Inhabitants of interesting subsets of the Bousfield lattice

Andrew D. Brooke-Taylor[a], Benedikt Löwe[b,c,d], Birgit Richter[c]

[a]School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom; a.d.brooke-taylor@leeds.ac.uk
[b]Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus 94242, 1090 GE Amsterdam, The Netherlands; b.loewe@uva.nl
[c]Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany; birgit.richter@uni-hamburg.de
[d]Churchill College & Faculty of Mathematics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, United Kingdom

Abstract

In 1979, Bousfield defined an equivalence relation on the stable homotopy category. The set of Bousfield classes has some important subsets such as the distributive lattice, **DL**, of all classes $\langle E \rangle$ which are smash idempotent and the complete Boolean algebra, **cBA**, of closed classes. We provide examples of spectra that are in **DL**, but not in **cBA**; in particular, for every prime p, the Bousfield class of the Eilenberg-MacLane spectrum $\langle H\mathbb{F}_p \rangle$ is in $\mathbf{DL} \setminus \mathbf{cBA}$.

1. Introduction

An important tool for understanding structural and computational phenomena in the stable homotopy category (*i.e.*, the homotopy category of spectra) is the Bousfield localization at a spectrum E, L_E [2]. In [1], Bousfield defines an equivalence relation on spectra such that the localization functor L_E depends only on the equivalence class of the spectrum E. These equivalence classes, called *Bousfield classes*, form a lattice.

*The first author acknowledges the financial support of the Research Networking Programme INFTY funded by the *European Science Foundation* (ESF), the *Japan Society for the Promotion of Science* (JSPS) via a Postdoctoral Fellowship for Foreign Researchers and JSPS Grant-in-Aid 2301765, and the *Engineering and Physical Sciences Research Council* (EPSRC) via the Early Career Fellowship *Bringing Set Theory & Algebraic Topology Together* (EP/K035703/1).

The second author acknowledges financial support in the form of a VLAC fellowship-in-residence at the *Koninklijke Vlaamse Academie von België voor Wetenschappen en Kunsten* and an International Exchanges grant of the *Royal Society* (reference IE141198).

All three authors were Visiting Fellows of the research programme *Semantics & Syntax* (SAS) at the *Isaac Newton Institute for Mathematical Sciences* in Cambridge, England whilst part of this research was done.

In the original paper [1] introducing the Bousfield lattice **B**, Bousfield also introduces its subsets **BA** and **DL** and identifies the location of many explicit Bousfield classes. In [5, Definition 6.3], Hovey and Palmieri add a third interesting subset, denoted by **cBA**. (We shall give definitions below.) It is easy to see that

$$\mathbf{BA} \subseteq \mathbf{cBA} \subseteq \mathbf{DL} \subseteq \mathbf{B}.$$

In this paper, we deal with the question of which and how many classes of spectra live in the various parts of **B** defined by this chain of inclusions. We give lower bounds for the cardinality of $\mathbf{DL}\backslash\mathbf{cBA}$ and $\mathbf{cBA}\backslash\mathbf{BA}$ by identifying concrete examples of Bousfield classes in these complements. The cardinality results of this paper are graphically represented as in Figure 1 and concern the dark grey parts.

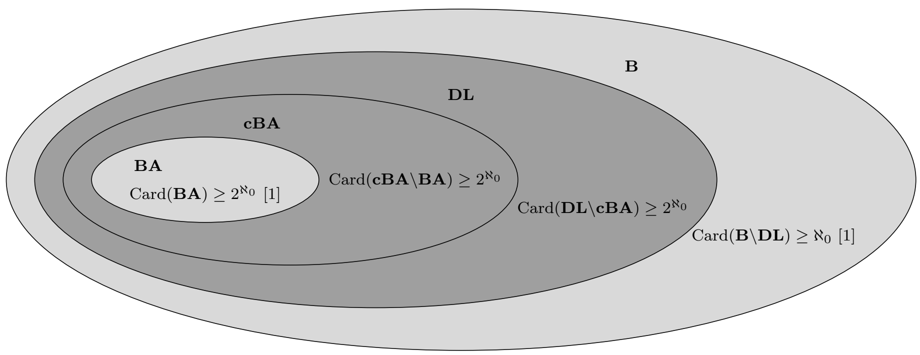

Figure 1: Lower bounds for the sizes of the four differences of subsets of **B**.

