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Ordinal notation systems corresponding to Friedman's linearized well-partial-orders with gap-condition

Michael Rathjen · Jeroen Van der Meeren · Andreas Weiermann

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Abstract In this article we investigate whether the following conjecture is true or not: does the addition-free theta functions form a canonical notation system for the linear versions of Friedman's well-partial-orders with the so-called gap-condition over a finite set of n labels. Rather surprisingly, we can show this is the case for two labels, but not for more than two labels. To this end, we determine the order type of the notation systems for addition-free theta functions in terms of ordinals less than ε_0. We further show that the maximal order type of the Friedman ordering can be obtained by a certain ordinal notation system which is based on specific binary theta functions.

Keywords Well-partial-orderings · Maximal order type · Gap-embeddability relation · Ordinal notation systems · Collapsing function

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1 Introduction

A major theme in proof theory is to provide natural independence results for formal systems for reasoning about mathematics. The most prominent system in this

Michael Rathjen
Department of Pure Mathematics, University of Leeds,
Leeds LS2 9JT, England
E-mail: rathjen@maths.leeds.ac.uk

Jeroen Van der Meeren
Department of Mathematics, Ghent University,
Krijgslaan 281, 9000 Gent, Belgium
E-mail: jeroenvandermeeren@gmail.com

Andreas Weiermann
Department of Mathematics, Ghent University,
Krijgslaan 281, 9000 Gent, Belgium
E-mail: andreas.weiermann@ugent.be

respect is first order Peano arithmetic, or almost equivalently its second order version ACA_0. Providing natural independence results for stronger systems turned out to be rather difficult. The strongest system considered in reverse mathematics [19] is Π^1_1-CA_0 which formalizes full Π^1_1-comprehension (with paramters) over RCA_0. Buchholz [2] provided a natural hydra game for Π^1_1-CA_0 but this follows closely a path which is delineated by the classification of the provably recursive functions in terms of a corresponding Hardy hierarchy. Harvey Friedman [18] obtained a spectacular independence result for Π^1_1-CA_0 by considering well-quasi-orders on labeled trees on which he imposed a so-called gap-condition. It is still an open problem to classify the strength of Friedman's assertion for the case that the set of labels consists of n elements where n is fixed from the outside. Nowadays it is known that the proof-theoretic strength of a well-quasi-order-principle can be measured in terms of the maximal order type of the well-quasi-order under consideration. Hence, the open problem can be translated into the following question: 'What are the maximal order types of Friedman's well-partial-orders?'. In [18], it is only shown that the maximal order type is bounded from below by $\psi_0\Omega_n$. Weiermann [21] conjectured that the collapsing functions ϑ_i can define a maximal linear extension of Friedman's well-partial-orders in a straightforward way. This would mean that the ordinal notation system based on the ϑ_i-functions defines the maximal order types of Friedman's well-partial-orders.

The maximal order type for the Friedman ordering is known for $n = 1$ by results of Schmidt and Friedman. Recently, Weiermann's conjecture has been proven to be true for the case $n = 2$ [13], meaning that the maximal order type of Friedman's well-partial-order for $n = 2$ can be expressed using higher collapsing functions ϑ_0 and ϑ_1.

As an intermediate step in classifying the general case it seems natural to classify the situation where trees are replaced by *sequences* over a finite set of cardinality n. The hope is that the simpler case indicates how to deal with the general case of trees. Investigations on finite sequences with respect to the Friedman ordering have been undertaken by Schütte and Simpson [17]. They showed how the Friedman ordering can be reduced to suitably nested versions of the Higman ordering [6]. Moreover they considered the corresponding Buchholz-style ordinal notation system in which the addition function has been dropped. Curiously this lead to an ordinal notation system which in the limit (for unbounded n) reached ε_0. It is quite natural to consider finite sequences as iterated applications of unary functions and it is quite natural to ask whether the ordinal notation system which is based on n collapsing functions (which in [17] are denoted by $\pi_0,\dots,\pi_{n-1}$) generates the maximal order type for the Friedman ordering for sequences over a set with n elements, denoted by $\mathbb{S}^w_n$ (see Definition 9). But it turns out that this is not the case: to produce the maximal order type for the $\mathbb{S}^w_n$ one needs the functions $\pi_0,\dots,\pi_{2n}$. It is known that the so called theta functions θ_i grow faster than the functions π_i and it is natural to ask whether their addition-free analogues $\vartheta_0,\dots,\vartheta_n$ generate the maximal order type of $\mathbb{S}^w_n$. This is actually Weiermann's conjecture [21] applied on sequences instead of trees. For $n = 2$, it turned out to be true and so one would expect that Weiermann's conjecture for sequences would generalize to $n \geq 3$. Quite surprisingly this is not the case: to obtain the maximal order type of $\mathbb{S}^w_n$ one requires the functions $\vartheta_0,\dots,\vartheta_{2\cdot n-3}$.

So the question remains whether $\mathbb{S}_n^w$ can be realized by a suitable choice of unary functions. It turns out that this, as we will show, is indeed possible using specific binary theta functions. However, with unary functions the question is still open.

In a sequel project, we intend to determine the relationship between other ordinal notation systems without addition (e.g. ordinal diagrams [20], Gordeev-style notation systems [5] and non-iterated ϑ-functions [3,22]) with the systems used in this article.

2 Preliminaries

2.1 Well-partial-orders

Well-partial-orders are the natural generalizations of well-orders. They have applications in computer science, commutative algebra and logic.

Definition 1 A **well-partial-order** (hereafter $\mathbb{wpo}$) is a partial order that is well-founded and does not admit infinite antichains. Hence, it is a partial order $(X, \leq_X)$ such that for every infinite sequence $(x_i)_{i<\omega}$ in X there exist two indices $i < j$ such that $x_i \leq_X x_j$. If the ordering is clear from the context, we do not write the subscript X.

$\mathbb{wpo}$'s appear everywhere in mathematics. For example, they are the main ingredients in Higman's theorem [6], Graph Minor theorem [4], Fraïssé's order type conjecture [9] and Kruskal's theorem [8]. The latter is used in field of term rewriting systems.

In this paper, we are interested in $\mathbb{wpo}$'s with the so-called gap-condition introduced in [18]. We are especially interested in the linearized version, which is already studied by Schütte and Simpson [17] (see subsection 2.2 for more information). With regard to these $\mathbb{wpo}$'s, we want to study ordinal notation systems which correspond to their maximal order types and maximal linear extensions.

Definition 2 The **maximal order type** of the $\mathbb{wpo}$ $(X, \leq_X)$ is equal to $\sup\{\alpha\colon \leq_X \subseteq \preceq,$ $\preceq$ is a well-order on X and $otype(X, \preceq) = \alpha\}$. We denote this ordinal by $o(X, \leq_X)$ or by $o(X)$ if the ordering is obvious from the context.

The following theorem by de Jongh and Parikh [7] shows that this supremum is actually a maximum.

Theorem 1 (de Jongh and Parikh [7]) *Assume that $(X, \leq_X)$ is a $\mathbb{wpo}$. Then there exists a well-order $\preceq$ on X which is an extension of $\leq_X$ such that $otype(X, \preceq) = o(X, \leq_X)$.*

Definition 3 Let X be a $\mathbb{wpo}$. Every well-order $\preceq$ on X that satisfies Theorem 1 is called a **maximal linear extension**.

The following definition and lemma are very useful.

Definition 4 A **quasi-embedding** e from the partial order $(X, \leq_X)$ to the partial order $(Y, \leq_Y)$ is a mapping such that for all $x, x' \in X$, if $e(x) \leq_Y e(x')$, then $x \leq_X x'$ holds.

Lemma 1 *Assume that e is a quasi-embedding from the partial order X to the partial order Y. If Y is a* wpo*, then X is also a* wpo *and* $o(X) \leq o(Y)$.

Definition 5 Let α be an ordinal. Define $\omega_0[\alpha]$ as α and $\omega_{n+1}[\alpha]$ as $\omega^{\omega_n[\alpha]}$. Write ω_n for the ordinal $\omega_n[1]$.

2.2 Well-partial-orders with the gap-condition

In 1982, Harvey Friedman introduced a well-partial-order of finite rooted trees with labels in $\{0,\dots,n-1\}$ with a gap-embeddability relation on it. This was later published by Simpson in [18]. This wpo was very important, because it was one of the first natural examples of statements not provable in the strongest theory of the Big Five in Reverse Mathematics, Π^1_1-CA_0.

Definition 6 Let $\mathbb{T}_n$ be the set of finite rooted trees with labels in $\{0,\dots,n-1\}$. An element of $\mathbb{T}_n$ is of the form (T,l), where T is a finite rooted tree, which we see as a partial order on the set of nodes, and l is a labeling function, a mapping from T to the set $\{0,\dots,n-1\}$. Define $(T_1,l_1) \leq_{gap} (T_2,l_2)$ if there exists an injective order- and infimum-preserving mapping f from T_1 to T_2 such that

1. $\forall \tau \in T_1$, we have $l_1(\tau) = l_2(f(\tau))$.
2. $\forall \tau \in T_1$ and for all immediate successors $\tau' \in T_1$ of τ, we have that if $\overline{\tau} \in T_2$ is strictly between $f(\tau)$ and $f(\tau')$, then $l_2(\overline{\tau}) \geq l_2(f(\tau')) = l_1(\tau')$.

Theorem 2 (Simpson/Friedman[18]) *For all n,* $(\mathbb{T}_n, \leq_{gap})$ *is a* wpo *and* Π^1_1-$\mathsf{CA}_0 \nvdash$ $\forall n < \omega$ *'*$(\mathbb{T}_n, \leq_{gap})$ *is a* wpo*'.*

We are interested in the linearized versions of these wpo's, which have been studied extensively by Schütte and Simpson [17]. Before we give the definition of these linearized wpo's, we introduce the disjoint sum and cartesian product between wpo's and the Higman ordering.

Definition 7 Let X_0 and X_1 be two wpo's. Define the **disjoint sum** $X_0 + X_1$ as the set $\{(x,0) : x \in X_0\} \cup \{(y,1) : y \in X_1\}$ with the following ordering:

$$(x,i) \leq (y,j) \Leftrightarrow i = j \text{ and } x \leq_{X_i} y.$$

For an arbitrary element (x,i) in $X_0 + X_1$, we omit the second coordinate i if it is clear from the context to which set the element x belongs to. Define the **cartesian product** $X_0 \times X_1$ as the set $\{(x,y) : x \in X_0, y \in X_1\}$ with the following ordering:

$$(x,y) \leq (x',y') \Leftrightarrow x \leq_{X_0} x' \text{ and } y \leq_{X_1} y'.$$

Definition 8 Let X^* be the **set of finite sequences** over the partial order $(X, \leq_X)$. Denote $x_0 \dots x_{k-1} \leq^*_X y_0 \dots y_{l-1}$ if there exists a strictly increasing function $f : \{0,\dots,k-1\} \to \{0,\dots,l-1\}$ such that for all $0 \leq i \leq k-1$, $x_i \leq_X y_{f(i)}$ holds. If the ordering on X is clear from the context, we write X^* instead of $(X^*, \leq^*_X)$.

Hence, if we write X^*, we mean the set of of finite sequences over X or the partial order $(X^*, \leq_X^*)$. The context will make clear what we mean. Define $\mathbb{S}_n$ as $\{0,\ldots,n-1\}^*$ and $\mathbb{S}$ as $\mathbb{N}^*$. $\mathbb{S}_n$ and $\mathbb{S}$ are either sets of finite sequences or partial orders.

Theorem 3 (de Jongh-Parikh[7], Schmidt[16]) *If X_0, X_1 and X are* $\mathbb{wpo}$*'s, then X_0+X_1, $X_0\times X_1$ and X^* are still* $\mathbb{wpo}$*'s, and*

$$o(X_0+X_1)=o(X_0)\oplus o(X_1),$$
$$o(X_0\times X_1)=o(X_0)\otimes o(X_1),$$

where $\oplus$ and $\otimes$ is the natural sum and product between ordinals, and

$$o(X^*)=\begin{cases}\omega^{\omega^{o(X)-1}} & \text{if } o(X) \text{ is finite,}\\ \omega^{\omega^{o(X)+1}} & \text{if } o(X)=\varepsilon+n, \text{ with } \varepsilon \text{ an epsilon number and } n<\omega,\\ \omega^{\omega^{o(X)}} & \text{otherwise.}\end{cases}$$

Now, we define the linearized versions of the gap-embeddability relation.

Definition 9 In this context, let $\mathbb{S}_n$ be the set of the finite sequences over $\{0,\ldots,n-1\}$. We say that $s=s_0\ldots s_{k-1}\leq^w_{gap} s'_0\ldots s'_{l-1}=s'$ if there exists a strictly increasing function $f:\{0,\ldots,k-1\}\to\{0,\ldots,l-1\}$ such that

1. for all $0\leq i\leq k-1$, we have $s_i=s'_{f(i)}$,
2. for all $0\leq i<k-1$ and all j between $f(i)$ and $f(i+1)$, the inequality $s'_j\geq s'_{f(i+1)}=s_{i+1}$ holds.

This ordering on $\mathbb{S}_n$ is called the **weak gap-embeddability relation**. The partial order $(\mathbb{S}_n,\leq^w_{gap})$ is also denoted by $\mathbb{S}^w_n$. The **strong gap-embeddability relation** fulfills the extra condition

3. for all $j<f(0)$, we have $s'_j\geq s'_{f(0)}=s_0$.

This ordering on $\mathbb{S}_n$ is denoted by $\leq^s_{gap}$ We also write $\mathbb{S}^s_n$ for the partial order $(\mathbb{S}_n,\leq^s_{gap}$).

We now give an overview of the results in the article of Schütte and Simpson [17].

Theorem 4 (Schütte-Simpson[17], Simpson/Friedman[18]) *For all n, $(\mathbb{S}_n,\leq^w_{gap})$ and $(\mathbb{S}_n,\leq^s_{gap})$ are* $\mathbb{wpo}$*'s.*

Theorem 5 (Schütte-Simpson[17])
$\mathsf{ACA}_0 \nvdash \forall n<\omega$ '$(\mathbb{S}_n,\leq^w_{gap})$ *is a* $\mathbb{wpo}$',
$\mathsf{ACA}_0 \nvdash \forall n<\omega$ '$(\mathbb{S}_n,\leq^s_{gap})$ *is a* $\mathbb{wpo}$'.

Theorem 6 (Schütte-Simpson[17])
For all n, $\mathsf{ACA}_0 \vdash$ '$(\mathbb{S}_n,\leq^w_{gap})$ *is a* $\mathbb{wpo}$',
For all n, $\mathsf{ACA}_0 \vdash$ '$(\mathbb{S}_n,\leq^s_{gap})$ *is a* $\mathbb{wpo}$'.

Definition 10 Denote the subset of $\mathbb{S}_n$ of elements $s_0 \dots s_k$ that fulfill the extra condition $s_0 \leq i$ by $\mathbb{S}_n[i]$. Accordingly as in Definition 9, $(\mathbb{S}_n[i], \leq^w_{gap})$, respectively $(\mathbb{S}_n[i], \leq^s_{gap})$, is denoted by $\mathbb{S}^w_n[i]$, respectively $\mathbb{S}^s_n[i]$.

Definition 11 Take two partial orders X_0 and X_1. We say that X_0 and X_1 are order-isomorphic if there exists a bijective function f such that $x \leq_{X_0} y \Leftrightarrow f(x) \leq_{X_1} f(y)$ for all x and y in X_0. We denote this by $X_0 \cong X_1$.

If $X_0 \cong X_1$ and X_0 or X_1 is a $\mathbb{wpo}$, then the other one is also a $\mathbb{wpo}$ with the same maximal order type.

The proofs by Schütte and Simpson [17] also yield results on the maximal order types of the sequences with the gap-embeddability relation. More specifically, they prove the next lemma (which is in Lemma 5.5 in [17]). However, there is a small error in their proof, although we believe that this can actually be seen as a typo. For clarity reasons, the proof is given here. ε denotes the empty string in $\mathbb{S}^s_n$ or $\mathbb{S}^w_n$, whereas $()$ denotes the empty string in $(\mathbb{S}^s_n)^*$ or $(\mathbb{S}^w_n)^*$.

