $\mathcal{G}(\mathrm{Sym}^N(\mathbb{C}))$ which maps the generator L_{2q} to the Newton field $\mathcal{L}^0_{2q} = 2\sum x_i^{q+1}\partial_{x_i}$. The image of this representation belongs to the Lie algebra $W_{-1}(N)$, which has the structure of a free left module over the polynomial ring $\mathbb{C}(\mathrm{Sym}^N(\mathbb{C}))$. We obtain an explicit expression for the vector fields V_i with symmetric matrix $V_{i,j}(\mathbf{e})$ in terms of the Newton fields $\mathcal{L}^0_{2q}$ (see Corollary 7). An important role in our calculations is played by a grading of variables and operators. In this connection, we introduce variables y_{2m} and $\mathcal{N}_{2k}$ by setting $e_m = y_{2m}$ and $p_k = \mathcal{N}_{2k}$ and bear in mind that $\deg x_i = 2$.

Note that the Jacobi identity in the $\mathbb{C}[\mathcal{N}_2, \ldots, \mathcal{N}_{2N}]$-polynomial Lie algebra $W_{-1}(N)$ implies the nontrivial differential relation

$$\sum_{m=1}^{N} m\left(\mathcal{N}_{2(k+m)}\frac{\partial \mathcal{N}_{2(q+n)}}{\partial \mathcal{N}_{2m}} - \mathcal{N}_{2(q+m)}\frac{\partial \mathcal{N}_{2(k+n)}}{\partial \mathcal{N}_{2m}}\right) = (q-k)\mathcal{N}_{2(k+q+n)}$$

for the Newton polynomials. We show that there exists a *unique* representation of the Lie algebra W_{-1} in the Lie algebra of *horizontal* vector fields $L_{-2}, L_0, L_2, \ldots$ of the Lie algebroid $\mathcal{G}(\mathcal{E}^2_{N,0})$ (see Theorem 6). The proof of the uniqueness of this representation uses the fact that, relative to the Lie bracket $[\cdot,\cdot]$, the Lie algebra W_{-1} is defined by the generators L_{-2} and L_4 and the relation $[L_2, [L_2, [L_2, L_4]]] = 12[L_4, [L_2, L_4]]$.

On the Lie algebroid $\mathcal{G}(\mathcal{E}^2_{N,0})$ there exist two commuting *vertical* vector fields. For each point $\mathbf{x}$, the restrictions of these fields to $\mathrm{Sym}^2(V_{\mathbf{x}})$ are the images of obviously commuting fields on $V_{\mathbf{x}} \times V_{\mathbf{x}}$. In our upcoming publication we shall give an explicit description of such commuting fields on $\mathrm{Sym}^k(V_{\mathbf{x}})$ for any $k \geq 2$. In Section 4 of this article we give an explicit description of these fields in the case of interest to us, namely, for $k = 2$.

Similar operators on $\mathrm{Sym}^m(\mathbb{C}^2)$ were constructed in [9] on the basis of a construction of the spectral curve and the Poisson structure. A proof of the commutativity of operators in [9] uses a method that differs from ours.

The key results of our work are a formula for the generating series $L(t)$ (see Theorem 7) of the horizontal vector fields $\mathcal{L}^0_{-2}, \ldots, \mathcal{L}^0_{2k}, \ldots$ that determine a representation of the Lie algebra W_{-1} and a commutation formula for the vertical and horizontal vector fields (see Theorem 8).

In Section 6 we construct an N-dimensional algebraic variety $W(N)$ in the $(N+2)$–dimensional algebraic variety $\mathcal{E}^2_{N,0}$ and homeomorphisms $f_N\colon \mathbb{C}^{N+2} \setminus \{u_4 = 0\} \to \mathcal{E}^2_{N,0} \setminus W(N)$, where $\{u_4 = 0\}$ is a hyperplane in $\mathbb{C}^{N+2}$ with graded coordinates $u_2, u_4, v_{N-2}, v_N; y_2, \ldots, y_{2(N-2)}$. One of the main results of the present work is an explicit construction, which uses the homeomorphisms f_N, of the polynomial Lie algebras on $\mathbb{C}^{N+2}$ that are determined by the Lie algebroids $\mathcal{G}(\mathcal{E}^2_{N,0})$, $N = 3, 4, \ldots$ (see Theorem 9).

The article concludes with a description of polynomial Lie algebras on $\mathbb{C}^{N+1}$ for $N = 3, 4, 5$. In the case $N = 5$, we obtain a polynomial Lie algebra isomorphic to the Lie algebra of vector fields on the universal bundle of *Jacobians* of nonsingular hyperelliptic curves of genus 2, which was constructed in [10] by using the theory of two-dimensional σ-functions.

1 The Space of Symmetric Squares of Hyperelliptic Curves

Consider a family of plane curves

$$V_{N,0} = \{(X, Y; \mathbf{x}) \in \mathbb{C}^2 \times \mathbb{C}^N : \pi(X, Y; \mathbf{x}) = 0\}, \tag{1}$$

where

$$\pi(X, Y; \mathbf{x}) = Y^2 - \prod_{k=1}^{N} (X - x_k). \tag{2}$$

The vector $\xi_N(\mathbf{x}) = \mathbf{e}$ is the *parameter set* for a curve in family (1). By V_N we denote the subfamily of curves satisfying $e_1 = p_1 = 0$.

In this paper we use the following grading of variables: $\deg x_k = 2$, $k = 1, \ldots, N$, $\deg X = 2$, and $\deg Y = N$. With respect to this grading the polynomial $\pi(X, Y; \mathbf{x})$ is homogeneous of degree $2N$.

The *discriminant variety* of family (1) is the algebraic variety

$$\mathrm{Disc}(V_{N,0}) = \{\xi_N(\mathbf{x}) \in \mathrm{Sym}^N(\mathbb{C}) : \Delta_N = 0\}, \quad \text{where } \Delta_N = \prod_{i<j} (x_i - x_j)^2.$$

The discriminant variety $\mathrm{Disc}(V_N)$ of the family of curves V_N is defined similarly. The variety $\mathrm{Disc}(V_{N,0}) \subset \mathbb{C}^N$ is the image under the projection $\mathbb{C}^N \to \mathrm{Sym}^N(\mathbb{C}) = \mathbb{C}^N$ of the union of the so-called mirrors, that is, the hyperplanes $\{x_i = x_j,\ i \neq j\}$, and $\mathrm{Disc}(V_N) \subset \mathbb{C}^{N-1}$ is the image of the intersection of the space $\mathbb{C}^{N-1} = \{\mathbf{e} \in \mathbb{C}^n : e_1 = 0\}$ with the union of mirrors.

For $N = 3$, the variety $\mathrm{Disc}(V_3) \subset \mathbb{C}^2$ in the coordinates (e_2, e_3) is determined by the equation $\Delta_3 = 27e_3^2 - 4e_2^3 = 0$, i.e., is the well-known swallowtail in $\mathbb{C}^2$. In the book [5] it was proposed to refer to the varieties $\mathrm{Disc}(V_N)$ as generalized swallowtails in $\mathbb{C}^{N-1}$.

Let $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$ denote the open varieties $\mathbb{C}^N \setminus \mathrm{Disc}(V_{N,0})$ and $\mathbb{C}^{N-1} \setminus \mathrm{Disc}(V_N)$, respectively. The curves of the families $V_{N,0}$ and V_N with parameters in the spaces $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$ are said to be nonsingular for obvious reasons. They have genus $\left[\frac{N-1}{2}\right]$. For example, in the cases $N = 3$ and 4, these are elliptic curves.

We set $\widehat{\mathcal{E}}_{N,0} = \{(X, Y; \mathbf{x}) \in \mathbb{C}^2 \times \mathbb{C}^N : \pi(X, Y; \mathbf{x}) = 0,\ \xi(\mathbf{x}) \in \mathcal{B}_{N,0}\}$. The group S_N acts freely on $\widehat{\mathcal{E}}_{N,0}$ by permutations of the coordinates $x_1, \ldots, x_N$, and therefore a regular $N!$-sheeted covering $\widehat{\mathcal{E}}_{N,0} \to \mathcal{E}_{N,0} = \widehat{\mathcal{E}}_{N,0}/S_N$ is defined.

Definition 1 *The universal bundle of nonsingular hyperelliptic curves of family (1) is the bundle* $\mathcal{E}_{N,0} \to \mathcal{B}_{N,0} \colon (X, Y; \xi(\mathbf{x})) \mapsto \xi(\mathbf{x})$.

The space $\mathcal{E}_{N,0}$ is the *universal space* of nonsingular hyperelliptic curves of genus $\left[\frac{N-1}{2}\right]$. The fiber over a point of the base $\mathcal{B}_{N,0}$ is the curve with parameters determined by this point. The universal bundle $\mathcal{E}_N \to \mathcal{B}_N$ of nonsingular hyperelliptic curves is defined similarly.

We set

$$\widehat{\mathcal{E}}^2_{N,0} = \{(X_1, Y_1; X_2, Y_2; \mathbf{x}) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^N : \pi(X_k, Y_k; \mathbf{x}) = 0,\ k = 1, 2;\ X_1 - X_2 \neq 0,\ \xi(\mathbf{x}) \in \mathcal{B}_{N,0}\}.$$

The group $G = S_2 \times S_N$ acts freely on $\widehat{\mathcal{E}}^2_{N,0}$, so that the generator of S_2 determines the permutation $(X_1, Y_1) \leftrightarrow (X_2, Y_2)$, and the elements of the group S_N determine permutations of the coordinates of the vector $\mathbf{x}$. Therefore, a regular covering $\mathcal{E}^2_{N,0} = \widehat{\mathcal{E}}^2_{N,0}/G \to \widehat{\mathcal{E}}^2_{N,0}$ is defined.

Definition 2 *The universal bundle of symmetric squares of nonsingular hyperelliptic curves of family (1) is the bundle*

$$\mathcal{E}^2_{N,0} \to \mathcal{B}_{N,0} \colon ([X_1, Y_1; X_2, Y_2]; [\mathbf{x}]) \mapsto [\mathbf{x}],$$

where $[X_1, Y_1; X_2, Y_2] = \xi_2(X_1, Y_1; X_2, Y_2)$, $[\mathbf{x}] = \xi(\mathbf{x})$, *and* $\xi_2 \colon \mathbb{C}^2 \times \mathbb{C}^2 \to Sym^2(\mathbb{C}^2)$ *is the canonical projection onto the symmetric square of* $\mathbb{C}^2$.

The space $\mathcal{E}^2_{N,0}$ is called the *universal space* of symmetric squares of nonsingular hyperelliptic curves of genus $\left[\frac{N-1}{2}\right]$. The fiber over a point of the base $\mathcal{B}_{N,0}$ is the variety $(\mathrm{Sym}^2 V)\backslash(X_1 - X_2 = 0)$ with parameters determined by this point.

The universal bundle $\mathcal{E}^2_N \to \mathcal{B}_N$ is defined similarly.

2 The Lie Algebra of Newton Vector Fields

In the paper [6] the theory of polynomial Lie algebras was constructed. Important examples of such infinite-dimensional Lie algebras are the Lie algebras of vector fields on $\mathbb{C}^N$ and $\mathbb{C}^{N-1}$ tangent to the varieties $\mathrm{Disc}(V_{N,0})$ and $\mathrm{Disc}(V_N)$ and, therefore, the Lie algebras of vector fields on $\mathcal{B}_{N,0}$ and $\mathcal{B}_N$.

In this section we give an explicit description of the Lie algebras $\mathcal{G}_{P_0}(N)$ and $\mathcal{G}_P(N)$ of vector fields in coordinates $(p_1, \ldots, p_N)$ and $(p_2, \ldots, p_N)$ determined by the Newton polynomials. We have $\deg x_i = 2$, $i = 1, \ldots, N$. To control grading, we introduce the notation

$$\mathcal{N}_{2k} = p_k(\mathbf{x}) = \sum_{i=1}^{N} x_i^k, \qquad k = 0, 1, \ldots .$$

For graded generators of the polynomial ring $\mathcal{P}(\mathrm{Sym}^N(\mathbb{C}))$ we take the polynomials $\mathcal{N}_2, \ldots, \mathcal{N}_{2N}$. Then $\mathcal{P}(\mathrm{Sym}^N(\mathbb{C})) \simeq \mathbb{C}[\mathcal{N}_2, \ldots, \mathcal{N}_{2N}]$.

