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# Dynamic demand and mean-field games

Dario Bauso

**Abstract**—Within the realm of *smart buildings* and *smart cities*, dynamic response management is playing an ever-increasing role thus attracting the attention of scientists from different disciplines. Dynamic demand response management involves a set of operations aiming at decentralizing the control of loads in large and complex power networks. Each single appliance is fully responsive and readjusts its energy demand to the overall network load. A main issue is related to mains frequency oscillations resulting from an unbalance between supply and demand. In a nutshell, this paper contributes to the topic by equipping each signal consumer with strategic insight. In particular, we highlight three main contributions and a few other minor contributions. First, we design a mean-field game for a population of thermostatically controlled loads (TCLs), study the mean-field equilibrium for the deterministic mean-field game and investigate on asymptotic stability for the microscopic dynamics. Second, we extend the analysis and design to uncertain models which involve both stochastic or deterministic disturbances. This leads to robust mean-field equilibrium strategies guaranteeing stochastic and worst-case stability, respectively. Minor contributions involve the use of stochastic control strategies rather than deterministic, and some numerical studies illustrating the efficacy of the proposed strategies.

**Index Terms**—mean-field games, stochastic stability, power networks

## I. INTRODUCTION

Demand response involves a set of operations aiming at decentralizing load control in power networks [1], [12], [13], [28]. In particular, it calls for the alteration of the timing or of the total electricity by end-use customers from their normal consumption patterns in response to changes in the price of electricity. This is possible also through the design of incentive payments to induce lower electricity use at off-peak times.

A communication protocol aggregates information on past, current and forecasted demand and transmits it to each load controller, which will increase or decrease the proper load. The novelty of this paper is that *fully responsive load control* is obtained by enhancing the intelligence on the demand side of the grid. This leads to a less-prescriptive environment in which the loads, rather than being pre-programmed to adopt specific switching behaviors, are designed as intelligent appliances selecting their switching behaviors as best-responses to the population behavior. The population behavior is sensed by the individual appliances through the mains frequency state. In this

paper, fully responsive load control is reviewed in the context of thermostatically controlled loads (TCLs), in smart buildings or plug-in electric vehicles [2], [21], [22], [26].

A first idea is to use stochastic strategies rather than deterministic as in [2], [4]. Each TCL selects a probability with which to switch *on* and *off*. Thus a probability value of  $\frac{1}{2}$  means that the TCL is 50% *on* and 50% *off*. It has been shown in [2], [4] that stochastic response strategies outperform deterministic ones, especially in terms of attenuating the *mains frequency* oscillations. These are due to the unbalance between energy demand and supply (see e.g. [23]). The mains frequency usually needs to be stabilized around a nominal value (50 Hz in Europe). If electricity demand exceeds generation then frequency will decline, and vice versa.

The model used in this paper is as follows. Each single TCL is a player and is characterized by two state variables, the temperature and the functioning mode. The state dynamics of a TCL — henceforth referred to as *microscopic* dynamics — describes the time evolution of its temperature and mode in the form of a linear ordinary differential equation in the deterministic case, and of a stochastic differential equation in the stochastic case. Such dynamics is different from the dynamics of the aggregate temperature and functioning mode of the whole population, which is henceforth referred to as *macroscopic* dynamics. In addition to the state dynamics, each TCL is programmed with a given finite-horizon cost functional that accounts for i) energy consumption, ii) deviation of mains frequency from the nominal one, and iii) deviation of the TCL's temperature from a reference value. Bringing together in the objective functional both individual costs (in the form of energy consumption and deviation from a reference temperature) and common costs (in the form of the individual contribution to the deviation of the mains frequency from the nominal one) is original to the best of the author's knowledge. More formally, the mains frequency involved in specifics ii) mentioned above is used in a cross-coupling mean-field term that incentivizes the TCL to switch to *off* if the mains frequency is below the nominal value and to switch to *on* if the mains frequency is above the nominal value. In other words, the cross-coupling mean-field term models all kinds of incentive payments, benefits, or smart pricing policies aiming at shifting demand from high-peak to off-peak periods.

### A. Highlights of contributions

This paper provides three main results. First, in the spirit of prescriptive game theory and mechanism design [3] we design a mean-field game for the TCLs application, study the mean-field equilibrium for the deterministic mean-field game and investigate on asymptotic stability for the microscopic dynamics. Asymptotic stability means that both the temperature and the

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mode functioning of each TCL converge to the reference value. A second result relates to the stochastic case, characterized by a stochastic disturbance in the form of a Brownian motion in the microscopic dynamics. After establishing a mean-field equilibrium, we provide some results on stochastic stability. In particular, we focus on two distinct scenarios. In one case, we assume that the stochastic disturbance expires in a neighborhood of the origin. This reflects in having the Brownian motion coefficients linear in the state. The resulting dynamics is well-known in the literature as geometric Brownian motion. As for any geometric Brownian motion, we can study conditions for it to be stochastically stable. This means that the state trajectories are moment bounded. In a second case, the stochastic disturbance is independent on the state and the Brownian motion coefficients are constant. This leads to a dynamics which resembles the Langevin equation. Following well-known results on the Langevin equation, the dynamics is proven to be stochastically stable in the second-moment. An expository work on stochastic analysis and stability is [20]. A third result deals with robustness for the microscopic dynamics. The dynamics is now influenced by an additional adversarial disturbance, with bounded resource or energy. Even for this case, we study the mean-field equilibrium and investigate on conditions that guarantee worst-case stability. The stochastic stability analysis and the worst-case analysis under adversarial disturbances add originality to the mean-field game approach.

## B. Literature overview

We introduce next two streams of literature. One is related to dynamic response management, while the second one is about the theory of differential games with a large number of indistinguishable players, also known as mean-field games.

1) *Related literature on demand response*: Examples of papers developing the idea of dynamic demand management are [10], [11], [21], [22]. In particular, [10] provides an overview on the redistribution of the load away from peak hours and the design of decentralized strategies to produce a predefined load trajectory. This idea is further developed in [11]. To understand the role of game theory in respect to this specific context the reader is referred to [21]. There, the authors present a large population game where the agents are plug-in electric vehicles and the Nash-equilibrium strategies (see [6]) correspond to distributed charging policies that redistribute the load away from peaks. The resulting strategies are known with the name of *valley-filling strategies*. In this paper we adopt the same perspective in that we show that network frequency stabilization can be achieved by giving incentives to the agents to adjust their strategies in order to converge to a mean-field equilibrium. To do this, in the spirit of *prescriptive game theory* [3], a central planner or game designer has to design the individual objective function so to penalize those agents that are in *on* state in peak hours, as well as those who are in *off* state in off-peak hours. Valley-filling and coordination strategies have been shown particularly efficient in thermostatically controlled loads such as refrigerators, air conditioners and electric water heaters [22].

2) *Related literature on mean-field games*: A second stream of literature related to the problem at hand is on mean-field games. Mean-field games were formulated by Lasry and Lions in [19] and independently by M.Y. Huang, P. E. Caines and R. Malhamé in [17], [18]. The mean-field theory of dynamical games is a modeling framework at the interface of differential game theory, mathematical physics, and  $H_\infty$ -optimal control that tries to capture the mutual influence between a crowd and its individuals. From a mathematical point of view the mean-field approach leads to a system of two PDEs. The first PDE is the Hamilton-Jacobi-Bellman (HJB) equation. The second PDE is the Fokker-Planck-Kolmogorov (FPK) equation which describes the density of the players. Explicit solutions in terms of mean-field equilibria are available for linear-quadratic mean-field games [5], and have been recently extended to more general cases in [14].

The idea of extending the state space, which originates in optimal control [24], [25], has been also used to approximate mean-field equilibria in [8].

More recently, robustness and risk-sensitivity have been brought into the picture of mean-field games [9], [27] where the first PDE is now the Hamilton-Jacobi-Isaacs (HJI) equation. For a survey on mean-field games and applications we refer the reader to [15]. A first attempt to apply mean-field games to demand response is in [4]. Mean-field based control in power systems is studied also in [29] and [30] with focus on energy storage devices and electric water heating loads respectively. Regarding the computational investigation for mean-field game theory, a similar algorithm to the one presented in this paper is presented in [31].

The paper is organized as follows. In Section II we state the problem and introduce the model. In Section III we review some preliminary results. In Section IV we state and discuss the main results. In Section V we carry out some numerical studies. In Section VI we provide some discussion. Finally, in Section VII we provide some conclusions.