Theorem 7 (Schütte-Simpson[17])
$\mathbb{S}^s_{n+1} \cong \mathbb{S}^s_n \times (\mathbb{S}^s_n)^*$.

Proof Assume $n \geq 0$. We define an order-preserving bijection h_n from $\mathbb{S}^s_{n+1}$ to the partial order $\mathbb{S}^s_n \times (\mathbb{S}^s_n)^*$. Let $h_n(\varepsilon)$ be $(\varepsilon, ())$. Take an arbitrary element $s \in \mathbb{S}^s_{n+1} \backslash \{\varepsilon\}$. Then $s = t_0 \dots t_l \in \{0, \dots, n\}^*$. s is of the form $t'_0 0 t'_1 0 \dots 0 t'_k$ with $k \geq 0$ and $t'_i \in \{1, \dots, n\}^*$ as follows. Define $i_0 = \min\{l : t_l = 0\}$, $i_1 = \min\{l : t_l = 0 \text{ and } l > i_0\}, \dots$, $i_{k-1} = \min\{l : t_l = 0 \text{ and } l > i_{k-2}\}$. k is the least number where i_k is undefined. Then $t'_1 = t_0 \dots t_{i_0 - 1}$, $t'_2 = t_{i_0+1} \dots t_{i_1-1}$, etc. Remark that is possible that t'_i is the empty sequence. There exist unique $s_i \in \{0, \dots, n-1\}^*$ such that $s^+_i = t'_i$ for every i, where s^+_i is the result of replacing every number j in s_i by $j+1$. Hence, every element $s \in \mathbb{S}^s_{n+1} \backslash \{\varepsilon\}$ has a unique representation $s = s^+_0 0 s^+_1 0 \dots 0 s^+_k$ with $s_i \in \{0, \dots, n-1\}^*$. For example if $s = 021003$, then $s_0 = \varepsilon$, $s_1 = 10$, $s_2 = \varepsilon$ and $s_3 = 2$, as $s = s^+_0 0 s^+_1 0 s^+_2 0 s^+_3$. Define then $h_n(s)$ as $(s_0, (s_1, \dots, s_k))$. Note that for the sequence $s = 0$, k is not zero, i.e. $s = s^+_0 0 s^+_1$ with $s_0 = s_1 = \varepsilon$. In other words, k represents the number of 0's occurring in s. Because s can be written in an unique way as $s = s^+_0 0 s^+_1 0 \dots 0 s^+_k$, h_n is a injection. It's also easy to see that h_n is a surjection from $\mathbb{S}^s_{n+1}$ to the partial order $\mathbb{S}^s_n \times (\mathbb{S}^s_n)^*$.

We know prove that $s \leq s'$ yields $h_n(s) \leq h_n(s')$ by induction on $lh(s) + lh(s')$. If s or s' are ε, then this is trivial. So assume $s = s^+_0 0 \dots 0 s^+_k$ and $s' = s'^+_0 0 \dots 0 s'^+_l$. If $s = s'$, the claim is trivial. Assume $s < s'$. If $k = 0$, then $l = 0$ and $s^+_0 < s'^+_0$, or $l > 0$ and $s^+_0 \leq s'^+_0$. In both cases, $h_n(s) < h_n(s')$. Assume $k > 0$. Then $s < s'$ yields $l > 0$, $s^+_0 \leq s'^+_0$ and $s^+_1 0 \dots 0 s^+_k \leq s'^+_j 0 \dots 0 s'^+_l$ for a certain $j \geq 1$. From $s^+_1 0 \dots 0 s^+_k \leq s'^+_j 0 \dots 0 s'^+_l$, one can prove as before (or by an additional induction argument on k) that $s^+_1 \leq s'^+_j$ and $s^+_2 0 \dots 0 s^+_k \leq s'^+_{j_2} 0 \dots 0 s'^+_l$ for a certain $j_2 \geq j+1$. In the end, we have $s^+_0 \leq s'^+_0$ and $(s^+_1, \dots, s^+_k) \leq^* (s'^+_1, \dots, s'^+_l)$. This yields $h_n(s) \leq h_n(s')$. The reverse direction $h_n(s) < h_n(s') \to s < s'$ can be proven in a similar way by induction.

Corollary 1 $o(\mathbb{S}^s_{n+1}) = o(\mathbb{S}^s_n) \otimes o((\mathbb{S}^s_n)^*)$.

Hence, from the maximal order type of $\mathbb{S}_1^s$, which is the ordinal ω, one can calculate the maximal order types of all $\mathbb{S}_n^s$. Following the same template, one also has the following lemma.

Lemma 2 $o(\mathbb{S}_{n+1}^w) = o(\mathbb{S}_{n+1}^w[0]) = o(\mathbb{S}_{n+1}^s[0]) = o((\mathbb{S}_n^s)^*)$.

Proof The equality $o(\mathbb{S}_{n+1}^s[0]) = o((\mathbb{S}_n^s)^*)$ follows from the proof of Theorem 7. $o(\mathbb{S}_{n+1}^w[0]) = o(\mathbb{S}_{n+1}^s[0])$ is trivial as they refer to the same ordering. To prove $o(\mathbb{S}_{n+1}^w) = o(\mathbb{S}_{n+1}^w[0])$, note that $\mathbb{S}_{n+1}^w[0] \subseteq \mathbb{S}_{n+1}^w$, hence $o(\mathbb{S}_n^w[0]) \leq o(\mathbb{S}_n^w)$. Furthermore, the mapping e which plots $s_0 \dots s_{k-1}$ to $0s_0 \dots s_{k-1}$ is a quasi-embedding from $\mathbb{S}_n^w$ to $\mathbb{S}_n^w[0]$. Hence, $o(\mathbb{S}_n^w) \leq o(\mathbb{S}_n^w[0])$.

These results yield for example

$$o(\mathbb{S}_2^w) = \omega^{\omega^{\omega}}.$$

de Jongh, Parikh, Schmidt, Schütte and Simpson's results imply an easy calculation of the maximal order types of $\mathbb{S}_n^w$ and $\mathbb{S}_n^s$. In this article, we want to connect these maximal order types with well-known addition-free ordinal notation systems. We expected that addition-free ordinal notation system based on the collapsing functions $\vartheta_0, \dots, \vartheta_{n-1}$ defines a maximal linear extension on $\mathbb{S}_n^w$ and $\mathbb{S}_n^s$. However, we show that this is only the case for $n = 2$.

Definition 12 Let $\overline{\mathbb{S}}_n$ be the subset of $\mathbb{S}_n$ which consists of all sequences $s_0 \dots s_{k-1}$ in $\mathbb{S}_n$ such that for all $i < k-1$, $s_i - s_{i+1} \geq -1$. This means that if $s_i = j$, then s_{i+1} is an element in $\{0, \dots, j+1\}$. For example $02 \notin \overline{\mathbb{S}}_3$. Like in Definition 9, we denote the subset of $\overline{\mathbb{S}}_n$ that fulfill the extra condition $s_0 \leq i$ by $\overline{\mathbb{S}}_n[i]$. We denote $(\overline{\mathbb{S}}_n, \leq_{gap}^w)$ by $\overline{\mathbb{S}}_n^w$, $(\overline{\mathbb{S}}_n, \leq_{gap}^s)$ by $\overline{\mathbb{S}}_n^s$, $(\overline{\mathbb{S}}_n[i], \leq_{gap}^w)$ by $\overline{\mathbb{S}}_n^w[i]$ and $(\overline{\mathbb{S}}_n[i], \leq_{gap}^s)$ by $\overline{\mathbb{S}}_n^s[i]$.

The reason why we define these substructures $\overline{\mathbb{S}}_n^w$ and $\overline{\mathbb{S}}_n^s$ of the $\mathbb{wpo}$'s $\mathbb{S}_n^w$ and $\mathbb{S}_n^s$ is that the addition-free ordinal notation system based on the ϑ_i-functions has the same property if you look to the indices of the occurring ϑ_i in the terms of the notation system (see Definitions 24 and 25). Furthermore, it turns out that the maximal order types of these substructures are exactly equal to an ω-tower. It is expected that the ordinals of the addition-free ordinal notation systems with the ϑ_i-functions are also ω-towers.

Lemma 3 $o(\overline{\mathbb{S}}_{n+1}^w) = o(\overline{\mathbb{S}}_{n+1}^w[0]) = o(\overline{\mathbb{S}}_{n+1}^s[0]) = o((\overline{\mathbb{S}}_n^s[0])^*) = o((\overline{\mathbb{S}}_n^w[0])^*)$
$= o((\overline{\mathbb{S}}_n^w)^*)$.

Proof Completely similar as in Theorem 7, Corollary 1 and Lemma 2: one can use the same embeddings and mappings between the partial orders.

Corollary 2 *For all n,* $o(\overline{\mathbb{S}}_n^w) = \omega_{2n-1}$.

We remark that the maximal order type of $\mathbb{S}_n^w$ and $\mathbb{S}_n^s$ are in general not ω-towers.

2.3 Ordinal notation systems

In this subsection, we introduce several ordinal notation systems for ordinals smaller than ε_0. All of them do not use the addition operator.

2.3.1 The Veblen hierarchy

Assume that $(T,<)$ is a notation system with $otype(T) \in \varepsilon_0 \backslash \{0\}$. Define the representation system $\varphi_T 0$ recursively as follows.

Definition 13 – $0 \in \varphi_T 0$,
– if $\alpha \in \varphi_T 0$ and $t \in T$, then $\varphi_t \alpha \in \varphi_T 0$.

Define on $\varphi_T 0$ the following total order.

Definition 14 For $\alpha, \beta \in \varphi_T 0$, $\alpha < \beta$ is valid if

– $\alpha = 0$ and $\beta \neq 0$,
– $\alpha = \varphi_{t_1} \alpha'$, $\beta = \varphi_{t_2} \beta'$ and one of the following cases holds:
  1. $t_1 < t_2$ and $\alpha' < \beta$,
  2. $t_1 = t_2$ and $\alpha' < \beta'$,
  3. $t_1 > t_2$ and $\alpha \leq \beta'$.

Theorem 8 *Assume $otype(T) = \alpha \in \varepsilon_0 \backslash \{0\}$. Then $(\varphi_T 0, <)$ is a notation system for the ordinal $\omega^{\omega^{-1+\alpha}}$.*

Proof A proof of this fact can be found in [10].

2.3.2 Using the π_i-collapsing functions

We use an ordinal notation system that employs the π_i-collapsing functions. These functions are based on Buchholz's ψ_i-functions [1]. We state some basic facts that the reader can find in [1,17].

Definition 15 Let $\Omega_0 := 1$ and define Ω_i as the i^{th} regular ordinal number strictly above ω. Define Ω_ω as $\sup_i \Omega_i$.

Define the sets $B_i^m(\alpha)$ and $B_i(\alpha)$ and the ordinal numbers $\pi_i \alpha$ as follows.

Definition 16 – If $\gamma = 0$ or $\gamma < \Omega_i$, then $\gamma \in B_i^m(\alpha)$,
– if $i \leq j$, $\beta < \alpha$, $\beta \in B_j(\beta)$ and $\beta \in B_i^m(\alpha)$, then $\pi_j \beta \in B_i^{m+1}(\alpha)$,
– define $B_i(\alpha)$ as $\bigcup_{m<\omega} B_i^m(\alpha)$,
– $\pi_i \alpha := \min\{\eta : \eta \notin B_i(\alpha)\}$.

Lemma 4 *1. if $i \leq j$ and $\alpha \leq \beta$, then $B_i(\alpha) \subseteq B_j(\beta)$ and $\pi_i \alpha \leq \pi_j \beta$,*
2. $\Omega_i \leq \pi_i \alpha < \Omega_{i+1}$,
3. $\pi_i 0 = \Omega_i$,
4. $\alpha \in B_i(\alpha)$ and $\alpha < \beta$ yields $\pi_i \alpha < \pi_i \beta$,
5. $\alpha \in B_i(\alpha)$, $\beta \in B_i(\beta)$ and $\pi_i \alpha = \pi_i \beta$ yields $\alpha = \beta$.

Definition 17 For ordinals $\alpha \in B_0(\Omega_\omega)$, define $G_i(\pi_j\alpha)$ as

$$\begin{cases} \emptyset & \text{if } j < i, \\ G_i\alpha \cup \{\alpha\} & \text{otherwise.} \end{cases}$$

Define $G_i(0)$ as $\emptyset$.

This is well-defined, because one can prove that $\pi_j\alpha \in B_0(\Omega_\omega)$ yields $\alpha \in B_0(\Omega_\omega)$. For a set of ordinals A and an ordinal α, we write $A < \alpha$ if for all $\beta \in A(\beta < \alpha)$.

Lemma 5 *If $\alpha \in B_0(\Omega_\omega)$, then $G_i(\alpha) < \beta$ iff $\alpha \in B_i(\beta)$.*

Proof We prove this by induction on the length of construction of α. If $\alpha = 0$ or $\alpha = \pi_j\delta$ with $j < i$, then this is trivial. Assume $\alpha = \pi_j\delta$ with $j \geq i$. $\alpha = \pi_j\delta \in B_0(\Omega_\omega)$ yields $\delta \in B_j(\delta)$. Now, $G_i(\alpha) < \beta$ is valid iff $G_i(\delta) < \beta$ and $\delta < \beta$. By the induction hypothesis, this is equivalent with $\delta \in B_i(\beta)$ and $\delta < \beta$, which is equivalent with $\alpha = \pi_j\delta \in B_i(\beta)$ because $\delta \in B_j(\delta)$.

Now we define the ordinal notation systems $\pi(\omega)$ and $\pi(n)$, but first, we have to define a set of terms $\pi(\omega)'$ and $\pi(n)'$.

Definition 18
- $0 \in \pi(\omega)'$ and $0 \in \pi(n)'$,
- if $\alpha \in \pi(\omega)'$, then $D_j\alpha \in \pi(\omega)'$,
- if $\alpha \in \pi(n)'$ and $j < n$, then $D_j\alpha \in \pi(n)'$.

Definition 19 Let $\alpha, \beta \in \pi(\omega)'$ or $\alpha, \beta \in \pi(n)'$. Then define $\alpha < \beta$ if

1. $\alpha = 0$ and $\beta \neq 0$,
2. $\alpha = D_j\alpha'$, $\beta = D_k\beta'$ and $i < j$ or $i = j$ and $\alpha' < \beta'$.

Lemma 6 *$<$ is a linear order on $\pi(\omega)'$ and $\pi(n)'$.*

Proof Similar as Lemma 2.1 in [1].

Definition 20 For $\alpha \in \pi(\omega)', \pi(n)'$, define $G_i(\alpha)$ as follows.

1. $G_i(0) = \emptyset$,
2. $G_i(D_j\alpha') := \begin{cases} G_i(\alpha') \cup \{\alpha'\} & \text{if } i \leq j, \\ \emptyset & \text{if } i > j. \end{cases}$

Now, we are ready to define to ordinal notation systems $\pi(\omega) \subseteq \pi(\omega)'$ and $\pi(n) \subseteq \pi(n)'$.

Definition 21 $\pi(\omega)$ and $\pi(n)$ are the least sets such that

1. $0 \in \pi(\omega)$, $0 \in \pi(n)$,
2. if $\alpha \in \pi(\omega)$ and $G_i(\alpha) < \alpha$, then $D_i\alpha \in \pi(\omega)$,
3. if $\alpha \in \pi(n)$, $i < n$ and $G_i(\alpha) < \alpha$, then $D_i\alpha \in \pi(n)$.

Apparently, the $D_j\alpha$'s correspond to the ordinal functions $\pi_j\alpha$:

Definition 22 For $\alpha \in \pi(\omega)$ or $\alpha \in \pi(n)$, define

1. $o(0) := 0$,
2. $o(D_j\alpha') := \pi_j(o(\alpha'))$.