Definition 3 *The gradient homogeneous polynomial vector fields*

$$\mathcal{L}^0_{2q} = 2\sum_{i=1}^{N} x_i^{q+1} \partial_{x_i}, \qquad q = -1, 0, 1, \ldots, \tag{3}$$

on $\mathbb{C}^N$ *of degree* $2q$ *are called the Newton derivations of the ring* $\mathbb{C}[x_1, \ldots, x_N]$.

Lemma 1 *The operators* $\mathcal{L}^0_{2q}$, $q = -1, 0, 1, \ldots$ *are derivations of the ring* $\mathbb{C}[\mathcal{N}_2, \ldots, \mathcal{N}_{2N}]$, *and they are uniquely determined by the formula*

$$\mathcal{L}^0_{2q} \mathcal{N}_{2k} = 2k\mathcal{N}_{2(k+q)}, \qquad k = 1, 2, \ldots . \tag{4}$$

Corollary 1 *The operators $\mathcal{L}^0_{2q}$ act on the ring $\mathbb{C}[\mathcal{N}_2,\dots,\mathcal{N}_{2N}]$ as*

$$\mathcal{L}^0_{2q} = \sum_{k=1}^{N} 2k\mathcal{N}_{2(q+k)}\,\frac{\partial}{\partial\mathcal{N}_{2k}}\,. \tag{5}$$

Lemma 1 and Corollary 1 are verified directly.

Let us write the equation $\prod_{i=1}^{N}(x-x_i)=0$ in the form $x^N=\sum_{j=1}^{N}(-1)^{j+1}y_{2j}x^{N-j}$.

Lemma 2 *For any $k\geq 0$,*

$$x^{N+k} = \sum_{j=1}^{N}(-1)^{j+1}y_{2k,2j}x^{N-j}, \tag{6}$$

where the $\{y_{2k,2j},\ k=1,\dots\}$, $j=1,\dots,N$, are the sets of symmetric functions in $x_1,\dots,x_N$ with generating series

$$\mathcal{Y}_{2j} = \sum_{k=0}^{\infty} y_{2k,2j}t^k = \frac{1}{E(t)}\sum_{s=0}^{N-j}(-1)^s y_{2(j+s)}t^s, \qquad E(t)=\prod_{i=1}^{N}(1-x_it). \tag{7}$$

Proof: According to (6), for any $k\geq 0$, we have

$$\begin{aligned}\sum_{j=1}^{N}(-1)^{j+1}y_{2(k+1),2j}x^{N-j} &= x^{N+k+1} = x\cdot x^{N+k}\\ &= y_{2k,2}\left[\sum_{j=1}^{N}(-1)^{j+1}y_{2j}x^{N-j}\right]+\sum_{j=2}^{N}(-1)^{j+1}y_{2k,2j}x^{N-j+1}.\end{aligned}$$

Hence $y_{2(k+1),2N}=y_{2N}y_{2k,2}$ and $y_{2(k+1),2j}=y_{2j}y_{2k,2}-y_{2k,2(j+1)}$. We obtain the following system of equations for the generating series:

$$\begin{aligned}\mathcal{Y}_{2N} &= y_{2N}(1+t\mathcal{Y}_2),\\ \mathcal{Y}_{2j} &= y_{2j}(1+t\mathcal{Y}_2)-t\mathcal{Y}_{2(j+1)}, \qquad j=1,\dots,N-1.\end{aligned}$$

Solving this system, we obtain (7).
□

Corollary 2 *For any $k\geq 0$,*

$$\mathcal{N}_{2(N+k)} = \sum_{j=1}^{N}(-1)^{j+1}y_{2k,2j}\mathcal{N}_{2(N-j)}, \tag{8}$$

$$\mathcal{L}^0_{2(N+k-1)} = \sum_{j=1}^{N}(-1)^{j+1}y_{2k,2j}\mathcal{L}^0_{2(N-j-1)}. \tag{9}$$

Let us introduce the generating series $\mathcal{L}^0(t)=\sum_{q=-1}^{\infty}\mathcal{L}^0_{2q}t^{q+1}$.

Corollary 3 *The following relation holds:*

$$\mathcal{L}^0(t) = \frac{1}{E(t)}\sum_{m=1}^{N}E(t;m)\mathcal{L}^0_{2(m-2)}t^{m-1}, \tag{10}$$

where

$$E(t;m) = \sum_{k=0}^{N-m}(-1)^k y_{2k}\,t^k \quad \text{and} \quad E(t)=E(t;0). \tag{11}$$

Poof Let us write the series $\mathcal{L}^0(t)$ in the form $\mathcal{L}^0(t) = \sum_{q=-1}^{N-2} \mathcal{L}^0_{2q} t^{q+1} + t^N \sum_{k=0}^{\infty} \mathcal{L}^0_{2(N+k-1)} t^k$. According to (9), we obtain

$$\begin{aligned}\mathcal{L}^0(t) &= \sum_{q=-1}^{N-2} \mathcal{L}^0_{2q} t^{q+1} + t^N \sum_{j=1}^{N} (-1)^{j+1} \Big(\sum_{k=0}^{\infty} y_{2k,2j}\, t^k \Big) \mathcal{L}^0_{2(N-j-1)} \\ &= \sum_{j=1}^{N} t^{N-j} (1 + (-1)^{j+1} t^j \mathcal{Y}_{2j}) \mathcal{L}^0_{2(N-j-1)}.\end{aligned}$$

It remains to use (7).
□

Lemma 3 *The following relation holds:*

$$[\mathcal{L}^0_{2q_1}, \mathcal{L}^0_{2q_2}] = 2(q_2 - q_1) \mathcal{L}^0_{2(q_1+q_2)}. \tag{12}$$

Proof: The required relation follows directly from (4).
□

Corollary 4 *For all $k, q \in \mathbb{N}$ and $n = 1, \dots,$ the polynomials $\mathcal{N}_{2k}$, $k = 0, 1, \dots,$ are related by*

$$\sum_{m=1}^{N} m \Big(\mathcal{N}_{2(k+m)} \frac{\partial \mathcal{N}_{2(q+n)}}{\partial \mathcal{N}_{2m}} - \mathcal{N}_{2(q+m)} \frac{\partial \mathcal{N}_{2(k+n)}}{\partial \mathcal{N}_{2m}} \Big) = (q - k) \mathcal{N}_{2(k+q+n)}. \tag{13}$$

Proof: The required relations follow directly from (5) and (12).
□

Example 1 *For $N = 3$, the polynomials $\mathcal{N}_2$, $\mathcal{N}_4$, and $\mathcal{N}_6$ are algebraically independent and $\mathcal{N}_0 = 3$. For $k = 0$, $q = 1$, and $n = 3$, relation (13) gives the Euler differential equation*

$$2\mathcal{N}_2 \frac{\partial \mathcal{N}_8}{\partial \mathcal{N}_2} + 4\mathcal{N}_4 \frac{\partial \mathcal{N}_8}{\partial \mathcal{N}_4} + 6\mathcal{N}_6 \frac{\partial \mathcal{N}_8}{\partial \mathcal{N}_6} = 8\mathcal{N}_8.$$

Lemma 4 *For all $q = -1, 0, 1, \dots,$*

$$\mathcal{L}^0_{2q} \Delta(N) = 4\gamma_{2q}(N) \Delta(N), \tag{14}$$

where

$$\Delta(N) = \prod_{i<j} (x_i - x_j)^2 \quad \textit{and} \quad \gamma_{2q}(N) = \sum_{i<j} \frac{x_i^{q+1} - x_j^{q+1}}{x_i - x_j}. \tag{15}$$

For any $k \geq 0$,

$$\gamma_{2(N+k)}(N) = \sum_{s=1}^{N-1} (-1)^{s+1} y_{2(k+1),2s} \gamma_{2(N-s-1)}.$$

Proof: We have

$$\mathcal{L}^0_{2q} \Delta(N) = 4\Delta(N) \sum_{k=1}^{N} x_k^{q+1} \partial_k \ln \prod_{i<j} (x_i - x_j) = 4\Delta(N) \Big(\sum_{i<j} \frac{x_i^{q+1} - x_j^{q+1}}{x_i - x_j} \Big).$$

The formula for $\gamma_{2(N+k)}(N)$ follows from (6).
□

Example 2 *$\gamma_{-2}(N) = 0$, $\gamma_0(N) = 2N(N-1)$, and $\gamma_1(N) = 4(N-1)\mathcal{N}_2$.*

Corollary 5 *The vector fields $\mathcal{L}^0_{2q}$, $q = -1, 0, 1, \ldots$, on $\mathbb{C}^N$ determine vector fields on $Sym^N(\mathbb{C})$ tangent to the algebraic variety $Disc(V_{N,0}) \subset Sym^N(\mathbb{C})$.*

Theorem 1 *The Lie algebra $\mathcal{G}_{P_0}(N)$ of vector fields on the variety $\mathcal{B}_{N,0}$ in the coordinates $\mathcal{N}_2, \ldots, \mathcal{N}_{2N}$ has the structure of a free N-dimensional module over the ring $\mathbb{C}[\mathcal{N}_2, \ldots, \mathcal{N}_{2N}]$ with generators $\mathcal{L}^0_{2q}$, $q = -1, 0, 1, \ldots, N-2$. The set of generators extends to an infinite set $\{\mathcal{L}^0_{2q}\}$, where the elements $\mathcal{L}^0_{2q}$ for $q > N-2$ are given by (9). The operators $\mathcal{L}^0_{2q}$ act on $\mathcal{N}_{2k}$ by (4).*

The structure of the Lie algebra $\mathcal{G}_{P_0}(N)$ is determined by (12) and (4), where the $\mathcal{N}_{2k}$ for $k > N$ are the polynomials $\mathcal{N}_{2k}(\mathcal{N}_2, \ldots, \mathcal{N}_{2N})$ defined recursively by (8).

We set $\mathcal{L}^0_A(t) = tE(t)\mathcal{L}^0(t)$. According to Corollary 3, operators $\mathcal{L}^0_{A,2(k-2)}$ for which

$$\mathcal{L}^0_A(t) = \sum_{m=1}^{N} \mathcal{L}^0_{A,2(m-2)} t^m = \sum_{m=1}^{N} E(t;m)\mathcal{L}^0_{2(m-2)} t^m$$

are defined.

Lemma 5 *The following relation holds: $\mathcal{L}^0_A(t) = \sum_{k=1}^{N} t \prod_{j\neq k}(1 - x_j t)\frac{\partial}{\partial x_k}$.*

Proof: We have

$$\mathcal{L}^0_A(t) = tE(t)\sum_{q=-1}^{\infty} \mathcal{L}^0_{2q} t^{q+1} = E(t)\sum_{k=1}^{N} \frac{t}{1 - x_k t}\frac{\partial}{\partial x_k} = \sum_{k=1}^{N} t \prod_{j\neq k}(1 - x_j t)\frac{\partial}{\partial x_k}\,. \qquad \square$$

$\square$

Example 3 *$\mathcal{L}^0_{A,-2} = \mathcal{L}^0_{-2}$ and $\mathcal{L}^0_{A,2(N-2)} = y_{2N}\mathcal{L}^0_{-4}$.*

Lemma 6 *The generating polynomial*

$$\mathcal{L}^0_A(t)E(s) = \sum_{m=1}^{N} (-1)^m (\mathcal{L}_A(t) y_{2m}) s^m$$

in s is symmetric with respect to the permutation $t \leftrightarrow s$.