2. Definitions

In order to fix notation, we give the relevant definitions, following closely the exposition in [5]. We consider the Bousfield equivalence of spectra [1]: two spectra X and Y are equivalent if for all spectra E, $X_*(E) = 0$ if and only $Y_*(E) = 0$ (alternatively put: $X \wedge E \simeq *$ if and only if $Y \wedge E \simeq *$). For a spectrum X, we write $\langle X \rangle$ for the class of all spectra E with $X_*(E) = 0$. The class of all Bousfield classes is denoted by **B**. By a theorem of Ohkawa [6, 3], it is known that **B** is a set and

$$2^{\aleph_0} \leq \mathrm{Card}(\mathbf{B}) \leq 2^{2^{\aleph_0}}.$$

This set is a poset with respect to reverse inclusion: $\langle X \rangle \leq \langle Y \rangle$ if and only if for all spectra Z, $Y_*Z = 0$ implies $X_*Z = 0$. The poset $(\mathbf{B}, \leq)$ has a largest element $\mathbf{1} := \langle S \rangle$ where S is the sphere spectrum and we denote by $\mathbf{0}$ the minimal element which is the Bousfield class of the trivial spectrum. We work at a fixed but arbitrary prime p, *i.e.*, we consider p-local spectra.

For every prime p, $K(n)$ denotes the nth Morava K-theory spectrum with coefficients $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$ where the degree of v_n is $2p^n - 2$. We use the convention that $K(\infty)$ is the mod p Eilenberg-MacLane spectrum, $H\mathbb{F}_p$. For any subset $S \subseteq \mathbb{N} \cup \{\infty\}$, we denote by $K(S)$ the spectrum $\bigvee_{n \in S} K(n)$.

The topological operations $\wedge$ and $\vee$ of taking smash products and wedges, respectively, are well-defined on $\mathbf{B}$; the class $\langle \bigvee_{i \in I} X_i \rangle$ is the least upper bound ("join") in the structure $(\mathbf{B}, \leq)$ of the classes $\langle X_i \rangle$ [1, (2.2)], but in general, $\wedge$ does not produce the greatest lower bound. We can define the greatest lower bound ("meet") by

$$\curlywedge \mathcal{X} := \bigvee \{\langle Z \rangle \, ; \, \forall \langle X \rangle \in \mathcal{X}(\langle Z \rangle \leq \langle X \rangle)\},$$

and observe that $\wedge$ and $\curlywedge$ can differ quite a bit: the Brown-Comenetz dual I of the p-local sphere spectrum satisfies $\langle I \rangle \wedge \langle I \rangle = \mathbf{0} \neq \langle I \rangle = \langle I \rangle \curlywedge \langle I \rangle$ [1, Lemma 2.5].

The complete lattice $(\mathbf{B}, \curlywedge, \vee)$ is endowed with a pseudo-complementation function

$$aX := \bigvee \{Z \, ; \, Z \wedge X = 0\}$$

which is well-defined on Bousfield classes, *i.e.*, $a\langle X \rangle := \langle aX \rangle$ is independent of the choice of representative X of $\langle X \rangle$. The function a is not in general a complement. While $a^2 = \mathrm{id}$ and $a\langle X \rangle \wedge \langle X \rangle = \mathbf{0}$, we may not have $a\langle X \rangle \vee \langle X \rangle = \mathbf{1}$ [1, Lemma 2.7]. Bousfield defined two subclasses of $\mathbf{B}$ as follows:

$$\mathbf{BA} := \{\langle X \rangle \, ; \, \langle X \rangle \vee a\langle X \rangle = \mathbf{1}\}, \text{ and}$$
$$\mathbf{DL} := \{\langle X \rangle \, ; \, \langle X \rangle \wedge \langle X \rangle = \langle X \rangle\}.$$

Many examples for classes in $\mathbf{BA}$ or $\mathbf{DL}$ are known. Bousfield showed in [1] that every Moore spectrum of an abelian group is in $\mathbf{BA}$ and so are the periodic topological K-theory spectra $\langle KO \rangle = \langle KU \rangle$; furthermore, he shows that (arbitrary joins of) finite CW spectra also give classes in $\mathbf{BA}$. Every class of a ring spectrum is in $\mathbf{DL}$ but not necessarily in $\mathbf{BA}$ [1, § 2.6]; in particular, all Eilenberg-MacLane spectra of rings are in $\mathbf{DL}$, but, *e.g.*, the class of the Eilenberg-MacLane spectrum of the integers, $\langle H\mathbb{Z} \rangle$, is in $\mathbf{DL} \backslash \mathbf{BA}$ [1, Lemma 2.7]. However, the Brown-Comenetz duals of (p-local) spheres are not in $\mathbf{DL}$ [1, Lemma 2.5].

