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CLARK MEASURES AND A THEOREM OF RITT

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Abstract. We determine when a finite Blaschke product B can be written, in a non-trivial way, as a composition of two finite Blaschke products (Ritt's problem) in terms of the Clark measure for B. Our tools involve the numerical range of compressed shift operators and the geometry of certain polygons circumscribing the numerical range of the relevant operator. As a consequence of our results, we can determine, in terms of Clark measures, when two finite Blaschke products commute.

1. Introduction

A *finite Blaschke product* is an analytic function on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$B(z) = \mu \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } |\mu| = 1 \text{ and } |a_j| < 1.$$

Note that B maps $\mathbb{D}$ onto itself n-times and that B maps the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ onto itself n-times. The *degree* of B is the number of zeros $\{a_1, \ldots, a_n\}$ of B repeated according to multiplicity. It is well known that the composition of two finite Blaschke products is another finite Blaschke product with degree equal to the product of the degrees of the two composites. We say that B is *decomposable* if

$$B = C \circ D,$$

where B and C are (finite) Blaschke products of degree greater than one. Otherwise we say that B is *indecomposable*. The condition "degree greater than one" is to avoid trivial decompositions such as $B = (B \circ \varphi) \circ \varphi^{-1}$ or $B = \varphi \circ (\varphi^{-1} \circ B)$, where φ is a disk automorphism (which is a Blaschke product of degree one). Clearly a Blaschke product of prime degree is indecomposable.

The complete answer to this question of decomposability has been known for some time now, dating back to a 1922 paper of Ritt [19] (see also [17, 18] for a more recent treatment) where it was shown that B is decomposable if and only if the associated monodromy group for B^{-1} is imprimitive. Though an alternate, perhaps more readable, rendition of Ritt's theorem was given in [6], both treatments involve computing the monodromy group associated with B^{-1} which, though quite beautiful and a complete answer to the

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question, is impractical. Various, more practical, criteria were given in [8] that also yield further insight as to what really makes a Blaschke product decomposable and how to decompose it.

In this paper, we relate this decomposability problem to the Clark measure naturally associated with a finite Blaschke product and convert the problem into one of expressing this Clark measure in a certain way. The main result of this paper is Theorem 8, though in order to state it, we need a few preliminaries as well as interpolation results that are both interesting and useful on their own. The interpolation results will be presented in Section 3. As a corollary of Theorem 8 we also obtain results about commuting Blaschke products with a fixed point in $\mathbb{D}$ and their associated Clark measures (Theorem 8).

2. Some preliminaries

We first make some normalizing assumptions. If φ is an automorphism of $\mathbb{D}$, i.e.,

$$\varphi(z) = \mu \frac{a - z}{1 - \overline{a}z}, \quad \mu \in \partial\mathbb{D},\ a \in \mathbb{D},$$

then B is decomposable if and only if $\varphi \circ B$ is decomposable. Indeed,

$$B = C \circ D \implies \varphi \circ B = (\varphi \circ C) \circ D$$

and

$$\varphi \circ B = C \circ D \implies B = (\varphi^{-1} \circ C) \circ D.$$

Moreover, an automorphism composed (pre or post) with a finite Blaschke product is again a Blaschke product. Therefore, we may assume that $B(0) = 0$.

For $\alpha \in \mathbb{D}$ let ψ_α denote the involutive ($\psi_\alpha(\psi_\alpha(z)) = z$) disk automorphism defined by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}, \qquad z \in \mathbb{D}. \tag{2.1}$$

Thus we have

$$B = C \circ D \implies B = (C \circ \psi_{D(0)}) \circ \psi_{D(0)} \circ D. \tag{2.2}$$

Hence, whenever B is a composition of two finite Blaschke products C and D, we can always assume that $C(0) = D(0) = 0$.

Next, we define the Clark measure associated with a finite Blaschke product. For each $\lambda \in \partial\mathbb{D}$, the function

$$z \mapsto \Re\left(\frac{\lambda + B(z)}{\lambda - B(z)}\right) = \frac{1 - |B(z)|^2}{|\lambda - B(z)|^2}$$

is a positive harmonic function on $\mathbb{D}$ and thus, by a classical theorem of Herglotz [9, p. 2], there is a unique, finite, positive Borel measure μ_B^λ on $\partial\mathbb{D}$ satisfying

$$\frac{1 - |B(z)|^2}{|\lambda - B(0)|^2} = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu_B^\lambda(\xi), \quad z \in \mathbb{D}. \tag{2.3}$$

The integral on the right-hand side of (2.3) is the Poisson integral of the measure μ_B^λ. With the additional assumption that $B(0) = 0$, one can show [4, p. 204] that μ_B^λ is a probability measure and

$$\mu_B^\lambda = \sum_{j=1}^{n} \frac{1}{|B'(\beta_j)|}\delta_{\beta_j}, \tag{2.4}$$

where $\{\beta_j : 1 \leqslant j \leqslant n\} \subset \partial\mathbb{D}$ are the distinct solutions to the equation $B(\beta) - \lambda = 0$ and δ_β is the unit point mass at β. Observe that the identified points $\{\beta_j : 1 \leqslant j \leqslant n\}$ are distinct since B is analytic in a neighborhood of $\overline{\mathbb{D}}$ (the closure of $\mathbb{D}$) and a computation shows that

$$|B'(e^{i\theta})| = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|e^{i\theta} - a_k|^2} > 0, \quad \theta \in [0, 2\pi]. \tag{2.5}$$

Thus the multiplicity of each of the zeros of $B - \lambda$ is one. The fact that the β_j are distinct (and certainly that $B'(e^{i\theta}) \neq 0$ for all θ) will be important throughout this paper.

The family of measures

$$\{\mu_B^\lambda : \lambda \in \partial\mathbb{D}\}$$

is called the *Clark measures* associated with B and appears in many contexts (perturbation theory, mathematical physics, and composition operators to name a few [4]). As we will see shortly, being able to write μ_B^λ in a certain way will determine whether or not the given Blaschke product B is decomposable.