Proof: We have

$$\mathcal{L}^0_A(t)E(s) = tE(t)\mathcal{L}^0(t)E(s) = -E(t)E(s)\sum_{i=1}^{N} \frac{t}{1 - x_i t}\,\frac{s}{1 - x_i s}\,. \qquad \square$$

$\square$

Corollary 6 *The action of the operators $\mathcal{L}^0_{A,2(k-2)}$, $k = 1, \ldots, N$, on the elementary symmetric polynomials $e_m = y_{2m}$ is given by a symmetric matrix $T^0_{k,m} = T^0_{k,m}(y_2, \ldots, y_{2N})$, that is,*

$$\mathcal{L}^0_{A,2(k-2)} y_{2m} = \mathcal{L}^0_{A,2(m-2)} y_{2k}.$$

Proof: We have

$$\mathcal{L}_A^0(t)E(s) = \left(\sum_{k=1}^{N}(-1)^k\mathcal{L}^0_{A,2(k-2)}t^k\right)\left(\sum_{m=1}^{N}(-1)^m y_{2m}s^m\right)$$
$$= \sum_{k=1}^{N}\sum_{m=1}^{N}(-1)^{k+m}(\mathcal{L}^0_{A,2(k-2)}y_{2m})t^k s^m.$$

It remains to use Lemma 6.
□

The Lie algebra $\mathcal{G}_P(N)$ of vector fields on $\mathcal{B}_N$ in the coordinates $\mathcal{N}_4, \dots, \mathcal{N}_{2N}$ is a Lie subalgebra of $\mathcal{G}_{P_0}(N)$. It consists of the fields that leave the ideal $J_2 = \langle\mathcal{N}_2\rangle \subset \mathbb{C}[\mathcal{N}_2, \dots, \mathcal{N}_{2N}]$ invariant.

We have $\mathcal{L}^0_{-2}\mathcal{N}_2 = 2N$ and $\mathcal{L}^0_{2q}\mathcal{N}_2 = 2\mathcal{N}_{2(q+1)}$. We set $\mathcal{L}_0 = \mathcal{L}^0_0$ and $\mathcal{L}_{2q} = \mathcal{L}^0_{2q} - \frac{1}{N}\mathcal{N}_{2(q+1)}\mathcal{L}^0_{-2}$. By construction $\mathcal{L}_0\mathcal{N}_2 = 2\mathcal{N}_2$ and $\mathcal{L}_{2q}\mathcal{N}_2 = 0$ for $q \neq 0$.

Theorem 2 *The Lie algebra $\mathcal{G}_P(N)$ of vector fields on $\mathcal{B}_N$ in the coordinates $\mathcal{N}_4, \dots, \mathcal{N}_{2N}$ has the structure of a free $(N-1)$-dimensional module over the ring $\mathbb{C}[\mathcal{N}_4, \dots, \mathcal{N}_{2N}]$ with generators $\mathcal{L}_{2q}$, $q = 0, 1, \dots, N-2$. The set of generators extends to an infinite set $\{\mathcal{L}_{2q}\}$, and the elements $\mathcal{L}_{2q}$ for $q = N+k-1$, $k \geq 0$, are given by (see (9))*

$$\mathcal{L}^0_{2(N+k-1)} = \sum_{j=1}^{N}(-1)^{j+1}\widehat{y}_{2k,2j}\mathcal{L}^0_{2(N-j-1)}, \tag{16}$$

where $\widehat{y}_{2k,2j} \equiv y_{2k,2j} \bmod \langle\mathcal{N}_2\rangle$ and the $y_{2k,2j}$ are the polynomials determined by the generating series (7).

The structure of the Lie algebra on $\mathcal{G}_P(N)$ is introduced directly by the condition that this is a Lie subalgebra of the Lie algebra $\mathcal{G}_{P_0}(N)$.

Let $\mathcal{L}_A(t) = tE(t)\mathcal{L}(t)$, where

$$\mathcal{L}(t) = \sum_{q=-1}^{\infty}\mathcal{L}_{2q}t^{q+1} = \sum_{q=-1}^{\infty}\left(\mathcal{L}_{2q} - \frac{1}{N}\mathcal{N}_{2(q+1)}\mathcal{L}^0_{-2}\right)t^{q+1}.$$

Lemma 7 *The following relation holds:*

$$\mathcal{L}_A(t) = \mathcal{L}^0_A(t) + \frac{1}{N}\left(\mathcal{L}^0_{-2}E(t)\right)\mathcal{L}^0_{-2} = \left[\mathcal{L}_0 - \left(1 - \frac{4}{N}\right)y_2\mathcal{L}_{-2}\right]t^2 + \cdots. \tag{17}$$

Lemmas 6 and 7 give the following result.

Lemma 8 *The generating polynomial $\mathcal{L}_A(t)E(s) = \sum_{m=1}^{N}(-1)^m(\mathcal{L}_A(t)y_{2m})s^m$ is symmetric with respect to the permutation $t \leftrightarrow s$.*

Let us introduce operators $\mathcal{L}^0_{A,2(k-2)}$ such that $\mathcal{L}_A(t) = \sum_{k=2}^{N}\mathcal{L}_{A,2(k-2)}t$.

Corollary 7 *The operators $\mathcal{L}_{A,2(k-2)}$, $k = 2, \dots, N$, leave the ideal $\langle\mathcal{N}_2\rangle$ invariant. Their action on the elementary symmetric polynomials $e_m = y_{2m}$ is determined up to the ideal $\langle\mathcal{N}_2\rangle$ by a symmetric matrix $T_{k,m} = T_{k,m}(y_4, \dots, y_{2N})$.*

Proof: The proof of this assertion is similar to that of Corollary 6.
□

The Lie algebra $\mathcal{G}_P(N)$ of vector fields on $\mathcal{B}_N$ in the coordinates $y_4, \ldots, y_{2N}$ has the structure of a free $(N-1)$-dimensional module over the ring $\mathbb{C}[y_4, \ldots, y_{2N}]$ with generators $\mathcal{L}_{A,2(k-2)}$, $k = 2, \ldots, N$. The action of these generators on y_{2m} is given by a symmetric matrix $(T_{k,m}) = (T_{k,m}(N))$.

Remark 1 *For each N, we give an explicit construction of the fields $\mathcal{L}_{A,2(k-2)}$ and the symmetric matrix $(T_{k,m}(N))$. The notation $\mathcal{L}_A$ is suggested by Arnold's monograph [5] (see also [8]). These fields will be used in Section 9 to construct the Lie algebroids $\mathcal{G}(\mathcal{E}^2_{N,0})$ and explicitly describe an isomorphism between the Lie algebroid $\mathcal{G}(\mathcal{E}^2_{5,0})$ and the algebroid constructed in [10] from the universal bundle of Jacobians of genus 2 curves.*

3 Representations of the Witt Algebra W_{-1} in Lie Algebras with the Structure of a Free N-Dimensional Module over the Polynomial Ring

Let us introduce the following notion.

Definition 4 *We define an N-polynomial Lie algebra $W_{-1}(N)$ as the graded Lie algebra with*

- *the structure of a free left module over the graded ring $A(N) = \mathbb{C}[v_2, \ldots, v_{2N}]$, $\deg v_{2k} = 2k$;*
- *an infinite set of generators L^0_{2q}, $q = -1, 0, 1, \ldots, \deg L^0_{2q} = 2q$;*
- *a skew-symmetric operation $[\cdot, \cdot]$ such that*

$$\begin{aligned}
[L^0_{2q_1}, L^0_{2q_2}] &= 2(q_2 - q_1) L^0_{2(q_1+q_2)}, \\
[L_{2q_1}, v_{2k} L_{2q_2}] &= v_{2q_1,2k} L_{2q_2} + v_{2k}[L_{2q_1}, L_{2q_2}], \\
[v_{2k_1} L_{2q_1}, v_{2k_2} L_{2q_2}] &= v_{2k_1} v_{2q_1,2k_2} L_{2q_2} - v_{2k_2}[L_{2q_2}, v_{2k_1} L_{2q_1}],
\end{aligned}$$

where $v_{2q,2k} \in A(N)$ is a homogeneous polynomial $v_{2q,2k}(v_2, \ldots, v_{2N})$ of degree $2(q+k)$.

Using the identity $v_{k_1}(v_{k_2} L_{2q}) = (v_{k_1} v_{k_2}) L_{2q}$ and Leibniz' rule, we see that the skew-symmetric operation $[\cdot, \cdot]$ on the Lie algebra $W_{-1}(N)$ is completely determined by the set of homogeneous polynomials $v_{2q,2k} = v_{2q,2k}(v_2, \ldots, v_{2N})$.

Theorem 3 *The set of polynomials $v_{2q,2k} = v_{2q,2k}(v_2, \ldots, v_{2N}) \in A(N)$ determines a skew-symmetric operation on an N-polynomial Lie algebra $W_{-1}(N)$ if and only if the homomorphism*

$$\gamma \colon W_{-1}(N) \to Der A(N), \qquad \gamma(L^0_{2q}) = \sum_{k=1}^{N} v_{2q,2k} \frac{\partial}{\partial v_{2k}},$$

of $A(N)$-modules is a homomorphism of the N-polynomial Lie algebra $W_{-1}(N)$ to the Lie algebra of polynomial derivations of the ring $A(N) = \mathbb{C}[v_2, \ldots, v_{2N}]$.

Proof: The theorem is proved by a direct verification of its statements.
□

The Lie algebra W_{-1} with generators $\mathcal{L}^0_{2q}$, $q = -1, 0, 1, \dots,$ contains the Lie subalgebra generated by the three operators $\mathcal{L}^0_{-2}$, $\mathcal{L}^0_0$, and $\mathcal{L}^0_2$, where $[\mathcal{L}^0_{-2}, \mathcal{L}^0_2] = 4\mathcal{L}^0_0$. The Lie algebra W_{-1} with respect to the bracket $[\cdot,\cdot]$ is generated by only two generators, $\mathcal{L}^0_{-2}$ and $\mathcal{L}^0_4$.

Example 4 $6\mathcal{L}^0_2 = [\mathcal{L}^0_{-2}, \mathcal{L}^0_4]$, $4\mathcal{L}^0_0 = [\mathcal{L}^0_{-2}, \mathcal{L}^0_2]$, *and* $2\mathcal{L}^0_6 = [\mathcal{L}^0_2, \mathcal{L}^0_4]$.

The generators $\mathcal{L}_{2q}$, where $q \geq 1$, are given by the recurrence relation $2q\mathcal{L}^0_{2(q+2)} = [\mathcal{L}^0_2, \mathcal{L}^0_{2(q+1)}]$. Moreover, the operators $\mathcal{L}^0_{-2}$, $\mathcal{L}^0_4$ are related by commutation relations, the first of which is

$$[\mathcal{L}^0_2, [\mathcal{L}^0_2, [\mathcal{L}^0_2, \mathcal{L}^0_4]]] = 12[\mathcal{L}^0_4, [\mathcal{L}^0_2, \mathcal{L}^0_4]]. \tag{18}$$

Corollary 8 *The representations* $\gamma^j(\mathcal{L}^0_{2q}) = \sum_{k=1}^N v^j_{2q,2k}\frac{\partial}{\partial v_{2k}}$, $j = 1, 2$, *of the* N*–polynomial algebra* W_{-1} *coincide if and only if* $v^1_{2q,2k} \equiv v^2_{2q,2k}$ *for* $q = -1$ *and* 2.

By construction there is an embedding of the Lie algebra W_{-1} into the Lie algebra $W_{-1}(N)$. On the other hand, the ring homomorphism $\varphi\colon A(N) \to \mathbb{C}$, $\varphi(v_k) = 0$, $k = 1, \dots, N$, induces a projection $W_{-1}(N) \to W_{-1}$ of Lie algebras.

Corollary 9 *The homomorphism*

$$\gamma\colon W_{-1}(N) \to \mathcal{G}_{P,0}(N), \qquad \gamma(L^0_{2q}) = \mathcal{L}^0_{2q} = \sum_{k=1}^N 2k\mathcal{N}_{2(q+k)}\frac{\partial}{\partial \mathcal{N}_{2k}}, \quad \gamma(v_{2k}) = \mathcal{N}_{2k},$$

extends to an epimorphism of Lie algebras.