## C. Notation

The symbol  $\mathbb{E}$  indicates the expectation operator. We use  $\partial_x$  and  $\partial_{xx}^2$  to denote the first and second partial derivatives with respect to  $x$ , respectively. Given a vector  $x \in \mathbb{R}^n$  and a matrix  $a \in \mathbb{R}^{n \times n}$  we denote by  $\|x\|_a^2$  the weighted two-norm  $x^T a x$ . The symbol  $a_{i\bullet}$  means the  $i$ th row of a given matrix  $a$ . We denote by  $Diag(x)$  the diagonal matrix in  $\mathbb{R}^{n \times n}$  whose entries in the main diagonal are the components of  $x$ . We denote by  $dist(X, X^*)$  the distance between two points  $X$  and  $X^*$  in  $\mathbb{R}^n$ . We denote by  $\Pi_{\mathcal{M}}(X)$  the projection of  $X$  onto set  $\mathcal{M}$ . The symbol “ $\cdot$ ” denotes the Frobenius product. We denote by  $]\xi, \zeta[$  the open interval for any pair of real numbers  $\xi \leq \zeta$ .

## II. POPULATION OF TCLS THROUGH MEAN-FIELD GAMES

In this section, in the spirit of prescriptive game theory [3], we design a mean-field game for the TCLs application, with the aim of incentivizing cooperation among the TCLs through an opportune design of cost functionals, one per each TCL.

Consider a population of hybrid controlled thermostat loads (TCLs) and a time horizon window  $[0, T]$ . Each TCL is

characterized by a continuous state, namely the temperature  $x(t)$ , and a binary state  $\pi_{on}(t) \in \{0, 1\}$ , which represents the condition *on* or *off* at time  $t \in [0, T]$ . When the TCL is set to *on* the temperature decreases exponentially up to a fixed lower temperature  $x_{on}$  whereas in the *off* position the temperature increases exponentially up to a higher temperature  $x_{off}$ . Then, the temperature of each appliance evolves according to the following differential equations, for all  $t \in [0, T]$ :

$$\dot{x}(t) = \begin{cases} -\alpha(x(t) - x_{on}) & \text{if } \pi_{on}(t) = 1 \\ -\beta(x(t) - x_{off}) & \text{if } \pi_{on}(t) = 0 \end{cases}, \quad (1)$$

where the initial state is  $x(0) = x$  and where the rates  $\alpha, \beta$  are given positive scalars.

In accordance with [2], [4] we set the problem in a stochastic framework where each TCL is in one of the two states *on* or *off* with given probabilities  $\pi_{on} \in [0, 1]$  and  $\pi_{off} \in [0, 1]$ . The control variable is the transitioning rate  $u_{on}$  from *off* to *on* and the transitioning rate  $u_{off}$  from *on* to *off*. This is illustrated in the automata in Fig. 1.

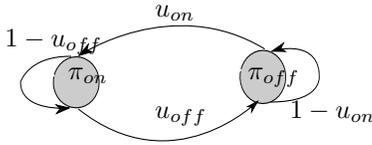


Fig. 1: Automata describing transition rates from *on* to *off* and vice versa.

The corresponding dynamics is then given by

$$\begin{cases} \dot{\pi}_{on}(t) = u_{on}(t) - u_{off}(t), & t \in [0, T], \\ \dot{\pi}_{off}(t) = u_{off}(t) - u_{on}(t), & t \in [0, T], \\ 0 \leq \pi_{on}(t), \pi_{off}(t) \leq 1, & t \in [0, T]. \end{cases} \quad (2)$$

As  $\dot{\pi}_{on}(t) + \dot{\pi}_{off}(t) = 0$ , we can simply consider only one of the above dynamics. Then, let us denote  $y(t) = \pi_{on}(t)$  and introduce a stochastic disturbance in the form of a Brownian motion, denote it  $\mathcal{B}(t)$ , and a deterministic disturbance  $w(t) = [w_1(t) \ w_2]^T$ . For any  $x, y$  in the

$$\text{“set of feasible states” } \mathcal{S} := ]x_{on}, x_{off}[ \times ]0, 1[,$$

the resulting dynamics in a very general form is given by

$$\left\{ \begin{array}{l} dx(t) = \left( y(t) \begin{bmatrix} -\alpha(x(t) - x_{on}) \\ + (1 - y(t)) \begin{bmatrix} -\beta(x(t) - x_{off}) \end{bmatrix} \end{bmatrix} \right. \\ \quad \left. + d_{11}w_1(t) + d_{12}w_2(t) \right) dt + \sigma_{11}(x) d\mathcal{B}(t), \\ =: \left( f(x(t), y(t)) + d_{11}w_1(t) + d_{12}w_2(t) \right) dt \\ \quad + \sigma_{11}(x) d\mathcal{B}(t), \quad t \in [0, T], \\ x(0) = x, \\ dy(t) = \left( u_{on}(t) - u_{off}(t) + d_{21}w_1(t) \right. \\ \quad \left. + d_{22}w_2(t) \right) dt + \sigma_{22}(y) d\mathcal{B}(t) \\ =: \left( g(u(t)) + d_{21}w_1(t) \right. \\ \quad \left. + d_{22}w_2(t) \right) dt + \sigma_2(y) d\mathcal{B}(t), \quad t \in [0, T], \\ y(0) = y, \end{array} \right. \quad (3)$$

where  $\sigma_{ij}$  and  $d_{ij}$ ,  $i, j = 1, 2$  are positive scalar coefficients.

For a mean-field game formulation, consider a probability density function  $m : [x_{on}, x_{off}] \times [0, 1] \times [t, T] \rightarrow [0, +\infty[$ ,  $(x, y, t) \mapsto m(x, y, t)$ , which satisfies  $\int_{x_{on}}^{x_{off}} \int_{[0, 1]} m(x, y, t) dx dy = 1$  for every  $t$ . Let us also define as  $m_{on}(t) := \int_{x_{on}}^{x_{off}} \int_{[0, 1]} y m(x, y, t) dx dy$ . Likewise we denote by  $m_{off}(t) = 1 - m_{on}(t)$ .

At every time  $t$  the mains frequency depends linearly on the discrepancy between the percentage of TCLs in *on* position and a nominal value. We call such a discrepancy as error and denote it by  $e(t) = m_{on}(t) - \bar{m}_{on}$ , where  $\bar{m}_{on}$  is the nominal value (the higher the percentage of TCLs in *on* position with respect to the nominal value, the lower the network frequency). Note that the grid frequency is related to the power mismatch between supply and demand. Here we assume that the power supply is equal to the nominal power consumption all the time.

We then consider the running cost below, which depends on the distribution  $m(x, y, t)$  through the error  $e(t)$ :

$$c(x(t), y(t), u(t), m(x, y, t)) = \frac{1}{2} \left( qx(t)^2 + r_{on}u_{on}(t)^2 + r_{off}u_{off}(t)^2 \right) + y(t)(Se(t) + W), \quad (4)$$

where  $q, r_{on}, r_{off}$ , and  $S$  are opportune positive scalars.

Note that cost (4) includes four terms. The term  $\frac{1}{2}qx(t)^2$  penalizes the deviation of the TCLs' temperature from the nominal value, which we set to zero. Setting the nominal temperature to a nonzero value would simply imply a translation of the origin of the axes. The terms  $\frac{1}{2}r_{on}u_{on}(t)^2$  introduces a cost for fast switching; i.e. this cost is zero when either  $u_{on}(t) = 0$  (no switching) and is maximal when  $u_{on}(t) = 1$  (probability 1 of switching). A similar comment applies to  $\frac{1}{2}r_{off}u_{off}(t)^2$ . A positive error  $e(t) > 0$ , means that demand exceeds supply. Thus, the term  $y(t)Se(t)$  penalizes the appliances that are *on* when demand exceeds supply ( $e(t) > 0$ ). When supply exceeds demand, we have a negative error  $e(t) < 0$ , and the term  $y(t)Se(t)$  penalizes the appliances that are *off*. Finally, the term  $y(t)W$  minimizes the power consumption, i.e., whenever the TCL is *on* the consumption is  $W$ . Also consider a terminal cost  $g : \mathbb{R} \rightarrow [0, +\infty[$ ,  $x \mapsto g(x)$  to be yet designed.

**Problem statement.** Given a finite horizon  $T > 0$  and an initial distribution  $m_0 : [x_{on}, x_{off}] \times [0, 1] \rightarrow [0, +\infty[$ , minimize over  $\mathcal{U}$  and maximize over  $\mathcal{W}$ , subject to the controlled system (3), the cost functional

$$J(x, y, t, u(\cdot), w(\cdot)) = \mathbb{E} \int_0^T (c(x(t), y(t), u(t), m(x, y, t)) - \frac{1}{2}\gamma^2 \|w(t)\|^2) dt + g(X(T)),$$

where  $\gamma$  is a positive scalar,  $\mathcal{U}$  and  $\mathcal{W}$  are the sets of all measurable state feedback closed-loop policies  $u(\cdot) : [0, +\infty[ \rightarrow \mathbb{R}$  respectively, and  $w(\cdot) : [0, +\infty[ \rightarrow \mathbb{R}$  and  $m(\cdot)$  is the time-dependent function describing the evolution of the mean of the distribution of the TCLs' states.