Given two measure spaces (X_1, Σ_1) and (X_2, Σ_2) and a measurable mapping f from X_1 to X_2, the *push forward measure* $f_*\mu$ associated with a measure $\mu : \Sigma_1 \to [0, \infty]$ is the measure on Σ_2 defined by

$$f_\star\mu(E) = \mu(f^{-1}(E)), \quad E \in \Sigma_2. \tag{2.6}$$

Following Gau and Wu [11], we let $\mathcal{S}_n$ denote the class of all completely non-unitary contractions on $\mathbb{C}^n$ with $\operatorname{rank}(I - T^\star T) = 1$. Via unitary equivalence, $\mathcal{S}_n$ is the same as the class of operators S_ϕ, where ϕ is a finite Blaschke product of degree n,

$$S_\phi : H^2 \ominus \phi H^2 \to H^2 \ominus \phi H^2, \quad S_\phi f = P_\phi(zf),$$

and P_ϕ is the orthogonal projection of H^2 (the Hardy space) onto the *model space* $H^2 \ominus \phi H^2$. The operator S_ϕ is called the *compression of the shift* (multiplication by the independent variable) to the model space $H^2 \ominus \phi H^2$. Note that the spectrum $\sigma(S_\phi)$ of S_ϕ is

$$\sigma(S_\phi) = \phi^{-1}(\{0\}). \tag{2.7}$$

In other words, the eigenvalues of S_ϕ are the zeros of the finite Blaschke product ϕ.

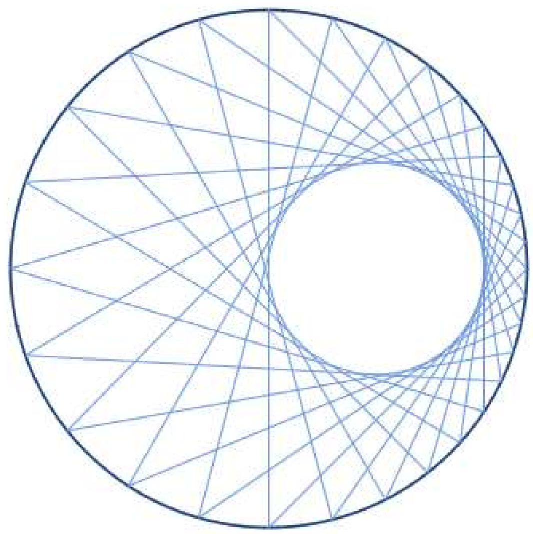

FIGURE 1. For $B(z) = \left(\frac{z-1/2}{1-z/2}\right)\left(\frac{z-1/3}{1-z/3}\right)$, we plot the intersections of the convex hulls of the solutions to $zB(z) = e^{i\theta}$, $\theta \in [0, 2\pi]$ (in this case triangles). The result is $W(S_B)$, the numerical range of the compressed shift S_B.

For a bounded linear transformation (operator) T on a Hilbert space, the *numerical range* $W(T)$ of T is defined to be

$$W(T) := \{\langle Tx, x\rangle : \|x\| = 1\}.$$

The Toeplitz-Hausdorff theorem says that $W(T)$ is a convex subset of $\mathbb{C}$ and, since we will be working in finite dimensions, $W(T)$ will also be compact. These facts along with other basic information about the numerical range can be found in [13, Chapter 1].

If ϕ_1 is a finite Blaschke product, set $\phi(z) := z\phi_1(z)$. For $\lambda \in \partial\mathbb{D}$, let

$$P_\lambda := \text{convex hull of } \phi^{-1}(\{\lambda\}),$$

(which is just an $(n+1)$-gon whose vertices are at the distinct $n+1$ identified points $\phi^{-1}(\{\lambda\})$). Then

$$W(S_{\phi_1}) = \bigcap_{\lambda \in \partial\mathbb{D}} P_\lambda. \tag{2.8}$$

See Figure 1 for a demonstration. This fact comes from [11, Corollary 2.8] along with the following three other important facts.

Theorem 1. [11, Theorem 3.1] *For any $(n+1)$-gon P inscribed in $\partial\mathbb{D}$ and any n points $\alpha_1, \ldots, \alpha_n$, one from the interior of each side of P, there is a $T \in \mathcal{S}_n$ such that $W(T)$ is inscribed in P with tangent points at the α_j.*

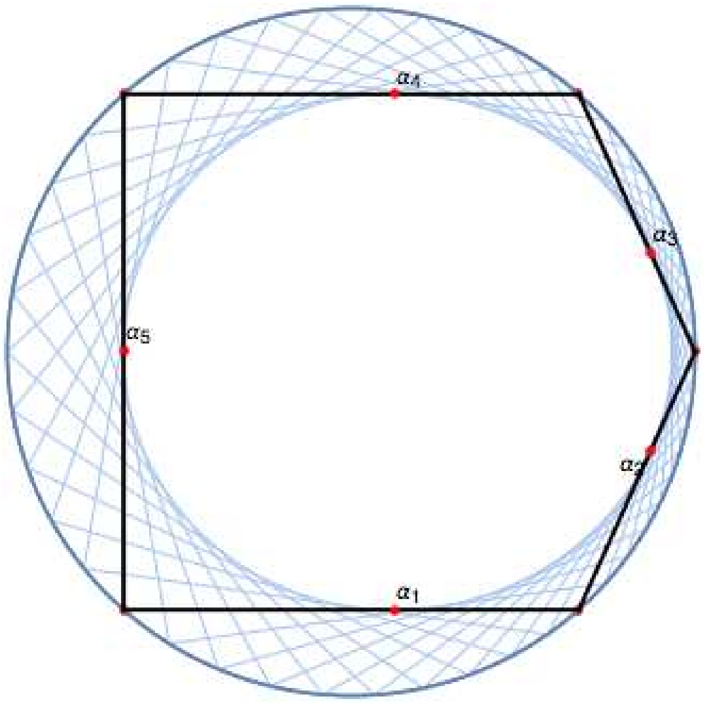

FIGURE 2. A demonstration of Theorem 1: A given 5-gon and an inscribed $W(T)$ for $T \in \mathcal{S}_4$.

See Figure 2 for a demonstration of this theorem.

Theorem 2. [11, Theorem 3.2] *The following statements are equivalent for operators* $T_1, T_2 \in \mathcal{S}_n$.

(1) T_1 *is unitarily equivalent to* T_2*;*
(2) $W(T_1) = W(T_2)$*;*
(3) $W(T_1)$ *and* $W(T_2)$ *have a common circumscribing* $(n+1)$*-gon circumscribed by* $\partial\mathbb{D}$ *and tangent to* $W(T_1)$ *and* $W(T_2)$ *at the same points.*

Given an operator T on a Hilbert space H and a Hilbert space K containing H, we say that $\widetilde{T}$ is a *dilation* (in the sense of Halmos [14]) if

$$T = P_H \widetilde{T}|_H,$$

where P is the orthogonal projection of K onto H. Given a Blaschke product B of degree n, the corresponding compressed shift S_B has a unitary dilation to an $(n+1)$-dimensional space (e.g., $(H^2 \ominus BH^2) \oplus \mathbb{C}$) [3] (see also [4, p. 196]). These unitary dilations are said to be *unitary* 1*-dilations of* S_B.

