Proceedings Paper:
Boncompagni, V, Penev, I and Vuskovic, K (2017) Clique cutsets beyond chordal graphs. In: Electronic Notes in Discrete Mathematics. IX Latin and American Algorithms, Graphs and Optimization Sympoisium, 11-15 Sep 2017, Marseille, France. Elsevier , pp. 81-86.

https://doi.org/10.1016/j.endm.2017.10.015

Clique cutsets beyond chordal graphs

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Abstract

Truemper configurations (thetas, pyramids, prisms, and wheels) have played an important role in the study of complex hereditary graph classes (e.g. the class of perfect graphs and the class of even-hole-free graphs), appearing both as excluded configurations, and as configurations around which graphs can be decomposed. In this paper, we study the structure of graphs that contain (as induced subgraphs) no Truemper configurations other than (possibly) universal wheels and twin wheels. We also study several subclasses of this class. We use our structural results to analyze the complexity of the recognition, maximum weight clique, maximum weight stable set, and optimal vertex coloring problems for these classes. We also obtain polynomial χ-bounding functions for these classes.

Keywords: clique, stable set, vertex coloring, structure, algorithms.

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[4] Partially supported by EPSRC grant EP/K016423/1
[5] Partially supported by EPSRC grant EP/N0196660/1.
[6] Partially supported by Serbian Ministry of Education and Science projects 174033 and III44006.

1 Introduction

All graphs that we consider are finite, simple, and nonnull. We use standard terminology and notation. Given graphs G and H, we say that G is *H-free* if G does not contain (an isomorphic copy of) H as an induced subgraph. Given a family $\mathcal{H}$ of graphs, we say that a graph G is *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$. A class of graphs is *hereditary* if it is closed under induced subgraphs. A *hole* in a graph is an induced cycle of length at least four. A *chordal graph* is a graph that contains no holes.

Configurations known as thetas, pyramids, prisms, and wheels (defined below) have played an important role in the study of such diverse (and important) classes as the classes of regular matroids, balanceable matrices, perfect graphs, and even-hole-free graphs (for a survey, see [6]). These configurations are also called *Truemper configurations*, as they appear in a theorem due to Truemper [5] that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities.

A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$. A *pyramid* is any subdivision of the complete graph K_4 in which one triangle remains unsubdivided, and of the remaining three edges, at least two edges are subdivided at least once. A *prism* is any subdivision of $\overline{C_6}$ in which the two triangles remain unsubdivided. A *3-path-configuration* (or *3PC* for short) is any theta, pyramid, or prism (see Fig. 1). A *wheel* (H, x) is a graph that consists of a hole H and a vertex x that has at least three neighbors in $V(H)$. A *universal wheel* is a wheel (H, x) such that x is adjacent to all vertices in $V(H)$. A *twin wheel* is a wheel (H, x) such that x has precisely three neighbors in $V(H)$, and those neighbors are consecutive vertices of H. A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel. A *cap* is a graph that consists of a chordless cycle of length at least four and a vertex adjacent to two consecutive vertices of the cycle (and to no other vertices of the cycle).

Here, we are interested in the hereditary classes $\mathcal{G}_{UT}$, $\mathcal{G}_U$, $\mathcal{G}_T$, and $\mathcal{G}_{UT}^{cap-free}$, defined as follows. $\mathcal{G}_{UT}$ is the class of all (3PC, proper wheel)-free graphs (so the only Truemper configurations that graphs in $\mathcal{G}_{UT}$ may contain are universal wheels and twin wheels); $\mathcal{G}_U$ is the class of all (3PC, proper wheel, twin wheel)-free graphs; $\mathcal{G}_T$ is the class of all (3PC, proper wheel, universal wheel)-free graphs; and $\mathcal{G}_{UT}^{cap-free}$ is the class of all (3PC, proper wheel, cap)-free graphs. Clearly, $\mathcal{G}_U$, $\mathcal{G}_T$, and $\mathcal{G}_{UT}^{cap-free}$ are proper subclasses of $\mathcal{G}_{UT}$; furthermore, the class of chordal graphs is a proper subclass of each of these four classes.