Lemma 7 *For $\alpha, \beta \in \pi(\omega)$ or $\alpha, \beta \in \pi(n)$, we have:*

1. $o(\alpha) \in B_0(\Omega_\omega)$,
2. $G_i(o(\alpha)) = \{o(x) : x \in G_i(\alpha)\}$,
3. $\alpha < \beta \rightarrow o(\alpha) < o(\beta)$.

Proof A similar proof can be found in [1].

Lemma 8 *1.* $\{o(x) : x \in \pi(\omega)\} = B_0(\Omega_\omega)$,
2. $\{o(x) : x \in \pi(\omega)$ *and* $x < D_1 0\} = \pi_0\Omega_\omega$,
3. $\{o(x) : x \in \pi(n)$ *and* $x < D_1 0\} = \pi_0\Omega_n$ *if* $n > 0$.

Proof A similar proof can be found in [1].

Define $\pi(\omega) \cap D_1 0$ as $\pi_0(\omega)$ and $\pi(n) \cap D_1 0$ as $\pi_0(n)$. It is very important to see that we work with two different contexts: one context is at the level of ordinals, i.e. if we use the π_i's. The other context at the syntactical level, i.e. if we use the D_i's (because it is an ordinal notation system). The previous results actually indicate that D_i and π_i play the same role and for notational convenience, we will identify these two notations: from now on, we write π_i instead of D_i. The context will make clear what we mean. If we use Ω_i in the ordinal context, it is interpreted as in Definition 15. In the other context, at the level of ordinal notation systems, we define Ω_i as $D_i 0$ (which is now also denoted by $\pi_i 0$).

We could also have defined $\pi(\omega)$ in the following equivalent way.

Definition 23 Define $\pi(\omega)$ as the least set of ordinals such that

1. $0 \in \pi(\omega)$,
2. If $\alpha \in \pi(\omega)$ and $\alpha \in B_i(\alpha)$, then $\pi_i\alpha \in \pi(\omega)$.

Define $\pi(n)$ in the same manner, but with the restriction that $i < n$.

In [17], the following theorem is shown. Therefore, $\pi_0(n)$ is an ordinal notation system for $\omega_n[1]$ if $n > 0$ and $\pi_0(\omega)$ is a system for ε_0.

Theorem 9 *1.* $\pi_0\Omega_n = \omega_n[1]$ *if* $n > 0$,
2. $\pi_0\Omega_\omega = \varepsilon_0$.

2.3.3 Using the ϑ_i-collapsing functions

In this subsection, we give an ordinal representation system that is based on the ϑ_i-functions. For more information about this system that includes the addition-operator, see [14, 15]. In this subsection, we introduce them without the addition-operator.

Definition 24 Define T and the function S simultaneously as follows. T is the least set such that $0 \in T$, where $S(0) := -1$ and if $\alpha \in T$ with $S(\alpha) \leq i+1$, then $\vartheta_i \alpha \in T$ and $S(\vartheta_i \alpha) := i$. We call the number of occurrences of symbols ϑ_j in $\alpha \in T$, the **length** of α and denote this by $lh(\alpha)$. Furthermore, let $\Omega_i := \vartheta_i 0$.

Like in the D_i-case, Ω_i is defined as something syntactically because T is an ordinal notation system. However, the usual interpretation of Ω_i in the context of ordinals is as in Definition 15. $S(\alpha)$ represents the index i of the first occurring ϑ_i in α, if $\alpha \neq 0$.

Definition 25 Let $n < \omega$. Define T_n as the set of elements α in T such that for all ϑ_j in α, we have $j < n$. Let $T[m]$ be the set of elements α in T such that $S(\alpha) \leq m$. Define $T_n[m]$ accordingly.

For example $T_1 = T_1[0] = \{0, \vartheta_0 0, \vartheta_0 \vartheta_0 0, \dots\}$. For every element α in T, we define its *coefficients*. The definition is based on the usual definition of the coefficients in a notation system *with* addition.

Definition 26 Let $\alpha \in T$. If $\alpha = 0$, then $k_i(0) := 0$. Assume $\alpha = \vartheta_j(\beta)$. Let $k_i(\alpha)$ then be

$$\begin{cases} \vartheta_j(\beta) = \alpha & \text{if } j \leq i, \\ k_i(\beta) & \text{if } j > i. \end{cases}$$

Using this definition, we introduce a well-order on T (and its substructures). This ordering is based on the usual ordering between the ϑ_i-functions defined with addition.

Definition 27 1. If $\alpha \neq 0$, then $0 < \alpha$,
2. if $i < j$, then $\vartheta_i \alpha < \vartheta_j \beta$,
3. if $\alpha < \beta$ and $k_i \alpha < \vartheta_i \beta$, then $\vartheta_i \alpha < \vartheta_i \beta$,
4. if $\alpha > \beta$ and $\vartheta_i \alpha \leq k_i \beta$, then $\vartheta_i \alpha < \vartheta_i \beta$.

Definition 28 If $\alpha, \beta \in T$ and $\beta < \Omega_1$, let $\alpha[\beta]$ be the element in T where the last zero in α is replaced by β.

The following lemma gives some useful properties of this ordinal notation system.

Lemma 9 *For all α, β and γ in T and for all $i < \omega$,*

1. *$k_i(\alpha) \leq \alpha$,*
2. *if $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} t$ with $j_1, \dots, j_n \geq i$ and ($t = 0$ or $t = \vartheta_k t'$ with $k \leq i$), then $t < \vartheta_i(\alpha)$,*
3. *$k_i(\alpha) < \vartheta_i \alpha$,*
4. *$k_i(\alpha)[\gamma] = k_i(\alpha[\gamma])$ for $\gamma < \Omega_1$,*
5. *if $\gamma < \Omega_1$, then $\gamma \leq \beta[\gamma]$ and the last inequality is only an equality if $\beta = 0$,*
6. *if $\alpha < \beta$ and $\gamma < \Omega_1$, then $\alpha[\gamma] < \beta[\gamma]$.*

Proof 1. The first assertion is easy to see.

2. By induction on $lh(\alpha)$ and sub-induction on $lh(t)$. If $\alpha = 0$, then the claim is trivial. Assume from now on $\alpha > 0$. If $t = 0$ or $t = \vartheta_k t'$ with $k < i$, then this is trivial. Assume $t = \vartheta_i t'$. Then $t = \vartheta_i \vartheta_{l_1} \dots \vartheta_{l_m} k_i(t')$ with $l_1, \dots, l_m > i$. The sub-induction hypothesis, $lh(k_i(t')) < lh(t)$ and $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} \vartheta_i \vartheta_{l_1} \dots \vartheta_{l_m} k_i(t')$ yield $k_i(t') < \vartheta_i \alpha$. If $t' < \alpha$, then $t = \vartheta_i t' < \vartheta_i \alpha$. Assume $t' > \alpha$. Note that equality is impossible because t' is a strict subterm of α. We claim that $t = \vartheta_i t' \leq k_i(\alpha)$, hence we are done. We know that $k_i(\alpha) = \vartheta_{j_p} \dots \vartheta_{j_n} \vartheta_i t'$ for a certain p with $j_p = i$ or $k_i(\alpha) = \vartheta_i t'$. In the latter case, the claim is trivial. In the former case, the main induction hypothesis on $\vartheta_{j_{p+1}} \dots \vartheta_{j_n} \vartheta_i t'$ yields $t < \vartheta_i \vartheta_{j_{p+1}} \dots \vartheta_{j_n} \vartheta_i t' = k_i(\alpha)$.
3. This follows easily from the second assertion because $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} k_i(\alpha)$ with $j_1, \dots, j_n > i$.
4. Follows easily by induction on $lh(\alpha)$.
5. By induction on $lh(\gamma)$ and sub-induction on $lh(\beta)$. If $\gamma = 0$, the statement is trivial to see. From now on, let $\gamma = \vartheta_0 \gamma'$. If $\beta = 0$ or $\beta = \vartheta_i \beta'$ with $i > 0$, the statement also easily follows. Assume $\beta = \vartheta_0 \beta'$. We see $\beta[\gamma] = \vartheta_0(\beta'[\gamma])$. Suppose $\gamma' < \beta'[\gamma]$. Assume $\gamma' = \vartheta_{j_1} \dots \vartheta_{j_k} k_0(\gamma')$ with $j_1, \dots, j_k > 0$ and define $\overline{\beta}$ as $\beta[\vartheta_0 \vartheta_{j_1} \dots \vartheta_{j_k} 0]$. The main induction hypothesis yields $k_0(\gamma') \leq \overline{\beta}[k_0(\gamma')] = \beta[\gamma] = \vartheta_0(\beta'[\gamma])$. Note that equality is not possible because $k_0(\gamma')$ is a strict subterm of $\overline{\beta}[k_0(\gamma')]$, hence $\gamma = \vartheta_0 \gamma' < \vartheta_0(\beta'[\gamma]) = \beta[\gamma]$. Assume $\gamma' > \beta'[\gamma]$. The sub-induction hypothesis yields $\gamma \leq k_0(\beta')[\gamma] \overset{\gamma < \Omega_1}{=} k_0(\beta'[\gamma])$. Hence, $\gamma \leq k_0(\beta'[\gamma]) < \vartheta_0(\beta'[\gamma]) = \beta[\gamma]$.
6. By induction on $lh(\alpha) + lh(\beta)$. If $\alpha = 0$ and $\beta \neq 0$, then the previous assertion yields $\alpha[\gamma] = \gamma < \beta[\gamma]$. Assume $\alpha = \vartheta_i \alpha' < \vartheta_j \beta' = \beta$. If $i < j$, then also $\alpha[\gamma] < \beta[\gamma]$. Suppose $i = j$. Then either $\alpha' < \beta'$ and $k_i(\alpha') < \vartheta_j \beta'$, or $\alpha \leq k_j(\beta')$. In the former case, the induction hypothesis yields $\alpha'[\gamma] < \beta'[\gamma]$ and $k_i(\alpha'[\gamma]) \overset{\gamma < \Omega_1}{=} k_i(\alpha')[\gamma] < (\vartheta_j \beta')[\gamma] = \vartheta_j(\beta'[\gamma])$. Hence, $\alpha[\gamma] = (\vartheta_i \alpha')[\gamma] = \vartheta_i(\alpha'[\gamma]) < \vartheta_j(\beta'[\gamma]) = (\vartheta_j \beta')[\gamma] = \beta[\gamma]$. In the latter case, the induction hypothesis yields $\alpha[\gamma] \leq k_j(\beta')[\gamma] \overset{\gamma < \Omega_1}{=} k_j(\beta'[\gamma]) < \vartheta_j(\beta'[\gamma]) = (\vartheta_j \beta')[\gamma] = \beta[\gamma]$.

On T and its substructures, we define the following partial order $\trianglelefteq$, which can be seen as a natural sub-order of the ordering $<$ on T (see Lemma 11).

Definition 29 1. $0 \trianglelefteq \alpha$,
2. if $\alpha \trianglelefteq k_i \beta$, then $\alpha \trianglelefteq \vartheta_i \beta$,
3. if $\alpha \trianglelefteq \beta$, then $\vartheta_i \alpha \trianglelefteq \vartheta_i \beta$.

Apparently, T_n with this natural sub-ordering is the same as $\overline{\mathbb{S}}_n^s$.

Lemma 10 $(T_n, \trianglelefteq) \cong (\overline{\mathbb{S}}_n, \leq^s_{gap})$.

Proof Define $e : T_n \to \overline{\mathbb{S}}_n$ as follows. $e(0)$ is the empty sequence ε. Let $e(\vartheta_i \alpha)$ be $(i)^\frown e(\alpha)$. For example $e(\vartheta_2 \vartheta_1 0)$ is the finite sequence 21. From the definitions of e, T_n and $\overline{\mathbb{S}}_n$, it is trivial to see that e is a bijection: remark that there are $\overline{\mathbb{S}}_n$-like restrictions on T_n. So the only thing we still need to show is that for all α and β in T_n, $e(\alpha) \leq^s_{gap} e(\beta)$ if and only if $\alpha \trianglelefteq \beta$. We show this by induction on the sum of the lengths of α and β. If α or β are equal to 0, then this is trivial. Assume α and β are different from 0. Hence, $\alpha = \vartheta_i\alpha'$ and $\beta = \vartheta_j\beta'$. Assume $\alpha \trianglelefteq \beta$. Then $\alpha \trianglelefteq k_j\beta'$ or $i = j$ and $\alpha' \trianglelefteq \beta'$. In the latter case, the induction hypothesis yields $e(\alpha') \leq^s_{gap} e(\beta')$, hence $e(\alpha) = (i)^\frown e(\alpha') \leq^s_{gap} (i)^\frown e(\beta') = e(\beta)$. In the former case, assume $\beta' = \vartheta_{l_1} \ldots \vartheta_{l_k}\beta''$, with $l_1, \ldots, l_k > j$ and $S(\beta'') \leq j$ such that $k_j(\beta') = \beta''$. The induction hypothesis yields $e(\alpha) \leq^s_{gap} e(\beta'')$. From the strong gap-embeddability relation we obtain $i \leq S(\beta'') \leq j$, hence $e(\alpha) \leq^s_{gap} (jl_1, \ldots l_k)^\frown e(\beta'')$ because $j, l_1, \ldots, l_k \geq i$. The reverse direction can be proved in a similar way.

The previous proof also yields $(T_n[0], \trianglelefteq) \cong (\overline{\mathbb{S}}_n[0], \leq^s_{gap}) = (\overline{\mathbb{S}}_n[0], \leq^w_{gap})$. We prove that the linear order $<$ on T_n is a linear extension of $\triangleleft$. Let $\alpha \triangleleft \beta$ if $\alpha \trianglelefteq \beta$ and $\alpha \neq \beta$.

Lemma 11 *If $\alpha \trianglelefteq \beta$, then $\alpha \leq \beta$.*

Proof We prove this by induction on the sum of the lengths of α and β Assume $\alpha \trianglelefteq \beta$. If $\alpha = 0$, then trivially $\alpha \leq \beta$. Assume $\alpha = \vartheta_i\alpha'$. $\alpha \trianglelefteq \beta$ yields $\beta = \vartheta_i\beta'$ and either $\alpha \trianglelefteq k_i\beta'$ or $\alpha' \trianglelefteq \beta'$. In the first case, the induction hypothesis yields $\alpha \leq k_i\beta' < \vartheta_i\beta' = \beta$. Assume that $\alpha' \trianglelefteq \beta'$. The induction hypothesis yields $\alpha' \leq \beta'$. if $\alpha' = \beta'$, we can finish the proof, so assume $\alpha' < \beta'$. We want to prove that $k_i\alpha' < \beta$. Using the induction hypothesis, it is sufficient to prove that $k_i\alpha' \triangleleft \beta$. This follows from $\alpha = \vartheta_i\vartheta_{j_1} \ldots \vartheta_{j_l}k_i\alpha' \trianglelefteq \beta$ (with $j_1, \ldots, j_l > i$) and Lemma 10.

The previous lemmata imply that the linear ordering on $T_n[0]$ is a linear extension of $\overline{\mathbb{S}}_n[0]$ with the strong (and weak) gap-embeddability relation and furthermore,

$$o(T_n[0], \trianglelefteq) = o(\overline{\mathbb{S}}^s_n[0]) = o(\overline{\mathbb{S}}^w_n[0]) = o(\overline{\mathbb{S}}^w_n).$$

These results also hold in the case if we allow the addition-operator: the ordinal notation systems using ϑ_i and the addition-operator corresponds to a linear extension of Friedman's $\mathbb{wpo}$ $\overline{\mathbb{T}}_n[0]$ with the strong and weak gap-embeddability relation ($\overline{\mathbb{T}}_n[0]$ is defined in a similar way as $\overline{\mathbb{S}}_n[0]$, but with trees). It is our general belief that this is a **maximal** linear extension. In [11, 12] we already obtained partial results concerning this conjecture. In this paper, we want to investigate whether this is also true for the linearized version of the gap-embeddability relation, i.e. if the well-order $(T_n[0], <)$ is a maximal linear extension of $(T_n[0], \trianglelefteq) \cong (\overline{\mathbb{S}}_n[0], \leq^s_{gap}) = (\overline{\mathbb{S}}_n[0], \leq^w_{gap})$. This can be shown by proving that the order type of $(T_n[0], <)$ is equal to the maximal order type of $(\overline{\mathbb{S}}_n[0], \leq^s_{gap})$, which is ω_{2n-1}.