In what follows, let B be a Blaschke product of degree n. Given $\lambda \in \partial\mathbb{D}$, let $z_1, \ldots, z_{n+1}$ denote the (distinct) solutions to $zB(z) - \lambda = 0$. If we use partial fractions to write

$$\frac{B(z)}{zB(z)-\lambda} = \sum_{j=1}^{n+1} \frac{m_j}{z-z_j}, \tag{2.9}$$

then the following is true:

Theorem 3. [10, Theorem 2.1, part (10)] *Let U be a unitary 1-dilation of S_B where S_B is the compression of the shift operator corresponding to the Blaschke product B with zeros at $b_1, \ldots, b_n$. If $z_1, \ldots, z_{n+1}$, which turn out to be the eigenvalues of U, are listed in terms of increasing argument, then the points of tangency of the line segment joining z_j and z_{j+1} to the boundary of $W(S_B)$ are given by*

$$\frac{m_{j+1}z_j + m_j z_{j+1}}{m_j + m_{j+1}}, \qquad j = 1, 2, \ldots, n+1, \tag{2.10}$$

where the m_j are as in (2.9).

In (2.10) note that the indices above are taken mod $(n+1)$, that is, $m_{j+2} = m_1$ and $z_{n+2} = z_1$. We note that the $m_j > 0$ and $\sum_{j=1}^{n+1} m_j = 1$ [7, Lemma 4]. See Figure 3 for a demonstration of this theorem.

This should be contrasted with Siebeck's theorem [16]. Theorem 4 is more general than what we have here, but closely connected. One difference between it and the result as applied to Blaschke products is that the curves we consider will be tangent to line segments joining consecutive points, while that is not necessarily the case in Theorem 4.

Theorem 4. [16, Theorem 4.2] *For distinct points $z_1, \ldots, z_p \in \mathbb{C}$ and $m_1, \ldots, m_p \in \mathbb{R} \setminus \{0\}$, the zeros of the function*

$$F(z) = \sum_{j=1}^{p} \frac{m_j}{z-z_j}$$

determine a curve that touches each line segment $[z_j, z_k]$ at a point dividing the line segment in the ratio $m_j : m_k$.

See [16] for more on the curve mentioned in the theorem above.

If we consider the Blaschke product B in Theorem 3 and $\lambda \in \partial\mathbb{D}$ and let F be the function

$$F(z) = \frac{B(z)}{zB(z)-\lambda},$$

then Siebeck's theorem says that the line segments are tangent to some curve at the points

$$\frac{m_{j+1}z_j + m_j z_{j+1}}{m_j + m_{j+1}} \text{ for } j = 1, 2, \ldots, n+1,$$

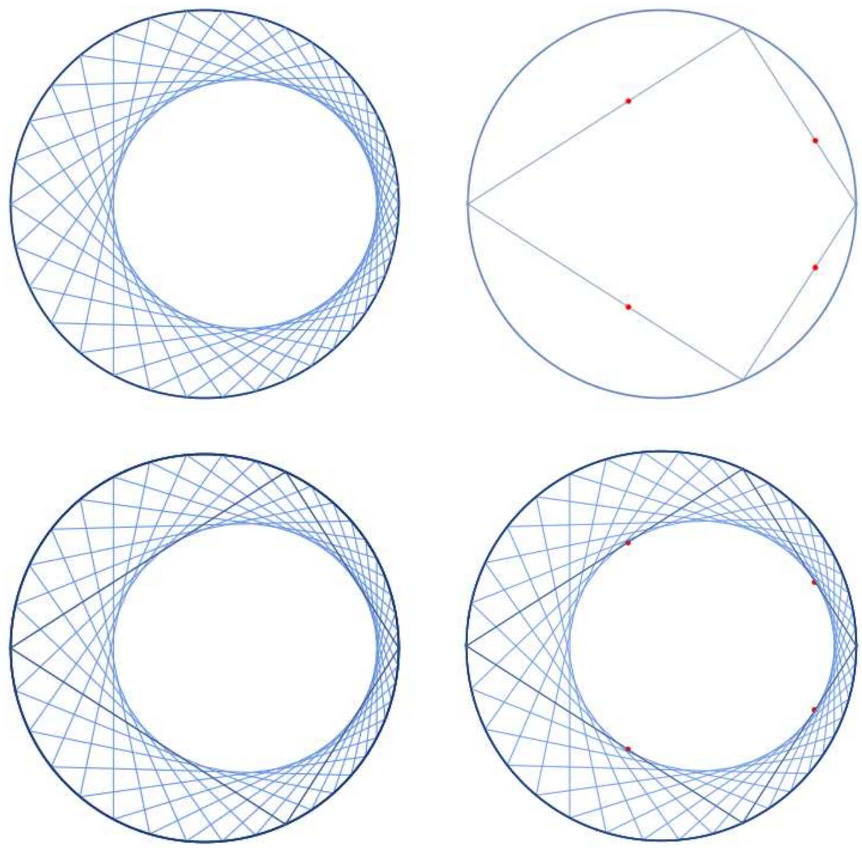

FIGURE 3. For $B(z) = z\left(\frac{z-1/2}{1-z/2}\right)\left(\frac{z-1/3}{1-z/3}\right)$ we form $W(S_B)$ from intersections of the convex hulls of the four solutions of $zB(z)-\lambda = 0$, where $\lambda \in \mathbb{D}$. Then we form the line segments $\{[z_j, z_{j+1}], j = 1, 2, 3, 4\}$ along with the points from (2.10). The last two images are the line segments along with $W(S_B)$ (notice the predicted tangent properties) and the same image but with the predicted tangent points from (2.10).

where $z_{n+2} = z_1$ and $m_{n+2} = m_1$. However, since the definition of F depends on λ, Siebeck's theorem allows for the possibility that the curve depends on λ. Theorem 3 shows that, in fact, it is the same curve that is circumscribed for each $\lambda \in \partial\mathbb{D}$.