We have that $\mathbf{BA} \subseteq \mathbf{DL}$; on $\mathbf{DL}$, $\wedge$ and $\curlywedge$ coincide, and $(\mathbf{DL}, \wedge, \vee)$ is a distributive lattice. Furthermore, on $\mathbf{BA}$, a is a true complement, so $(\mathbf{BA}, \wedge, \vee, \mathbf{0}, \mathbf{1}, a)$ is a Boolean algebra, but not complete.

There is a retraction from $\mathbf{B}$ to $\mathbf{DL}$ defined by

$$r\langle X \rangle := \bigvee \{\langle Z \rangle \, ; \, \langle Z \rangle \in \mathbf{DL} \text{ and } \langle Z \rangle \leq \langle X \rangle\}.$$

The pseudo-complementation function a may not respect $\mathbf{DL}$, *i.e.*, it could be that $\langle X \rangle \in \mathbf{DL}$, but $a\langle X \rangle \notin \mathbf{DL}$. On $\mathbf{DL}$, we therefore define a new pseudo-complement by

$$A\langle X \rangle := ra\langle X \rangle.$$

While $A^3 = A$ and $\langle X \rangle \leq A^2\langle X \rangle$, it is not in general the case that $A^2 = \mathrm{id}$. It is known [5, Lemma 6.2(d)] that A converts joins to meets, *i.e.*,

$$A(\bigvee \mathcal{X}) = \bigcurlywedge\{A\langle X \rangle\,;\,\langle X \rangle \in \mathcal{X}\}.$$

Following [5, Definition 6.3], we define

$$\mathbf{cBA} := \{\langle X \rangle \in \mathbf{DL}\,;\, A^2\langle X \rangle = \langle X \rangle\}.$$

If $\langle X \rangle \in \mathbf{BA}$, then $A^2\langle X \rangle = a^2\langle X \rangle = \langle X \rangle$, thus $\mathbf{BA} \subseteq \mathbf{cBA}$. The set $\mathbf{cBA}$ carries a complete Boolean algebra structure [5, Theorem 6.4]; however, it is not $(\mathbf{cBA}, \wedge, \vee, \mathbf{0}, \mathbf{1}, A)$, but instead $(\mathbf{cBA}, \wedge, \curlyvee, \mathbf{0}, \mathbf{1}, A)$ with $\curlyvee$ defined by

$$\bigcurlyvee \mathcal{X} := A^2 \bigvee \mathcal{X}.$$

Note that $\bigcurlyvee \mathcal{X}$ is perfectly well-defined for subsets $\mathcal{X}$ of $\mathbf{DL}$ or even $\mathbf{B}$, it just will not in general produce the least upper bound in these contexts.

3. Results

We start with an observation on joins of elements in $\mathbf{BA}$ and use this to derive lower bounds for the size of $\mathbf{DL}\backslash\mathbf{cBA}$ and $\mathbf{cBA}\backslash\mathbf{BA}$.

Lemma 1. *If $\mathcal{X} \subseteq \mathbf{BA}$, then $\bigcurlyvee \mathcal{X} = \bigvee \mathcal{X}$. In particular, $\bigvee \mathcal{X} \in \mathbf{cBA}$.*

Proof. We have that

$$\bigcurlyvee \mathcal{X} = A^2 \bigvee \mathcal{X} = rara \bigvee \mathcal{X},$$

and as a converts joins to meets, the latter is equal to

$$rar \bigcurlywedge\{a\langle X \rangle\,;\,\langle X \rangle \in \mathcal{X}\}.$$

Since every $a\langle X \rangle$ is in $\mathbf{BA}$, it is also in $\mathbf{DL}$, and as $\mathbf{DL}$ is complete,

$$\Xi := \bigcurlywedge\{a\langle X \rangle\,;\,\langle X \rangle \in \mathcal{X}\} \in \mathbf{DL}$$

and hence $r\Xi = \Xi$. Therefore, as a sends meets to joins,

$$\begin{aligned} rar\Xi &= ra\Xi \\ &= r\bigvee\{a^2\langle X \rangle\,;\,\langle X \rangle \in \mathcal{X}\} \\ &= r\bigvee\{\langle X \rangle\,;\,\langle X \rangle \in \mathcal{X}\} \\ &= \bigvee \mathcal{X}. \end{aligned}$$

q.e.d.