Note that the nontrivial relation (13) between Newton polynomials in $x_1, \dots, x_N$ ensures the fulfillment of the condition

$$\gamma([L_{2k}, L_{2k}]) = [\gamma(L_{2k}), \gamma(L_{2k})].$$

The kernel of the homomorphism γ is described by (9). The restriction of the homomorphism γ to the Lie subalgebra W_{-1} gives a representation of the Lie algebra W_{-1} in the Lie algebra $\mathcal{G}_{P,0}(N)$ with the structure of a free N-dimensional $\mathbb{C}[\mathcal{N}_2, \dots, \mathcal{N}_{2N}]$-module.

4 Commuting Vector Fields on the Symmetric Square of a Plane Curve

Consider the symmetric square of the curve $V = \{(X, Y) \in \mathbb{C}^2 : F(X, Y) = 0\}$, where $F(X, Y)$ are polynomials in X and Y. Let $\mathcal{D}_k = F(X_k, Y_k)_{Y_k}\partial_{X_k} - F(X_k, Y_k)_{X_k}\partial_{Y_k}$, $k = 1, 2$. We introduce the operators

$$\mathcal{L}^1 = \frac{1}{X_1 - X_2}(\mathcal{D}_1 - \mathcal{D}_2), \quad \mathcal{L}^2 = \frac{1}{X_1 - X_2}(X_2\mathcal{D}_1 - X_1\mathcal{D}_2). \tag{19}$$

Lemma 9 *1. The operators $\mathcal{L}^1$ and $\mathcal{L}^2$ are derivations of the function ring on Sym$^2(\mathbb{C}^2) \setminus \{X_1 - X_2 = 0\}$.*

2. The operators $\mathcal{L}^1$ and $\mathcal{L}^2$ annihilate the polynomials $F(X_1, Y_1)$ and $F(X_2, Y_2)$.

3. $[\mathcal{L}^1, \mathcal{L}^2] \equiv 0$.

Proof: The operators $\mathcal{L}^1$ and $\mathcal{L}^2$ are derivations of the function ring on $(\mathbb{C}^2 \times \mathbb{C}^2) \setminus \{X_1 - X_2 = 0\}$. Statements 1 and 2 are verified directly. A standard calculation shows that

$$[(X_1 - X_2)\mathcal{L}^1, (X_1 - X_2)\mathcal{L}^2] = -F(X_2, Y_2)_{Y_2}\mathcal{D}_1 - F(X_1, Y_1)_{Y_1}\mathcal{D}_2 + (X_1 - X_2)^2[\mathcal{L}^1, \mathcal{L}^2].$$

On the other hand, $[\mathcal{D}_1 - \mathcal{D}_2, X_2\mathcal{D}_1 - X_1\mathcal{D}_2] = -F(X_2, Y_2)_{Y_2}\mathcal{D}_1 - F(X_1, Y_1)_{Y_1}\mathcal{D}_2$. The coincidence of the left-hand sides of the equations and the relation $X_1 - X_2 \neq 0$ imply the lemma. □

5 Lie Algebroids on the Space of Nonsingular Hyperelliptic Curves

Consider the bundle $f\colon \mathcal{E}_{N,0} \to \mathcal{B}_{N,0}$ (see Definition 1). In Section 2 we described the Lie algebra of vector fields on $\mathcal{B}_{N,0}$ generated by the Newton fields $\mathcal{L}_{2k}^0$, $k = -1, 0, 1, \ldots, N-2$. In this section we construct a Lie algebroid on the space $\mathcal{E}_{N,0}$. We set $\pi = \pi(X, Y; \mathbf{x}) = Y^2 - P$, where $P = P(X; \mathbf{x}) = \prod_{i=1}^{N}(X - x_i)$. By $\mathcal{G}(\mathbb{C}[X, Y; \mathbf{x}])$ we denote the Lie algebra of derivations of the ring $\mathbb{C}[X, Y; \mathbf{x}]$. Let us introduce the operator $\mathcal{L}_{N-2}^* = 2Y\partial_X + P_X\partial_Y \in \mathcal{G}(\mathbb{C}[X, Y; \mathbf{x}])$. We have $\mathcal{L}_{N-2}^*\pi \equiv 0$. Hence, for fixed $\mathbf{x}$, the operator $\mathcal{L}_{N-2}^*$ determines a vector field on $\mathbb{C}^2$ that is tangent to the curve $V = \{(X, Y) \in \mathbb{C}^2 : \pi(X, Y; \mathbf{x}) = 0\}$. The field $\mathcal{L}_{N-2}^*$ determines the vertical field of the bundle $f\colon \mathcal{E}_{N,0} \to \mathcal{B}_{N,0}$.

Lemma 10 *Let $\mathcal{D}$ be a derivation of the form $a\partial_X + b\partial_Y$ of the ring $\mathbb{C}[X, Y; \mathbf{x}]$, where $\mathbf{x} \in \mathcal{B}_{N,0}$. Then $\mathcal{D}\pi = \Phi\pi$, where $\Phi \in \mathbb{C}[X, Y; \mathbf{x}]$, implies $\mathcal{D} = \psi\mathcal{L}_{N-2}^* + \pi\mathcal{D}^1$, where $\psi \in \mathbb{C}[X, Y; \mathbf{x}]$ and $\mathcal{D}^1 \in \mathcal{G}(\mathbb{C}[X, Y; \mathbf{x}])$.*

Proof: We shall carry out calculations in the ring $K = \mathbb{C}[X, Y; \mathbf{x}]/\langle\pi\rangle$. In this ring $Y^2 = P$, and thus K is a free $\mathbb{C}[X; \mathbf{x}]$-module with generators 1 and Y. We set $a = a_1 + a_2Y$ and $b = b_1 + b_2Y$, where $a_l, b_l \in \mathbb{C}[X; \mathbf{x}]$, $l = 1, 2$. The condition that $\mathcal{D}\pi = 0$ in the ring K implies $(a_1 + a_2Y)P_X = (b_1 + b_2Y)2Y$. Hence $a_1P_X = 2b_2P$ and $a_2P_X = 2b_1$. On the other hand, the condition $\mathcal{D} = \psi\mathcal{L}_{N-2}^*$, where $\psi = \psi_1 + \psi_2Y$, implies

$$a_1 + a_2Y = 2\psi_2P + 2\psi_1Y, \qquad b_1 + b_2Y = \psi_1P_X + \psi_2P_XY.$$

Hence $2\psi_1 = a_2$, $2\psi_2P = a_1$, and $\psi_2P_X = b_2$. Since $\mathbf{x} \in \mathcal{B}_{N,0}$, it follows that the polynomials $P(X; \mathbf{x})$ and $P_X(X; \mathbf{x})$ are coprime, and this system has a polynomial solution $\psi_2 = \psi_2(X; \mathbf{x})$. □

Consider the following sequence of derivations of the ring $\mathbb{C}[X, Y; \mathbf{x}]$:

$$L_{2k}^0 = \mathcal{L}_{2k}^0 + 2X^{k+1}\partial_X + C_{2k}Y\partial_Y, \quad \text{where } C_{2k} = \sum_{i=1}^{N} \frac{X^{k+1} - x_i^{k+1}}{X - x_i}. \tag{20}$$

Theorem 4 *The homogeneous fields L^0_{2k}, $k=-1,0,1,\dots,$ of degree $2k$ are uniquely determined by the condition that they are lifts of the Newton fields $\mathcal{L}^0_{2k}$ and generate the Lie algebra of Newton horizontal vector fields on the space of the bundle $\mathcal{E}_{N,0}$, that is,*

$$[L^0_{2q_1}, L^0_{2q_2}] = 2(q_2-q_1)L^0_{2(q_1+q_2)}.$$

Proof: We set $\widehat{\mathcal{L}}^0_{2k} = \mathcal{L}^0_{2k} + 2X^{k+1}\partial_X = 2(\sum_{i=1}^N x_i^{k+1}\partial_{x_i} + X^{k+1}\partial_X)$. The operator $\widehat{\mathcal{L}}^0_{2k}$ determines a Newton derivation of the ring $\mathbb{C}[X;\mathbf{x}]$. It is easy to check that (20) can be written as

$$L^0_{2k} = \widehat{\mathcal{L}}^0_{2k} + \frac{1}{2}(\widehat{\mathcal{L}}^0_{2k}\ln P)Y\partial_Y. \tag{21}$$

Hence $L^0_{2k}(Y^2-P) = P(\widehat{\mathcal{L}}^0_{2k}\ln P - \widehat{\mathcal{L}}^0_{2k}\ln P) \equiv 0$. Thus, formula (20) determines horizontal vector fields L^0_{2k}, $k=-1,0,1,\dots,$ on $\mathcal{E}_{N,0}$, which are lifts of the fields $\mathcal{L}^0_{2k}$ on the base $\mathcal{B}_{N,0}$.

Now let $L^{0,1}_{2k}$ and $L^{0,2}_{2k}$ be two homogeneous horizontal vector fields on $\mathcal{E}_{N,0}$ that are lifts of the field $\mathcal{L}^0_{2k}$ on the base $\mathcal{B}_{N,0}$. Then, according to Lemma 10, $L^{0,2}_{2k} = L^{0,1}_{2k} + \psi_{2k+2-N}\mathcal{L}^*_{N-2}$, where $\psi_{2k+2-N} = \psi_1 + \psi_2 Y$ and ψ_1, $\psi_2 \in \mathbb{C}[X;\mathbf{x}]$ are homogeneous polynomials such that $\deg\psi_1 = 2m = 2k+2-N$ and $\deg\psi_2 = 2(k+1-N)$.

Note that the degree of the function ψ_{2k+2-N} cannot be negative. Hence the condition $\psi_{2k+2-N} \neq 0$ implies $N \le 2k+2$. On the other hand, according to Corollary 6, the generators of the algebra W_{-1} are completely determined by the operators L^0_{-2} and L^0_4. As a result, we obtain the following conditions: $N\le 0$ for $k=-1$, $N\le 4$ for $k=1$, and $N \le 6$ for $k=2$. In the case where $k=2$ and $N=5$, we obtain $\deg\psi_1 = 1$, which contradicts $\deg\psi_1 = 2m$. Thus, we have proved that the lift of the fields $\mathcal{L}^0_{2k}$, $k=-1,0,1,\dots,$ is unique for $N=5$ and $N>6$.

It remains to consider the cases $N=3,4,6$. As shown above, in the case $N=6$, the lifts of the fields $\mathcal{L}^0_{-2}$ and $\mathcal{L}^0_2$ are unique, and any lift of $\mathcal{L}^0_4$ must have the form $L^0_4 + \alpha\mathcal{L}^*_4$, where $\alpha\in\mathbb{C}$. A direct verification shows that the commutation relation (see (18)) in the Witt algebra holds only for $\alpha=0$. Thus, in the case $N=6$, the lift of the fields $\mathcal{L}^0_{2k}$ is unique. Similar arguments show that this is also true in the cases $N=3$ and 4.

The commutation rule $[L^0_{2q_1}, L^0_{2q_2}] = 2(q_2-q_1)L^0_{2(q_1+q_2)}$ follows from the fact that $\widehat{\mathcal{L}}^0_{2k}$ is a Newton operator and from (21). This completes the proof of the theorem.
□

Corollary 10 *The generating function for the operators (20) has the form*

$$L^0(t) = \widehat{\mathcal{L}}^0(t) + \frac{1}{2}(\widehat{\mathcal{L}}^0(t)\ln P)Y\partial Y, \quad \textit{where } \widehat{\mathcal{L}}^0(t) = \mathcal{L}^0(t) + 2\,\frac{1}{1-Xt}\,\partial_X. \tag{22}$$

Consider the space $\mathbb{C}^{N+1}$ with the graded coordinates $(X,Y;\mathcal{N}_2,\dots,\mathcal{N}_{2(N-1)})$. Using the equation $Y^2 = P(X;\mathbf{x})$, we can identify the space $\mathcal{E}_{N,0}$ with an open dense subvariety in $\mathbb{C}^{N+1}$. The Lie algebra of vector fields on $\mathcal{E}_{N,0}$ described above determines a polynomial Lie algebra generated by the field $\mathcal{L}^*_{N-2}$ and the fields $L^0_{-2}, L^0_0, L^0_2, \dots, L^0_{2(N-2)}$.