### III. PRELIMINARY RESULTS

This section reviews first- and second-order mean-field games in preparation to apply the game to the problem at hand.

In the first case, the microscopic dynamics is deterministic and the resulting mean-field game involves only the first derivatives of the value function and of the density function. In the second case, the microscopic dynamics is a stochastic differential equation driven by a Brownian motion, which leads to the involvement of second derivatives of the value function and density function. In addition to this, this section specializes the model to the application introduced in the previous section, involving a population of TCLs.

#### A. First- and second-order mean-field games

This section streamlines some preliminary results on mean-field games. Consider a generic cost and dynamics

$$\begin{aligned} J(X, 0, U(\cdot)) &= \inf_{U(\cdot)} \int_{t=0}^T c(X(t), m, U(\cdot)) dt \\ &\quad + g(X(T)), \\ \dot{X}(t) &= F(X(t), U(\cdot)) \text{ in } \mathbb{R}^n, \end{aligned} \quad (5)$$

where  $c(\cdot)$  is the running cost,  $g(X) \forall X \in \text{in } \mathbb{R}^n$  is the terminal penalty, and where  $U(\cdot)$  is any state-feedback closed-loop control policy. Let  $v(X, t)$  be the value function, i.e., the optimal value of  $J(X, t, U(\cdot))$ . Then from [19] it is well-known that the problem results in the following mean-field game system

$$\begin{cases} -\partial_t v(X, t) - F(X, U^*(X)) \partial_X v(X, t) \\ -c(X, m, U^*(X)) = 0 \text{ in } \mathbb{R}^n \times ]0, T], & \text{(a)} \\ v(X, T) = g(X) \forall X \in \text{in } \mathbb{R}^n, & \text{(b)} \\ U^*(X, t) = \operatorname{argmax}_{U \in \mathbb{R}} \{-F(X, U) \\ \cdot \partial_X v(X, t) - c(X, m, U)\}, & \text{(b)} \end{cases} \quad (6)$$

$$\begin{cases} \partial_t m(X, t) + \operatorname{div}(F(X, U^*(X))m(X, t)) = 0 \\ \text{in } \mathbb{R}^n \times ]0, T], \\ m(X, 0) = m_0(X), \forall X \in \text{in } \mathbb{R}^n. \end{cases} \quad (7)$$

The partial differential equation (PDE) 6 (a) is the Hamilton-Jacobi-Bellman equation which returns the value function  $v(X, t)$  once we fix the distribution  $m(X, t)$ ; This PDE has to be solved backwards with boundary conditions at final time  $T$ , represented by the last line in 6 (a). In 6 (b) we have the optimal closed-loop control  $U^*(X, t)$  as maximizer of the Hamiltonian function in the RHS. The PDE (7) represents the transport equation of the measure  $m$  immersed in a vector field  $F(X, U^*(X))$ ; It returns the distribution  $m(X, t)$  once fixed the optimal closed-loop control  $U^*(X, t)$  and consequently the vector field  $F(X, U^*(X))$ . Such a PDE has to be solved forwards with boundary condition at the initial time (see the last line of (7)).

In a second-order mean-field game, the dynamics is a stochastic differential equation driven by a Brownian motion, and the cost function is considered through its expected value, namely,

$$\begin{aligned} J(X, 0, U(\cdot)) &= \inf_{U(\cdot)} \mathbb{E} \int_{t=0}^T c(X(t), m, U(X(t))) dt + g(X(T)) \\ dX(t) &= F(X(t), U(\cdot)) dt + \sigma(X) dB(t) \text{ in } \mathbb{R}^n, \end{aligned} \quad (8)$$

where  $B(t) \in \mathbb{R}^n$  is the Brownian motion and  $\sigma(X) \in \mathbb{R}^{n \times n}$  is the coefficient matrix.

From [19] the second-order mean-field game system is then given by

$$\begin{cases} -\partial_t v(X, t) - F(X, U^*(X)) \partial_X v(X, t) \\ -c(X, m, U^*(X)) \\ -\frac{1}{2} \sigma(X) \sigma(X)^T : \partial_{XX} v(X, t) = 0 \\ \text{in } \mathbb{R}^n \times ]0, T], \\ v(X, T) = g(X) \forall X \in \text{in } \mathbb{R}^n, & \text{(a)} \\ U^*(X, t) = \operatorname{argmax}_{U \in \mathbb{R}} \{-F(X, U) \\ \cdot \partial_X v(X, t) - c(X, m, U)\}, & \text{(b)} \end{cases} \quad (9)$$

$$\begin{cases} \partial_t m(X, t) + \operatorname{div}(F(X, U^*(X))m(X, t)) \\ -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{X_i X_j}^2 (\tilde{\sigma}_{ij} m(X, t)) = 0 \\ \text{in } \mathbb{R}^n \times ]0, T], \\ m(X, 0) = m_0(X), \forall X \in \text{in } \mathbb{R}^n, \end{cases} \quad (10)$$

where the symbol “ $\cdot$ ” denotes the Frobenius product and  $\tilde{\sigma}_{ij} = \sum_{k=1}^n \sigma_{ik}(X) \sigma_{jk}(X)$ .

In a second-order mean-field game the Hamilton-Jacobi-Bellman equation, as in 9 (a), involves the second-order derivatives of the value function in the additional term represented by the Frobenius product; Likewise, also the transport equation as in (10) involves the second-order derivatives of the density function. The rest of the system is similar to the first-order case. Let us now specialize the above model to the TCLs application introduced in the previous section.

#### B. Mean-field game for the TCL application

Specializing to our TCLs application, let  $v(x, y, m, t)$  be the value function, i.e., the optimal value of  $J(x, y, t, u(\cdot))$ . Let us denote by

$$k(x(t)) = x(t)(\beta - \alpha) + (\alpha x_{on} - \beta x_{off}).$$

Then, the problem at hand can be rewritten in terms of the state, control and disturbance vectors

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_{on}(t) \\ u_{off}(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

and yields the linear quadratic problem:

$$\begin{aligned} \inf_{u(\cdot)} \sup_{w(\cdot)} \mathbb{E} \int_0^T & \left[ \frac{1}{2} (\|X(t)\|_Q^2 + \|u(t)\|_R^2 - \gamma^2 \|w(t)\|^2) \right. \\ & \left. + L^T X(t) \right] dt + g(X(T)), \\ dX(t) &= (AX(t) + Bu(t) + C + Dw(t)) dt \\ & \quad + \Sigma dB(t), \text{ in } \mathcal{S} \end{aligned} \quad (11)$$

where

$$\begin{aligned} Q &= \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, & R &= r = \begin{bmatrix} r_{on} & 0 \\ 0 & r_{off} \end{bmatrix}, \\ L(e) &= \begin{bmatrix} 0 \\ Se + W \end{bmatrix}, & A(x) &= \begin{bmatrix} -\beta & k(x) \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, & C &= \begin{bmatrix} \beta x_{off} \\ 0 \end{bmatrix}, \\ D &= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, & \Sigma &= \begin{bmatrix} \sigma_{11}(x) & 0 \\ 0 & \sigma_{22}(y) \end{bmatrix}. \end{aligned} \quad (12)$$