3. THE CONNECTION WITH INTERPOLATION

Interpolation on the boundary of the unit disk has been considered by several authors [5, 12, 15, 20]. Here we consider a mixture of interpolation on $\partial\mathbb{D}$ and $\mathbb{D}$. The interpolation results presented here will play a key role in our analysis of Ritt's problem. As we will show in Section 5.1, given the numerical range of an operator $T \in \mathcal{S}_n$, Corollary 7 provides an algorithm for constructing a Blaschke product for which $W(S_B) = W(T)$ – which is interesting in its own right.

We first state a rational interpolation theorem for the real line and then, via conformal mapping, state an analogous interpolation problem for the unit circle.

Proposition 5. *For distinct points $x_1, \dots, x_N \in \mathbb{R}$ and $c_1, \dots, c_N > 0$, there is a rational function F of degree N on $\mathbb{C}$ mapping $\mathbb{C}^\pm$ to $\mathbb{C}^\pm$ and the real axis to the extended real axis, with $F(x_j) = 0$ and $F'(x_j) = c_j$ for each $1 \leqslant j \leqslant N$.*

Proof. Define

$$\begin{aligned} F(z) &= \frac{1}{1 + \sum_{j=1}^N \frac{1}{c_j} \frac{1}{z - x_j}} \\ &= \frac{\prod_{k=1}^N (z - x_k)}{\prod_{k=1}^N (z - x_k) + \sum_{j=1}^N \frac{1}{c_j} \prod_{k \neq j} (z - x_k)}. \end{aligned}$$

From the second expression above it is clear that F has zeros at $x_1, \dots, x_N$ (and nowhere else). Furthermore, a calculation shows that

$$F'(z) = \frac{-\left(\sum_{j=1}^N \frac{1}{c_j} \frac{-1}{(z - x_j)^2}\right)}{(1 + \sum_{j=1}^N \frac{1}{c_j} \frac{1}{z - x_j})^2}.$$

Thus $F'(x_j) = c_j$. If $z \in \mathbb{C}^\pm$, then $1/(z - x_j) \in \mathbb{C}^\mp$ for each j and so $F(z) \in \mathbb{C}^\pm$. $\square$

For example, if we take just one zero, at the origin, then the function F above becomes

$$F(z) = \frac{1}{1 + c^{-1} z^{-1}} = \frac{cz}{cz + 1}$$

and $F'(z) = c/(cz + 1)^2$. Note that $F'(0) = c$.

Of independent interest here is the following version for infinitely many interpolation points.

Corollary 6. *Let $\{x_n\}_{n \geqslant 1} \subset \mathbb{R}$ be a sequence of distinct points, and let $\{c_n\}_{n \geqslant 1} \subset (0, \infty)$ such that*

$$\sum_{j=1}^{\infty} \frac{1}{c_j(1 + |x_j|)} < \infty. \tag{3.1}$$

Then

(1) *the series*

$$\frac{1}{1 + \sum_{j=1}^{\infty} \frac{1}{c_j} \frac{1}{z - x_j}} \tag{3.2}$$

converges uniformly on compact sets disjoint from $\mathbb{R}$ and defines a function F that is an analytic self-map of $\mathbb{C}^+$, respectively $\mathbb{C}^-$;

(2) *if $x_k \in \mathbb{R}$ is an isolated point of the sequence $\{x_n\}_{n\geqslant 1}$, then the series (3.2) converges uniformly in z in a neighborhood of x_k, yielding an analytic function F such that $F(x_k)=0$ and $F'(x_k)=c_k$.*

Proof. (1): On a compact subset K of $\mathbb{C}$ disjoint from $\mathbb{R}$ the quantities

$$\frac{1}{1+|x_j|} \quad \text{and} \quad \frac{1}{|z-x_j|}$$

are uniformly equivalent (independently of j) and thus the series

$$1+\sum_{j=1}^{\infty}\frac{1}{c_j}\frac{1}{z-x_j} \tag{3.3}$$

converges uniformly on K. Moreover,

$$\Im\left(1+\sum_{j=1}^{\infty}\frac{1}{c_j}\frac{1}{z-x_j}\right)=\sum_{j=1}^{\infty}\frac{1}{c_j}\frac{\Im\overline{z}}{|z-x_j|^2}$$

which is strictly negative in $\mathbb{C}^+$ and positive in $\mathbb{C}^-$. Thus the series in (3.3) is bounded away from zero on K. It now follows that the series (3.2) converges locally uniformly, as claimed.

(2): Similarly, for a bounded open neighborhood of x_k, an isolated point of $\{x_n\}_{n\geqslant 1}$, the series

$$1+\sum_{j\neq k}\frac{1}{c_j}\frac{1}{z-x_j}$$

converges to an analytic function. Upon adding in the missing term for $j=k$, and taking a smaller neighborhood if necessary, we conclude that the series converges to a meromorphic function that is nonzero and has a pole at x_k. Thus the function F, defined by the series in (3.2), is analytic and has a zero at x_k. Since (as is easily checked) the partial sums of the series (3.2) converge uniformly near x_k, it follows from Proposition 5 that $F'(x_k)=c_k$. $\square$

Here is a version of Proposition 5 for the unit disk.

Corollary 7. *Let $z_1,\ldots,z_N$ be distinct points of $\partial\mathbb{D}$ and $c_1,\ldots,c_N>0$. Then there is a Blaschke product B of degree N such that $B(z_j)=-1$ and $|B'(z_j)|=c_j$ for $j=1,\ldots,N$.*

Proof. Without loss of generality, we may assume $z_j\neq 1$ for $j=1,\ldots,n$. Let

$$\varphi:\mathbb{D}\to\mathbb{C}^+, \quad \varphi(z)=i\left(\frac{1+z}{1-z}\right),$$

$$x_j=\varphi(z_j), \quad 1\leqslant j\leqslant N.$$

By Proposition 5 there is a rational function F such that

$$F(x_j) = 0, \quad |F'(x_j)| = \frac{c_j}{|(\varphi^{-1})'(0)\varphi'(z_j)|}, \quad 1 \leqslant j \leqslant N. \tag{3.4}$$

Since F is also a degree-N rational self-map of $\mathbb{C}^+$ that maps $\mathbb{R}$ to the extended reals, the function B defined on $\mathbb{D}$ by

$$B := \varphi^{-1} \circ F \circ \varphi$$

is a Blaschke product of degree N. Furthermore, for $1 \leqslant j \leqslant N$, we have

$$B(z_j) = \varphi^{-1}(0) = -1,$$

and

$$\begin{aligned} |B'(z_j)| &= |(\varphi^{-1})'(F(\varphi(z_j)))F'(x_j)\varphi'(z_j)| \\ &= \frac{c_j|(\varphi^{-1})'(0)\varphi'(z_j)|}{|(\varphi^{-1})'(0)\varphi'(z_j)|} = c_j. \end{aligned}$$ □