Proposition 2. *If $S \subseteq \mathbb{N}$ is infinite, then $\langle K(S)\rangle = \bigvee_{i\in S}\langle K(i)\rangle \in$ $\mathbf{cBA}\backslash\mathbf{BA}$ and $\langle K(S)\rangle \geq \langle I\rangle$.*

Proof. Hovey and Palmieri [5, §5] proved that for each n, $\langle K(n)\rangle$ is in $\mathbf{BA}$, so by Lemma 1, $\langle K(S)\rangle$ is in $\mathbf{cBA}$. Hovey showed [4, Proof of Theorem 3.6] that the mod-p Moore spectrum, $M(p)$ is $K(S)$-local, so in particular $K(S)$ has a finite local and [5, Proposition 7.2] gives that $\langle K(S)\rangle \geq \langle I\rangle$. If $K(S)$ were in $\mathbf{BA}$, having a finite local implies [5, Lemma 7.9] that $\langle K(S) \wedge I\rangle \neq \mathbf{0}$. But we know that $\langle K(n) \wedge I\rangle = \mathbf{0}$ and hence using distributivity we get that $\langle K(S) \wedge I\rangle = \mathbf{0}$. q.e.d.

Corollary 3. *We have a proper inclusion $\mathbf{BA} \subsetneqq \mathbf{cBA}$; in fact, the set $\mathbf{cBA}\backslash\mathbf{BA}$ has at least $2^{\aleph_0}$ elements.*

Proof. As noted above, $\mathbf{BA} \subseteq \mathbf{cBA}$. For the non-equality, if $S \neq S'$ are infinite subsets of $\mathbb{N}$, then Dwyer and Palmieri showed that $\langle K(S)\rangle \neq \langle K(S')\rangle$ [3, Lemma 3.4], so there are continuum many elements in the complement. q.e.d.

To sum up, we have

$$\mathbf{BA} \subsetneqq \mathbf{cBA} \subseteq \mathbf{DL} \subsetneqq \mathbf{B}.$$

Hovey and Palmieri argue that the middle inclusion is also proper:

> This argument also implies that A^2 is not the identity—indeed, if A^2 were the identity, one can check that A would have to convert meets to joins. However, we do not know a specific spectrum X in $\mathbf{DL}$ for which $A^2\langle X\rangle \neq \langle X\rangle$. [5, p. 185]

We analyse the argument sketched in the above quote:

Lemma 4. *Let $\mathcal{X} \subseteq \mathbf{DL}$ be any set such that A^2 is the identity for each $\langle X\rangle \in \mathcal{X}$ and for $\bigvee\{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}$. Then*

$$A(\bigwedge \mathcal{X}) = \bigvee\{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}.$$

Proof. Since A converts joins to meets, under the assumption of the lemma, we have

$$\begin{aligned} A(\bigwedge \mathcal{X}) &= A\bigwedge\{A^2\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\} \\ &= A^2\bigvee\{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\} \\ &= \bigvee\{A\langle X\rangle\,;\,\langle X\rangle \in \mathcal{X}\}. \end{aligned}$$

q.e.d.

Corollary 5 (Hovey-Palmieri). *The operation A^2 is not the identity on* **DL***; i.e.,* $\mathbf{cBA} \subsetneq \mathbf{DL}$.

Proof. Let $X := K(\mathbb{N})$, $Y := H\mathbb{F}_p = K(\infty)$, and $\mathcal{X} := \{\langle X\rangle, \langle Y\rangle\} \subseteq \mathbf{DL}$. We assume towards a contradiction that A^2 is the identity on $\mathbf{DL}$, so in particular, the assumptions of Lemma 4 are satisfied for $\mathcal{X}$. But $\langle X\rangle \curlywedge \langle Y\rangle = \langle X\rangle \wedge \langle Y\rangle = \mathbf{0}$, hence $A(\langle X\rangle \curlywedge \langle Y\rangle) = \mathbf{1}$. On the other hand, $A\langle X\rangle \vee A\langle Y\rangle \leq a\langle I\rangle < \mathbf{1}$, in contradiction to Lemma 4. q.e.d.