The resulting mean-field game is given by

$$\left\{ \begin{array}{l} \partial_t \mathcal{V}_t(X) + \inf_u \sup_w \left\{ \partial_X \mathcal{V}_t(X)^T \right. \\ \cdot (AX + Bu + C + Dw) + \frac{1}{2} \left( \|X\|_Q^2 \right. \\ \left. + \|u\|_R^2 - \gamma^2 \|w\|^2 \right) + L^T X \left. \right\} \\ + \frac{1}{2} (\sigma_{11}(x))^2 \partial_{xx} v(X, t) \\ + \sigma_{22}(y)^2 \partial_{yy} v(X, t) = 0, \text{ in } \mathcal{S} \times [0, T[, \\ v(X, T) = g(X), \text{ in } \mathcal{S} \end{array} \right. \quad (a) \quad (13)$$

$$\left\{ \begin{array}{l} u^*(x, t) = \operatorname{argmin}_{u \in \mathbb{R}} \left\{ \partial_X \mathcal{V}_t(X)^T \right. \\ \cdot (AX + Bu + C + Dw^*) + \frac{1}{2} \|u(t)\|_R^2 \left. \right\}, \\ w^*(x, t) = \operatorname{argmax}_{w \in \mathbb{R}} \left\{ \partial_X \mathcal{V}_t(X)^T \right. \\ \cdot (AX + Bu^* + C + Dw) - \frac{1}{2} \gamma^2 \|w(t)\|^2 \left. \right\} \end{array} \right. \quad (b)$$

and

$$\left\{ \begin{array}{l} \partial_t m(x, y, t) + \operatorname{div}[(AX + Bu \\ + C + Dw) m(x, y, t)] \\ - \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \partial_{X_i X_j}^2 (\tilde{\sigma}_{ij} m(X, t)) = 0 \\ \text{in } \mathcal{S} \times ]0, T[, \\ m(x_{on}, y, t) = m(x_{off}, y, t) = 0 \\ \forall y \in [0, 1], t \in [0, T], \\ m(x, y, 0) = m_0(x, y) \\ \forall x \in [x_{on}, x_{off}], y \in [0, 1] \\ \int_{x_{on}}^{x_{off}} \int_{[0,1]} m(x, y, t) dx dy = 1 \forall t \in [0, T], \end{array} \right. \quad (14)$$

where  $\tilde{\sigma}_{ij} = \sum_{k=1}^n \sigma_{ik}(X) \sigma_{jk}(X)$ .

Essentially, the partial differential equation (PDE) (13) (a) is the Hamilton-Jacobi-Isaacs equation which returns the value function  $v(x, y, m, t)$  once we fix the distribution  $m(x, y, t)$ ; This PDE has to be solved backwards with boundary conditions at final time  $T$ , represented by the last line in 13 (a). In 13 (b) we have the optimal closed-loop control  $u^*(x, t)$  and worst-case disturbance  $w^*(x, t)$  as minimaximizers of the Hamiltonian function in the RHS. The PDE (14) represents the transport equation of the measure  $m$  immersed in a vector field  $AX + Bu + C + Dw$ ; It returns the distribution  $m(x, y, t)$  once fixed both  $u^*(x, t)$  and  $w^*(x, t)$  and consequently the vector field  $AX + Bu^* + C + Dw^*$ . Such a PDE has to be solved forwards with boundary condition at the initial time (see the fourth line of (14)). Finally, once given  $m(x, y, t)$  from (c) and entered into the running cost  $c(x, y, m, u)$  in (a), we obtain the error

$$\left\{ \begin{array}{l} m_{on}(t) := \int_{x_{on}}^{x_{off}} \int_{[0,1]} y m(x, y, t) dx dy \\ \forall t \in [0, T], \\ e(t) = m_{on}(t) - \bar{m}_{on}. \end{array} \right. \quad (15)$$

Note that

$$\begin{aligned} \bar{X}(t) &= \begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix} = \begin{bmatrix} \bar{x}(t) \\ m_{on} \end{bmatrix} \\ &= \begin{bmatrix} \int_{x_{on}}^{x_{off}} \int_{[0,1]} x m(x, y, t) dx dy \\ \int_{x_{on}}^{x_{off}} \int_{[0,1]} y m(x, y, t) dx dy \end{bmatrix}, \end{aligned}$$

and therefore, henceforth we can refer to as mean-field equilibrium solutions any pair  $(v(X, t), \bar{X}(t))$  which is solution of (13)-(14).

## IV. MAIN RESULTS

The contribution of this paper to the TCLs application introduced earlier is three-fold. First, we analyze and compute the mean-field equilibrium for the deterministic mean-field game and we prove that under certain conditions the microscopic dynamics is asymptotically stable. We repeat the analysis for the stochastic case, assuming that the microscopic dynamics is uncertain. Even for this case, a mean-field equilibrium is computed, and stochastic stability is studied. We distinguish two cases. In the first case, we consider a state dependent stochastic disturbance which vanishes around the origin. The Brownian motion coefficients are linear in the state and the resulting dynamics is also known as geometric Brownian motion. In the second case, we take the stochastic disturbance being independent on the state. The Brownian motion coefficients are constant and the resulting dynamics mirrors the Langevin equation. In both cases we prove stochastic stability of second-moment for the stochastic process at hand. This section ends with a detailed analysis of robustness properties. The microscopic dynamics is now subject to an additional exogenous input, the disturbance, with bounded energy. We conclude our study by obtaining the mean-field equilibrium and investigating conditions that guarantee stability even in the presence of such a disturbance.

### A. Mean-field equilibrium and stability

In this section we establish an explicit solution in terms of mean-field equilibrium for the deterministic case and study stability of the microscopic dynamics. This case is obtained by fixing to zero the coefficients of both stochastic and adversarial disturbance.

The linear quadratic problem we wish to solve is then:

$$\begin{aligned} \inf_{u(\cdot)} \int_0^T & \left[ \frac{1}{2} \left( X(t)^T Q X(t) + u(t)^T R u(t)^T \right) \right. \\ & \left. + L^T X(t) \right] dt + g(X(T)), \\ \dot{X}(t) &= AX(t) + Bu(t) + C \text{ in } \mathcal{S}. \end{aligned} \quad (16)$$

The next result shows that the problem reduces to solving three matrix equations.

**Theorem 1: (Mean-field equilibrium)** Let  $D, \Sigma = 0$  in the game (13)-(14). A mean-field equilibrium for (13)-(14) is given by

$$\left\{ \begin{array}{l} v(X, t) = \frac{1}{2} X^T P(t) X + \Psi(t)^T X + \chi(t), \\ \dot{\bar{X}}(t) = [A(x) - BR^{-1}B^T P] \bar{X}(t) \\ - BR^{-1}B^T \bar{\Psi}(t) + C, \end{array} \right. \quad (17)$$

where

$$\left\{ \begin{array}{l} \dot{P} + PA(x) + A(x)^T P - PBR^{-1}B^T P + Q = 0 \\ \text{in } [0, T[, P(T) = \phi, \\ \dot{\Psi} + A(x)^T \Psi + PC - PBR^{-1}B^T \Psi + L = 0 \\ \text{in } [0, T[, \Psi(T) = 0, \\ \dot{\chi} + \Psi^T C - \frac{1}{2} \Psi^T BR^{-1}B^T \Psi = 0 \text{ in } [0, T[, \\ \chi(T) = 0, \end{array} \right. \quad (18)$$

and  $\bar{\Psi}(t) = \int_{x_{on}}^{x_{off}} \int_{[0,1]} \Psi(t) m(x, y, t) dx dy$  and where the boundary conditions are obtained by imposing that

$$v(X, T) = \frac{1}{2} X^T P(T) X + \Psi(T)^T X + \chi(T) = \frac{1}{2} X^T \phi X =: g(X).$$

Furthermore, the mean-field equilibrium strategy is

$$u^*(X, t) = -R^{-1}B^T[PX + \Psi]. \quad (19)$$

**Proof.** Given in the appendix.  $\square$

Let us note that by substituting the mean-field equilibrium strategies  $u^* = -R^{-1}B^T[PX + \Psi]$  given in (19) in the open-loop microscopic dynamics  $\dot{X}(t) = AX(t) + Bu(t) + C$  as defined in (16), the closed-loop microscopic dynamics is

$$\begin{aligned} \dot{X}(t) &= [A(x) - BR^{-1}B^T P]X(t) \\ &\quad - BR^{-1}B^T \Psi(x, e, t) + C. \end{aligned} \quad (20)$$

In the above, and occasionally in the following, we highlight the dependence of  $\Psi$  on  $x$ ,  $e$ , and  $t$ . Such a dependence is shown in the proof of Theorem 1. Now, let  $\mathcal{X}$  be the set of equilibrium points for (20), namely, the set of  $X$  such that

$$\mathcal{X} = \{(X, e) \in \mathbb{R}^2 \times \mathbb{R} \mid [A(x) - BR^{-1}B^T P]X(t) - BR^{-1}B^T \Psi(x, e, t) + C = 0\},$$

and let  $V(X, t) = \text{dist}(X, \mathcal{X})$ . The next result establishes a condition under which the above dynamics converges asymptotically to the set of equilibrium points.

*Corollary 1: (Asymptotic stability)* If it holds

$$\begin{aligned} \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P]X(t) \right. \\ \left. - BR^{-1}B^T \Psi + C \right) < -\|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \end{aligned} \quad (21)$$

then dynamics (20) is asymptotically stable, namely,  $\lim_{t \rightarrow \infty} V(X(t)) = 0$ .

**Proof.** Given in the appendix.  $\square$

## B. Stochastic case

In this section we study the case where the dynamics is given by a stochastic differential equation driven by a Brownian motion. In other words, the model is uncertain and the uncertainty is described by a stochastic disturbance.