Quite surprisingly, the maximal linear extension principle is not true in this sequential version: if $n > 2$, then the order type of $(T_n[0], <)$ is equal to ω_{n+1}. We remark that the maximal linear extension principle is true if $n = 1$ and $n = 2$. We prove these claims in the next sections.

3 Maximal linear extension of gap-sequences with one and two labels

It is trivial to show that the order type of $(T_1[0],<)$ is equal to ω, hence $(T_1[0],<)$ corresponds to a maximal linear extension of $\overline{\mathbb{S}}_1^s[0]$. So we can concentrate on the case of $T_2[0]$. We show that the order type of $(T_2[0],<)$ is equal to ω^{ω^ω}. This implies that $(T_2[0],<)$ corresponds to a maximal linear extension of $\overline{\mathbb{S}}_2^w[0]$ and that the order type of $(T_2[0],<)$ is equal to $o(\overline{\mathbb{S}}_2^w)$. More specifically, we show that

$$\sup_{n_1,\dots,n_k} \vartheta_0\vartheta_1^{n_1}\dots\vartheta_0\vartheta_1^{n_k}(0) = \omega^{\omega^\omega}.$$

The supremum is equal to $\vartheta_0\vartheta_1\vartheta_2(0)$ and knowing that Ω_i is defined as $\vartheta_i(0)$, we thus want to show

$$\vartheta_0\vartheta_1\Omega_2 = \omega^{\omega^\omega}.$$

Theorem 10 $\vartheta_0\vartheta_1\Omega_2 = \omega^{\omega^\omega}$

Proof We present a order-preserving bijection from $\varphi_\omega 0$ to $\vartheta_0\vartheta_1\Omega_2$. Lemma 8 then yields the assertion.

Define $\chi 0 := 0$ and $\chi\varphi_n\alpha := \vartheta_0\vartheta_1^n\chi\alpha$. Then χ is order preserving. Indeed, we show $\alpha<\beta \Rightarrow \chi\alpha<\chi\beta$ by induction on $lh(\alpha)+lh(\beta)$. If $\alpha=0$ and $\beta\neq 0$, then trivially $\chi\alpha<\chi\beta$. Let $\alpha=\varphi_n\alpha'<\beta=\varphi_m\beta'$. If $\alpha'<\beta$ and $n<m$ then the induction hypothesis yields $\chi\alpha'<\vartheta_0\vartheta_1^m\chi\beta'$ and then $n<m$ yields $\chi\alpha=\vartheta_0\vartheta_1^n\chi\alpha'<\vartheta_0\vartheta_1^m\chi\beta'=\chi\beta$. If $n=m$ and $\alpha'<\beta'$ then $\chi\alpha=\vartheta_0\vartheta_1^n\chi\alpha'<\vartheta_0\vartheta_1^n\chi\beta'=\chi\beta$. If $\alpha\le\beta'$, then $\chi\alpha\le\chi\beta'<\vartheta_0\vartheta_1^m\chi\beta'$.

It might be instructive, although it is in fact superfluous, to redo the argument for the standard representation for ω^{ω^ω}. First, we need an additional lemma.

Lemma 12 *Let α,β and γ be elements in T.*

1. *$\alpha<\beta<\Omega_1$ and $l_i<n$, $k_i>0$ for all $i\le r$ yield* $\vartheta_0^{k_0}\vartheta_1^{l_1}\vartheta_0^{k_1}\dots\vartheta_1^{l_r}\vartheta_0^{k_r}\vartheta_1^n\alpha<\vartheta_0\vartheta_1^n\beta$,
2. *$\alpha<\beta<\Omega_1$ and $l_{ij}<n$, $k_{ij}>0$ for all i,j yield* $\vartheta_0^{k_{00}}\vartheta_1^{l_{01}}\vartheta_0^{k_{01}}\dots\vartheta_1^{l_{0m_0}}\vartheta_0^{k_{0m_0}}\vartheta_1^n\dots\vartheta_0^{k_{r0}}\vartheta_1^{l_{r1}}\vartheta_0^{k_{r1}}\dots\vartheta_1^{l_{rm_r}}\vartheta_0^{k_{rm_r}}\vartheta_1^n\alpha<$ $\vartheta_0^{p_{00}}\vartheta_1^{q_{01}}\vartheta_0^{p_{01}}\dots\vartheta_1^{q_{0s_0}}\vartheta_0^{p_{0s_0}}\vartheta_1^n\dots\vartheta_0^{p_{r0}}\vartheta_1^{q_{r1}}\vartheta_0^{p_{r1}}\dots\vartheta_1^{q_{rs_r}}\vartheta_0^{p_{rs_r}}\vartheta_1^n\beta$,
3. *$l_i<n$ and $k_i>0$ for all $i\le r$ yield* $\vartheta_0^{k_0}\vartheta_1^{l_1}\vartheta_0^{k_1}\dots\vartheta_1^{l_r}\vartheta_0^{k_r}0<\vartheta_0\vartheta_1^n 0$.

Proof The first assertion follows by induction on r: if $r=0$, then $\vartheta_0^{k_0}\vartheta_1^n\alpha<\vartheta_0\vartheta_1^n\beta$ follows by induction on k_0. If $r>0$, then the induction hypothesis yields $\xi=\vartheta_0^{k_1}\dots\vartheta_1^{l_r}\vartheta_0^{k_r}\vartheta_1^n\alpha<\vartheta_0\vartheta_1^n\beta$. We have $\xi<\vartheta_1^{n-l_1}\beta$ because $k_1>0$, and thus $\vartheta_1^{l_1}\xi<\vartheta_1^n\beta$. We prove $\vartheta_0^{k_0}\vartheta_1^{l_1}\xi<\vartheta_0\vartheta_1^n\beta$ by induction on k_0. First note that we know $k_0(\vartheta_1^{l_1}\xi)=\xi<\vartheta_0\vartheta_1^n\beta$, hence the induction base $k_0=1$ easily follows. The induction step is straightforward.

The second statement follows from the first by induction on the number of involved blocks.

The third assertion follows by induction on r.

Proof (Another proof of Theorem 10) Define $\chi : \omega^{\omega^\omega} \to \vartheta_0\vartheta_1\Omega_2$ as follows. Take $\alpha < \omega^{\omega^\omega}$. Let n be the least number such that $\alpha < \omega^{\omega^n}$. Let m then be minimal such that

$$\alpha = \omega^{\omega^{n-1}\cdot m} \cdot \alpha_m + \cdots + \omega^{\omega^{n-1}\cdot 0} \cdot \alpha_0,$$

with $\alpha_m \neq 0$ and $\alpha_0, \ldots, \alpha_m < \omega^{\omega^{n-1}}$. Put $\chi\alpha$ as the element

$$\vartheta_0\vartheta_1^n\chi(\alpha_0)\cdots\vartheta_0\vartheta_1^n\chi(\alpha_m).$$

It is trivial to see that χ is surjective. We claim that $\alpha < \beta$ yields $\chi(\alpha) < \chi(\beta)$. We prove the claim by induction on $lh(\alpha) + lh(\beta)$.
Let $\alpha = \omega^{\omega^{n-1}\cdot m} \cdot \alpha' + \tilde{\alpha}$ and $\beta = \omega^{\omega^{n'-1}\cdot m'} \cdot \beta' + \tilde{\beta}$ with $\alpha', \beta' > 0$, $\tilde{\alpha} < \omega^{\omega^{n-1}\cdot m}$ and $\tilde{\beta} < \omega^{\omega^{n'-1}\cdot m'}$. If $n < n'$, then $\chi(\beta)$ contains a consecutive sequence of $\vartheta_1^{n'}$ which has no counterpart in $\chi(\alpha)$. Hence, $\chi\alpha < \chi\beta$ follows from a combination of the second and third assertion of the previous lemma. If $n = n'$ and $m < m'$ then $\chi(\beta)$ contains at least one more consecutive sequence of ϑ_1^n than the ones occurring in $\chi(\alpha)$. Thus again $\chi\alpha < \chi\beta$ using the second and third assertion of the previous lemma. If $n = n'$ and $m = m'$ and $\alpha' < \beta'$ then the induction hypothesis yields $\chi(\alpha') < \chi(\beta')$. We know $\chi(\alpha) = \chi(\tilde{\alpha})\vartheta_0\vartheta_1^n\chi(\alpha')$ and $\chi(\beta) = \chi(\tilde{\beta})\vartheta_0\vartheta_1^n\chi(\beta')$. So, the second assertion of the previous lemma yields the assertion. If $n = n'$ and $m = m'$ and $\alpha' = \beta'$ then $\tilde{\alpha} < \tilde{\beta}$ and the induction hypothesis yield $\chi(\tilde{\alpha}) < \chi(\tilde{\beta})$ and $\chi(\alpha) = \chi(\tilde{\alpha})\vartheta_0\vartheta_1^n\chi(\alpha')$ and $\chi(\beta) = \chi(\tilde{\beta})\vartheta_0\vartheta_1^n\chi(\beta')$. The assertion follows from the sixth assertion of Lemma 9.

4 The order type of $(T_n[0], <)$ with $n > 2$

As mentioned before, we expected that $(T_n[0], <)$ corresponds to a maximal linear extension of $\overline{\mathbb{S}}_n^w[0]$ and $\overline{\mathbb{S}}_n^s[0]$. This could have been shown by proving that the order type of $(T_n[0], <)$ is equal to ω_{2n-1}. However, by calculations of the second author, we saw that $(T_n[0], <)$ does not correspond to a maximal linear extension. Instead we now show that the order type of $(T_n[0], <)$ is equal to ω_{n+1} for $n \geq 2$. We will show that

$$\omega_{n+2} = \vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1},$$

for $n \geq 1$. The next Lemma shows that this is sufficient to prove. To prove the lower bound ($\leq$) of

$$\omega_{n+2} = \vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1},$$

we use results by Schütte and Simpson [17]. The other direction will be shown by turning the already convincing sketch of the second author into a general argument.

Lemma 13 *The order type of* $(T_{n+1}[0], <)$ *is equal to*

$$\vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1} = \vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\vartheta_{n+1}0.$$

Proof We show that

$$\vartheta_0\vartheta_1\vartheta_2\dots\vartheta_n\vartheta_{n+1}0 = \sup_{n_1,\dots,n_k<n+1 \text{ with } \vartheta_0\vartheta_{n_1}\dots\vartheta_{n_k}0\in T} \vartheta_0\vartheta_{n_1}\dots\vartheta_{n_k}0$$

$$= \sup_{\alpha\in T_{n+1}[0]} \alpha.$$

1. We prove by induction on the length of α, that $\forall k\le n\forall\alpha\in T_{n+1}[k]$, $\alpha<\vartheta_k\vartheta_{k+1}\dots\vartheta_n\vartheta_{n+1}0$. If $\alpha=0$, the claim is trivial. Assume $\alpha=\vartheta_l\alpha'$ with $l\le k$. If $l<k$, then the claim is trivial. Assume $l=k$. α' is a term in $T_{n+1}[k+1]$, hence $\alpha'<\vartheta_{k+1}\dots\vartheta_n\vartheta_{n+1}0$. Furthermore, $k_k\alpha'\in T_{n+1}[k]$ is a term of length strictly smaller than α, hence $k_k\alpha'<\vartheta_k\vartheta_{k+1}\dots\vartheta_n\vartheta_{n+1}0$. Hence, $\vartheta_k\alpha'<\vartheta_k\vartheta_{k+1}\dots\vartheta_n\vartheta_{n+1}0$.
2. To prove that $\vartheta_0\vartheta_1\vartheta_2\dots\vartheta_n\vartheta_{n+1}0$ is a least upper bound[1], we show by induction on the length of β that $\forall k\le n\forall\beta\in T$, if $\beta<\vartheta_k\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}$, then $\beta\in T_{n+1}[k]$. The only thing we have to show is that β does not contain ϑ_j's for $j\ge n+1$. If $\beta=0$, this is trivial. Assume $\beta=\vartheta_l\beta'$. If $l<k\le n$, then $\beta'\in T[k]$. Hence, $\beta'<\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}$ because $S(\beta')<k+1$. Therefore, β' and β are in $T_{n+1}[k]$. Suppose that $l=k\le n$. $\beta<\vartheta_k\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}$ yields $\beta\le k_k\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}=0$ or ($\beta'<\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}$ and $k_k\beta'<\vartheta_k\dots\vartheta_n\Omega_{n+1}$). In the former case, the claim follows trivially. Assume the latter. $\beta'<\vartheta_{k+1}\dots\vartheta_n\Omega_{n+1}$ yields $\beta'\in T_{n+1}[k+1]$. Hence $\beta=\vartheta_k\beta'\in T_{n+1}[k]$. This ends the proof.

4.1 Lower bound

In this subsection, we prove $\omega_{n+2}\le\vartheta_0\vartheta_1\vartheta_2\dots\vartheta_n\Omega_{n+1}$, where $n\ge 1$.

Definition 30 1. If $\alpha\in T$, define

$$d_i\alpha := \begin{cases}\vartheta_i\alpha & \text{if } S\alpha\le i,\\ \vartheta_i d_{i+1}\alpha & \text{otherwise.}\end{cases}$$

2. For ordinals in $\pi(\omega)$, define $\bar{\cdot}$ as follows:
   - $\overline{0}:=0$,
   - $\overline{\pi_i\alpha}:=d_{i+1}\overline{\alpha}$.
3. On T, define $0[\beta]:=\beta$ and $(\vartheta_i\alpha)[\beta]:=\vartheta_i(\alpha[\beta])$.
4. Let ψ be the function from $\varphi_{\pi_0(n)}0$ to T which is defined as follows:
   - $\psi 0:=0$,
   - $\psi\varphi_{\pi_0\alpha}\beta:=d_0\overline{\alpha}[\psi\beta]$.

It is easy to see that the image of ψ lies in $T_{n+1}[0]$. We show that ψ is order-preserving in order to obtain a lower bound for the order type of $T_{n+1}[0]$.

Lemma 14 *Let α,β be elements in $\pi(\omega)$ and γ,δ elements in T.*

1. $\alpha<\beta$ and $\gamma,\delta<\Omega$ yields $\overline{\alpha}[\gamma]<\overline{\beta}[\delta]$,

[1] Remark that $T_{n+1}[0]$ does not have a maximum: one can prove by induction on the length of α that $\alpha<\vartheta_0\alpha$.

2. *$\gamma < \delta < \Omega$ yields $\overline{\alpha}[\gamma] < \overline{\alpha}[\delta]$,*
3. *$G_k\alpha < \beta$ and $\gamma,\delta < \Omega$ yield $k_{k+1}\overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$,*
4. *$\alpha < \beta$, $G_k\alpha < \beta$ and $\gamma,\delta < \Omega$ yields $d_{k+1}\overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$,*
5. *If $\zeta,\eta \in \varphi_{\pi_0(n)}0$, then $\zeta < \eta$ yields $\psi\zeta < \psi\eta$.*

Proof We prove assertions 1.–4. simultaneously by induction on $lh(\alpha)$. If $\alpha = 0$, then 1. and 2. are trivial. Assertion 3. is also easy to see because $k_{k+1}\overline{\alpha}[\gamma] = \gamma < \Omega \le d_{k+1}\overline{\beta}[\delta]$. In assertion 4., $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\gamma$. Now, $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}\zeta$ for a certain $\zeta \ge \Omega$. Therefore, $\gamma < \zeta$ and $k_{k+1}\gamma = \gamma < d_{k+1}\overline{\beta}[\delta]$, which yields $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\gamma < d_{k+1}\overline{\beta}[\delta]$.