The proof of Corollary 5 due to Hovey and Palmieri yields a trichotomy result: at least one of $\langle K(\mathbb{N})\rangle$, $\langle H\mathbb{F}_p\rangle$, and $A\langle K(\mathbb{N})\rangle \vee A\langle H\mathbb{F}_p\rangle$ is not in **cBA**. We improve this in our Dichotomy Lemma 7 to a dichotomy which will allow us to identify concrete elements in $\mathbf{DL}\backslash\mathbf{cBA}$, including in particular $\langle H\mathbb{F}_p\rangle$ (Corollary 10).

Lemma 6. *For any spectrum, the condition $A\langle E\rangle < \mathbf{1}$ is equivalent to $\langle E\rangle \neq \mathbf{0}$.*

Proof. If $\langle E\rangle = \mathbf{0}$, then clearly $A\langle E\rangle = \mathbf{1}$. Conversely, if $A\langle E\rangle = \mathbf{1}$, then $a\langle E\rangle \geq A\langle E\rangle = \mathbf{1}$, and so

$$\langle E\rangle = \mathbf{1} \wedge \langle E\rangle = a\langle E\rangle \wedge \langle E\rangle = \mathbf{0}.$$

q.e.d.

Lemma 7 (Dichotomy Lemma). *Let X and Y be spectra, and let E be a spectrum such that $\langle E\rangle \neq \mathbf{0}$. Suppose that the following conditions hold:*

1. $\langle X\rangle \in \mathbf{DL}$,
2. $\langle Y\rangle \in \mathbf{DL}$,
3. $\langle X\rangle \wedge \langle Y\rangle = \mathbf{0}$,
4. $\langle E\rangle \leq \langle X\rangle$, *and*
5. $\langle E\rangle \leq \langle Y\rangle$.

Then $\langle X\rangle$ or $\langle Y\rangle$ is not in **cBA**.

Note that conditions 4 and 5 are equivalent to saying that $\langle X\rangle \curlywedge \langle Y\rangle \neq \mathbf{0}$, and thus the Dichotomy Lemma extracts the failure of $A^2 = \mathrm{id}$ from the discrepancy between $\curlywedge$ and $\wedge$ in **B**.

Proof. Assume that $A^2\langle X\rangle = \langle X\rangle$ and $A^2\langle Y\rangle = \langle Y\rangle$. Since A converts joins to meets, we get by our assumption on X and Y

$$\mathbf{1} = A\mathbf{0} = A(\langle X\rangle \wedge \langle Y\rangle) = A(A^2\langle X\rangle \wedge A^2\langle Y\rangle) = A^2(A\langle X\rangle \vee A\langle Y\rangle)$$

and the latter is $A\langle X\rangle \curlyvee A\langle Y\rangle$ by definition of $\curlyvee$. As A is order-reversing we get $A\langle X\rangle \leq A\langle E\rangle$ and $A\langle Y\rangle \leq A\langle E\rangle$ and hence (using Lemma 6)

$$\mathbf{1} = A^2(A\langle X\rangle \vee A\langle Y\rangle) = A\langle X\rangle \curlyvee A\langle Y\rangle \leq A\langle E\rangle \curlyvee A\langle E\rangle = A\langle E\rangle < \mathbf{1},$$

a contradiction, showing that our assumption that both $\langle X\rangle$ and $\langle Y\rangle$ are in **cBA** cannot hold. q.e.d.

As usual, we call a set $S \subset \mathbb{N}\cup\{\infty\}$ *coinfinite* if its complement $(\mathbb{N}\cup\{\infty\})\backslash S$ is infinite.

Theorem 8. *For any coinfinite set $S \subseteq \mathbb{N}\cup\{\infty\}$ with $\infty \in S$, we have that $\langle K(S)\rangle$ is not in* **cBA**.