The problem at hand is then:

$$\begin{aligned} \inf_{u(\cdot)} \mathbb{E} \int_0^T \left[ \frac{1}{2} \left( X(t)^T Q X(t) + u(t)^T R u(t)^T \right) \right. \\ \left. + L^T X(t) \right] dt + g(X(T)), \end{aligned} \quad (22)$$

$$dX(t) = (AX(t) + Bu(t) + C)dt + \Sigma dB_t,$$

where all matrices are as in (12) and

$$\Sigma = \begin{bmatrix} \sigma_{11}(x) & 0 \\ 0 & \sigma_{22}(y) \end{bmatrix}.$$

This section investigates on the solution of the HJI equation under the assumption that the time evolution of the common state is given. We show that the problem reduces to solving three matrix equations. To see this, by isolating the HJI part of (13) for fixed  $m_t$ , for  $t \in [0, T]$ , we have

$$\begin{cases} -\partial_t v(X, t) - \sup_u \left\{ -\partial_X v(X, t)^T (AX + Bu + C) \right. \\ \left. - \frac{1}{2} \left( X^T Q X - u^T R u \right) - L^T X \right\} + \frac{1}{2} (\sigma_{11}(x))^2 \partial_{xx} v(X, t) \\ \quad + \sigma_{22}(y)^2 \partial_{yy} v(X, t) = 0, \text{ in } \mathcal{S} \times [0, T], \\ v(X, T) = g(X) \text{ in } \mathcal{S}, \\ u^*(x, t) = -R^{-1}B^T \partial_y v(X, t). \end{cases}$$

Let us consider the following value function

$$v(X, t) = \frac{1}{2} X^T P(t) X + \Psi(t)^T X + \chi(t),$$

and

$$u^* = -R^{-1}B^T[PX + \Psi],$$

so that (13) can be rewritten as

$$\begin{cases} \frac{1}{2} X^T \dot{P}(t) X + \dot{\Psi}(t)^T X + \dot{\chi}(t) \\ \quad + (P(t) X + \Psi(t))^T \left[ -BR^{-1}B^T \right. \\ \quad \quad \left. \cdot (P(t) X + \Psi(t)) \right. \\ \quad \quad \left. + (P(t) X + \Psi(t))^T (AX + C) \right. \\ \quad \left. + \frac{1}{2} \left( X(t)^T Q X(t) + u(t)^T R u(t)^T \right) \right. \\ \quad \quad \left. + L^T X(t) + \frac{1}{2} (\sigma_{11}(x))^2 P_{11}(t) \right. \\ \quad \quad \left. + \sigma_{22}(y)^2 P_{22}(t) \right) = 0 \text{ in } \mathcal{S} \times [0, T], \\ P(T) = \phi, \quad \Psi(T) = 0, \quad \chi(T) = 0. \end{cases} \quad (23)$$

The boundary conditions are obtained by imposing that

$$v(X, T) = \frac{1}{2} X^T P(T) X + \Psi(T)^T X + \chi(T) = \frac{1}{2} X^T \phi X =: g(X).$$

1) *Case I: state dependent variance:* The first case we consider involves coefficients for the Brownian motion which are linear in the state, namely for given positive  $\hat{\sigma}_{11}$  and  $\hat{\sigma}_{22}$

$$\Sigma(X) = \begin{bmatrix} \hat{\sigma}_{11} x & 0 \\ 0 & \hat{\sigma}_{22} y \end{bmatrix}. \quad (24)$$

**Theorem 2: (Stochastic mean-field equilibrium: case I)** A mean-field equilibrium for the game (13)-(14) with  $\Sigma(X)$  as in (24) is given by

$$\begin{cases} v(X, t) = \frac{1}{2} X^T P(t) X + \Psi(t)^T X + \chi(t), \\ \dot{X}(t) = [A - BR^{-1}B^T P] \bar{X}(t) \\ - BR^{-1}B^T \Psi(\bar{X}(t)) + C, \end{cases} \quad (25)$$

where

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P - PBR^{-1}B^T P \\ \quad + Q + \tilde{P} = 0 \text{ in } [0, T], \quad P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC - PBR^{-1}B^T \Psi \\ \quad + L = 0 \text{ in } [0, T], \quad \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C - \frac{1}{2} \Psi^T BR^{-1}B^T \Psi = 0 \\ \quad \text{in } [0, T], \quad \chi(T) = 0, \end{cases} \quad (26)$$

and

$$\tilde{P} = \text{Diag}((\hat{\sigma}_{ii}^2 P_{ii})_{i=1,2}) = \begin{bmatrix} \hat{\sigma}_{11}^2 P_{11} & 0 \\ 0 & \hat{\sigma}_{22}^2 P_{22} \end{bmatrix}. \quad (27)$$

Furthermore, the mean-field equilibrium strategy is

$$u^* = -R^{-1}B^T[PX + \Psi] \quad (28)$$

**Proof.** Given in the appendix.  $\square$

Based on the above result, let us now substitute the expression of the mean-field equilibrium strategy  $u^* = -R^{-1}B^T[PX + \Psi]$  as in (28) in the open-loop microscopic dynamics  $dX(t) = (AX(t) + Bu(t) + C)dt + \Sigma dB(t)$  given in (22) to obtain the closed-loop microscopic dynamics

$$\begin{aligned} dX(t) &= \left[ (A(x) - BR^{-1}B^T P)X(t) \right. \\ &\quad \left. - BR^{-1}B^T \Psi + C \right] dt + \Sigma dB(t). \end{aligned} \quad (29)$$

Now, let  $\mathcal{X}$  be the set of equilibrium points for (29), namely, the set of  $X$  such that

$$\mathcal{X} = \{(X, e) \in \mathbb{R}^2 \times \mathbb{R} \mid (A(x) - BR^{-1}B^T P)X(t) - BR^{-1}B^T \Psi + C = 0\}, \quad (30)$$

and let  $V(X, t) = \text{dist}(X, \mathcal{X})$ . We are in a position to state conditions under which the distance from the set of equilibrium points has bounded variance.

**Corollary 2: (2nd moment boundedness)** Let a compact set  $\mathcal{M} \subset \mathbb{R}^2$  be given. Suppose that for all  $X \notin \mathcal{M}$

$$\begin{aligned} & \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P]X(t) \right. \\ & \left. - BR^{-1}B^T \Psi + C \right) \\ & < -\frac{1}{2}(\sigma_{11}^2(x)\partial_{xx}V(X, t) + \sigma_{22}^2(x)\partial_{yy}V(X, t)) \end{aligned} \quad (31)$$

then dynamics (29) is a stochastic process and the distance  $V(X(t))$  is 2nd moment bounded.

**Proof.** Given in the appendix.  $\square$

2) *Case II: state independent variance and Langevin equation:* The second case we consider involves coefficients for the Brownian motion which are constant, namely

$$\Sigma = \begin{bmatrix} \hat{\sigma}_{11} & 0 \\ 0 & \hat{\sigma}_{22} \end{bmatrix}. \quad (32)$$

**Theorem 3: (Stochastic mean-field equilibrium: case II)**

Let  $\Sigma$  be as in (32). A mean-field equilibrium for the game (13)-(14) is given by

$$\begin{cases} v(X, t) = \frac{1}{2}X^T P(t)X + \Psi(t)^T X + \chi(t), \\ \dot{\bar{X}}(t) = [A - BR^{-1}B^T P]\bar{X}(t) \\ - BR^{-1}B^T \Psi(\bar{X}(t)) + C, \end{cases} \quad (33)$$

where

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P - PBR^{-1}B^T P \\ + Q = 0 \text{ in } [0, T[, P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC - PBR^{-1}B^T \Psi \\ + L = 0 \text{ in } [0, T[, \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C - \frac{1}{2}\Psi^T BR^{-1}B^T \Psi \\ + \dot{P} = 0 \text{ in } [0, T[, \chi(T) = 0, \end{cases} \quad (34)$$

and

$$\tilde{P} = \begin{bmatrix} \hat{\sigma}_{11}^2 & 0 \\ 0 & \hat{\sigma}_{22}^2 \end{bmatrix}. \quad (35)$$

Furthermore, the mean-field equilibrium strategies are given by

$$u^*(X, t) = -R^{-1}B^T [PX + \Psi]. \quad (36)$$

**Proof.** Given in the appendix.  $\square$

Based on the above result, let us now substitute the expression of the mean-field equilibrium strategy  $u^* = -R^{-1}B^T [PX + \Psi]$  as in (36) in the open-loop microscopic dynamics  $dX(t) = (AX(t) + Bu(t) + C)dt + \Sigma dB(t)$  given in (22) to obtain the closed-loop microscopic dynamics

$$\begin{aligned} dX(t) = & \left[ (A(x) - BR^{-1}B^T P)X(t) \right. \\ & \left. - BR^{-1}B^T \Psi + C \right] dt + \Sigma dB(t). \end{aligned} \quad (37)$$

Now, let  $\mathcal{X}$  be the set of equilibrium points for (37), namely, the set of  $X$  such that

$$\mathcal{X} = \{(X, e) \in \mathbb{R}^2 \times \mathbb{R} \mid (A(x) - BR^{-1}B^T P)X(t) - BR^{-1}B^T \Psi + C = 0\}, \quad (38)$$

and let  $V(X, t) = \text{dist}(X, \mathcal{X})$ . The next result establishes a condition under which the distance from the set of equilibrium points is 2nd moment bounded.