From now on, assume $\alpha = \pi_i\alpha'$.

Assertion 1.: $\alpha < \beta$ yields $\beta = \pi_j\beta'$ with $i \le j$. If $i < j$, then the assertion follows. Assume $i = j$. Then $\alpha' < \beta'$. We know that $G_i(\alpha') < \alpha'$ because $\pi_i\alpha' \in \pi(\omega)$. Assertion 4. and $\alpha' < \beta'$ yield $d_{i+1}\overline{\alpha'}[\gamma] < d_{i+1}\overline{\beta'}[\delta]$, which is $\overline{\alpha}[\gamma] < \overline{\beta}[\delta]$.

Assertion 2.: We know that $G_i(\alpha') < \alpha'$, hence $G_l(\alpha') < \alpha'$ for all $l \ge i$. Assertion 3. then yields $k_{l+1}\overline{\alpha'}[\gamma] < d_{l+1}\overline{\alpha'}[\delta]$ for all $l \ge i$. If $\alpha' = 0$, then assertion 2. easily follows from $\gamma < \delta$. Assume $\alpha' \neq 0$.

If $S(\overline{\alpha'}) \le i+1$, then $\overline{\alpha}[\gamma] = d_{i+1}\overline{\alpha'}[\gamma] = \vartheta_{i+1}\overline{\alpha'}[\gamma]$. Therefore, assertion 2. follows if $\overline{\alpha'}[\gamma] < \overline{\alpha'}[\delta]$ and $k_{i+1}\overline{\alpha'}[\gamma] < \vartheta_{i+1}\overline{\alpha'}[\delta] = d_{i+1}\overline{\alpha'}[\delta]$. We already know that the second inequality is valid. The first inequality follows from the main induction hypothesis.

Assume now $S(\overline{\alpha'}) > i+1$. We claim that $d_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ for all $j \in \{i+1,\dots,S(\overline{\alpha'})\}$. Assertion 2. then follows from $j = i+1$. We prove our claim by induction on $l = S(\overline{\alpha'}) - j \in \{0,\dots,S(\overline{\alpha'}) - i - 1\}$. If $l = 0$, then $j = S(\overline{\alpha'}) > i+1$. Then the claim follows if $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ and $\overline{\alpha'}[\gamma] < \overline{\alpha'}[\delta]$. The first inequality follows from assertion 3. and the fact that $G_{j-1}(\alpha') < \alpha'$. The second inequality follows from the main induction hypothesis. Now, assume that the claim is true for l. We want to prove that it is true for $l+1 = S(\overline{\alpha'}) - j$. Hence, $l = S(\overline{\alpha'}) - (j+1)$. The induction hypothesis yields $d_{j+1}\overline{\alpha'}[\gamma] < d_{j+1}\overline{\alpha'}[\delta]$. We also see that $j \ge i+1$, so $j-1 \ge i$, hence $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$. Because $S(\overline{\alpha'}) - j = l+1 > 0$, we have $S(\overline{\alpha'}) > j$. Hence, $d_j\overline{\alpha'}[\gamma] = \vartheta_j d_{j+1}\overline{\alpha'}[\gamma]$. The claim follows if $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ and $d_{j+1}\overline{\alpha'}[\gamma] < d_{j+1}\overline{\alpha'}[\delta]$, but we already know that both inequalities are true.

Assertion 3.: If $i < k$, then $k_{k+1}\overline{\alpha}[\gamma] = \overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$ because $S(\overline{\alpha}[\gamma]) = i+1 < k+1$.
If $i > k$, then $k_{k+1}\overline{\alpha}[\gamma] = k_{k+1}\overline{\alpha'}[\gamma]$. Therefore, $G_k(\alpha) = G_k(\alpha') \cup \{\alpha'\} < \beta$ and the induction hypothesis yield the assertion.
Assume that $i = k$. Then $k_{k+1}\overline{\alpha}[\gamma] = \overline{\alpha}[\gamma] = d_{k+1}\overline{\alpha'}[\gamma]$ and $G_k(\alpha) = G_k(\alpha') \cup \{\alpha'\} < \beta$. The induction hypothesis on assertion 4. yields $d_{k+1}\overline{\alpha'}[\gamma] < d_{k+1}\overline{\beta}[\delta]$, from which we can conclude the assertion.

Assertion 4.: $\alpha < \beta$ yields $\beta = \pi_j\beta'$ with $i \le j$.

If $i+1 = S(\overline{\alpha}) \le k+1$, then $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\overline{\alpha}[\gamma]$. There are two sub-cases: either $j+1 = S(\overline{\beta}[\delta]) \le k+1$ or not. In the former case, we obtain $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}\overline{\beta}[\delta]$. Assertion 4. then follows from assertions 1. and 3. and the induction hypothesis. In the latter case, we have $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}d_{k+2}\overline{\beta}[\delta]$. Assertion 4. follows from $\overline{\alpha}[\gamma] <$

$d_{k+2}\overline{\beta}[\delta]$ and assertion 3. The previous strict inequality is valid because $S(\overline{\alpha}[\gamma]) = i+1 \leq k+1 < k+2$.

From now on assume that $i+1 = S(\overline{\alpha}) > k+1$. Actually, we only assume that $S(\overline{\alpha}) \geq k$.

$G_k\alpha < \beta$ yields $G_l\alpha < \beta$ for all $l \geq k$. We claim that $d_{j+1}\overline{\alpha}[\gamma] < d_{j+1}\overline{\beta}[\delta]$ for all $j \in \{k,\ldots,S(\overline{\alpha})\}$ and show this by induction on $l = S(\overline{\alpha}) - j \in \{0,\ldots,S(\overline{\alpha})-k\}$. The assertion then follows from taking $l = S(\overline{\alpha}) - k$.

If $l = 0$ or $l = 1$, then $S(\overline{\alpha}) = k$ or equals $k+1$, hence the claim follows from the case $S(\overline{\alpha}) \leq k+1$. Assume that the claim is true for $l \geq 1$. We want to prove that this is also true for $l+1 = S(\overline{\alpha}) - j$. The induction hypothesis on $l = S(\overline{\alpha}) - (j+1)$ yields $d_{j+2}\overline{\alpha}[\gamma] < d_{j+2}\overline{\beta}[\delta]$. Now because $l \geq 1$, we have $S(\overline{\beta}) \geq S(\overline{\alpha}) \geq j+2 > j+1$. So, $d_{j+1}\overline{\alpha}[\gamma] = \vartheta_{j+1}d_{j+2}\overline{\alpha}[\gamma]$ and $d_{j+1}\overline{\beta}[\delta] = \vartheta_{j+1}d_{j+2}\overline{\beta}[\delta]$. Then the claim is valid if $d_{j+2}\overline{\alpha}[\gamma] < d_{j+2}\overline{\beta}[\delta]$ and $k_{j+1}\overline{\alpha}[\gamma] < d_{j+1}\overline{\beta}[\delta]$. We already know the first strict inequality. The second one follows from assertion 3. and $j \geq k$.

Assertion 5.: We prove this by induction on $lh(\zeta)+lh(\eta)$. Assume $\zeta = \varphi_{\pi_0\alpha}\gamma < \varphi_{\pi_0\beta}\delta = \eta$. There are three cases.

Case 1: $\pi_0\alpha < \pi_0\beta$ and $\gamma < \eta$. The induction hypothesis yields $\psi(\gamma) < \psi(\eta)$. Furthermore, we know that $\alpha < \beta$. If $\alpha = 0$, then $d_0\overline{\alpha}[\psi(\gamma)] = \vartheta_0\psi(\gamma)$. We want to check if this is strictly smaller than $\psi(\eta) = d_0\overline{\beta}[\psi(\delta)] = \vartheta_0 d_1\overline{\beta}[\psi(\delta)]$. Trivially $\psi(\gamma) < d_1\overline{\beta}[\psi(\delta)]$. Furthermore, $k_0(\psi(\gamma)) = \psi(\gamma) < \psi(\eta)$. Hence $\psi(\zeta) = \vartheta_0\psi(\gamma) < \vartheta_0 d_1\overline{\beta}[\psi(\delta)] = \psi(\eta)$. Assume now $0 < \alpha < \beta$. We want to prove that

$$\begin{aligned} d_0\overline{\alpha}[\psi(\gamma)] &= \vartheta_0 d_1\overline{\alpha}[\psi(\gamma)] \\ &< d_0\overline{\beta}[\psi(\delta)] = \vartheta_0 d_1\overline{\beta}[\psi(\delta)]. \end{aligned}$$

Assertion 4., $\alpha < \beta$ and $G_0(\alpha) < \alpha < \beta$ yield $d_1\overline{\alpha}[\psi(\gamma)] < d_1\overline{\beta}[\psi(\delta)]$. Additionally,

$$k_0 d_1\overline{\alpha}[\psi(\gamma)] = \psi(\gamma) < \psi(\eta) = \vartheta_0 d_1\overline{\beta}[\psi(\delta)],$$

hence $d_0\overline{\alpha}[\psi(\gamma)] < d_0\overline{\beta}[\psi(\delta)]$.

Case 2: $\pi_0\alpha = \pi_0\beta$ and $\gamma < \delta$. The induction hypothesis yields $\psi(\gamma) < \psi(\delta)$. Assertion 2. on $\pi_0\alpha$ then yields $\overline{\pi_0\alpha}[\psi(\gamma)] < \overline{\pi_0\alpha}[\psi(\delta)]$. Hence, $d_1\overline{\alpha}[\psi(\gamma)] < d_1\overline{\alpha}[\psi(\delta)] = d_1\overline{\beta}[\psi(\delta)]$. Additionally,

$$k_0 d_1\overline{\alpha}[\psi(\gamma)] = \psi(\gamma) < \psi(\delta) = k_0(d_1\overline{\beta}[\psi(\delta)]) \leq \vartheta_0(d_1\overline{\beta}[\psi(\delta)]),$$

hence $d_0\overline{\alpha}[\psi(\gamma)] < d_0\overline{\beta}[\psi(\delta)]$.

Case 3.: $\pi_0\alpha > \pi_0\beta$ and $\zeta < \delta$. Then $\psi(\zeta) < \psi(\delta) \leq k_0(d_1\overline{\beta}[\psi(\delta)]) \leq \vartheta_0(d_1\overline{\beta}[\psi(\delta)]) = \psi(\eta)$.

Corollary 3 $\omega_{n+2} \leq \vartheta_0\vartheta_1\ldots\vartheta_n\Omega_{n+1}$

Proof From the Theorems 8 and 9, we know that the order type of $\varphi_{\pi_0(n)}0$ is ω_{n+2}. Therefore, using assertion 5 in Lemma 14, we obtain $\omega_{n+2} \leq otype(T_{n+1}[0]) = \vartheta_0\ldots\vartheta_n\Omega_{n+1}$.

4.2 Upper bound

In this subsection, we prove $\vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1} = otype(T_{n+1}[0]) \le \omega_{n+2}$. For this purpose, we introduce a new notation system with the same order type as T_n.

Definition 31 Let $n < \omega$. Define T'_{n+1} as the least subset of T_{n+1} such that

- $0 \in T'_{n+1}$,
- if $\alpha \in T'_{n+1}$, $S\alpha = i+1$ and $i < n$, then $\vartheta_i\alpha \in T'_{n+1}$,
- if $\alpha \in T'_{n+1}$, then $\vartheta_n\alpha \in T'_{n+1}$.

Note that for all $\alpha \in T'_{n+1}$, we have $S\alpha \le n$. Let T'_0 be $\{0\}$ and define $T'_n[m]$ accordingly as $T_n[m]$.

Lemma 15 *The order types of T'_n and T_n are equal.*

Proof Trivially, $T'_n \subseteq T_n$, hence $otype(T'_n) \le otype(T_n)$. Now, we give an order-preserving function ψ from T_n to T'_n. If $n = 0$, this function appears trivially. So assume $n = m+1 > 0$.

$$\begin{aligned} \psi : T_{m+1} &\to T'_{m+1}, \\ 0 &\mapsto 0, \\ \vartheta_i\alpha &\mapsto \vartheta_i\vartheta_{i+1}\ldots\vartheta_m\psi(\alpha). \end{aligned}$$

Let us first prove the following claim: for all $i \le m$, if $\psi(\xi) < \psi(\zeta) < \Omega_{i+1} = \vartheta_{i+1}0$, then $\psi(\vartheta_i\xi) < \psi(\vartheta_i\zeta)$. We prove this claim by induction on $m-i$. $i = m$, then $\psi(\vartheta_m\xi) = \vartheta_m\psi(\xi)$ and $\psi(\vartheta_m\zeta) = \vartheta_m\psi(\zeta)$. Hence, $\psi(\vartheta_m\xi) < \psi(\vartheta_m\zeta)$ easily follows because $k_m(\psi(\xi)) = \psi(\xi) < \psi(\zeta) = k_m(\psi(\zeta)) < \vartheta_m(\psi(\zeta))$. Let $i < m$. Then

$$\begin{aligned} \psi(\vartheta_i\xi) &= \vartheta_i\ldots\vartheta_m\psi(\xi), \\ \psi(\vartheta_i\zeta) &= \vartheta_i\ldots\vartheta_m\psi(\zeta). \end{aligned}$$

Using the induction hypothesis, we obtain $\psi(\vartheta_{i+1}\xi) = \vartheta_{i+1}\ldots\vartheta_m\psi(\xi) < \psi(\vartheta_{i+1}\zeta) = \vartheta_{i+1}\ldots\vartheta_m\psi(\zeta)$. Furthermore, $k_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\xi)) = k_i(\psi(\xi)) = \psi(\xi) < \psi(\zeta) = k_i(\psi(\zeta)) = k_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\zeta)) < \vartheta_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\zeta))$. Hence, $\psi(\vartheta_i\xi) = \vartheta_i\ldots\vartheta_m\psi(\xi) < \psi(\vartheta_i\zeta) = \vartheta_i\ldots\vartheta_m\psi(\zeta)$. This finishes the proof of the claim.

Now we prove by main induction on $lh(\alpha)+lh(\beta)$ that $\alpha < \beta$ yields $\psi(\alpha) < \psi(\beta)$. If $\alpha = 0$, then the claim trivially holds. Assume $0 < \alpha < \beta$. Then $\alpha = \vartheta_i\alpha'$ and $\beta = \vartheta_j\beta'$. If $i < j$, then $\psi(\alpha) < \psi(\beta)$ is also trivial. Assume $i = j \le m$ and let $\alpha' = \vartheta_{j_1}\ldots\vartheta_{j_k}k_i\alpha'$ and $\beta' = \vartheta_{n_1}\ldots\vartheta_{n_l}k_i\beta'$ with $j_1,\ldots,j_k,n_1,\ldots,n_l > i$. $\alpha < \beta$ either yields $\alpha \le k_i\beta'$ or $\alpha' < \beta'$ and $k_i\alpha' < \beta$. In the former case, the induction hypothesis yields $\psi(\alpha) \le \psi(k_i\beta') = k_i(\psi(\vartheta_{n_1}\ldots\vartheta_{n_l}k_i\beta')) = k_i(\psi(\beta')) = k_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\beta'))$ $< \vartheta_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\beta')) = \psi(\beta)$.

Assume that we are in the latter case, meaning $\alpha' < \beta'$ and $k_i\alpha' < \beta$. The induction hypothesis yields $\psi\alpha' < \psi\beta'$ and $\psi(k_i\alpha') < \psi\beta$. Like before, we attain $\psi(k_i\alpha') = k_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\alpha')) < \psi\beta = \vartheta_i(\vartheta_{i+1}\ldots\vartheta_m\psi(\beta'))$. So if we can prove $\vartheta_{i+1}\ldots\vartheta_m\psi(\alpha') < \vartheta_{i+1}\ldots\vartheta_m\psi(\beta')$, we are done. But this follows from the claim: if $i = j < m$, then $S(\alpha'), S(\beta') \le i+1 \le m$, hence $\psi(\alpha') < \psi(\beta') < \Omega_{i+2}$, so $\vartheta_{i+1}\ldots\vartheta_m\psi(\alpha') =$

$\psi(\vartheta_{i+1}\alpha') < \psi(\vartheta_{i+1}\beta') = \vartheta_{i+1}\ldots\vartheta_m\psi(\beta')$. If $i = j = m$, then $\vartheta_{i+1}\ldots\vartheta_m\psi(\alpha')$ and $\vartheta_{i+1}\ldots\vartheta_m\psi(\beta')$ are actually $\psi(\alpha')$ and $\psi(\beta')$ and we know that $\psi(\alpha') < \psi(\beta')$ holds.