Example 5 *Case $N=3$. The coordinates in $\mathbb{C}^4$ are X, Y, $\mathcal{N}_2$, and $\mathcal{N}_4$. We have*

$$\frac{1}{3}\mathcal{N}_6 = -Y^2 + X^3 - \mathcal{N}_2X^2 + \frac{1}{2}(\mathcal{N}_2^2 - \mathcal{N}_4)X + \left(\frac{1}{2}\mathcal{N}_2\mathcal{N}_4 - \frac{1}{6}\mathcal{N}_2^3\right).$$

Using this formula, we obtain an explicit expression for the basis polynomial fields $\mathcal{L}^*_1$, L^0_{-2}, L^0_0, and L^0_2 in $\mathbb{C}^4$.

6 Coordinate Rings of Spaces of Symmetric Squares of Hyperelliptic Curves

Consider the space $\mathbb{C}^2 \times \mathbb{C}^2$ with coordinates (X_1, Y_1) and (X_2, Y_2) graded as above, i.e., so that $\deg X_k = 2$ and $\deg Y_k = N$, $k = 1, 2$, and the space $\mathbb{C}^5$ with graded coordinates u_2, u_4, v_N, v_{N+2}, and v_{2N}. Here the subscript corresponds to the degree of variables.

Lemma 11 *The algebraic homogeneous map*

$$\xi\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^5, \qquad \xi((X_1, Y_1), (X_2, Y_2)) = (u_2, u_4, v_N, v_{N+2}, v_{2N}),$$

where $u_2 = X_1 + X_2$, $u_4 = (X_1 - X_2)^2$, $v_N = Y_1 + Y_2$, $v_{N+2} = (X_1 - X_2)(Y_1 - Y_2)$, *and* $v_{2N} = (Y_1 - Y_2)^2$, *makes it possible to identify the algebraic variety* $(\mathbb{C}^2 \times \mathbb{C}^2)/S_2$ *with the hypersurface in* $\mathbb{C}^5$ *determined by the equation* $u_4 v_{2N} - v_{N+2}^2 = 0$.

Proof: The lemma is proved directly.
□

For what follows we need the homogeneous polynomials $a_{2k}(u_2, u_4)$ of degree $2k$ determined by the generating series

$$\begin{aligned}\frac{1}{(1 - X_1 t)(1 - X_2 t)} &= a(t; u_2, u_4) = \sum_{k=0}^{\infty} a_{2k}(u_2, u_4) t^k \\ &= 4\,\frac{1}{(2 - u_2 t)^2 - u_4 t^2} = 1 + u_2 t + (t^2). \end{aligned} \tag{23}$$

In the notation of Lemma 11 we have

$$\sum_{k=0}^{\infty} (X_1^k + X_2^k) t^k = \frac{1}{1 - X_1 t} + \frac{1}{1 - X_2 t} = (2 - u_2 t) a(t; u_2, u_4). \tag{24}$$

Moreover,

$$\begin{aligned} &\sum_{k=2}^{\infty} (X_1 - X_2)(X_1^{k-1} - X_2^{k-1}) t^{k-2} \\ &\qquad = \sum_{k=2}^{\infty} [(X_1^k + X_2^k) t^{k-2} - X_1 X_2 (X_1^{k-2} + X_2^{k-2}) t^{k-2}] = u_4 a(t; u_2, u_4), \end{aligned} \tag{25}$$

$$\sum_{k=0}^{\infty} (Y_1 - Y_2)(X_1^k - X_2^k) t^k = (Y_1 - Y_2)\left[\frac{1}{1 - X_1 t} - \frac{1}{1 - X_2 t}\right] = v_{N+2} t a(t; u_2, u_4). \tag{26}$$

We have $Y_j^2 = X_j^N + \sum_{k=2}^{N} (-1)^k y_{2k} X_j^{N-k}$. Hence

$$\begin{aligned}(Y_1^2 + Y_2^2) &= \frac{1}{2}(v_N^2 + v_{2N}) \\ &= (2a_{2N} - u_2 a_{2N-2}) + \sum_{k=2}^{N-1} (-1)^k y_{2k} (2a_{2(N-k)} - u_2 a_{2(N-k-1)}) + (-1)^N 2 y_{2N}. \end{aligned}$$

We also have

$$(X_1 - X_2)(Y_1^2 - Y_2^2) = v_N v_{N+2} = u_4\left(a_{2(N-2)} + \sum_{k=2}^{N-2}(-1)^k y_{2k}a_{2(N-k-2)}\right),$$

$$(Y_1 - Y_2)(Y_1^2 - Y_2^2) = v_N v_{2N} = v_{N+2}\left(a_{2(N-1)} + \sum_{k=2}^{N-1}(-1)^k y_{2k}a_{2(N-k-1)}\right).$$

The graded coordinate ring $\mathcal{R}_0(N)$ of the space $\widehat{\mathcal{E}}^2_{N,0}$ in $(\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{X_1 - X_2 = 0\}) \times \mathcal{B}_{N,0}$ (see Section 1) has the form

$$\mathbb{C}[X_1, Y_1; X_2, Y_2; x_1, \ldots, x_N]/J,$$
$$\deg x_j = \deg X_k = 2,\ \deg Y_k = N,\ j = 1, \ldots, N,\ k = 1, 2,$$

where $J = \langle \pi_1, \pi_2\rangle$ is the ideal generated by the polynomials $\pi_k = \pi_k(X_k, Y_k; \mathbf{x})$. Let $\mathcal{R}_0^G(N) \subset \mathcal{R}_0(N)$ denote the invariant ring of the free action of G on $\mathcal{R}_0(N)$. Consider the graded ring $\mathcal{R}(N) = \mathcal{R}_0(N)/\langle y\rangle$, where $y = x_1 + \cdots + x_N$. Let $\mathcal{R}^G(N) \subset \mathcal{R}(N)$ denote the invariant ring of the free action of G on $\mathcal{R}(N)$. We shall treat the ring $\mathcal{R}^G(N)$ as the coordinate ring of the universal space $\mathcal{E}^2_N$.

Lemma 12 *The ring $\mathcal{R}^G(N)$ is isomorphic to the graded ring*

$$\mathcal{R}_U^G = \mathbb{C}[u_2, u_4, v_N, v_{N+2}, v_{2N}, \mathbf{y}]/J^G,$$

where $\mathbf{y} = (y_4, \ldots, y_{2N})$ and the ideal J^G has Gröbner basis

$$P_{2N+4} = v_{N+2}^2 - u_4 v_{2N},$$

$$P_{2N+2} = v_N v_{N+2} - u_4\left(a_{2N-2} + \sum_{k=2}^{N-1}(-1)^k y_{2k}a_{2(N-k-1)}\right),$$

$$P_{2N} = v_N^2 + v_{2N} - (a_{2N} - u_2 a_{2(N-1)}) - \sum_{k=2}^{N-1}(-1)^k y_{2k}(2a_{2(N-k)} - u_2 a_{2(N-k-1)}) - (-1)^N 2y_{2N},$$

$$P_{3N} = v_N v_{2N} - v_{N+2}\left(a_{2(N-1)} + \sum_{k=2}^{N-1}(-1)^k y_{2k}a_{2(N-k-1)}\right).$$

The relation $v_N P_{2N+4} - v_{N+2}P_{2N+2} + u_4 P_{3N} = 0$ holds.

Proof: The lemma follows easily from the relations obtained above.
□

Let us introduce the ring $\mathcal{A}(N) = \mathbb{C}[u_2, u_4, v_{N-2}, v_N, \widehat{\mathbf{y}}]$, where $\widehat{\mathbf{y}} = (y_4, \ldots, y_{2(N-2)})$.

Lemma 13 *The following ring homomorphism holds:*

$$\varphi\colon \mathcal{R}_U^G \to \mathcal{A}(N),$$
$$\varphi(u_{2k}) = u_{2k},\quad k = 1, 2,\qquad \varphi(v_N) = v_N,\quad \varphi(y_{2k}) = y_{2k},\qquad k = 2, \ldots, N-2,$$
$$\varphi(v_{N+2}) = u_4 v_{N-2},\quad \varphi(v_{2N}) = u_4 v_{N-2}^2,$$

$$\varphi(y_{2(N-1)}) = (-1)^{N-1}\left[v_{N-2}v_N - a_{2(N-1)} - \sum_{k=2}^{N-2}(-1)^k y_{2k}a_{2(N-k-1)}\right],$$

$$\varphi(y_{2N}) = (-1)^N \frac{1}{2}\left[(v_N^2 + v_{2N}) - (2a_{2N} - u_2 a_{2(N-1)}) - \sum_{k=2}^{N-1}(-1)^k y_{2k}(2a_{2(N-k)} - u_2 a_{2(N-k-1)})\right].$$

Proof: A direct verification shows that the homomorphism φ maps the ideal J^G to 0.
□

Corollary 11 *The homomorphism $\varphi[u_4^{-1}]\colon \mathcal{R}_U^G[u_4^{-1}] \to \mathcal{A}(N)[u_4^{-1}]$ is an isomorphism.*

Proof: The ring homomorphism

$$\eta\colon \mathcal{A}(N)[u_4^{-1}] \to \mathcal{R}_U^G[u_4^{-1}],$$
$$\eta(u_{2k}) = u_{2k}, \quad k = 1, 2, \qquad \eta(v_N) = v_N, \quad \eta(y_{2k}) = y_{2k}, \quad k = 2, \dots, N-2,$$
$$\eta(v_{N-2}) = u_4^{-1} v_{N+2}$$

is inverse to the homomorphism $\varphi[u_4^{-1}]$.
□

Consider the space $\mathbb{C}^{N+4}$ with the graded coordinates $(u_2, u_4, v_N, v_{N+2}, v_{2N}; \mathbf{y})$ and the space $\mathbb{C}^{N+1}$ with the graded coordinates $(u_2, u_4, v_{N-2}, v_N, \widehat{\mathbf{y}})$. As mentioned above, the space $\mathcal{E}_N^2$ can be identified with the algebraic subvariety in $\mathbb{C}^{N+4}$ determined by the equations $P_{2N+k} = 0$, $k = 0, 2, 4, N$ (see Lemma 12). We set

$$b_{2N}(u_2; \widehat{\mathbf{y}}) = \frac{1}{2^{N-1}} \left(u_2^N + \sum_{k=2}^{N-1} (-1)^k 2^k y_{2k} u_2^{N-k} + (-1)^N 2^N y_{2N} \right).$$

Let $W_s(N)$, $s = 1, 2$, denote the algebraic subvarieties in $\mathbb{C}^{N+4}$ determined by the equations

$$\begin{array}{lllll} \text{for } s = 1: & u_4 = 0, & v_N = 0, & v_{N+2} = 0, & v_{2N} = b_{2N}(u_2; \vec{\mathbf{y}}), \\ \text{for } s = 2: & u_4 = 0, & v_{N+2} = 0, & v_{2N} = 0, & v_N^2 = b_{2N}(u_2; \vec{\mathbf{y}}). \end{array}$$

We set $W(N) = W_1(N) \cup W_2(N)$. Note that, for given $\mathbf{y}$, the intersection $W_1(N) \cap W_2(N)$ is the set of roots of the equation $b_{2N}(u_2; \vec{\mathbf{y}}) = 0$. Clearly, $W(N) \subset \mathcal{E}_N^2$.