**Corollary 3: (2nd moment boundedness)** Let a compact set  $\mathcal{M} \subset \mathbb{R}^2$  be given. Suppose that for all  $X \notin \mathcal{M}$

$$\begin{aligned} & \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P]X(t) \right. \\ & \left. - BR^{-1}B^T \Psi + C \right) \\ & < -\frac{1}{2}(\hat{\sigma}_{11}^2 \partial_{xx}V(X, t) + \hat{\sigma}_{22}^2 \partial_{yy}V(X, t)) \end{aligned} \quad (39)$$

then dynamics (37) is a stochastic process and  $V(X(t))$  is 2nd moment bounded.

**Proof.** Given in the appendix.  $\square$

### C. Model misspecification

This section deals with model misspecification, which is represented by an additional exogenous and adversarial disturbance. The disturbance is supposed to be of bounded energy. Thus, the linear quadratic problem we wish to solve is:

$$\begin{aligned} & \inf_{u(\cdot)} \sup_{w(\cdot)} \mathbb{E} \int_0^T \left[ \frac{1}{2} \left( X(t)^T Q X(t) + u(t)^T R u(t) \right. \right. \\ & \left. \left. - \gamma^2 w(t)^T w(t) \right) + L^T X(t) \right] dt + g(X(T)), \\ & \dot{X}(t) = AX(t) + Bu(t) + C + Dw(t) \text{ in } \mathcal{S}. \end{aligned} \quad (40)$$

This section investigates the solution of the HJI equation under the assumption that the time evolution of the common state is given. We show that the problem reduces to solving three matrix equations. To see this, by isolating the HJI part of (13) for fixed  $m_t$ , for  $t \in [0, T]$ , we have the following result.

**Theorem 4: (Worst-case mean-field equilibrium)**

A mean-field equilibrium for (13)-(14) is given by

$$\begin{cases} v(X, t) = \frac{1}{2}X^T P(t)X + \Psi(t)^T X + \chi(t), \\ \dot{\bar{X}}(t) = [A - BR^{-1}B^T P]\bar{X}(t) \\ - BR^{-1}B^T \Psi(\bar{X}(t)) + C, \end{cases} \quad (41)$$

where

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P + P(-BR^{-1}B^T \\ + \frac{1}{\gamma^2}DD^T)P + Q = 0 \text{ in } [0, T[, P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC + (-BR^{-1}B^T \\ + \frac{1}{\gamma^2}DD^T)\Psi + L = 0 \text{ in } [0, T[, \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C + \frac{1}{2}\Psi^T (-BR^{-1}B^T \\ + \frac{1}{\gamma^2}DD^T)\Psi = 0 \text{ in } [0, T[, \chi(T) = 0. \end{cases} \quad (42)$$

Furthermore, the mean-field equilibrium control and disturbance are

$$\begin{aligned} u^* &= -R^{-1}B^T [PX + \Psi], \\ w^* &= \frac{1}{\gamma^2}D^T [PX + \Psi]. \end{aligned} \quad (43)$$

**Proof.** Given in the appendix.  $\square$

Note that by substituting the mean-field equilibrium strategies  $u^* = -R^{-1}B^T [PX + \Psi]$  and  $w^* = \frac{1}{\gamma^2}D^T [PX + \Psi]$  as given in (43) in the open-loop microscopic dynamics

$\alpha$	$\beta$	$x_{on}$	$x_{off}$	$r_{on}, r_{off}$	$q$	$std(m_0)$	$\bar{m}_0$
1	1	-10	10	10	1	1	0

TABLE I: Simulation parameters

$\dot{X}(t) = AX(t) + Bu(t) + C + Dw$  as defined in (40), the closed-loop microscopic dynamics is

$$\dot{X}(t) = [A(x) + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)P]X(t) + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)\Psi + C. \quad (44)$$

Now, let  $\mathcal{X}$  be the set of equilibrium points for (44), namely, the set of  $X$  such that

$$\mathcal{X} = \{(X, e) | [A(x) + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)P]X(t) + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)\Psi + C = 0\}, \quad (45)$$

and let  $V(X, t) = dist(X, \mathcal{X})$ . The next result establishes a condition under which the above dynamics converges asymptotically to the set of equilibrium points.

*Corollary 4: (Worst-case stability)* If it holds

$$\begin{aligned} & \partial_X V(X, t)^T \left( [A + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)P]X(t) \right. \\ & \left. + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)\Psi + C \right) \\ & < -\|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \end{aligned} \quad (46)$$

then dynamics (44) is asymptotically stable, namely,  $\lim_{t \rightarrow \infty} dist(X(t), \mathcal{X}) = 0$ .

**Proof.** Given in the appendix.  $\square$

## V. NUMERICAL STUDIES

Consider a system consisting of  $n = 10^2$  TCLs. The size of the population is large enough to highlight mass interaction. Simulations are carried out with MATLAB on an Intel(R) Core(TM)2 Duo, CPU P8400 at 2.27 GHz and a 3GB of RAM. The number of iterations is  $T = 30$ . Consider a discrete time version of (16)

$$X(t + dt) = X(t) + (A(x(t))X(t) + Bu(t) + C)dt. \quad (47)$$

The parameter are as shown in Table I and in particular the step size  $dt = 0.1$ , the cooling and heating rates are  $\alpha = \beta = 1$ , the lowest and highest temperatures are  $x_{on} = -10$ , and  $x_{off} = 10$ , respectively, the penalty coefficients are  $r_{on} = r_{off} = 1$ , and  $q = 1$ , and the initial distribution is normal with zero mean and standard deviation  $std(m(0)) = 1$ .

The numerical results are obtained using the algorithm in Table II for a discretized set of states.

The optimal control is taken as

$$u^* = -R^{-1}B^T[PX + \Psi],$$

where  $P$  is obtained from running the MATLAB command `[P]=care(A,B,Q,R)`, which receives the matrices as input and returns the solution  $P$  to the algebraic Riccati equation. Assuming  $BR^{-1}B^T\Psi \approx C$  we get the closed-loop dynamics

$$X(t + dt) = X(t) + [A - BR^{-1}B^T P]X(t)dt.$$

Figure 2 displays the time plot of the state of each TCL, namely its temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row). The TCLs show a stable behavior. The simulation is

**Input:** Set of parameters as in Table I.

**Output:** TCLs' states  $X(t)$

```

1 : Initialize. Generate  $X(0)$  given  $\bar{m}_0$  and  $std(m_0)$ 
2 : for time  $iter = 0, 1, \dots, T - 1$  do
3 :   if  $iter > 0$ , then compute  $m_t, \bar{m}_t$ , and  $std(m_t)$ 
4 :   end if
5 :   for player  $i = 1, \dots, n$  do
6 :     Set  $t = iter \cdot dt$  and
       compute control  $\tilde{u}(t)$  using current  $\bar{m}(t)$ 
7 :     compute  $X(t + dt)$  from (47)
8 :   end for
9 : end for
10 : STOP

```

TABLE II: Simulation algorithm

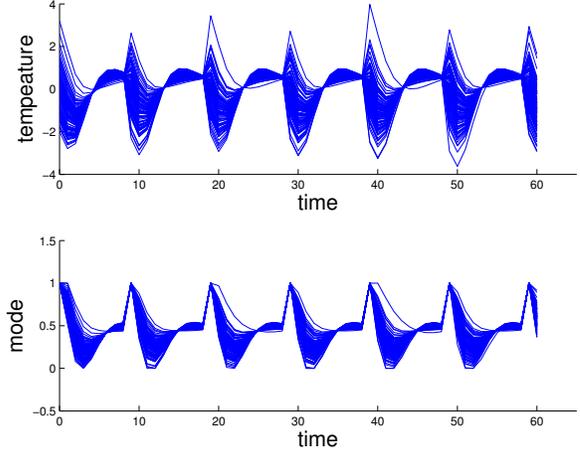


Fig. 2: Time plot of the state of each TCL, namely temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row) in the deterministic case.

carried out assuming that any 10 seconds the states are subject to an impulse. The TCLs react to the impulse fast and converge to the equilibrium point before a new impulse is activated.