The previous proof also yields that the order types of $T'_n[m]$ and $T_n[m]$ are equal.

4.2.1 The instructive part: $\vartheta_0\vartheta_1\vartheta_2\Omega_3 \le \omega^{\omega^{\omega^\omega}}$

In this subsection, we prove that $\omega^{\omega^{\omega^\omega}}$ is an upper bound for $\vartheta_0\vartheta_1\vartheta_2\Omega_3$ as an instructive instance for the general case

$$\vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1} = otype(T_{n+1}[0]) \le \omega_{n+2}.$$

We will show this by proving that $otype(T'_3[0]) \le \omega^{\omega^{\omega^\omega}}$. We start with two simple lemmata, where we interpret Ω_i as usual as the i^{th} uncountable cardinal number for $i > 0$.

Lemma 16 *If $\Omega_2\cdot\alpha+\beta < \Omega_2\cdot\gamma+\delta$ and $\alpha,\gamma < \varepsilon_0$ and $\beta,\delta < \Omega_2$ and if $\beta = \xi\cdot\beta'$ where $\beta' < \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$ and $\xi < \omega^{\omega^\gamma}$, then $\Omega_1\cdot\omega^\alpha+\omega^{\omega^\alpha}\cdot\beta < \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$.*

Proof Note that it is possible that $\beta,\delta \ge \Omega_1$. If $\alpha = \gamma$ then $\beta < \delta$ and the assertion is obvious. So assume $\alpha < \gamma$. $\beta' < \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$ yields $\beta = \xi\beta' < \xi(\Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta) = \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$ since Ω_1 and ω^{ω^γ} are multiplicatively closed. By the same argument $\omega^{\omega^\alpha}\beta < \omega^{\omega^\alpha}(\Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta) = \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$. Finally, $\Omega_1\cdot\omega^\alpha+\omega^{\omega^\alpha}\cdot\beta < \Omega_1\cdot\omega^\alpha+\Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta = \Omega_1\cdot\omega^\gamma+\omega^{\omega^\gamma}\cdot\delta$.

Lemma 17 *If $\Omega_1\cdot\alpha+\beta < \Omega_1\cdot\gamma+\delta$ and $\alpha,\gamma < \varepsilon_0$ and $\beta,\delta < \Omega_1$ and if $\beta < \omega^{\omega^\gamma}\cdot\delta$, then $\omega^{\omega^\alpha}\cdot\beta < \omega^{\omega^\gamma}\cdot\delta$.*

Proof If $\alpha = \gamma$, then $\beta < \delta$ and the assertion is obvious. So assume $\alpha < \gamma$. Then $\omega^{\omega^\alpha}\cdot\beta < \omega^{\omega^\alpha}\omega^{\omega^\gamma}\cdot\delta = \omega^{\omega^\gamma}\cdot\delta$.

The last two lemmata indicate how one might replace iteratively terms in ϑ_i (starting with the highest level i) by terms in $\omega,+,\Omega_i$ in an order-preserving way such that terms of level 0 are smaller than ε_0.

Definition 32 Define E as the least set such that

- $0 \in E$,
- $\alpha \in E$, then $\omega^\alpha \in E$,
- $\alpha,\beta \in E$, then $\alpha+\beta \in E$.

Define the subset P of E as the set of all elements of the form ω^α for $\alpha \in E$. This actually means that P is the set of the additively closed ordinals strictly below ε_0.

A crucial role is played by the following function f.

Definition 33 Let $f(0) := 0$ and $f(\omega^{\alpha_1}+\alpha_2) := \omega^{\alpha_1}+f(\alpha_1)+f(\alpha_2)$.

This definition even works (by magic) also for non Cantor normal forms. So if $\omega^{\alpha_1}+\alpha_2=\alpha_2$ we still have $f(\omega^{\alpha_1}+\alpha_2)=\omega^{\alpha_1}+f(\alpha_1)+f(\alpha_2)[=f(\alpha_2)]$. The function f is easily shown to be order-preserving. Moreover, one finds $\omega^{\alpha_1}\leq f(\omega^{\alpha_1}+\alpha_2)<\omega^{\alpha_1+1}$ if $\alpha_2<\omega^{\alpha_1+1}$.

Fix a natural number n. We formally work with 4-tuples $(\alpha,\beta,\gamma,\delta)\in E\times T[n-1]\times P\times E$ with $\alpha,\delta\in E$, $\gamma\in P$, $\beta\in T[n-1]$ and $\delta<\gamma$. Let $T[-1]:=\{0\}$. We order these tuples lexicographically. Intuitively, we interpret such a tuple as the ordinal

$$\Omega_n\cdot\alpha+\gamma\cdot\beta+\delta,$$

where Ω_i is as usual the i^{th} uncountable ordinal for $i>0$, but now Ω_0 is interpreted as 0.

We remark that the interpretation of $(\alpha,\beta,\gamma,\delta)$ as an ordinal number is not entirely correct: the lexicographic order on the tuples is not the same as the induced order by the ordering on the class of ordinals On. But in almost all applications, we know that $\gamma=\omega^{f(\alpha)}$. And if this is true, we know that the order induced by the ordering on On is the same as the defined lexicographic one. Additionally, the encountered cases where $\gamma\neq\omega^{f(\alpha)}$, we know that if we compare two tuples $(\alpha,\beta,\gamma,\delta)$ and $(\alpha',\beta',\gamma',\delta')$ such that $\alpha=\alpha'$, then we already know that $\gamma=\gamma'$. Hence, the order induced by the ordering on On between these terms is also the same as the lexicographic one.

β is either 0 or of the form $\vartheta_j\beta'$ with $j<n$, hence we can interpret that $\beta<\Omega_n$ for $n>0$. Assume that $\zeta\in P$. Then we know that $\zeta\cdot\Omega_n=\Omega_n$. Hence using all of these interpretations, $\zeta\cdot(\alpha,\beta,\gamma,\delta)$ is still a 4-tuple, namely it is equal to $(\alpha,\beta,\zeta\cdot\gamma,\zeta\cdot\delta)$. We can also define the sum between 4-tuples: assume $n>0$. If $\alpha'>0$, then

$$\begin{aligned}(\alpha,\beta,\gamma,\delta)+(\alpha',\beta',\gamma',\delta') &= \Omega_n\cdot\alpha+\gamma\cdot\beta+\delta+\Omega_n\cdot\alpha'+\gamma'\cdot\beta'+\delta'\\ &= \Omega_n\cdot(\alpha+\alpha')+\gamma'\cdot\beta'+\delta'\\ &= (\alpha+\alpha',\beta',\gamma',\delta')\end{aligned}$$

If $\alpha'=0$ and $\beta'=0$, then

$$\begin{aligned}(\alpha,\beta,\gamma,\delta)+(\alpha',\beta',\gamma',\delta') &= \Omega_n\cdot\alpha+\gamma\cdot\beta+\delta+\Omega_n\cdot\alpha'+\gamma'\cdot\beta'+\delta'\\ &= \Omega_n\alpha+\gamma\cdot\beta+(\delta+\delta')\\ &= (\alpha,\beta,\gamma,\delta+\delta')\end{aligned}$$

We do not need the case $\alpha'=0$ and $\beta'\neq 0$. If $n=0$, then

$$\begin{aligned}(\alpha,\beta,\gamma,\delta)+(\alpha',\beta',\gamma',\delta') &= \Omega_n\cdot\alpha+\gamma\cdot\beta+\delta+\Omega_n\cdot\alpha'+\gamma'\cdot\beta'+\delta'\\ &= \delta+\delta'\\ &= (0,0,0,\delta+\delta')\end{aligned}$$

From now on, we write

$$\Omega_n\cdot\alpha+\gamma\cdot\beta+\delta,$$

instead of the 4-tuple $(\alpha,\beta,\gamma,\delta)$, although we know that the induced order by the ordering on On is not entirely the same as the lexicographic one.

Definition 34 Define T_n^{all} as the set consisting of $\Omega_n \cdot \alpha + \omega^{f(\alpha)} \cdot \delta + \gamma$, where $\alpha, \gamma \in E$ with $\gamma < \omega^{f(\alpha)}$ and $\delta \in T[n-1]$.

Note that after an obvious translation, $T_0^{all} = E$ and $T_n \subseteq T[n-1] \subseteq T_n^{all}$.

Lemma 18 *Assume* $\alpha', \beta' \in T[0]$. *If*

$$\alpha = \vartheta_1 \vartheta_2^{n_1} \dots \vartheta_1 \vartheta_2^{n_p} \alpha' < \beta = \vartheta_1 \vartheta_2^{l_1} \dots \vartheta_1 \vartheta_2^{l_q} \beta'$$

with $n_i, l_i > 0$, *then*

$$\begin{aligned} &\Omega_1 \cdot (\omega^{n_1} + \dots + \omega^{n_p}) + \omega^{\omega^{n_1} + \dots + \omega^{n_p} + n_p} \cdot \alpha' + \omega^{\omega^{n_1} + \dots + \omega^{n_p}} \\ &+ \omega^{\omega^{n_1} + \dots + \omega^{n_{p-1}}} + \dots + \omega^{\omega^{n_1}} \\ < &\,\Omega_1 \cdot (\omega^{l_1} + \dots + \omega^{l_q}) + \omega^{\omega^{l_1} + \dots + \omega^{l_q} + l_q} \cdot \beta' + \omega^{\omega^{l_1} + \dots + \omega^{l_q}} \\ &+ \omega^{\omega^{l_1} + \dots + \omega^{l_{q-1}}} + \dots + \omega^{\omega^{l_1}}. \end{aligned}$$

Proof Note that $f(\omega^{n_1} + \dots + \omega^{n_p}) = \omega^{n_1} + \dots + \omega^{n_p} + n_p$ and that $\omega^{n_1} + \dots + \omega^{n_p}$ is not necessarily in Cantor normal form. We prove by induction on $lh(\alpha) - lh(\alpha') + lh(\beta) - lh(\beta')$ that the assumption yields

$$\begin{aligned} &(\omega^{n_1} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1}) \\ <_{lex} &(\omega^{l_1} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1}). \end{aligned}$$

From this inequality, the lemma follows.

If $lh(\alpha) = lh(\alpha')$, then $p = 0$. If $q > 0$, then this is trivial, so we can assume that q is also 0. But then $\omega^{n_1} + \dots + \omega^{n_p} = \omega^{l_1} + \dots + \omega^{l_q} = 0$ and $\alpha' = \alpha < \beta = \beta'$. Now assume that $p > 0$. It is impossible that $q = 0$. $\alpha < \beta$ yields either $\vartheta_1 \vartheta_2^{n_1} \dots \vartheta_1 \vartheta_2^{n_p} \alpha' < \vartheta_1 \vartheta_2^{l_2} \dots \vartheta_1 \vartheta_2^{l_q} \beta'$ or $(\vartheta_2^{n_1} \dots \vartheta_1 \vartheta_2^{n_p} \alpha' < \vartheta_2^{l_1} \dots \vartheta_1 \vartheta_2^{l_q} \beta'$ and $\vartheta_1 \vartheta_2^{n_2} \dots \vartheta_1 \vartheta_2^{n_p} \alpha' < \vartheta_1 \vartheta_2^{l_1} \dots \vartheta_1 \vartheta_2^{l_q} \beta')$.

In the former case, the induction hypothesis yields

$$\begin{aligned} &(\omega^{n_1} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1}) \\ <_{lex} &(\omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_2}). \end{aligned}$$

If $l_2 \leq l_1$, then trivially

$$\begin{aligned} &(\omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_2}, \omega^{l_1}) \\ <_{lex} &(\omega^{l_1} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}, \omega^{l_1}). \end{aligned}$$

If $l_2 > l_1$, then

$$\begin{aligned} &(\omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_2}) \\ = &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}) \\ <_{lex} &(\omega^{l_1} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}, \omega^{l_1}). \end{aligned}$$

Assume that we are in the latter case. $\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ yields $n_1 < l_1$ or $n_1 = l_1$ and $\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_2}\dots\vartheta_1\vartheta_2^{l_q}\beta'$.

Suppose $n_1 < l_1$. The induction hypothesis on

$$\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$$

implies

$$\begin{aligned}&(\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ <_{lex}\, &(\omega^{l_1}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_1}).\end{aligned}$$

Let

$$\begin{aligned}s &:= (\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ s' &:= (\omega^{l_1}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_1}).\end{aligned}$$

Note that $lh(s) = p$ and $lh(s') = q+1$. If $lh(s) < lh(s')$ and $s_i = s'_i$ for all $i < lh(s)$, then

$$\begin{aligned}&(\omega^{n_1}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_1})\\ =\, &(\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2},\omega^{n_1})\\ <_{lex}\, &(\omega^{l_1}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_1}),\end{aligned}$$

where for the last inequality we need $n_1 < l_1$ if $p = q$. If there exists an index $j < \min\{lh(s), lh(s')\}$ such that $s_j < s'_j$ and $s_i = s'_i$ for all $i < j$, then

$$\begin{aligned}&(\omega^{n_1}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_1})\\ &(\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2},\omega^{n_1})\\ <_{lex}\, &(\omega^{l_1}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_1}).\end{aligned}$$

Now assume $n_1 = l_1$. The induction hypothesis on $\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_2}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ implies

$$\begin{aligned}&(\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ <_{lex}\, &(\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_2}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_2}).\end{aligned}$$

Let

$$\begin{aligned}s &:= (\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_2}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ s' &:= (\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_2}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_2}).\end{aligned}$$

Note that $lh(s) = p$ and $lh(s') = q$. If $lh(s) < lh(s')$ and $s_i = s'_i$ for all $i < lh(s)$, then one can easily prove

$$\begin{aligned}&(\omega^{n_1}+\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\dots,\omega^{n_1}+\omega^{n_2})\\ <_{lex}\, &(\omega^{l_1}+\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\dots,\omega^{l_1}+\omega^{l_2}),\end{aligned}$$

hence

$$\begin{aligned}
&(\omega^{n_1}+\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\ldots,\omega^{n_1}+\omega^{n_2},\omega^{n_1})\\
<_{lex}\,&(\omega^{l_1}+\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\ldots,\omega^{l_1}+\omega^{l_2})\\
<_{lex}\,&(\omega^{l_1}+\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\ldots,\omega^{l_1}+\omega^{l_2},\omega^{l_1}).
\end{aligned}$$

If there exists an index $j<\min\{lh(s),lh(s')\}$ such that $s_j<s'_j$ and $s_i=s'_i$ for all $i<j$, then also

$$\begin{aligned}
&(\omega^{n_1}+\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\ldots,\omega^{n_1}+\omega^{n_2})\\
<_{lex}\,&(\omega^{l_1}+\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\ldots,\omega^{l_1}+\omega^{l_2}),
\end{aligned}$$

hence

$$\begin{aligned}
&(\omega^{n_1}+\omega^{n_2}+\cdots+\omega^{n_p},\alpha',\omega^{n_1}+\cdots+\omega^{n_{p-1}},\ldots,\omega^{n_1}+\omega^{n_2},\omega^{n_1})\\
<_{lex}\,&(\omega^{l_1}+\omega^{l_2}+\cdots+\omega^{l_q},\beta',\omega^{l_1}+\cdots+\omega^{l_{q-1}},\ldots,\omega^{l_1}+\omega^{l_2},\omega^{l_1}).
\end{aligned}$$

Define τ_0 as the mapping from $T'_3[0]$ to $T_0^{all}=E$ as follows: let $\tau_0 0:=0$. If $\alpha=\vartheta_0\vartheta_1\vartheta_2^{n_1}\ldots\vartheta_1\vartheta_2^{n_p}\alpha'$ with $\alpha'\in T'_3[0]$ and $n_1,\ldots,n_p,p>0$, define $\tau_0\alpha$ as

$$\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\omega^{\omega^{n_1}+\cdots+\omega^{n_{p-1}}}+\cdots+\omega^{\omega^{n_1}}).$$

Lemma 19 *Assume $\alpha,\beta\in T'_3[0]$. If $\alpha<\beta$, then $\tau_0\alpha<\tau_0\beta$.*

Proof We prove this by induction on the length of α and β. If $\alpha=0$, then this is trivial. So we can assume that $0<\alpha<\beta$. Hence,

$$\alpha=\vartheta_0\vartheta_1\vartheta_2^{n_1}\ldots\vartheta_1\vartheta_2^{n_p}\alpha'$$

and

$$\beta=\vartheta_0\vartheta_1\vartheta_2^{l_1}\ldots\vartheta_1\vartheta_2^{l_q}\beta'$$

with $\alpha',\beta'\in T'_3[0]$ and $n_1,\ldots,n_p,l_1,\ldots,l_q,p,q>0$.