Theorem 5 *The mapping $f\colon \mathbb{C}^{N+1} \to \mathbb{C}^{N+4}$ defined by $f(u_2, u_4, v_{N-2}, v_N, \widehat{\mathbf{y}}) = (u_2, u_4, v_N, v_{N+2}, v_{2N}; \mathbf{y})$, where $v_{N+2} = u_4 v_{N-2}$, $v_{2N} = u_4 v_{N-2}^2$,*

$$y_{2(N-1)} = (-1)^{N-1} \left[v_N v_{N-2} - \left(a_{2(N-1)} + \sum_{k=2}^{N-2} (-1)^k y_{2k} a_{2(N-k-1)} \right) \right],$$

and

$$y_{2N} = (-1)^N \frac{1}{2} \left[v_N^2 + v_{2N} - (2a_{2N} - u_2 a_{2(k-1)}) - \sum_{k=2}^{N-1} (-1)^k y_{2k} (2a_{2(N-k)} - u_2 a_{2(N-k-1)}) \right],$$

determines a homomorphism $f\colon \mathbb{C}^{N+1} \setminus \{u_4 = 0\} \to \mathcal{E}_N^2 \setminus W(N)$.

Proof: The required assertion follows directly from Lemmas 12 and 13 and Corollary 11.
□

7 Lie Algebroids on the Space of Symmetric Squares of Nonsingular Hyperelliptic Curves

Consider the bundle $\mathcal{E}^2_{N,0} \to \mathcal{B}_{N,0}$ (see Definition 2). We have defined an action of the Witt algebra of Newton fields $\mathcal{L}^0_{2q}$ (see Definition 3) on the base $\mathcal{B}_{N,0}$. Let us introduce the following derivations of the ring $\mathbb{C}[X_1, Y_1; X_2, Y_2; \mathbf{x}]$:

$$L^0_{2k} = \mathcal{L}^0_{2k} + 2(X_1^{k+1}\partial_{X_1} + X_2^{k+1}\partial_{X_2}) + C^1_{2k}Y_1\partial_{Y_1} + C^2_{2k}Y_2\partial_{Y_2}, \tag{27}$$

where

$$C^j_{2k} = \sum_{i=1}^{N} \frac{X_j^{k+1} - x_i^{k+1}}{X_j - x_i}, \qquad j = 1, 2.$$

We set $\widehat{\mathcal{L}}^0_{2k} = \mathcal{L}^0_{2k} + 2(X_1^{k+1}\partial_{X_1} + X_2^{k+1}\partial_{X_2})$. It is verified directly that

$$L^0_{2k} = \widehat{\mathcal{L}}^0_{2k} + \frac{1}{2}\widehat{\mathcal{L}}^0_{2k}[(\ln P^{(1)})Y_1\partial_{Y_1} + (\ln P^{(2)})Y_2\partial_{Y_2}], \quad \text{where } P^{(j)} = \prod_{i=1}^{N}(X_j - x_i). \tag{28}$$

Now we are ready to obtain one of the main results of the present paper.

Theorem 6 *The homogeneous fields L^0_{2k}, $k = -1, 0, 1, \dots,$ of degree $2k$ (see (27)) are determined uniquely by the conditions that they are lifts of the Newton fields $\mathcal{L}^0_{2k}$ on the base of the bundle $\mathcal{E}^2_{N,0} \to \mathcal{B}_{N,0}$, generate the Lie algebra of Newton horizontal vector fields on $\mathcal{E}^2_{N,0}$, and determine a representation of the Lie algebra W_{-1}.*

Proof: The proof of the theorem uses explicit expressions and Lemma 10 by analogy with the proof of Theorem 4.
□

Using the operators $\mathcal{D}_k = 2Y_k\partial_{X_k} + P^{(k)}_{X_k}\partial Y_k$, $k = 1, 2$, we infer (see Lemma 9) that in the Lie algebroid of the bundle $\mathcal{E}^2_{N,0}$ there are the two commuting horizontal fields

$$\mathcal{L}^*_{N-4} = \frac{1}{X_1 - X_2}(\mathcal{D}_1 - \mathcal{D}_2) \qquad \text{and} \qquad \mathcal{L}^*_{N-2} = \frac{1}{X_1 - X_2}(X_2\mathcal{D}_1 - X_1\mathcal{D}_2). \tag{29}$$

Lemma 14 *For the curve $Y^2 = \prod_{i=1}^{N}(X - x_i)$,*

$$E(t) - (1 - tX)\sum_{m=1}^{N} E(t; m)X^{m-1}t^{m-1} = t^N Y^2.$$

Proof: We have

$$\begin{aligned} &E(t) - \sum_{m=1}^{N} E(t; m)X^{m-1}t^{m-1} + \sum_{m=1}^{N} E(t; m)X^m t^m \\ &\quad = (E(t) - E(t; 1)) + \sum_{m=1}^{N-1}(E(t; m) - E(t; m + 1))X^m t^m + E(t; N)X^N t^N \\ &\quad = (-1)^N y_{2N}t^N + \sum_{m=1}^{N-1}(-1)^m y_{2m}X^m t^N + X^N t^N = t^N Y^2. \qquad \square \end{aligned}$$

□

Let $L(t) = \sum_{k=-1}^{\infty} L^0_{2k}t^{k+1}$.

Theorem 7 *For the generating series $L(t)$ of the operators L^0_{2k}, $k = -1, 0, 1, \dots$ (see (27)), the relation*

$$E(t)L(t) = \sum_{m=1}^{N} E(t;m)t^{m-1}L^0_{2(m-2)} + \mathcal{A}_2(t)\mathcal{L}^*_{N-4} - \mathcal{A}_0(t)\mathcal{L}^*_{N-2} \tag{30}$$

holds on the variety $(X_1 - X_2 \neq 0, Y_1Y_2 \neq 0)$, where

$$\mathcal{A}_2(t) = t^N\left[Y_1\frac{X_1}{1-tX_1} + Y_2\frac{X_2}{1-tX_2}\right] = t^N a(t;u_2,u_4)\left[\frac{1}{2}(u_2v_N + v_{N+2}) - \frac{1}{4}t(u_2^2 - u_4)\right],$$
$$\mathcal{A}_0(t) = t^N\left[Y_1\frac{1}{1-tX_1} + Y_2\frac{1}{1-tX_2}\right] = t^N a(t;u_2,u_4)\left[v_N - \frac{1}{2}t(u_2v_N - v_{N+2})\right].$$

Proof: Let $\mathfrak{L} = E(t)L(t) - \sum_{m=1}^{N} E(t;m)t^{m-1}L^0_{2(m-2)}$. According to Corollary 3, we have $\mathfrak{L}x_i = 0$, $i = 1, \dots, N$. Thus, the field $\mathfrak{L}$ is vertical, and therefore $\mathfrak{L} = \mathcal{A}_2(t)\mathcal{L}^*_{N-4} - \mathcal{A}_0(t)\mathcal{L}^*_{N-2}$ for some series $\mathcal{A}_2(t)$ and $\mathcal{A}_0(t)$. On the other hand, according to Lemma 14,

$$\mathfrak{L}X_j = \frac{1}{1-tX_j}\left[E(t) - (1-tX_j)\sum_{m=1}^{N} E(t;m)X_j^{m-1}t^{m-1}\right] = \frac{t^NY_j^2}{1-tX_j}.$$

Using (29), we obtain the system of equations

$$\frac{1}{X_1 - X_2}(\mathcal{A}_2(t) - X_2\mathcal{A}_0(t)) = \frac{t^NY_1}{1-tX_1}, \qquad \frac{1}{X_2 - X_1}(\mathcal{A}_2(t) - X_1\mathcal{A}_0(t)) = \frac{t^NY_2}{1-tX_2},$$

provided that $Y_1Y_2 \neq 0$. Solving this system completes the proof of the theorem.
□

Corollary 12 *In the basis $\{L^0_{-2}, \dots, L^0_{2(N-2)}; \mathcal{L}^*_{N-4}, \mathcal{L}^*_{N-2}\}$ the following commutation relations hold:*

$$[L_{2p}, L_{2q}] = 2(q-p)L_{2(p+q)} = 2(q-p)\left(\sum_{m=1}^{N} \omega_{p+q,m}L_{2(m-2)} + \alpha_{p+q}\mathcal{L}^*_{N-4} - \beta_{p+q}\mathcal{L}^*_{N-2}\right),$$

where $\omega_{p+q,m}$, α_{p+q}, and β_{p+q} are the coefficients of t^{p+q} in the series $E(t;m)/E(t)$, $\mathcal{A}_2(t)/E(t)$, and $\mathcal{A}_0(t)/E(t)$. All these coefficients belong to the ring $\mathcal{R}^G(N)$ (see Lemma 12).

We set $\mathcal{N}(t) = \sum_{i=1}^{N} 1/(1 - x_it)$ and $\mathcal{D}_0(t) = \mathcal{N}(t) - 2(1/(1-tX_1) + 1/(1-tX_2)) = \mathcal{N}(t) - 2a(t)(2 - u_2t)$, where $a(t) = a(t;u_2,u_4)$ (see (23)).

Lemma 15 *The following relations hold:*

$$[L(t), \mathcal{L}^*_{N-4}]X_1 = \frac{t}{1-tX_1}\mathcal{D}_0(t)\mathcal{L}^*_{N-4}X_1, \tag{31}$$
$$[L(t), \mathcal{L}^*_{N-2}]X_1 = \left(\frac{2}{1-tX_2} + \frac{tX_2}{1-tX_1}\mathcal{D}_0(t)\right)\mathcal{L}^*_{N-4}X_1. \tag{32}$$

Proof: Using (27) and (29), we obtain

$$L(t)X_1 = \frac{2}{1-tX_1}, \qquad L(t)Y_1 = \frac{tY_1}{1-tX_1}\mathcal{N}(t),$$
$$\mathcal{L}^*_{N-4}X_1 = \frac{2Y_1}{X_1 - X_2}, \qquad \mathcal{L}^*_{N-2}X_1 = X_2\mathcal{L}^*_{N-4}X_1.$$

Hence

$$L(t)\mathcal{L}^*_{N-4}X_1 = \frac{t}{1-tX_1}\left(\mathcal{N}(t) - 2\,\frac{1}{1-tX_2}\right)\mathcal{L}^*_{N-4}X_1, \tag{33}$$

$$\mathcal{L}^*_{N-4}L(t)X_1 = \frac{2t}{(1-tX_1)^2}\,\mathcal{L}^*_{N-4}X_1. \tag{34}$$

Relations (33) and (34) imply (31). Further, we have $L(t)\mathcal{L}^*_{N-2}X_1 = L(t)(X_2\mathcal{L}^*_{N-4}X_1)$. Using (33), we obtain

$$L(t)\mathcal{L}^*_{N-2}X_1 = \left[\frac{2}{1-tX_2} + \frac{tX_2}{1-tX_1}\left(\mathcal{N}(t) - 2\,\frac{1}{1-tX_2}\right)\right]\mathcal{L}^*_{N-4}X_1, \tag{35}$$

$$\mathcal{L}^*_{N-2}L(t)X_1 = \frac{2tX_2}{(1-tX_1)^2}\,\mathcal{L}^*_{N-4}X_1. \tag{36}$$

Relations (35) and (36) imply (32).
□

We set

$$\mathcal{A}_{-2}(t) = ta(t)\mathcal{D}_0(t), \quad \mathcal{A}_{-4}(t) = t\mathcal{A}_{-2}(t), \tag{37}$$

$$\mathcal{B}_0(t) = a(t)\left[2(1-u_2t) + \frac{t^2}{4}(u_2^2 - u_4)\mathcal{D}_0(t)\right], \tag{38}$$

$$\mathcal{B}_{-2}(t) = ta(t)[2 + (1-u_2t)\mathcal{D}_0(t)]. \tag{39}$$

Theorem 8 *On the variety $\{X_1 - X_2 \neq 0,\, Y_1Y_2 \neq 0\}$ the commutation formulas for the generating series $L(t)$ of horizontal fields with the vertical fields $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$ are*

$$[L(t), \mathcal{L}^*_{N-4}] = \mathcal{A}_{-2}(t)\mathcal{L}^*_{N-4} - \mathcal{A}_{-4}(t)\mathcal{L}^*_{N-2}, \tag{40}$$

$$[L(t), \mathcal{L}^*_{N-2}] = \mathcal{B}_0(t)\mathcal{L}^*_{N-4} - \mathcal{B}_{-2}(t)\mathcal{L}^*_{N-2}. \tag{41}$$