Similarly, there is a version for infinitely-many points, which follows from Corollary 6. Using the expression for $F'(x_j)$ given in (3.4), and the fact that $|1 - z_j| + |1 + z_j|$ is uniformly bounded above and below for $z_j \in \partial\mathbb{D}$, we see that the convergence condition in equation (3.1) translates into

$$\sum_{j=1}^{\infty} \frac{1}{c_j|1 - z_j|} < \infty.$$

Our choice of φ gives a special role to the point 1, but clearly any other point on $\partial\mathbb{D}$ could replace it, so that the condition

$$\sum_{j=1}^{\infty} \frac{1}{c_j|\alpha - z_j|} < \infty$$

is sufficient for any $\alpha \in \partial\mathbb{D}$.

4. Clark measures

We are now ready to present our main theorem and its corollaries.

Theorem 8. *Let B be a finite Blaschke product of degree $n = mk$ with $m > 1$, $k > 1$ and $B(0) = 0$. Then the following conditions are equivalent:*

(1) *The Blaschke product is decomposable; i.e., there exist Blaschke products C of degree k and D of degree m with*

$$B = C \circ D \quad \text{and} \quad C(0) = D(0) = 0.$$

(2) *For every $\lambda \in \partial\mathbb{D}$ (or indeed for some $\lambda \in \partial\mathbb{D}$) there is a partition of the set $\{\beta : B(\beta) = \lambda\}$ into k sets of m points, denoted E_j, $1 \leqslant j \leqslant k$, and a Blaschke product D of degree m such that $D(0) = 0$ and both D and the function f on $\partial\mathbb{D}$ defined by*

$$f(\beta) := \frac{|B'(\beta)|}{|D'(\beta)|} \tag{4.1}$$

are constant on E_j.

(3) *There exist* $\lambda_1, \lambda_2 \in \partial\mathbb{D}$ *and Blaschke products* C *of degree* k *and* D *of degree* m *satisfying* $C(0) = D(0) = 0$ *with*

$$D_\star \mu_B^{\lambda_j} = \mu_C^{\lambda_j} \text{ for } j = 1, 2.$$

Proof. We begin by showing that (1) implies both (2) and (3). Suppose that B is decomposable and $\lambda \in \partial\mathbb{D}$. Let β_j denote the n (distinct) points in $\partial\mathbb{D}$ for which

$$B(\beta_j) = (C \circ D)(\beta_j) = \lambda.$$

Thus $C(D(\beta_j)) = \lambda$. Since C has degree k, there are k (distinct) points, $\alpha_1, \ldots, \alpha_k$ in $\partial\mathbb{D}$ for which $C(\alpha_k) = \lambda$, and since D has degree m, we see that D partitions the set $\beta_1, \ldots, \beta_n$ into k sets of m points, denoted E_j for $j = 1, \ldots, k$, with the property that $D(E_j) = \{\alpha_j\}$. We reindex the points β as

$$E_j = \{\beta_l^j : l = 1, \ldots, m\} \text{ for } j = 1, \ldots, k.$$

Consider the push forward measure $D_* \mu_B^\lambda$ from (2.6). For a Lebesgue measurable subset $E \subseteq \mathbb{D}$ we have, via (2.4),

$$\begin{aligned}
D_\star \mu_B^\lambda(E) &= \mu_B^\lambda(D^{-1}(E)) \\
&= \sum_{r=1}^{n} \frac{1}{|B'(\beta_r)|} \delta_{\beta_r}(D^{-1}(E)) \\
&= \sum_{s=1}^{k} \left(\sum_{l:\beta_l^s \in S_s} \frac{1}{|B'(\beta_l^s)|} \delta_{\beta_l^s}(D^{-1}(E)) \right) \\
&= \sum_{s=1}^{k} \left(\sum_{l:D(\beta_l^s)=\alpha_s} \frac{1}{|B'(\beta_l^s)|} \right) \delta_{\alpha_s}(E) \\
&= \sum_{s=1}^{k} \left(\sum_{l:D(\beta_l^s)=\alpha_s} \frac{1}{|C'(D(\beta_l^s)|\,|D'(\beta_l^s)|} \right) \delta_{\alpha_s}(E) \\
&= \sum_{s=1}^{k} \left(\sum_{l:D(\beta_l^s)=\alpha_s} \frac{1}{|C'(\alpha_s)|\,|D'(\beta_l^s)|} \right) \delta_{\alpha_s}(E).
\end{aligned}$$

Thus

$$D_\star \mu_B^\lambda(E) = \sum_{s=1}^{k} \frac{1}{|C'(\alpha_s)|} \left(\sum_{l:D(\beta_l^s)=\alpha_s} \frac{1}{|D'(\beta_l^s)|} \right) \delta_{\alpha_s}(E). \tag{4.2}$$

For a fixed value of s we look at the Clark measure $\mu_D^{\alpha_s}$ and use (2.4) again (note $D(0) = 0$) to see that

$$\mu_D^{\alpha_s} = \sum_{j=1}^{m} \frac{1}{|D'(\beta_j^s)|} \delta_{\beta_j^s}.$$

However, $\mu_D^{\alpha_s}$ is a probability measure and therefore

$$\sum_{j=1}^{m} \frac{1}{|D'(\beta_j^s)|} = 1.$$

Thus, since each of the inner terms in (4.2) equals 1, we see that $D_\star \mu_B^\lambda = \mu_C^\lambda$, establishing (3). Finally, since $B = C \circ D$ and D identifies the points of E_j, we see that for $\beta \in E_j$ we have

$$|B'(\beta)| = |C'(D(\beta))|\,|D'(\beta)| = |C'(\alpha_j)|\,|D'(\beta)|.$$

Hence the function f on $\partial\mathbb{D}$ defined by

$$f(\beta) = \frac{|B'(\beta)|}{|D'(\beta)|}$$

is constant on each E_j, $1 \leqslant j \leqslant k$. Thus we have shown (1) implies (2) and (3).