We want to prove that

$$\begin{aligned}
\tau_0\alpha&=\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\cdots+\omega^{\omega^{n_1}})\\
<\tau_0\beta&=\omega^{\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}}\cdot(\omega^{\omega^{l_1}+\cdots+\omega^{l_q}+l_q}\cdot\tau_0\beta'+\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}+\cdots+\omega^{\omega^{l_1}}).
\end{aligned}$$

$\alpha=\vartheta_0\vartheta_1\vartheta_2^{n_1}\ldots\vartheta_1\vartheta_2^{n_p}\alpha'<\beta=\vartheta_0\vartheta_1\vartheta_2^{l_1}\ldots\vartheta_1\vartheta_2^{l_q}\beta'$ yields two cases: either $\alpha\leq k_0(\vartheta_1\vartheta_2^{l_1}\ldots\vartheta_1\vartheta_2^{l_q}\beta')=\beta'$ or $(\vartheta_1\vartheta_2^{n_1}\ldots\vartheta_1\vartheta_2^{n_p}\alpha'<\vartheta_1\vartheta_2^{l_1}\ldots\vartheta_1\vartheta_2^{l_q}\beta'$ and $\alpha'<\beta)$. In the former case, the induction hypothesis yields $\tau_0\alpha\leq\tau_0\beta'<\tau_0\beta$.

So assume the latter case. Then the induction hypothesis yields $\tau_0\alpha'<\tau_0\beta$. Using Lemma 18, we know that

$$\begin{aligned}
&\Omega_1\cdot(\omega^{n_1}+\cdots+\omega^{n_p})+\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}\\
&\quad+\omega^{\omega^{n_1}+\cdots+\omega^{n_{p-1}}}+\cdots+\omega^{\omega^{n_1}}\\
<\,&\Omega_1\cdot(\omega^{l_1}+\cdots+\omega^{l_q})+\omega^{\omega^{l_1}+\cdots+\omega^{l_q}+l_q}\cdot\tau_0\beta'+\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}\\
&\quad+\omega^{\omega^{l_1}+\cdots+\omega^{l_{q-1}}}+\cdots+\omega^{\omega^{l_1}}.
\end{aligned}$$

If $\omega^{n_1}+\cdots+\omega^{n_p}<\omega^{l_1}+\cdots+\omega^{l_q}$, then

$$\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'<\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\tau_0\beta=\tau_0\beta.$$

Therefore,

$$\begin{aligned}&\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\cdots+\omega^{\omega^{n_1}})\\ <\;&\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'\\ &+\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\cdots+\omega^{\omega^{n_1}})\\ <\;&\tau_0\beta,\end{aligned}$$

because $\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\cdots+\omega^{\omega^{n_1}})<\omega^{\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}}$. We used the standard observation that $\xi<\rho+\omega^{\mu}$ and $\lambda<\mu$ imply $\xi+\omega^{\lambda}<\rho+\omega^{\mu}$.

Assume $\omega^{n_1}+\cdots+\omega^{n_p}=\omega^{l_1}+\cdots+\omega^{l_q}$ and $\tau_0\alpha'<\tau_0\beta'$. Then $\tau_0\alpha<\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot(\tau_0\alpha'+1)\le\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p}\cdot\tau_0\beta'\le\tau_0\beta$.

Assume $\omega^{n_1}+\cdots+\omega^{n_p}=\omega^{l_1}+\cdots+\omega^{l_q}$, $\tau_0\alpha'=\tau_0\beta'$ and $\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}+\omega^{\omega^{n_1}+\cdots+\omega^{n_{p-1}}}+\cdots+\omega^{\omega^{n_1}}<\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}+\omega^{\omega^{l_1}+\cdots+\omega^{l_{q-1}}}+\cdots+\omega^{\omega^{l_1}}$. Then trivially, $\tau_0\alpha<\tau_0\beta$.

4.2.2 The general part: $\vartheta_0\ldots\vartheta_n\Omega_{n+1}\le\omega_{n+2}$

We show that $otype(T'_{n+1}[0])\le\omega_{n+2}$. The previous section gives us the idea of how to deal with this question, however the order-preserving embeddings in this subsection are slightly different than the ones proposed in the previous Subsection 4.2.1 for technical reasons. Fix a natural number n strictly bigger than 0.

Definition 35 τ_m are functions from $T'_{n+1}[m]$ to T_m^{all}. We define $\tau_m\alpha$ for all m simultaneously by induction on the length of α. If $m\ge n+1$, then $T'_{n+1}[m]=T'_{n+1}$ and define $\tau_m\alpha=\alpha=\Omega_m0+\omega^0\alpha+0$ for all α. Note that $\alpha\in T'_{n+1}\subseteq T[n]\subseteq T[m-1]$. Assume $m\le n$. Define τ_m0 as 0. Define $\tau_m\vartheta_j\alpha$ as $\vartheta_j\alpha$ if $j<m$. Define $\tau_m\vartheta_m\alpha$ as $\Omega_m\omega^{\beta}+\omega^{\omega^{\beta}}(\omega^{f(\beta)}\cdot\tau_mk_m\alpha+\eta)+1$ if $\tau_{m+1}\alpha=\Omega_{m+1}\beta+\omega^{f(\beta)}k_m\alpha+\eta$.

First we prove that τ_m is well-defined.

Lemma 20 *For all $m>0$ and $\alpha\in T'_{n+1}[m]$, there exist uniquely determined β and η with $\eta<\omega^{f(\beta)}$ such that $\tau_m\alpha=\Omega_m\beta+\omega^{f(\beta)}k_{m-1}\alpha+\eta$. Furthermore, η is either zero or a successor.*

Proof We prove the first claim by induction on $lh(\alpha)$ and $n+1-m$. If $m\ge n+1$, then this is trivial by definition. Assume $0<m\le n$. From the induction hypothesis, we know that there exist $\beta,\eta,\beta_1,\eta_1$ such that $\tau_{m+1}\alpha=\Omega_{m+1}\beta+\omega^{f(\beta)}k_m\alpha+\eta$ with $\eta<\omega^{f(\beta)}$ and $\tau_mk_m\alpha=\Omega_m\beta_1+\omega^{f(\beta_1)}k_{m-1}k_m\alpha+\eta_1$ with $\eta_1<\omega^{f(\beta_1)}$. We want to prove that there exist β' and η' such that $\tau_m \vartheta_m \alpha = \Omega_m \beta' + \omega^{f(\beta')} k_{m-1} \vartheta_m \alpha + \eta'$ with $\eta' < \omega^{f(\beta')}$. Using the definition,

$$\begin{aligned}
&\tau_m \vartheta_m \alpha \\
&= \Omega_m \omega^\beta + \omega^{\omega^\beta}(\omega^{f(\beta)} \cdot \tau_m k_m \alpha + \eta) + 1 \\
&= \Omega_m \omega^\beta + \omega^{\omega^\beta}(\omega^{f(\beta)} \cdot (\Omega_m \beta_1 + \omega^{f(\beta_1)} k_{m-1} k_m \alpha + \eta_1) + \eta) + 1 \\
&= \Omega_m(\omega^\beta + \beta_1) + \omega^{\omega^\beta} \omega^{f(\beta)}(\omega^{f(\beta_1)} k_{m-1} k_m \alpha + \eta_1) + \omega^{\omega^\beta} \eta + 1 \\
&= \Omega_m(\omega^\beta + \beta_1) + \omega^{\omega^\beta} \omega^{f(\beta)} \omega^{f(\beta_1)} k_{m-1} k_m \alpha + \omega^{\omega^\beta} \omega^{f(\beta)} \eta_1 + \omega^{\omega^\beta} \eta + 1 \\
&= \Omega_m(\omega^\beta + \beta_1) + \omega^{f(\omega^\beta + \beta_1)} k_{m-1} k_m \alpha + \omega^{\omega^\beta} \omega^{f(\beta)} \eta_1 + \omega^{\omega^\beta} \eta + 1 \\
&= \Omega_m(\omega^\beta + \beta_1) + \omega^{f(\omega^\beta + \beta_1)} k_{m-1} \vartheta_m \alpha + \omega^{\omega^\beta} \omega^{f(\beta)} \eta_1 + \omega^{\omega^\beta} \eta + 1.
\end{aligned}$$

Define β' as $\omega^\beta + \beta_1 > 0$ and η' as $\omega^{\omega^\beta} \omega^{f(\beta)} \eta_1 + \omega^{\omega^\beta} \eta + 1$. Note that $\omega^{\omega^\beta} \omega^{f(\beta)} \eta_1 < \omega^{\omega^\beta} \omega^{f(\beta)} \omega^{f(\beta_1)} = \omega^{f(\beta')}$, $\omega^{\omega^\beta} \eta < \omega^{\omega^\beta + f(\beta)} \leq \omega^{f(\beta')}$ and $1 < \omega^{f(\beta')}$, hence $\eta' < \omega^{f(\beta')}$.

That η is either zero or a successor for all m and α follows by construction.

The argument in the proof of Lemma 20 is crucially based on the property of f regarding non-normal forms. The lemma implies that τ_m is well-defined for all $m > 0$ and it does not make sense for $m = 0$ because we did not define $k_{-1}\alpha$. But, looking to the definition of τ_0, it is easy to see that τ_0 is also well-defined.

Note that one can easily prove $\tau_0 \alpha \in T_0^{all}$ for all $\alpha \in T'_{n+1}[0]$. Furthermore, $\tau_0 \alpha$ is also either zero or a successor ordinal. For all m and α, define $(\tau_m \alpha)^-$ as $\tau_m \alpha$, if η is zero, and as $\tau_m \alpha$ but with $\eta - 1$ instead of η, if η is a successor. Additionally, note that if $m > 0$ and $\tau_m \alpha = \Omega_m \beta + \omega^{f(\beta)} k_{m-1} \alpha + \eta$ we have $\beta > 0$ iff $\eta > 0$.

In the next theorem, we will again use the standard observation that $\xi < \rho + \omega^\mu$ and $\lambda < \mu$ imply $\xi + \omega^\lambda < \rho + \omega^\mu$.

Theorem 11 *For all natural m and $\alpha, \beta \in T'_{n+1}[m]$, if $\alpha < \beta$, then $\tau_m \alpha < \tau_m \beta$.*

Proof We prove this theorem by induction on $lh\alpha + lh\beta$. If α and/or β are zero, this is trivial. So we can assume that $\alpha = \vartheta_i \alpha'$ and $\beta = \vartheta_j \beta'$. One can easily prove the statement if $i < j$, even if $j = m$. So we can assume that $i = j$. If $i = j < m$, then this is also easily proved. So suppose that $i = j = m$. If $m > n$, then $\tau_m \alpha = \alpha < \beta = \tau_m \beta$, hence we are done. So we can also assume that $m \leq n$.

$\alpha = \vartheta_m \alpha' < \vartheta_m \beta'$ yields $\alpha \leq k_m \beta'$ or $\alpha' < \beta'$ and $k_m \alpha' < \beta$. In the former case, the induction hypothesis yields $\tau_m \alpha \leq \tau_m k_m \beta' < \tau_m \vartheta_m \beta' = \tau_m \beta$, where $\tau_m k_m \beta' < \tau_m \vartheta_m \beta'$ follows from the definition of $\tau_m \vartheta_m \beta'$. (One can also look at the proof of Lemma 20 for $m > 0$. The case $m = 0$ is straightforward.) So we only have to prove the assertion in the latter case, i.e. if $\alpha' < \beta'$ and $k_m \alpha' < \beta$. The induction hypothesis yields $\tau_{m+1} \alpha' < \tau_{m+1} \beta'$ and $\tau_m k_m \alpha' < \tau_m \beta$. Assume

$$\begin{aligned}
\tau_{m+1} \alpha' &= \Omega_{m+1} \cdot \alpha_1 + \omega^{f(\alpha_1)} \cdot k_m \alpha' + \alpha_2, \\
\tau_{m+1} \beta' &= \Omega_{m+1} \cdot \beta_1 + \omega^{f(\beta_1)} \cdot k_m \beta' + \beta_2,
\end{aligned}$$

where $\alpha_2 < \omega^{f(\alpha_1)}$, $\beta_2 < \omega^{f(\beta_1)}$. Then

$$\begin{aligned}\tau_m\alpha =& \Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} \cdot \tau_m k_m \alpha' + \alpha_2) + 1,\\ \tau_m\beta =& \Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m \beta' + \beta_2) + 1.\end{aligned}$$

The inequality $\tau_{m+1}\alpha' < \tau_{m+1}\beta'$ yields $\alpha_1 \leq \beta_1$. Assume first that $\alpha_1 = \beta_1$. Then $\tau_{m+1}\alpha' < \tau_{m+1}\beta'$ yields $k_m\alpha' \leq k_m\beta'$. If $k_m\alpha' = k_m\beta'$, then $\alpha_2 < \beta_2$ and $\tau_m\alpha < \tau_m\beta$. If $k_m\alpha' < k_m\beta'$ then the induction hypothesis yields $\tau_m k_m\alpha' < \tau_m k_m\beta'$ and $\omega^{f(\alpha_1)} \cdot \tau_m k_m\alpha' + \alpha_2 < \omega^{f(\alpha_1)} \cdot \tau_m k_m\beta' + \beta_2$, since $\alpha_2 < \omega^{f(\alpha_1)}$. We then find that $\tau_m\alpha < \tau_m\beta$. So we may assume that $\alpha_1 < \beta_1$.

Case 1: $k_m\alpha' < \vartheta_m 0$. Then $\tau_m k_m\alpha' = k_m\alpha'$. Hence,

$$\begin{aligned}\tau_m\alpha &= \Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} \cdot k_m\alpha' + \alpha_2) + 1\\ &< \Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m\beta' + \beta_2) + 1\\ &= \tau_m\beta\end{aligned}$$

follows in a straightforward way.

Case 2: $k_m\alpha' \geq \vartheta_m 0$. Using the definition, we then have $(\tau_m k_m\alpha')^- + 1 = \tau_m k_m\alpha'$. We show that

$$\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m\alpha')^- + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1 < (\tau_m\beta)^-$$

holds, hence

$$\begin{aligned}\tau_m\alpha &= \Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m\alpha')^- + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1\\ &< \Omega_m \cdot \omega^{\alpha_1} + (\tau_m\beta)^-\\ &= (\tau_m\beta)^-\\ &< \tau_m\beta.\end{aligned}$$

We know $\tau_m k_m\alpha' < \tau_m\beta$, hence

$$(\tau_m k_m\alpha')^- < (\tau_m\beta)^- = \Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m\beta' + \beta_2).$$

Therefore, $\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m\alpha')^- < \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m\beta)^- = (\tau_m\beta)^-$ because $\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} = \omega^{f(\omega^{\alpha_1})}$ and $f(\omega^{\alpha_1}) < \omega^{\alpha_1+1} \leq \omega^{\beta_1}$.