We repeat the simulation for the two stochastic cases discussed earlier. Figure 3 displays the plot of the state of each TCL, i.e. its temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row) in the first case. The TCLs react to the impulse and converge to the equilibrium before a new impulse is activated. The Brownian motion enlarges the domain of attraction.

The experiment is repeated in Figure 4 for the geometric Brownian motion. As in the previous cases the plot displays the state of each TCL, i.e. its temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row) in the first case. As the Brownian motion is not weighted by the state, its effects are attenuated and the plot is more similar to the one in Fig. 2.

Note that except for the Langevin-type dynamics, in the remainder two cases the TCLs states are driven to zero. For the Langevin-type dynamics the state is confined within a neighborhood of zero.

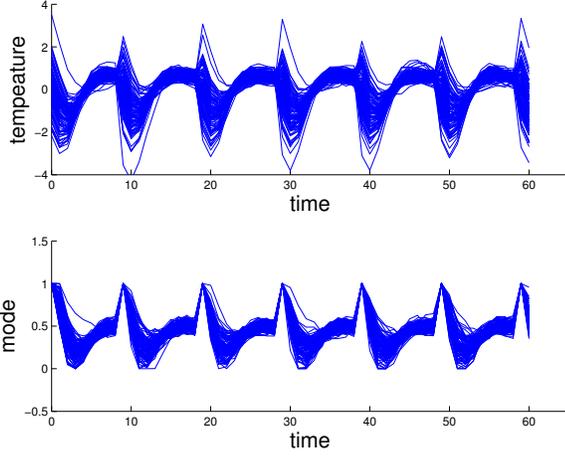


Fig. 3: Time plot of temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row) of each TCL in the stochastic case with state dependent variance.

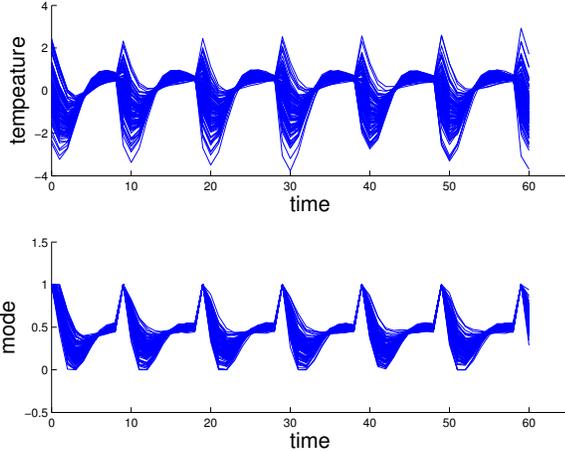


Fig. 4: Time plot of temperature  $x(t)$  (top row) and mode  $y(t)$  (bottom row) of each TCL in the stochastic case with state independent variance.

## VI. DISCUSSION

The topic of dynamic response has sparked attention from different disciplines. This is witnessed by the rapid growing of publications in different areas, from differential games [4], [11] to control and optimization [2], [10], [21], [22], [23], to computer science [26]. Actually, dynamic response intersects research programs in *smart buildings* and *smart cities*. The problem is relevant due to an increasing size of the systems and the consequent difficulties arising when centralizing the management.

The results of this paper are relevant for the following reasons. First, the game-theoretic approach presented here is a natural way to deal with large scale, distributed systems where no central planner can process all the information data. One way to deal with this issue, which is the main idea of dynamic demand, aims at assigning part of the regulation burden to

the consumers by using frequency responsive appliances. In other words, each appliance regulates automatically and in a decentralized fashion its power demand based on the mains frequency. In this respect, the provided model builds upon the strategic interaction among the electrical appliances. The model suits the case where the appliances are numerous and indistinguishable. Indistinguishable means that any appliance in the same condition will react in the same way. Indistinguishability is not a limitation, as in the case of heterogeneity of the electrical appliances, multi-population models may be derived based on the same approach used here.

The results provided in this paper shed light on the existence of mean-field equilibrium solutions. By this we mean strategies based on the forecasted demand, which attenuate mains frequency oscillations. Such strategies are stochastic, namely the TCL sets a probability with which to switch *on* or *off*. Stochastic linear strategies are designed as closed-loop strategies on the current state, temperature and switching mode. Such strategies are computed over a finite horizon and therefore are based on forecasted demand. The mean-field equilibrium strategies represent the asymptotic limit of Nash equilibrium strategies, and as such they are the best-response strategies of each player, for fixed behavior of the other players. The proven stability of the microscopic dynamics confirms the asymptotic convergence of the TCLs's states to an equilibrium, in terms of temperature and switching mode. The cases studied in the paper have shown that the strategies are robust as convergence occurs also with imperfect models. In the case of imperfect modeling, model misspecifications are considered both in a stochastic and deterministic scenario.

## VII. CONCLUDING REMARKS

We have developed a model based on mean-field games for a population of thermostatically controlled loads. The model integrates both stochastic or deterministic disturbances. We have studied robust equilibria and designed stabilizing stochastic control strategies.

Within the realm of mean-field games, we can extend our study in at least three directions. These include i) the analysis of the interplay between dynamic pricing and demand response, ii) the study of the benefits associated with coalitional aggregation of a large number of power producers, and iii) the design of incentives to stabilize aggregation of producers.

## APPENDIX

### Proof of Theorem 1

Let us start by isolating the HJI part of (13). For fixed  $m_t$  and for  $t \in [0, T]$ , we have

$$\begin{cases} -\partial_t v(x, y, t) - \left\{ y \left[ -\alpha(x - x_{on}) \right] \right. \\ \left. + (1 - y) \left[ -\beta(x - x_{off}) \right] \right\} \partial_x v(x, y, t) \\ + \sup_{u \in \mathbb{R}} \left\{ -Bu \partial_y v(x, y, t) - \frac{1}{2} q x^2 \right. \\ \left. + \frac{1}{2} u^T r u + y(Se + W) \right\} = 0 \\ \text{in } \mathcal{S} \times ]0, T], \\ v(x, y, T) = g(x) \text{ in } \mathcal{S}, \\ u^*(x, t) = -r^{-1} B^T \partial_y v(x, y, t), \end{cases} \quad (48)$$

which in a more compact form can be rewritten as

$$\begin{cases} -\partial_t v(X, t) - \sup_u \left\{ \partial_X v(X, t)^T (AX + Bu + C) \right. \\ \left. + \frac{1}{2} \left( X^T QX + u^T Ru^T \right) + L^T X \right\} = 0, \text{ in } \mathcal{S} \times [0, T[, \\ v(X, T) = g(X) \text{ in } \mathcal{S}, \\ u^*(x, t) = -r^{-1} B^T \partial_y v(X, t). \end{cases}$$

Let us consider the following value function

$$v(X, t) = \frac{1}{2} X^T P(t) X + \Psi(t)^T X + \chi(t),$$

and the corresponding optimal closed-loop state feedback strategy

$$u^* = -R^{-1} B^T [PX + \Psi].$$

Then (48) can be rewritten as

$$\begin{cases} \frac{1}{2} X^T \dot{P}(t) X + \dot{\Psi}(t)^T X + \dot{\chi}(t) \\ + (P(t)X + \Psi(t))^T \left[ -BR^{-1}B^T \right. \\ \left. \cdot (\dot{P}(t)X + \Psi(t)) \right. \\ \left. + (P(t)X + \Psi(t))^T (AX + C) \right. \\ \left. + \frac{1}{2} \left( X(t)^T QX(t) + u(t)^T Ru(t)^T \right) \right. \\ \left. + L^T X(t) \right] = 0 \text{ in } \mathcal{S} \times [0, T[, \\ P(T) = \phi, \quad \Psi(T) = 0, \quad \chi(T) = 0. \end{cases} \quad (49)$$

The boundary conditions are obtained by imposing that

$$v(X, T) = \frac{1}{2} X^T P(T) X + \Psi(T)^T X + \chi(T) = \frac{1}{2} X^T \phi X =: g(X).$$

Since (49) is an identity in  $X$ , it reduces to three equations:

$$\begin{cases} \dot{P} + PA(x) + A(x)^T P - PBR^{-1}B^T P \\ + Q = 0 \text{ in } [0, T[, \quad P(T) = \phi, \\ \dot{\Psi} + A(x)^T \Psi + PC - PBR^{-1}B^T \Psi + L = 0 \\ \text{in } [0, T[, \quad \Psi(T) = 0, \\ \dot{\chi} + \Psi^T C - \frac{1}{2} \Psi^T BR^{-1}B^T \Psi = 0 \\ \text{in } [0, T[, \quad \chi(T) = 0. \end{cases} \quad (50)$$