We will now show that (2) (even for a *single* value of λ) implies (1). We know that D partitions the points $\beta_1, \ldots, \beta_n$ identified by B into k sets $E_1, \ldots, E_k$ of m points each and that D sends 0 to 0. Since D has degree m, we have $D|_{E_s} = \alpha_s$ with the α_s distinct. Corollary 7 produces a Blaschke product C of degree k such that

$$C(\alpha_1) = C(\alpha_2) = \ldots = C(\alpha_k) = \lambda, \quad \text{and} \quad |C'(\alpha_j)| = \frac{|B'(\beta)|}{|D'(\beta)|}, \tag{4.3}$$

where $\alpha_j = D(\beta)$ for $\beta \in E_j$. Let

$$C_1 = \varphi_{C(0)} \circ C$$

and observe that $C_1 \circ D$ maps 0 to 0 and identifies β_j, $1 \leqslant j \leqslant n$. If we consider the function

$$F(z) = \frac{B(z)/z}{B(z) - \lambda} = \sum_{j=1}^{n} \frac{m_j}{z - \beta_j},$$

then, by Theorem 3, the point of tangency to the boundary of $W(S_B)$ of the line segment joining β_j and β_{j+1} is given by

$$\frac{m_j}{m_j + m_{j+1}} \beta_{j+1} + \frac{m_{j+1}}{m_j + m_{j+1}} \beta_j. \tag{4.4}$$

A computation (see also [7]; included below for completeness) shows that

$$\frac{B(z)}{z}\frac{z-z_j}{B(z)-B(z_j)}=\sum_{k=1}^{n}\frac{1}{m_j}\frac{z-z_j}{z-z_k}.$$

Taking the limit in the above as $z \to z_j$ and noting that $m_j > 0$ (see [7]) we see that

$$m_j = \frac{1}{|B'(z_j)|}. \tag{4.5}$$

With the notation $E_s = \{\beta_1^s, \ldots, \beta_m^s\}$ and $\gamma = C(D(\beta_j^s))$ we use equation (4.3) to conclude that

$$\begin{aligned}|\left(C_1 \circ D\right)'(\beta_j^s)| &= \frac{1-|C(0)|^2}{|1-\overline{C(0)}C(D(\beta_j^s))|^2}|C'(\alpha_s)\, D'(\beta_j^s)| \\ &= \frac{1-|C(0)|^2}{|1-\overline{C(0)}\gamma|^2}|B'(\beta_j^s)|.\end{aligned}$$

Using formula (4.4), the points of tangency associated with $C_1 \circ D$ are the same as those for B. By (2.8) and Theorem 1 the numerical range of the compressed shift

$$S_{C_1 \circ D(z)/z}$$

has a circumscribing polygon at the points identified by $C_1 \circ D$ that are the same points as those defined by B. Furthermore, the points of tangency are the same. Thus, by Theorem 2, we know that the compressed shifts $S_{C_1 \circ D(z)/z}$ and $S_{B(z)/z}$ are unitarily equivalent. Hence they have the same eigenvalues, which, by (2.7), are the zeros of corresponding the Blaschke products. Thus there exists a $\mu \in \partial\mathbb{D}$ such that

$$\frac{(C_1 \circ D)(z)}{z} = \mu \frac{B(z)}{z}.$$

Rotating C_1 we obtain C_2 and so $B = C_2 \circ D$. Thus (2) implies (1).

Now suppose that (3) holds. If $\lambda \in \partial\mathbb{D}$ and the two discrete measures, $D_\star \mu_B^\lambda$ and μ_C^λ, are equal, they must have the same atoms and the same weights. Denote the zeros of $C - \lambda$ by $\alpha_1, \alpha_2, \ldots, \alpha_k$ and the zeros of $B - \lambda$ by $\beta_1, \ldots, \beta_n$. Then

$$\frac{1}{|C'(\alpha_j)|} = D_\star \mu_B^\lambda(\{\alpha_j\}) = \sum_{r=1}^{n} \frac{1}{|B'(\beta_r)|}\delta_{\beta_r}(D^{-1}(\{\alpha_j\})).$$

Thus there is a $q \leqslant m$ such that

$$\beta_1, \ldots, \beta_q \in D^{-1}(\alpha_j).$$

Suppose there exists $\gamma \notin B^{-1}(\lambda)$ with $D(\gamma) = \alpha_j$.

Then

$$\sum_{j=1}^{k} \mu_C^\lambda(\{\alpha_j\}) = \sum_{j=1}^{k} \sum_{r:D(\beta_r)=\alpha_j} \frac{1}{|B'(\beta_r)|}.$$

However, the existence of γ implies that we are not summing over the n distinct β_r. Since each of the weights

$$\frac{1}{|B'(\beta_r)|}$$

is positive for each r, and we have omitted at least one, μ_C is not a probability measure, a contradiction. Therefore, D partitions the $n = mk$ zeros of B into k sets of m points, $E_1, \dots, E_k$ and we see that μ_B^λ and $\mu_{C \circ D}^\lambda$ have the same atoms, $\beta_1, \dots, \beta_n$.

In summary, if the equality holds for two distinct λ, there exist two sets of n points, β_j^1 and β_j^2 such that $(C \circ D)(\beta_j^k) = B(\beta_j^k)$. Multiplying by the denominators to obtain a polynomial of degree $2n$ and setting that equal to 0, we see that the β_j^k are $2n$ solutions to that problem. Since $C \circ D(0) = 0$ and $B(0) = 0$, the polynomial is also 0 at 0. Therefore, the polynomial is identically zero and so $B = C \circ D$. Hence (1) holds. □

5. Some Consequences

In this section, we provide two consequences of our results. The first of these provides an algorithm for constructing a finite Blaschke product B that corresponds to a given polygon and set of points of tangency. We then apply our results to obtain a statement about commuting Blaschke products, each of which have a fixed point in $\mathbb{D}$.

5.1. **An algorithm.** Suppose we have $W(T)$ where $T \in \mathcal{S}_n$ and a circumscribing polygon P with vertices $z_1, \dots, z_{n+1}$ and points of tangency at $t_j z_j + (1 - t_j) z_{j+1}$, where $0 < t_j < 1$ for all j (and the indices are understood mod $(n+1)$). We will show how one can use the computation in Theorem 8 (part 2) and Corollary 7 to provide an algorithm for constructing a Blaschke product B such that $W(S_B) = W(T)$.