Proof: The series $[L(t), \mathcal{L}^*_{N-4}]$ and $[L(t), \mathcal{L}^*_{N-2}]$ are generating series for the vertical fields. Let us find their representation in the form of linear combinations of the fields $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$. According to Lemma 15, the coefficients $\mathcal{A}_{-2}(t)$, $\mathcal{A}_{-4}(t)$, $\mathcal{B}_0(t)$, and $\mathcal{B}_{-2}(t)$ are solutions of the systems of equations

$$\mathcal{A}_{-2}(t) - X_2\mathcal{A}_{-4}(t) = \frac{t}{1-tX_1}\mathcal{D}_0(t),$$
$$\mathcal{A}_{-2}(t) - X_1\mathcal{A}_{-4}(t) = \frac{t}{1-tX_2}\mathcal{D}_0(t)$$

and

$$\mathcal{B}_0(t) - X_2\mathcal{B}_{-2}(t) = \frac{2}{1-tX_2} + \frac{tX_2}{1-tX_1}\mathcal{D}_0(t),$$
$$\mathcal{B}_0(t) - X_1\mathcal{B}_{-2}(t) = \frac{2}{1-tX_1} + \frac{tX_1}{1-tX_2}\mathcal{D}_0(t).$$

Solving these systems, we obtain (37)–(39). □

Corollary 13 *In the basis $\{L^0_{-2}, \dots, L^0_{2(N-2)}; \mathcal{L}^*_{N-4}, \mathcal{L}^*_{N-2}\}$ the following commutation relations hold:*

$$[L_{2q}, \mathcal{L}^*_{N-4}] = a_{-2,2q+2}\mathcal{L}^*_{N-4} - a_{-4,2q+2}\mathcal{L}^*_{N-2},$$
$$[L_{2q}, \mathcal{L}^*_{N-2}] = b_{0,2q+2}\mathcal{L}^*_{N-4} - b_{-2,2q+2}\mathcal{L}^*_{N-2},$$

where $a_{-2,2q+2}$, $a_{-4,2q+2}$, $b_{0,2q+2}$, and $b_{-2,2q+2}$ are the coefficients of t^{q+1} in the series $\mathcal{A}_{-2}(t)$, $\mathcal{A}_{-4}(t)$, $\mathcal{B}_0(t)$, and $\mathcal{B}_{-2}(t)$. All these coefficients lie in the ring $\mathcal{R}^G(N)$ (see Lemma 12).

8 Polynomial Lie Algebroids Determined by the Lie Algebroid on $\mathcal{E}^2_{N,0}$

In this section we give a description of the polynomial Lie algebroid $\mathcal{G}(N)$ on $\mathbb{C}^{N+1}$, which uses the homomorphism $f\colon \mathbb{C}^{N+1} \setminus \{u_4 = 0\} \to \mathcal{E}^2_{N,0} \setminus W(N)$ constructed in Section 6. As generators of the Lie algebroid $\mathcal{G}(N)$ we take the horizontal vector fields L^0_{2k} and the vertical fields $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$, which were constructed in Section 7. Without loss of generality, it is sufficient to consider the algebroid $\mathcal{G}$ as a module over the polynomial ring $\mathcal{A}(N) = \mathbb{C}[u_2, u_4, v_{N-2}, v_N; \mathbf{y}]$.

Lemma 16 *The action of the operators $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$ on the coordinate functions u_2 and u_4 has the form*

$$\begin{aligned} \mathcal{L}^\star_{N-4}u_2 &= 2v_{N-2}, & \mathcal{L}^\star_{N-2}u_2 &= u_2v_{N-2} - v_N,\\ \mathcal{L}^\star_{N-4}u_4 &= 4v_N, & \mathcal{L}^\star_{N-2}u_4 &= 2u_2v_N - 2u_4v_{N-2}. \end{aligned}$$

Proof: The required relations are derived directly from our results obtained above.
□

The action of the operators $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$ in the coordinates $X_1, Y_1; X_2, Y_2; \mathbf{x}$ has the form

$$\mathcal{L}^\star_{N-4}v_{N-2} = \mathcal{L}^\star_{N-4}\left(\frac{Y_1 - Y_2}{X_1 - X_2}\right) = \frac{2Y_2^2 - 2Y_1^2 + (X_1 - X_2)(P^{(1)}_{X_1} + P^{(2)}_{X_2})}{(X_1 - X_2)^3}, \tag{42}$$

$$\mathcal{L}^\star_{N-4}v_N = \mathcal{L}^\star_{N-4}(Y_1 + Y_2) = \frac{P^{(1)}_{X_1} - P^{(2)}_{X_2}}{X_1 - X_2}, \tag{43}$$

$$\mathcal{L}^\star_{N-2}v_{N-2} = \mathcal{L}^\star_{N-2}\left(\frac{Y_1 - Y_2}{X_1 - X_2}\right) = \frac{2(Y_2 - Y_1)(X_1Y_1 + X_2Y_1) + (X_1 - X_2)(X_2P^{(1)}_{X_1} + X_1P^{(2)}_{X_2})}{(X_1 - X_2)^3}, \tag{44}$$

$$\mathcal{L}^\star_{N-2}v_N = \mathcal{L}^\star_{N-2}(Y_1 + Y_2) = \frac{X_2P^{(1)}_{X_1} - X_1P^{(2)}_{X_2}}{X_1 - X_2}. \tag{45}$$

Our goal is to show that this action is polynomial in the coordinates $u_2, u_4, v_{N-2}, v_N, \mathbf{y}$. We shall use the polynomials $a_{2k}(u_2, u_4)$ (see (23)) and the polynomials $b_{2n}(u_2, u_4)$ for which $\sum b_{2n} \cdot t^n = a^2(t)$.

Lemma 17 *The action of the operators $\mathcal{L}^*_{N-4}$ and $\mathcal{L}^*_{N-2}$ on the coordinate functions v_{N-2} and v_N has the form*

$$\mathcal{L}^\star_{N-4}v_{N-2} = \sum_{k=0}^{N-3}(-1)^k y_{2k} b_{2N-2k-6}, \tag{46}$$

$$\mathcal{L}^\star_{N-4}v_N = \sum_{k=0}^{N-1}(-1)^k (N-k) y_{2k} a_{2N-2k-4}, \tag{47}$$

$$\mathcal{L}^\star_{N-2}v_{N-2} = v_{N-2}^2 + \frac{1}{2}\,u_2\sum_{k=0}^{N-3}(-1)^k y_{2k} b_{2N-2k-6} - \frac{1}{2}\sum_{k=0}^{N-1}(-1)^k (N-k) y_{2k} a_{2N-2k-4}, \tag{48}$$

$$\mathcal{L}^\star_{N-2}v_N = \sum_{k=0}^{N-1}(-1)^k (N-k) y_{2k}(u_2 a_{2N-2k-4} - a_{2N-2k-2}). \tag{49}$$

To prove this lemma, we need the following general statement.

Lemma 18 *The formula*

$$r(P) = \frac{2(P^{(2)} - P^{(1)}) + (X_1 - X_2)(P^{(1)}_{X_1} + P^{(2)}_{X_2})}{(X_1 - X_2)^3} \tag{50}$$

defines a linear map $r\colon \mathbb{C}[X;\mathbf{y}] \to \mathbb{C}[X_1;X_2;\mathbf{y}]$ *of* $\mathbb{C}[\mathbf{y}]$*-modules.*

Proof: The transform (50) is $\mathbb{C}[\mathbf{y}]$-linear; thus, it suffices to prove that $r(X^k) \in \mathbb{C}[X_1,X_2;\mathbf{y}]$, $k = 0,1,\dots$. Let us take the generating series $f(t;X) = \sum_{k=0}^{\infty} X^k t^k = (1-tX)^{-1}$. We obtain $r(f(t,X)) = t^3 a^2(t)$, where $a(t) = a(t;u_2,u_4)$ (see (23)). Thus, we have $r(1) = r(X) = r(X^2) = 0$ and $r(X^k) = b_{2k-6}$ for $k \geq 3$, where the b_{2n} are polynomials with generating series $\sum_{n=0}^{\infty} b_{2n} t^n = a^2(t)$.
□

We proceed to prove Lemma 17. Using (42), we derive (46). Relation (48) can be obtained by evaluating $\mathcal{L}^\star_{N-4} v_{N-2}$, since (44) can be rewritten as

$$\mathcal{L}^\star_{N-2} v_{N-2} = \left(\frac{Y_1 - Y_2}{X_1 - X_2}\right)^2 + \frac{(X_1 + X_2)}{2}\mathcal{L}^\star_{N-4} v_{N-2} - \frac{1}{2}\left(\frac{P^{(1)}_{X_1} - P^{(2)}_{X_2}}{X_1 - X_2}\right),$$

and applying the relation

$$P_X = \sum_{k=0}^{N-1} (-1)^k (N-k) y_{2k} X^{N-k-1}.$$

The expression (47) for $\mathcal{L}^\star_{N-4} v_N$ is obtained by using (25). Relation (45) can be rewritten as

$$\mathcal{L}^\star_{N-2} v_N = \frac{1}{2}(X_1 + X_2)\mathcal{L}^\star_{N-4} v_N - \frac{1}{2}(P^{(1)}_{X_1} + P^{(2)}_{X_2}).$$

Again applying (24), we obtain (49), which proves the lemma.

Thus, we have proved the following theorem, which is one of the main results of the present paper.

Theorem 9 *For each* $N \geq 3$, *a Lie* $\mathbb{C}[u_2,u_4,v_{N-2},v_N;\mathbf{y}]$*-algebra with generators* $L^0_{-2},\dots,L^0_{2(N-2)},\mathcal{L}^*_{N-4},\mathcal{L}^*_{N-2}$ *is defined. The commutation relations between these generators are described in Corollaries 12 and 13, and their action on* $u_2,u_4,v_{N-2},v_N,\mathbf{y}$, *in Lemmas 16 and 17.*

9 Examples of Polynomial Lie Algebras

In this section we give an explicit description of the polynomial Lie algebras $\mathcal{G}(N)$, $N = 3,4,5$, over the rings $\mathbb{C}[u_2,u_4,v_1,v_3]$ for $N = 3$, $\mathbb{C}[u_2,u_4,v_2,v_4;y_4]$ for $N = 4$, and $\mathbb{C}[u_2,u_4,v_3,v_5;y_4,y_6]$ for $N = 5$ with generators $\mathcal{L}_0,\dots,\mathcal{L}_{2(N-2)},\mathcal{L}^*_{N-4},\mathcal{L}^*_{N-2}$. Here L_0 is the Euler field and, therefore, $[L_0,L_{2k}] = 2kL_{2k}$.