Proof. In Lemma 7, choose E to be the Brown-Comenetz dual of the p-local sphere spectrum, I. We know by [5, Lemma 7.1(c)] that $\langle H\mathbb{F}_p\rangle \geq \langle I\rangle$, and hence $\langle K(S)\rangle \geq \langle I\rangle$. As the complement $\overline{S} := (\mathbb{N}\cup\{\infty\})\backslash S$ is infinite, we get by Proposition 2 that $\langle K(\overline{S})\rangle \geq \langle I\rangle$. Both $\langle K(S)\rangle$ and $\langle K(\overline{S})\rangle$ are in **DL** and $\langle K(S)\rangle \wedge \langle K(\overline{S})\rangle = \mathbf{0}$. Thus all conditions of the Dichotomy Lemma are satisfied, and we get that one of $\langle K(S)\rangle$ and $\langle K(\overline{S})\rangle$ is not in **cBA**. However, by Proposition 2, $\langle K(\overline{S})\rangle \in$ **cBA**, so $\langle K(S)\rangle \in$ **DL****cBA**. q.e.d.

Corollary 9. *There are at least $2^{\aleph_0}$ Bousfield classes in* **DL****cBA**.

Proof. This follows directly from Theorem 8 and [3, Lemma 3.4], as there are $2^{\aleph_0}$ many coinfinite subsets of $\mathbb{N}\cup\{\infty\}$. q.e.d.

By Corollaries 3 and 9, we get $2^{\aleph_0}$ as a lower bound for the cardinality for three of the four areas depicted in Figure 1; for **B****DL** we only get $\aleph_0$ as a lower bound. A natural project for future research would be to improve this to $2^{\aleph_0}$ by finding concrete inhabitants of that set. Getting even larger lower bounds than $2^{\aleph_0}$ is connected to the famous open question about the cardinality of **B**; as a consequence, we believe that this needs entirely novel ideas.

4. Applications

Several conjectures made by Hovey and Palmieri in [5] suggest that $\langle H\mathbb{F}_p\rangle$ is not in **cBA** [5, Proposition 6.14]. This follows directly from our Theorem 8:

Corollary 10. *For every prime p, we have that $\langle H\mathbb{F}_p\rangle \in$* **DL****cBA**.

Proof. This is clear from Theorem 8, as $\langle H\mathbb{F}_p\rangle = \langle K(\infty)\rangle = \langle K(\{\infty\})\rangle$ where $\{\infty\}$ is coinfinite in $\mathbb{N}\cup\{\infty\}$. q.e.d.

Our method also identifies several other explicit Bousfield classes in **DL****cBA**. The following examples exploit the fact that for any self-map of a spectrum X, $f\colon \Sigma^{|f|}X \to X$ one gets by [7, Lemma 1.34] that

$$\langle X\rangle = \langle C_f\rangle \vee \langle X[f^{-1}]\rangle.$$

Here, C_f denotes the cofiber of f and $X[f^{-1}]$ is the telescope. Then the Bousfield class of the Eilenberg-MacLane spectrum of the p-local integers, $H\mathbb{Z}_{(p)}$, is

$\langle K(\{0,\infty\})\rangle$. This is a special case of a truncated Brown-Peterson spectrum $BP\langle n\rangle$ with $\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1,\ldots,v_n]$ ($|v_i| = 2p^i - 2$). Multiplication by v_n is a self-map on $BP\langle n\rangle$ with cofiber $BP\langle n-1\rangle$ and $BP\langle n\rangle[v_n^{-1}] = E(n)$. An iteration then gives (cf. [7, Theorem 2.1]) $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle H\mathbb{F}_p\rangle$. As the Bousfield class of $E(n)$ is $\langle K(0)\rangle \vee \ldots \vee \langle K(n)\rangle$ we obtain $\langle BP\langle n\rangle\rangle = \langle K(\{0,\ldots,n,\infty\})\rangle$.

Corollary 11. *For every prime p and every natural number n, we have that $\langle H\mathbb{Z}_{(p)}\rangle$ and $\langle BP\langle n\rangle\rangle$ are in* $\mathbf{DL}\backslash\mathbf{cBA}$.

Proof. The subsets $\{0,\infty\}$ and $\{0,\ldots,n,\infty\}$ are coinfinite in $\mathbb{N}\cup\{\infty\}$. q.e.d.

For the connective Morava K-theory $k(n)$ (with $\pi_* k(n) = \mathbb{F}_p[v_n]$) we get $\langle k(n)\rangle = \langle K(n)\rangle \vee \langle H\mathbb{F}_p\rangle = \langle K(\{n,\infty\})\rangle$.