First we show that $t_1, \dots, t_n$ determine t_{n+1}. Because the polygon circumscribes $W(T)$, where $T \in \mathcal{S}_n$, we know that there is a Blaschke product B of degree $n+1$ with $B(0) = 0$ that identifies the z_j and satisfies

$$\frac{B(z)/z}{B(z) - B(z_1)} = \sum_{j=1}^{n+1} \frac{m_j}{z - z_j},$$

with $m_j > 0$ and $\sum_{j=1}^{n+1} m_j = 1$. Using formula (2.10) we also know that

$$\frac{m_j}{m_j + m_{j+1}} = t_j \text{ for all } j.$$

Solving this system of equations, we see that

$$\frac{1}{t_{n+1}} = 1 + \frac{m_1}{m_{n+1}} = 1 + \frac{1}{\prod_{j=1}^{n} (\frac{1}{t_j} - 1)}, \tag{5.1}$$

and so the points $t_1, \dots, t_n$ determine t_{n+1}.

Let

$$\widetilde{m}_j = \widetilde{m}_1 \prod_{k=1}^{j-1} (\frac{1}{t_j} - 1) \text{ for } j > 1$$

and

$$\widetilde{m}_1 = \frac{1}{1 + \left(\sum_{j=2}^{n+1} \prod_{k=1}^{j-1} (\frac{1}{t_j} - 1) \right)}.$$

It is clear that $\widetilde{m}_j > 0$ and a calculation shows that $\sum_{j=1}^{n+1} \widetilde{m}_j = 1$. Another computation shows that

$$\frac{\widetilde{m}_j}{\widetilde{m}_j + \widetilde{m}_{j+1}} = t_j \text{ for } j = 1, \ldots, n$$

and equation (5.1) shows that

$$\frac{\widetilde{m}_{n+1}}{\widetilde{m}_{n+1} + \widetilde{m}_1} = t_{n+1}.$$

We are now ready to construct our Blaschke product. Use Corollary 7 to produce a Blaschke product C of degree $n+1$ that identifies the $(n+1)$ vertices of the given polygon P and such that $|C'(z_j)| = 1/\widetilde{m}_j$ for all j.

We need our Blaschke product to take 0 to 0, so we consider the Blaschke product C_1 defined by

$$C_1 := \frac{C(0) - C}{1 - \overline{C(0)}C}.$$

Then one can see that $C_1(z_1) = \cdots = C_1(z_{n+1}) = \lambda$ and

$$|C_1'(z_j)| = \left| \frac{1 - |C(0)|^2}{(1 - \overline{C(0)}C(z_j))^2} C'(z_j) \right| = \left| \frac{1 - |C(0)|^2}{(1 - \overline{C(0)}\lambda)^2} \right| \frac{1}{\widetilde{m}_j}.$$

By (4.5), the numerators in the partial fraction expansion for

$$\frac{C_1(z)/z}{C_1(z) - C_1(z_1)}$$

are precisely $\widetilde{m}_j$, that is,

$$\frac{C_1(z)/z}{C_1(z) - C_1(z_1)} = \sum_{j=1}^{n+1} \frac{\widetilde{m}_j}{z - z_j}, \text{ where } |C_1'(z_k)| = \frac{1}{\widetilde{m}_k}.$$

Looking at formula (2.10) for the points of tangency to $W(S_{C_1(z)/z})$, we see that the points of tangency that we get from C_1 are the same as those obtained from C. We have also seen that

$$\frac{\widetilde{m}_j}{\widetilde{m}_j + \widetilde{m}_{j+1}} = t_j,$$

so the points of tangency to $W(T)$ determined by the polygon P and $W(C_1)$ are the same. Thus $W(S_B)$ and $W(T)$ have a common circumscribing $(n+1)$-gon circumscribed by $\partial\mathbb{D}$ that is tangent to $W(S_B)$ and $W(T)$ at the same points. By Theorem 2, S_B and T are unitarily equivalent and therefore have the same numerical range.

5.2. **Commuting Blaschke Products.** We turn to our results on commuting finite Blaschke products. Such Blaschke products were studied in [2] when there is a fixed point in the disk, and in [1] when there is no fixed point. Using the results above we may now give an alternative characterization for the first of these two cases. We begin with a simple lemma.

Lemma 9. *Let C and D be commuting finite Blaschke products, and $\alpha \in \mathbb{D}$. Then the following conditions are equivalent.*

(1) $C(\alpha) = \alpha$;
(2) $D(\alpha) = \alpha$;
(3) $(C \circ D)(\alpha) = \alpha$.

Proof. For (1) $\Rightarrow$ (2), note that $C \circ D = D \circ C$ implies that

$$C(D(\alpha)) = D(C(\alpha)) = D(\alpha).$$

However, a Blaschke product has at most one fixed point in $\mathbb{D}$ and so $D(\alpha) = \alpha$.

For (2) $\Rightarrow$ (3), we note that the argument used in (1) $\Rightarrow$ (2) shows that if two Blaschke products commute then any fixed point of one is a fixed point of the other. Now D commutes with $C \circ D$, so we have the result.

Finally, for (3) $\Rightarrow$ (1) we see that, likewise, $C \circ D$ commutes with C, and the implication follows. $\square$

Remark 10. *If C and D are finite Blaschke products such that $C(\alpha) = \alpha$ for some $\alpha \in \mathbb{D}$, then C and D commute if and only if $\widetilde{C}$ and $\widetilde{D}$ commute, where $\widetilde{C} = \psi_\alpha \circ C \circ \psi_\alpha$ and $\widetilde{D} = \psi_\alpha \circ D \circ \psi_\alpha$ (recall that ψ_α is the automorphism defined in (2.1)); by Lemma 9 we also have $\widetilde{C}(0) = \widetilde{D}(0) = 0$.*

Combining these ideas we have the following result, with $\widetilde{C}$ and $\widetilde{D}$ as in Remark 10.

Theorem 11. *Let C and D be finite Blaschke products and $\alpha \in \mathbb{D}$ such that $C(\alpha) = \alpha$. Then the following are equivalent.*

(1) $C \circ D = D \circ C$;
(2) *There exist $\lambda_1, \lambda_2 \in \partial\mathbb{D}$ with*
$$\widetilde{D}_* \mu^{\lambda_j}_{\widetilde{D}\circ\widetilde{C}} = \mu^{\lambda_j}_{\widetilde{C}}, \qquad j = 1, 2.$$
(3) *There exist $\lambda_1, \lambda_2 \in \partial\mathbb{D}$ with*
$$\widetilde{C}_* \mu^{\lambda_j}_{\widetilde{C}\circ\widetilde{D}} = \mu^{\lambda_j}_{\widetilde{D}}, \qquad j = 1, 2.$$

Proof. This now follows immediately from the equivalence of (1) and (4) in Theorem 8, with $B = \widetilde{C} \circ \widetilde{D}$. $\square$

6. Examples

By computing monodromy groups, Cowen [6] presented two examples of Blaschke products, one of which is a composition and one of which is not. We now rework Cowen's two examples to see what happens to the composition in our algorithm if there is such a composition and what happens to our algorithm if there is not such a decomposition.