We first obtain decomposition theorems for the classes $\mathcal{G}_{UT}$, $\mathcal{G}_U$, $\mathcal{G}_T$, and $\mathcal{G}_{UT}^{cap-free}$, and then we use these theorems to analyze the complexity of the

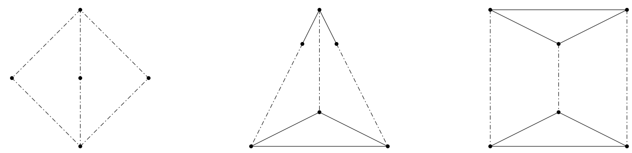

Fig. 1. Theta, pyramid and prism. (A full line represents an edge, and a dashed line represents a path of edge-length at least one.)

recognition, maximum clique, maximum stable set, and optimal vertex coloring problems for the four classes, as well as to give polynomial χ-bounding functions for these classes. In Section 2, we give all the necessary definitions, and in Section 3, we state and discuss our results.

2 Definitions

As usual, the chromatic number of a graph G is denoted by $\chi(G)$. An *optimal coloring* of G is a proper vertex coloring of G that uses precisely $\chi(G)$ colors. A *clique* (resp. *stable set*) of a graph G is a set of pairwise adjacent (resp. nonadjacent) vertices of G. The clique number (i.e. the maximum size of a clique) of G is denoted by $\omega(G)$, and the stability number (i.e. the maximum size of a stable set) of G is denoted by $\alpha(G)$. A *maximum clique* (resp. *maximum stable set*) of a graph G is a clique (resp. stable set) of G that is of size $\omega(G)$ (resp. $\alpha(G)$). A *weighted graph* is an ordered pair (G, w), where G is a graph and $w : V(G) \to \mathbb{R}$ is a weight function for G; a maximum weight clique and a maximum weight stable set of a weighted graph are defined in the obvious way. A hereditary class $\mathcal{G}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ (called a *χ-bounding function* for $\mathcal{G}$) such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$.

Given a graph G, a vertex $x \in V(G)$, and a set $Y \subseteq V(G) \setminus \{x\}$, we say that x is *complete* (resp. *anticomplete*) to Y in G if x is adjacent (resp. nonadjacent) to every vertex in Y. Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (resp. *anticomplete*) to Y in G if every vertex in X is complete (resp. anticomplete) to Y. The complement of a graph G is denoted by $\overline{G}$. An *antihole* in G is an induced subgraph of G whose complement is a hole in $\overline{G}$. A *long hole* (resp. *long antihole*) is a hole (resp. antihole) of length at least five.

A *hyperhole* of length $k \geq 4$ (or simply a *hyperhole*) is a graph H whose vertex-set $V(H)$ can be partitioned into k nonempty cliques, say $X_1, \dots, X_k$ (with subscripts in $\mathbb{Z}_k$), such that for each $i \in \mathbb{Z}_k$, X_i is a clique, complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$. Similarly, a *hyperantihole* of length $k \geq 4$ (or simply a *hyperantihole*) is a graph A whose vertex-set $V(A)$ can be partitioned into k nonempty cliques, say $X_1, \dots, X_k$ (with subscripts in $\mathbb{Z}_k$), such that for each $i \in \mathbb{Z}_k$, X_i is anticomplete to $X_{i-1} \cup X_{i+1}$ and complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$.

A *ring* of length $k \geq 4$ (or simply a *ring*) is a graph R whose vertex set can be partitioned into k nonempty sets, say $X_1, \dots, X_k$ (with subscripts in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i can be ordered as $X_i = \{u_1^i, \dots, u_{|X_i|}^i\}$ so that $X_i \subseteq N_R[u_{|X_i|}^i] \subseteq \dots \subseteq N_R[x_1^i] = X_{i-1} \cup X_i \cup X_{i+1}$. Note that every hyperhole is a ring.

A *cobipartite graph* is the complement of a bipartite graph. A *chordal cobipartite graph* is a graph that is both chordal and cobipartite. A *clique cutset* of a graph G is a (possibly empty) clique C of G such that $G \setminus C$ is disconnected. A *component* of G is a maximal connected induced subgraph of G, and an *anticomponent* of G is an induced subgraph of G whose complement is a component of $\overline{G}$. A component or anticomponent is *trivial* if it contains just one vertex, and it is *nontrivial* if it contains at least two vertices.

3 Results

In this section, we state our results. We begin with our decomposition theorems for the classes $\mathcal{G}_{UT}$, $\mathcal{G}_U$, $\mathcal{G}_T$, and $\mathcal{G}_{UT}^{cap-free}$. After that, we discuss our algorithmic and χ-boundedness results for these four classes.