Proof: Case $\boldsymbol{N=3}$ We have

$$y_4 = \tfrac{1}{4}(-3u_2^2 - u_4 + 4v_1v_3), \quad y_6 = \tfrac{1}{4}(-u_2^3 + u_2u_4 - u_4v_1^2 + 2u_2v_1v_3 - v_3^2).$$

The action of the generators L_0, L_2, and $\mathcal{L}^*_{-1}$, $\mathcal{L}^*_1$ of the free left $\mathbb{C}[u_2, u_4, v_1, v_3]$-module is as follows:

	u_2	u_4	v_1	v_3
L_0	$2u_2$	$4u_4$	v_1	$3v_3$
L_2	$\frac{1}{3}(3u_2^2 - u_4 - 8v_1v_3)$	$-4u_2u_4$	$\frac{1}{2}(u_2v_1 - 3v_3)$	$-\frac{3}{2}(u_4v_1 + u_2v_3)$
$\mathcal{L}^*_{-1}$	$2v_1$	$4v_3$	1	$3u_2$
$\mathcal{L}^*_1$	$u_2v_1 - v_3$	$-2(u_4v_1 - u_2v_3)$	$-u_2 + v_1^2$	$\frac{1}{2}(3u_2^2 - u_4 - 2v_1v_3)$

The commutation relations are

$$[\mathcal{L}^*_{-1}, L_2] = -3u_2\mathcal{L}^*_{-1} + \mathcal{L}^*_1, \qquad [\mathcal{L}^*_1, L_2] = \tfrac{1}{12}(9u_4 - 9u_2^2 + 16y_4)\mathcal{L}^*_{-1}.$$

□

Proof: Case $\boldsymbol{N = 4}$ We have

$$y_6 = \tfrac{1}{4}(2u_2^3 + 2u_2u_4 - 4v_2v_4 + 4u_2y_4),$$
$$y_8 = \tfrac{1}{16}(3u_2^4 - 2u_2^2u_4 - u_4^2 + 4u_4v_2^2 - 8u_2v_2v_4 + 4v_4^2 + 4u_2^2y_4 - 4u_4y_4).$$

The action of the generators L_0, L_2, L_4, and $\mathcal{L}^\star_0$, $\mathcal{L}^\star_2$ of the free left $\mathbb{C}[u_2, u_4, v_2, v_4, y_4]$-module is as follows:

	y_4	u_2	u_4
L_0	$4y_4$	$2u_2$	$4u_4$
L_2	$6y_6$	$-u_2^2 - u_4 - 2y_4$	$-4u_2u_4$
L_4	$8y_8$	$\frac{1}{2}(u_2^3 + 3u_2u_4 + 4u_2y_4 - 6y_6)$	$u_4(3u_2^2 + u_4 + 4y_4)$
$\mathcal{L}^\star_0$	0	$2v_2$	$4v_4$
$\mathcal{L}^\star_2$	0	$u_2v_2 - v_4$	$-2(u_4v_2 - u_2v_4)$

	v_2	v_4
L_0	$2v_2$	$4v_4$
L_2	$-2v_4$	$-2(u_4v_2 + u_2v_4)$
L_4	$\frac{1}{2}(-u_2^2v_2 + u_4v_2 + 4u_2v_4)$	$2u_2u_4v_2 + u_2^2v_4 + u_4v_4 + 2v_4y_4$
$\mathcal{L}^\star_0$	$2u_2$	$3u_2^2 + u_4 + 2y_4$
$\mathcal{L}^\star_2$	$\frac{1}{2}(-u_2^2 - u_4 + 2v_2^2 - 2y_4)$	$u_2^3 - u_2u_4 + y_6$

The commutation relations are

$$\begin{aligned}
[L_2, L_4] &= y_6L_0 - y_4L_2 - (u_4v_2 + u_2v_4)\mathcal{L}^\star_0 + 2v_4\mathcal{L}^\star_2,\\
[\mathcal{L}^\star_0, L_2] &= -2u_2\mathcal{L}^\star_0,\\
[\mathcal{L}^\star_0, L_4] &= (3u_2^2 + u_4 + 2y_4)\mathcal{L}^\star_0 - 2u_2\mathcal{L}^\star_2,\\
[\mathcal{L}^\star_2, L_2] &= \tfrac{1}{2}(u_4 - u_2^2 + 2y_4)\mathcal{L}^\star_0,\\
[\mathcal{L}^\star_2, L_4] &= \tfrac{1}{2}(2u_2^3 - 2u_2u_4 + 3y_6)\mathcal{L}^\star_0 - \tfrac{1}{2}(u_2^2 - u_4)\mathcal{L}^\star_2.
\end{aligned}$$

□

Proof: Case $\boldsymbol{N = 5}$ We have

$$\begin{aligned}
y_8 &= \tfrac{1}{16}(-5u_2^4 - 10u_2^2u_4 - u_4^2 + 16v_3v_5 - 12u_2^2y_4 - 4u_4y_4 + 16u_2y_6),\\
y_{10} &= \tfrac{1}{16}(-2u_2^5 + 2u_2u_4^2 - 4u_4v_3^2 + 8u_2v_3v_5\\
&\qquad - 4v_5^2 - 4u_2^3y_4 + 4u_2u_4y_4 + 4u_2^2y_6 - 4u_4y_6).
\end{aligned}$$

The action of the generators L_0, L_2, L_4, L_6, and $\mathcal{L}_1^\star$, $\mathcal{L}_3^\star$ of the free left $\mathbb{C}[u_2, u_4, v_3, v_5, y_4, y_6]$-module is as follows:

	y_4	y_6
L_0	$4y_4$	$6y_6$
L_2	$6y_6$	$-\frac{4}{5}(3y_4^2 - 10y_8)$
L_4	$8y_8$	$-\frac{2}{5}(4y_4y_6 - 25y_{10})$
L_6	$10y_{10}$	$-\frac{4}{5}y_4y_8$
$\mathcal{L}_1^*$	0	0
$\mathcal{L}_3^*$	0	0

	u_2	u_4
L_0	$2u_2$	$4u_4$
L_2	$\frac{1}{5}(-5u_2^2 - 5u_4 - 8y_4)$	$-4u_2u_4$
L_4	$\frac{1}{10}(5u_2^3 + 15u_2u_4 + 20u_2y_4 - 24y_6)$	$u_4(3u_2^2 + u_4 + 4y_4)$
L_6	$\frac{1}{20}(-5u_2^4 - 30u_2^2u_4 - 5u_4^2 - 20u_2^2y_4$ $-20u_4y_4 + 40u_2y_6 - 64y_8)$	$-2u_4(u_2^3 + u_2u_4 + 2u_2y_4 - 2y_6)$
$\mathcal{L}_1^*$	$2v_3$	$4v_5$
$\mathcal{L}_3^*$	$u_2v_3 - v_5$	$-2(u_4v_3 - u_2v_5)$

	v_3
L_0	$3v_3$
L_2	$\frac{1}{2}(-u_2v_3 - 5v_5)$
L_4	$\frac{1}{4}(10u_2v_5 - (u_2^2 - 3v_4 - 4y_4)v_3)$
L_6	$\frac{1}{8}(3u_2^3v_3 - 7u_2u_4v_3 - 15u_2^2v_5 - 5u_4v_5 + 4u_2v_3y_4 - 12v_5y_4)$
$\mathcal{L}_1^*$	$\frac{1}{2}(5u_2^2 + u_4 + 2y_4)$
$\mathcal{L}_3^*$	$\frac{1}{2}(-2u_2u_4 + 2v_3^2 - 2u_2y_4 + 2y_6)$

	v_5
L_0	$5v_5$
L_2	$-\frac{5}{2}(u_4v_3 + u_2v_5)$
L_4	$\frac{1}{4}(10u_2u_4v_3 + 5u_2^2v_5 + 5u_4v_5 + 12v_5y_4)$
L_6	$\frac{1}{8}(-15u_2^2u_4v_3 - 5u_4^2v_3 - 5u_2^3v_5 - 15u_2u_4v_5 - 12u_4v_3y_4 - 12u_2v_5y_4 + 16v_5y_6)$
$\mathcal{L}_1^*$	$\frac{1}{2}(5u_2^3 + 5u_2u_4 + 6u_2y_4 - 4y_6)$
$\mathcal{L}_3^*$	$\frac{1}{4}(5u_2^4 - u_4^2 - 4v_3v_5 + 6u_2^2y_4 - 2u_4y_4 - 4u_2y_6)$

The commutation relations are

$$
\begin{aligned}
[L_2, L_4] &= 2L_6 - \tfrac{8}{5}y_4L_2 + \tfrac{8}{5}y_6L_0,\\
[L_2, L_6] &= -4v_5\mathcal{L}_3^\star + 2(u_4v_3 + u_2v_5)\mathcal{L}_1^\star - \tfrac{4}{5}y_4L_4 + \tfrac{4}{5}y_8L_0,\\
[L_4, L_6] &= (u_4v_3 + u_2v_5)\mathcal{L}_3^\star - \tfrac{1}{2}(2u_2u_4v_3 + u_2^2v_5 + u_4v_5)\mathcal{L}_1^\star + 2y_4L_6 - \tfrac{6}{5}y_6L_4 + \tfrac{6}{5}y_8L_2 - 2y_{10}L_0,\\
[\mathcal{L}_1^\star, L_2] &= -\mathcal{L}_3^\star - u_2\mathcal{L}_1^\star,\\
[\mathcal{L}_1^\star, L_4] &= -u_2\mathcal{L}_3^\star + \tfrac{1}{4}(9u_2^2 + 3u_4 + 4y_4)\mathcal{L}_1^\star,\\
[\mathcal{L}_1^\star, L_6] &= \tfrac{1}{4}(9u_2^2 + 3u_4 + 4y_4)\mathcal{L}_3^\star - \tfrac{1}{2}(5u_2^3 + 5u_2u_4 + 6u_2y_4 - 4y_6)\mathcal{L}_1^\star,\\
[\mathcal{L}_3^\star, L_2] &= \tfrac{1}{20}(-5u_2^2 + 5u_4 + 16y_4)\mathcal{L}_1^\star,\\
[\mathcal{L}_3^\star, L_4] &= -\tfrac{1}{4}(u_2^2 - u_4 + 4y_4)\mathcal{L}_3^\star + \tfrac{3}{20}(5u_2^3 - 5u_2u_4 + 8y_6)\mathcal{L}_1^\star,\\
[\mathcal{L}_3^\star, L_6] &= -\tfrac{1}{80}(75u_2^4 - 50u_2^2u_4 - 25u_4^2 + 60u_2^2y_4 - 60u_4y_4 - 128y_8)\mathcal{L}_1^\star + \tfrac{3}{4}u_2(u_2^2 - u_4)\mathcal{L}_3^\star.
\end{aligned}
$$

□

Theorem 10 *There is an isomorphism of graded rings*

$$\varphi\colon \mathbb{C}[u_2, u_4, v_3, v_5; y_4, y_6] \to \mathbb{C}[x_2, x_3, x_4; z_4, z_5, z_6],$$

which determines an isomorphism of the polynomial Lie algebra described above (for $N = 5$) and the polynomial Lie algebra constructed in [10] on the basis of the theory of two-dimensional sigma-functions.

Proof: The isomorphism φ and its inverse are given by

$$\begin{array}{ll}
u_2 \to x_2, & x_2 \to u_2, \\
v_3 \to \frac{1}{2}x_3, & x_3 \to 2v_3, \\
u_4 \to x_2^2 + 4z_4, & x_4 \to 5u_2^2 + u_4 + 2y_4, \\
v_5 \to \frac{1}{2}(x_2x_3 + 2z_5), & z_4 \to \frac{1}{4}(u_4 - u_2^2), \\
y_4 \to \frac{1}{2}(-6x_2^2 + x_4 - 4z_4), & z_5 \to v_5 - u_2v_3, \\
y_6 \to -\frac{1}{4}(-8x_2z_4 + 8x_2^3 - 2x_4x_2 + x_3^2 + 2z_6), & z_6 \to 2(u_2y_4 + u_2u_4 - v_3^2 - y_6).
\end{array}$$

A direct verification shows that this isomorphism determines the required isomorphism of polynomial Lie algebras, which is the identity isomorohism at the generators L_0, L_2, L_4, L_6, $\mathcal{L}_1^0$, and $\mathcal{L}_3^0$. This proves the theorem.
□

Consider the sigma-function $\sigma = \sigma(u; \widehat{y})$, $u = (u_1, u_3)$ (see [11]) associated with the curve

$$\{(X, Y) \in \mathbb{C}^2 : Y^2 = X^5 + y_4X^3 - y_6X^2 + y_8X - y_{10}\}. \tag{51}$$

In the notation of [10] the functions

$$\wp_{i,3j} = \wp_{i,3j}(u; \widehat{y}) = -\frac{\partial^{i+j}}{\partial u_1^i \partial u_3^j} \ln \sigma,$$

which are Abelian in u, are defined. According to [10], the polynomial Lie algebra over $\mathbb{C}[x_2, x_3, x_4; z_4,$
$z_5, z_6]$ specified in Theorem 10 has a realization in terms of vector fields on the universal bundle of the Jacobians of curves of the form (51). In this realization $x_{q+2} = \wp_{q+2,0}$ and $z_{q+4} = \wp_{q+1,3}$ for $q = 0, 1, 2$, $\mathcal{L}_1^* = \partial/\partial u_1$, and $\mathcal{L}_3^* = \partial/\partial u_3$.

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