Example 1.

The finite Blaschke product

$$B(z) = z^2 \left(\frac{z - \frac{1}{3}}{1 - \frac{1}{3}z}\right)\left(\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}\right)$$

is indecomposable.

Proof. The above Blaschke product B is of degree 4. If it were a composition, it would be the composition of two degree-2 Blaschke products C and D. From (2.2) we may assume that $B = C \circ D$ with $C(0) = D(0) = 0$. This, in turn, implies that $C(z) = zC_1(z)$ and $B(z) = D(z)C_1(D(z))$ and so D must have its second zero at either 0, $\frac{1}{2}$ or $\frac{1}{3}$. Chasing down the three possibilities

$$D(z) = D_1(z) := z^2,$$

$$D(z) = D_2(z) := z\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z},$$

$$D(z) = D_3(z) := z\frac{z - \frac{1}{3}}{1 - \frac{1}{3}z},$$

one can see that $D(1) = D(-1) = 1$. A computation shows that

$$B'(1) = 7 \quad \text{and} \quad B'(-1) = -\frac{17}{6}.$$

If $D_1(z) = z^2$, then

$$\left|\frac{B'(1)}{D'(1)}\right| \neq \left|\frac{B'(-1)}{D'(-1)}\right|$$

and so, by Theorem 8, $B \neq C \circ D_1$. Checking D_2 and D_3 in a similar way, we see that B cannot be a composition of any of these and therefore B is indecomposable. $\square$

It is possible to run through an algorithm to determine a decomposable Blaschke product that is related to B. Consider the four points B sends to 1, namely

$$-1, \quad 1, \quad \frac{5 - i\sqrt{119}}{2} \quad \frac{5 + i\sqrt{119}}{2}. \tag{6.1}$$

We use the algorithm described in [12] to compute a Blaschke product D of degree-2 that identifies the points -1 and 1 and also identifies the points

$$\frac{1}{12}(5-i\sqrt{119}) \quad \text{and} \quad \frac{1}{12}(5+i\sqrt{119}).$$

We follow the algorithm in [12] (though there is an algorithm provided by Courtney and Sarason [5] that does not require mapping over to the upper-half plane). We map over to the upper-half plane, find a function F that maps the first pair to 0 and the other to ∞ and then map back to the unit circle. Doing this, we obtain the Blaschke product

$$D(z) = \frac{z(-5+12z)}{-12+5z}.$$

Then

$$D(1) = D(-1) = -1$$

and

$$D\left(\frac{5-i\sqrt{119}}{12}\right) = D\left(\frac{5+i\sqrt{119}}{12}\right) = 1.$$

If we let $C(z) = z^2$, then $C \circ D$ maps 0 to 0, is of degree 4, and $C \circ D$ is equal to B on the set of four points from (6.1), but no other set of four points. Thus, given a Blaschke product (B in this case) of degree 4, there is a decomposable Blaschke product ($C \circ D$ in this case) that agrees with it on four points, but that is not enough to guarantee that the two Blaschke products are equal up to composition with an automorphism.

Example 2.

Consider the finite Blaschke product

$$B(z) = z^2\left(\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right)^2.$$

This is obviously a composition $C \circ D$, where

$$C(z) = z^2 \quad \text{and} \quad D(z) = z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}.$$

The algorithm can be applied to this Blaschke product.

Assuming for the moment that we do not know C and D, we discuss a method for finding them. We know that D must be degree 2, map 0 to 0, and, if $B = C \circ D$, there must be a partition of the set of points that B identifies into two sets E_1 and E_2 with D constant on each. If we choose a set of points $\beta_1, \beta_2, \beta_3$ and β_4, ordered according to increasing argument, that B identifies, then the partition must be into two sets of two points. But, since D is a degree-2 Blaschke product, it must identify the points β_1 and β_3 and the points β_2 and β_4. One algorithm for constructing D is presented in [12]. Up to rotation, there is only one such Blaschke product D; see [7, Theorem 2]. It then remains to check the derivative condition.

We now compute $D_\star \mu_B^1$ and μ_C^1 and see they are equal. From (2.4) we have

$$\mu_C^1 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

Now the four points B sends to 1 are

$$\beta_1 = 1, \ \ \beta_2 = -1, \ \ \beta_3 = e^{-i\pi/6}, \ \ \beta_4 = e^{i\pi/6}.$$

Then

$$D'(\beta_1) = 4, \ \ D'(\beta_2) = -\frac{4}{3}, \ \ D'(\beta_3) = D'(\beta_4) = 2.$$

A computation shows that

$$B'(\beta_1) = 8, \ \ B'(\beta_2) = -\frac{8}{3}, \ \ B'(\beta_3) = 4\frac{-i+\sqrt{3}}{i+\sqrt{3}}, \ \ B'(\beta_4) = 4\frac{i+\sqrt{3}}{-i+\sqrt{3}}.$$

Thus

$$\left|\frac{B'(\beta_1)}{D'(\beta_1)}\right| = \left|\frac{B'(\beta_2)}{D'(\beta_2)}\right| \quad \text{and} \quad \left|\frac{B'(\beta_3)}{D'(\beta_3)}\right| = \left|\frac{B'(\beta_4)}{D'(\beta_4)}\right|.$$

Furthermore, notice that

$$D(\beta_1) = D(\beta_2) = 1, \quad D(\beta_3) = D(\beta_4) = -1$$

and so, using (4.2), we see that

$$\begin{aligned}
D_\star \mu_B^1 &= \sum_{s=1}^{2} \left(\sum_{l : D(\beta_l^s) = \alpha_s} \frac{1}{|C'(\alpha_s)|\,|D'(\beta_l^s)|} \right) \delta_{\alpha_s} \\
&= \left(\frac{1}{|C'(1)||D'(\beta_1)|} + \frac{1}{|C'(1)||D'(\beta_2)|} \right) \delta_1 \\
&\quad + \left(\frac{1}{|C'(-1)||D'(\beta_3)|} + \frac{1}{|C'(-1)|D'(\beta_4)|} \right) \delta_{-1} \\
&= \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \\
&= \mu_C^1.
\end{aligned}$$

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