The last term in the normal form of $\omega^{\omega^{\beta_1}} \cdot \beta_2$ is bigger than $\omega^{\omega^{\beta_1}}$. Note that $\tau_{m+1}\beta' = \Omega_{m+1} \cdot \beta_1 + \omega^{f(\beta_1)} \cdot k_m\beta' + \beta_2$. The observation just before this theorem yields $\beta_2 > 0$ otherwise β_1 is zero, a contradiction (because $\beta_1 > \alpha_1$). So if

$$\omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1 < \omega^{\omega^{\beta_1}},$$

we can finish the proof by the standard observation $\xi < \rho + \omega^\mu$ and $\lambda < \mu$ imply $\xi + \omega^\lambda < \rho + \omega^\mu$.

Now,

$$\begin{aligned}
&\omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)}+\alpha_2)+1\\
&=\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)}+\omega^{\omega^{\alpha_1}}\alpha_2+1\\
&<\omega^{\omega^{\beta_1}}
\end{aligned}$$

because $\omega^{\omega^{\alpha_1}}\alpha_2<\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)}=\omega^{f(\omega^{\alpha_1})}$ and $f(\omega^{\alpha_1})<\omega^{\alpha_1+1}\le\omega^{\beta_1}$.

Lemma 21 *For all $\alpha\in T'_{n+1}[m+1]$ we have that if $\tau_{m+1}\alpha=\Omega_{m+1}\beta+\omega^{f(\beta)}k_m\alpha+\eta$, then*

$$\begin{cases}\beta<\omega^0=\omega_0 & \text{if } m\ge n,\\ \beta<\omega_{n-m} & \text{if } m<n.\end{cases}$$

Proof We prove this by induction. If $m\ge n$, then $\tau_{m+1}\alpha=\Omega_{m+1}0+\omega^0\alpha$, hence we are done. Assume $m<n$. If $\alpha=\vartheta_j\alpha'$ with $j<m+1$, then $\beta=0<\omega_{n-m}$. Assume $\alpha=\vartheta_{m+1}\alpha'$. Assume $\tau_{m+2}\alpha'=\Omega_{m+2}\beta'+\omega^{f(\beta')}k_{m+1}\alpha'+\eta'$ and $\tau_{m+1}k_{m+1}\alpha'=\Omega_{m+1}\beta_1+\omega^{f(\beta_1)}k_mk_{m+1}\alpha'+\eta_1$. From the induction hypothesis, we know $\beta'<\omega_{n-m-1}$ and $\beta_1<\omega_{n-m}$. Then

$$\begin{aligned}
&\tau_{m+1}\alpha\\
&=\Omega_{m+1}\omega^{\beta'}+\omega^{\omega^{\beta'}}(\omega^{f(\beta')}(\Omega_{m+1}\beta_1+\omega^{f(\beta_1)}k_mk_{m+1}\alpha'+\eta_1)+\eta')+1\\
&=\Omega_{m+1}\omega^{\beta'}+\omega^{\omega^{\beta'}}\omega^{f(\beta')}(\Omega_{m+1}\beta_1+\omega^{f(\beta_1)}k_m\alpha'+\eta_1)+\omega^{\omega^{\beta'}}\eta'+1\\
&=\Omega_{m+1}(\omega^{\beta'}+\beta_1)+\omega^{\omega^{\beta'}}\omega^{f(\beta')}(\omega^{f(\beta_1)}k_m\alpha'+\eta_1)+\omega^{\omega^{\beta'}}\eta'+1.
\end{aligned}$$

Now, $\omega^{\beta'}+\beta_1<\omega_{n-m}$.

Lemma 22 *Let $n\ge1$. For all $\alpha\in T'_{n+1}[0]$ we have that $\tau_0\alpha<\omega_{n+2}$.*

Proof We prove this by induction on $lh(\alpha)$. If $\alpha=0$, this is trivial. Assume $\alpha\in T'_{n+1}[0]$, meaning $\alpha=\vartheta_0\alpha'$ with $\alpha'\in T'_{n+1}[1]$. Assume $\tau_1\alpha'=\Omega_1\beta'+\omega^{f(\beta')}k_0\alpha'+\eta'$ with $\eta'<\omega^{f(\beta')}$. Using Lemma 21, we know that $\beta'<\omega_{n-0}=\omega_n$. Additionally, the induction hypothesis yields $\tau_0k_0\alpha'<\omega_{n+2}$. Now,

$$\tau_0\vartheta_0\alpha'=\omega^{\omega^{\beta'}}(\omega^{f(\beta')}\tau_0k_0\alpha'+\eta')+1.$$

From the definition of f, one obtains that $f(\beta')\le\beta'\cdot\omega$. Hence, $\omega^{f(\beta')}\tau_0k_0\alpha'+\eta'<\omega^{f(\beta')}(\tau_0k_0\alpha'+1)<\omega_{n+2}$, so $\tau_0\vartheta_0\alpha'<\omega_{n+2}$.

Corollary 4 $otype(T'_{n+1})\le\omega_{n+2}$.

Proof By Theorem 11, τ_0 is an order preserving embedding from $T'_{n+1}[0]$ to $T_0^{all}=E$. Furthermore, from Lemma 22, we know $\tau_0\alpha<\omega_{n+2}$ for all $\alpha\in T'_{n+1}[0]$. Hence $otype(T'_{n+1})\le\omega_{n+2}$.

Corollary 5 $\vartheta_0\vartheta_1\ldots\vartheta_n\Omega_{n+1}\le\omega_{n+2}$.

Proof By Lemma 15, we know

$$\vartheta_0\vartheta_1\ldots\vartheta_n\Omega_{n+1}=otype(T_{n+1}[0])=otype(T'_{n+1}[0]),$$

hence the previous corollary yields $\vartheta_0\vartheta_1\ldots\vartheta_n\Omega_{n+1}\le\omega_{n+2}$.

5 Binary ϑ-functions

So the question remains whether a maximal linear extension of $\overline{\mathbb{S}}_n^w$ can be realized by a suitable choice of *unary* functions. It turns out that this, as we will show, is possible using specific *binary* theta-functions. However, the question if this is doable with unary functions remains open. Let n be a fixed non-negative integer. In this subsection, we also use the notation T_n, however it is different then the previous one.

Definition 36 Let T_n be the least set such that the following holds. On T_n, define S and K_i.

1. $0 \in T_n$, $S0 := -1$, $K_i0 := \emptyset$,
2. if $\alpha, \beta \in T_n$, $S\alpha \leq i+1$ and $S\beta \leq i < n$, then $\overline{\theta}_i\alpha\beta \in T_n$, $S\overline{\theta}_i\alpha\beta := i$ and

$$K_j\overline{\theta}_i\alpha\beta := \begin{cases} K_j\alpha \cup K_j\beta & \text{if } j < i, \\ \{\overline{\theta}_i\alpha\beta\} & \text{otherwise.} \end{cases}$$

Note that all indices in T_n are strictly smaller than n.

Definition 37 For $\overline{\theta}_i\alpha\beta, \overline{\theta}_j\gamma\delta \in T_n$, define $\overline{\theta}_i\alpha\beta < \overline{\theta}_j\gamma\delta$ iff either $i < j$ or $i = j$ and one of the following alternatives holds:

- $\alpha < \gamma$ & $K_i\alpha \cup \{\beta\} < \overline{\theta}_j\gamma\delta$,
- $\alpha = \gamma$ & $\beta < \delta$,
- $\alpha > \gamma$ & $\overline{\theta}_i\alpha\beta \leq K_i\gamma \cup \{\delta\}$.

Let $0 < \overline{\theta}_i\alpha\beta$ for all $\overline{\theta}_i\alpha\beta \in T_n \backslash \{0\}$.

Here $\overline{\theta}_i\alpha\beta \leq K_i\gamma \cup \{\delta\}$ means that $\overline{\theta}_i\alpha\beta \leq \xi$ for some $\xi \in K_i\gamma \cup \{\delta\}$.

Lemma 23 *For $\overline{\theta}_i\alpha\beta \in T_n$, we have $\beta < \overline{\theta}_i\alpha\beta$.*

Proof This can be proven by induction on $lh(\beta)$.

Definition 38 Define $OT_n \subseteq T_n$ as follows.

1. $0 \in OT_n$,
2. if $\alpha, \beta \in OT_n$, $S\alpha \leq i+1$, $S\beta \leq i < n$ and $K_i\alpha = \emptyset$, then $\overline{\theta}_i\alpha\beta \in OT_n$

Note that $K_i\alpha = \emptyset$ yields that α does not contain any $\overline{\theta}_j$ for $j \leq i$.

Definition 39 If $K_0\alpha = \emptyset$, let α^- be the result of replacing every occurence of $\overline{\theta}_i$ by $\overline{\theta}_{i-1}$.

Lemma 24 *If $\alpha < \beta$ & $K_0\alpha = K_0\beta = \emptyset$, then $\alpha^- < \beta^-$ and $(K_{i+1}\alpha)^- = K_i\alpha^-$.*

Proof This can be proven in a straightforward way by induction on $lh(\alpha) + lh(\beta)$.

Therefore, if $\overline{\theta}_i\alpha\beta \in OT_n$, then α^- is defined and it is an element of OT_{n-1}. Additionally, if $i = 0$, then $S(\alpha^-), S(\beta) \leq 0$.

Definition 40 Define $OT_n[0]$ as $OT_n \cap \Omega_1$, where $\Omega_1 := \overline{\theta}_0 00$

Definition 41 Define $o_1 : OT_1[0] \to \omega$ as follows. An arbitrary element of OT_1 is of the form $\overline{\theta}_0(0, \overline{\theta}_0(0, \ldots \overline{\theta}_0(0,0)\ldots))$. Define the image of this element under o_1 as k if $\overline{\theta}_0(\cdot,\cdot)$ occurs k many times. Define $o_n : OT_n[0] \to \omega_{2n-1}$ for $n > 1$ as follows.

1. $o_n(0) := 0$,
2. $o_n(\overline{\theta}_0\alpha\beta) := \varphi_{o_{n-1}(\alpha^-)}o_n(\beta)$.

Note that $S(\alpha^-), S(\beta) \le 0$ if $\overline{\theta}_0\alpha\beta \in OT_n[0]$.

Theorem 12 *For every $n \ge 1$, o_n is order-preserving and surjective.*

Proof The surjectivity of o_n is easy to prove. We prove that o_n is order-preserving. If $n = 1$, this is trivial. Assume $n > 1$ and assume that o_{n-1} is order preserving. We will show that for all $\alpha, \beta \in OT_n[0]$, $\alpha < \beta$ yields $o_n(\alpha) < o_n(\beta)$. If α and/or β are equal to zero, this is trivial. Assume $0 < \alpha < \beta$. Let $\alpha = \overline{\theta}_0\alpha_1\alpha_2$ and $\beta = \overline{\theta}_0\beta_1\beta_2$. Then $\alpha < \beta$ iff one of the following cases holds:

1. $\alpha_1 < \beta_1$ and $\alpha_2 < \overline{\theta}_0\beta_1\beta_2$,
2. $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$,
3. $\alpha_1 > \beta_1$ and $\overline{\theta}_0\alpha_1\alpha_2 \le \beta_2$.

Note that $\alpha_1 < \beta_1$ yields $\alpha_1^- < \beta_1^-$ by Lemma 24, hence $o_{n-1}(\alpha_1^-) < o_{n-1}(\beta_1^-)$. Furthermore, the induction hypothesis yields that the previous case i. is equivalent with the following case i. for all i.

1. $o_{n-1}\alpha_1^- < o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\overline{\theta}_0\beta_1\beta_2$,
2. $o_{n-1}\alpha_1^- = o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\beta_2$,
3. $o_{n-1}\alpha_1^- > o_{n-1}\beta_1^-$ and $o_n\overline{\theta}_0\alpha_1\alpha_2 \le o_n\beta_2$.

Hence the above case i. is equivalent with the following case i.:

1. $o_{n-1}\alpha_1^- < o_{n-1}\beta_1^-$ and $o_n\alpha_2 < \varphi_{o_{n-1}\beta_1^-}o_n\beta_2$,
2. $o_{n-1}\alpha_1^- = o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\beta_2$,
3. $o_{n-1}\alpha_1^- > o_{n-1}\beta_1^-$ and $\varphi_{o_{n-1}\alpha_1^-}o_n\alpha_2 \le o_n\beta_2$.

This is actually the definition of $\varphi_{o_{n-1}\alpha_1^-}o_n\alpha_2 < \varphi_{o_{n-1}\beta_1^-}o_n\beta_2$, so $o_n\overline{\theta}_0\alpha_1\alpha_2 < o_n\overline{\theta}_0\beta_1\beta_2$.

This yields the following corollary.

Corollary 6 *$otype(OT_n[0]) = \omega_{2n-1}$ if $n \ge 1$.*

This ordinal notation system corresponds to a maximal linear extension of $\overline{\mathbb{S}}_n^s[0] = \overline{\mathbb{S}}_n^w[0]$.

Definition 42 Define f from $\overline{\mathbb{S}}_n^s$ to OT_n as follows. $f(\varepsilon) := 0$ if ε is the empty sequence. $f(ii_1 \ldots i_k j\mathbf{s}) := \overline{\theta}_i(f(i_1 \ldots i_k))(f(j\mathbf{s}))$ if $i < i_1, \ldots, i_k$ and $j \le i$. This yields that $f(i)$ is defined as $\overline{\theta}_i(0,0)$.

Lemma 25 *OT_n is a linear extension of $\overline{\mathbb{S}}_n^s$.*

Proof We prove by induction on the length of s and t that $s \leq^s_{gap} t$ yields $f(s) \leq f(t)$. If s and/or t are ε, then this is trivial. Assume not, then $s = ii_1 \dots i_k j\mathbf{s}'$ and $t = pp_1 \dots p_r q\mathbf{t}'$ with $i_1, \dots, i_k > i \geq j$ and $p_1, \dots, p_r > p \geq q$. If $i < p$, then $f(s) \leq f(t)$ is trivial. Furthermore, $s \leq^s_{gap} t$ yields that $i > p$ is impossible. Therefore we can assume that $i = p$. If the first i of s is mapped into $q\mathbf{t}'$ according to the inequality $s \leq^s_{gap} t$, then $i = q$ and $s \leq^s_{gap} \mathbf{qt}'$, hence $f(s) \leq f(qt')$. From Lemma 23, we know $f(q\mathbf{t}') < f(t)$, hence we are done. Assume that the first i of s is mapped onto the first $i = p$ of t according to the $s \leq^s_{gap} t$ inequality. Then $j\mathbf{s}' \leq^s_{gap} q\mathbf{t}'$ and $i_1 \dots i_k \leq^s_{gap} p_1 \dots p_r$. The induction hypothesis yields $f(j\mathbf{s}') \leq f(q\mathbf{t}')$ and $f(i_1 \dots i_k) \leq f(p_1 \dots p_r)$. If $f(i_1 \dots i_k) = f(p_1 \dots p_r)$, then $f(s) \leq f(t)$ follows from $f(j\mathbf{s}') \leq f(q\mathbf{t}')$. If $f(i_1 \dots i_k) < f(p_1 \dots p_r)$, then $f(s) \leq f(t)$ follows from $f(j\mathbf{s}') \leq f(q\mathbf{t}') < f(t)$ and $K_i(f(i_1 \dots i_k)) = \emptyset$.

Corollary 7 *$OT_n[0]$ is a maximal linear extension of $\overline{\mathbb{S}}^w_n[0] = \overline{\mathbb{S}}^s_n[0]$.*

Proof The previous lemma yields that $OT_n[0]$ is a linear extension of $\overline{\mathbb{S}}_n[0]$. We also know that $otype(OT_n[0]) = \omega_{2n-1} = o(\overline{\mathbb{S}}_n[0])$.

In a sequel project, we intend to determine the relationship between other ordinal notation systems *without* addition with the systems studied here. More specifically, we intend to look at ordinal diagrams [20], Gordeev-style ordinal notation systems [5] and non-iterated ϑ-functions [3,22]. This will be published elsewhere.

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