Theorem 3.1 *Every graph $G \in \mathcal{G}_{UT}$ satisfies at least one of the following:*

- *G contains exactly one nontrivial anticomponent, and this anticomponent is a ring of length at least five;*
- *G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;*
- *$\alpha(G) = 2$, and every anticomponent of G is either a hyperhole of length five or a (long hole, $\overline{C_6}$)-free graph;*
- *G admits a clique cutset.*

Theorem 3.2 *Every graph $G \in \mathcal{G}_U$ satisfies at least one of the following:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a hole of length at least five;*
- *all nontrivial anticomponents of G are isomorphic to $\overline{K_2}$;*
- *G admits a clique cutset.*

Theorem 3.3 *Every graph $G \in \mathcal{G}_T$ satisfies one of the following:*

- *G is a complete graph, a ring, or a hyperantihole of length seven;*
- *G admits a clique cutset.*

Theorem 3.4 *Every graph $G \in \mathcal{G}_{UT}^{cap-free}$ satisfies at least one of the following:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a hyperhole of length at least six;*
- *each anticomponent of G is either a hyperhole of length five or a chordal cobipartite graph;*
- *G admits a clique cutset.*

We now turn to the algorithmic and χ-boundedness results for our four classes of graphs. We consider the following four algorithmic problems:

- the *recognition problem*, i.e. the problem of determining whether an input graph belongs to a given class;
- the *maximum weight stable set problem* (MWSSP), i.e. the problem of finding a maximum weight stable set in an input weighted graph (with real weights);
- the *maximum weight clique problem* (MWCP), i.e. the problem of finding a maximum weight clique in an input weighted graph (with real weights);
- the *optimal coloring problem* (ColP), i.e. the problem of finding an optimal coloring of an input graph.

We summarize our results in the table below. We note that all our algorithms are *robust*, that is, they either produce a correct solution to the problem in question for the input (weighted) graph, or they correctly determine that the graph does not belong to the class under consideration.

	recognition	MWSSP	MWCP	ColP	χ-bound.
$\mathcal{G}_{UT}$	$O(n^6)$	?	NP-hard	?	$\chi \leq 2\omega^4$
$\mathcal{G}_{U}$	$O(n^3)$	$O(n^3)$	$O(n^3)$	$O(n^3)$	$\chi \leq \omega + 1$
$\mathcal{G}_{T}$	$O(n^3)$	$O(n^4)$	$O(n^3)$	?	$\chi \leq \lfloor \frac{3}{2}\omega \rfloor$
$\mathcal{G}_{UT}^{cap-free}$	$O(n^5)$	$O(n^3)$	$O(n^3)$	$O(n^3)$	$\chi \leq \lfloor \frac{3}{2}\omega \rfloor$

We remark that we in fact show that the problem of finding the clique number of a (long hole, $K_{2,3}$, $\overline{C_6}$)-free graph is NP-hard (this easily follows from an observation of Poljak [3]); since all such graphs belong to $\mathcal{G}_{UT}$, we deduce that this problem (and consequently, the MWCP as well) is NP-hard for the class $\mathcal{G}_{UT}$. We do not know whether the MWSSP and ColP are solvable in polynomial time for (long hole, $K_{2,3}$, $\overline{C_6}$)-free graphs. Further, our polynomial-time algorithms are based on the decomposition theorems stated above, as well as on Tarjan's techniques for handling clique cutsets [4]. At this time, we do not know the complexity of coloring graphs in the class $\mathcal{G}_T$; this is because we do not know whether rings of odd length can be colored in polynomial time.

It follows from [1] that the class of theta-free graphs is χ-bounded; consequently, our four classes (classes $\mathcal{G}_{UT}$, $\mathcal{G}_U$, $\mathcal{G}_T$, and $\mathcal{G}_{UT}^{cap-free}$) are all χ-bounded. Unfortunately, the χ-bounding function from [1] is superexponential. Using our structural results, we obtain polynomial χ-bounding functions for our four classes (as shown in the table above). The bound given for the class $\mathcal{G}_U$ easily follows from Theorem 3.2. Next, a simple argument shows that every ring R satisfies $\chi(R) \leq \lfloor \frac{3}{2}\omega(R) \rfloor$; together with Theorems 3.3 and 3.4, this yields the bounds for the classes $\mathcal{G}_T$ and $\mathcal{G}_{UT}^{cap-free}$ given in the table. Finally, we show that every graph in the class $\mathcal{G}_{UT}$ either is cap-free (and therefore belongs to $\mathcal{G}_{UT}^{cap-free}$) or admits a "small" cutset (one whose size is bounded by a function of the clique number); using our χ-bounding function for $\mathcal{G}_{UT}^{cap-free}$, as well as a result of [2], we obtain the χ-bounding function for $\mathcal{G}_{UT}$ given in the table.

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