Corollary 12. *For every natural number n, $\langle k(n)\rangle \in \mathbf{DL}\backslash\mathbf{cBA}$.*

Proof. This follows from Theorem 8, as $\{n,\infty\}$ is coinfinite in $\mathbb{N}\cup\{\infty\}$. q.e.d.

Similar to the Morava K-theory spectra $K(n)$ we can consider the telescopes $T(n)$ of v_n-maps. (Cf. [5, §5] for details.) It is known that

$$\langle T(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle$$

where $A(n)$ is the spectrum describing the failure of the telescope conjecture. We set $\langle T(\infty)\rangle = \langle H\mathbb{F}_p\rangle$. The classes $\langle T(n)\rangle$ and $\langle A(n)\rangle$ are in $\mathbf{BA}$ but $\bigvee_{\mathbb{N}}\langle T(n)\rangle \notin \mathbf{BA}$ by [5, Corollary 7.10]. By Lemma 1, we know that for any $S\subseteq\mathbb{N}$, we have that $\bigvee_{n\in S}\langle T(n)\rangle \in \mathbf{cBA}$. An argument similar to the proof of Proposition 2 yields Proposition 13.

Proposition 13. *If $S\subseteq\mathbb{N}$ is infinite, then $\langle T(S)\rangle = \bigvee_{i\in S}\langle T(i)\rangle \in \mathbf{cBA}\backslash\mathbf{BA}$ and $\langle T(S)\rangle \geq \langle I\rangle$.*

Theorem 14. *Let $S\subseteq\mathbb{N}\cup\{\infty\}$ be a coinfinite subset with $\infty\in S$. Then $\langle T(S)\rangle$ is not in* $\mathbf{cBA}$.

Proof. Again, we use the Brown-Comenetz dual of the p-local sphere as E in the Dichotomy Lemma. Let $\overline{S}$ be the complement of S. As $\langle T(n)\rangle \geq \langle K(n)\rangle$ and as $\infty\in S$ we have that

$$\bigvee_{n\in S}\langle T(n)\rangle \geq \bigvee_{n\in S}\langle K(n)\rangle \geq \langle I\rangle$$

and $\bigvee_{n\in\overline{S}}\langle T(n)\rangle \geq \langle I\rangle$. The telescopes satisfy $\langle T(n)\rangle \wedge \langle T(m)\rangle = \mathbf{0}$ for $m\neq n$: cf. [5, §5] for the cases $n\neq\infty\neq m$ and the proof of [5, Proposition 6.14] for $\langle H\mathbb{F}_p\rangle \wedge \bigvee_{\mathbb{N}}\langle T(n)\rangle = \mathbf{0}$. Therefore we obtain that one of $\bigvee_{n\in S}\langle T(n)\rangle$ or $\bigvee_{n\in\overline{S}}\langle T(n)\rangle$ cannot be an element of $\mathbf{cBA}$, but $\bigvee_{n\in\overline{S}}\langle T(n)\rangle$ is in $\mathbf{cBA}$ by Proposition 13. q.e.d.

References

[1] Aldridge K. Bousfield, The Boolean algebra of spectra. Commentarii Mathematici Helvetici, 54(3) (1979) 368–377.

[2] Aldridge K. Bousfield, The localization of spectra with respect to homology. Topology 18(4) (1979) 257–281.

[3] William G. Dwyer and John H. Palmieri, Ohkawa's theorem: there is a set of Bousfield classes, Proceedings of the American Mathematical Society, 129(3) (2001) 881–886.

[4] Mark Hovey, Bousfield localization functors and Hopkins' chromatic splitting conjecture, The Čech Centennial, M. Cenkl and H. Miller, editors, Contemporary Mathematics, volume 181, Providence, RI, 1995, American Mathematical Society, pp. 225–250.

[5] Mark Hovey and John H. Palmieri. The structure of the Bousfield lattice. In: Jean-Pierre Meyer, Jack Morava, and W. Stephen Wilson, editors, Homotopy invariant algebraic structures. A conference in honor of J. Michael Boardman, Contemporary Mathematics, volume 239, Providence, RI, 1999, American Mathematical Society, pp. 175–196.

[6] Tetsusuke Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, Hiroshima Mathematical Journal, 19(3) (1989) 631–639.

[7] Douglas C. Ravenel, Localization with respect to certain periodic homology theories, American Journal of Mathematics, 106(2) (1984) 351–414.