To understand the influence of the congestion term on the value function, let us develop the expression for  $\Psi$  and obtain

$$\begin{aligned} & \begin{bmatrix} \dot{\Psi}_1 \\ \dot{\Psi}_2 \end{bmatrix} + \begin{bmatrix} -\beta & 0 \\ k(x(t)) & 0 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \\ & + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \beta x_{off} \\ 0 \end{bmatrix} \\ & - \begin{bmatrix} P_{12}(r_{on}^{-1} + r_{off}^{-1}) \Psi_2 \\ P_{22}(r_{on}^{-1} + r_{off}^{-1}) \Psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ Se + W \end{bmatrix}. \end{aligned} \quad (51)$$

The expression of  $\Psi$  then can be rewritten as

$$\begin{cases} \dot{\Psi}_1 - \beta \Psi_1 + P_{11} \beta x_{off} \\ -P_{12}(r_{on}^{-1} + r_{off}^{-1}) \Psi_2 = 0, \\ \dot{\Psi}_2 + k(x(t)) \Psi_1 - P_{22}(r_{on}^{-1} + r_{off}^{-1}) \Psi_2 \\ + (Se + W) = 0, \end{cases} \quad (52)$$

which is of the form

$$\begin{cases} \dot{\Psi}_1 + a\Psi_1 + b\Psi_2 + c = 0, \\ \dot{\Psi}_2 + a'\Psi_1 + b'\Psi_2 + c' = 0. \end{cases} \quad (53)$$

From the above set of inequalities, we obtain the solution  $\Psi(x(t), e(t), t)$ . Note that the term  $a'$  depends on  $x$  and  $c'$  depends on  $e(t)$ .

Substituting the expression of the mean-field equilibrium strategies  $u^* = -R^{-1}B^T[PX + \Psi]$  as in (19) in the open-loop microscopic dynamics  $\dot{X}(t) = AX(t) + Bu(t) + C$  introduced in (16), and averaging both LHS and RHS we obtain the following closed-loop macroscopic dynamics

$$\dot{\bar{X}}(t) = [A(x) - BR^{-1}B^T P] \bar{X}(t) - BR^{-1}B^T \bar{\Psi}(t) + C,$$

where  $\bar{\Psi}(t) = \int_{x_{on}}^{x_{off}} \int_{[0,1]} \Psi(x, e, t) m(x, y, t) dx dy$  and this concludes our proof.

### Proof of Corollary 1

Let  $X(t)$  be a solution of dynamics (20) with initial value  $X(0) \notin \mathcal{X}$ . Set  $t = \{\inf t > 0 \mid X(t) \in \mathcal{X}\} \leq \infty$ . For all  $t \in [0, t]$

$$\begin{aligned} & V(X(t+dt)) - V(X(t)) \\ & = \frac{1}{\|X(t)+dX(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) + dX(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \\ & \quad - \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2. \end{aligned}$$

Taking the limit of the difference above we obtain

$$\begin{aligned} \dot{V}(X(t)) & = \lim_{dt \rightarrow 0} \frac{V(X(t+dt)) - V(X(t))}{dt} \\ & = \lim_{dt \rightarrow 0} \frac{1}{dt} \left[ \frac{1}{\|X(t)+dX(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) + dX(t) \right. \\ & \quad \left. - \Pi_{\mathcal{X}}(X(t))\|^2 \right. \\ & \quad \left. - \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \right] \\ & \leq \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \left[ \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P] X(t) \right. \right. \\ & \quad \left. \left. - BR^{-1}B^T \Psi + C \right) + \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \right] < 0, \end{aligned}$$

which implies  $\dot{V}(X(t)) < 0$ , for all  $X(t) \notin \mathcal{X}$  and this concludes our proof.

### Proof of Theorem 2

This proof follows the same reasoning as the proof of Theorem 1. However, differently from there, here for the quadratic terms in (23) we have

$$\sigma_{11}(x)^2 P_{11}(t) + \sigma_{22}(y)^2 P_{22}(t) = \hat{\sigma}_{11}^2 x^2 P_{11}(t) + \hat{\sigma}_{22}^2 y^2 P_{22}(t).$$

Reviewing (23) as an identity in  $x$ , this leads to the following three equations to solve in the variable  $P(t)$ ,  $\Psi(t)$ , and  $\chi(t)$ :

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P - PBR^{-1}B^T P + Q \\ + \tilde{P} = 0 \text{ in } [0, T[, \quad P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC - PBR^{-1}B^T \Psi \\ + L = 0 \text{ in } [0, T[, \quad \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C - \frac{1}{2} \Psi^T BR^{-1}B^T \Psi = 0 \\ \text{in } [0, T[, \quad \chi(T) = 0, \end{cases} \quad (54)$$

where

$$\tilde{P} = \text{Diag}((\hat{\sigma}_{ii}^2 P_{ii})_{i=1,2}) = \begin{bmatrix} \hat{\sigma}_{11}^2 P_{11} & 0 \\ 0 & \hat{\sigma}_{22}^2 P_{22} \end{bmatrix}. \quad (55)$$

*Proof of Corollary 2*

Let  $X(t)$  be a solution of dynamics (29) with initial value  $X(0) \notin \mathcal{X}$ . Set  $t = \{\inf t > 0 | X(t) \in \mathcal{X}\} \leq \infty$  and let  $V(X(t)) = \text{dist}(X(t), \mathcal{X})$ . For all  $t \in [0, t]$

$$\begin{aligned} V(X(t+dt)) - V(X(t)) &= \|X(t+dt) - \Pi_{\mathcal{X}}(X(t))\| \\ &\quad - \|X(t) - \Pi_{\mathcal{X}}(X(t))\| \\ &= \frac{1}{\|X(t)+dX(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) + dX(t) - \Pi_{\mathcal{X}}(X(t))\|^2 - \\ &\quad \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2. \end{aligned}$$

From the definition of infinitesimal generator

$$\begin{aligned} \mathcal{L}V(X(t)) &= \lim_{dt \rightarrow 0} \frac{\mathbb{E}V(X(t+dt)) - V(X(t))}{dt} \\ &\leq \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \left[ \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P]X(t) \right. \right. \\ &\quad \left. \left. - BR^{-1}B^T \Psi + C \right) \right. \\ &\quad \left. + \frac{1}{2} (\sigma_{11}^2(x) \partial_{xx} V(X, t) + \sigma_{22}^2(y) \partial_{yy} V(X, t)) \right]. \end{aligned}$$

From (31) the above implies that  $\mathcal{L}V(X(t)) < 0$ , for all  $X(t) \notin \mathcal{M}$  and this concludes our proof.

*Proof of Theorem 3*

From (32), in the HJB equation (23) we now have constant terms

$$\frac{1}{2} \sum_{i=1}^2 \sigma_{ii}(\cdot)^2 P_{ii}(t) = \hat{\sigma}_{11}^2 P_{11}(t) + \hat{\sigma}_{22}^2 P_{22}(t).$$

Again, since the HJB equation (23) is an identity in  $x$ , it reduces to three equations:

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P - PBR^{-1}B^T P \\ \quad + Q = 0 \text{ in } [0, T[, \quad P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC - PBR^{-1}B^T \Psi \\ \quad + L = 0 \text{ in } [0, T[, \quad \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C - \frac{1}{2} \Psi^T BR^{-1}B^T \Psi \\ \quad + \tilde{P} = 0 \text{ in } [0, T[, \quad \chi(T) = 0, \end{cases} \quad (56)$$

where

$$\tilde{P} = \begin{bmatrix} \hat{\sigma}_{11}^2 & 0 \\ 0 & \hat{\sigma}_{22}^2 \end{bmatrix}. \quad (57)$$

Substituting the expression of the mean-field equilibrium strategy  $u^* = -R^{-1}B^T[PX + \Psi]$  as in (36) in the open-loop microscopic dynamics  $dX(t) = (AX(t) + Bu(t) + C)dt + \Sigma dB_t$  given in (22) and averaging both LHS and RHS we obtain the following closed-loop macroscopic dynamics

$$\dot{\bar{X}}(t) = [A - BR^{-1}B^T P] \bar{X}(t) - BR^{-1}B^T \Psi(\bar{X}(t)) + C,$$

and this concludes our proof.

*A. Proof of Corollary 3*

Let  $X(t)$  be a solution of dynamics (37) with initial value  $X(0) \notin \mathcal{X}$ . Set  $t = \{\inf t > 0 | X(t) \in \mathcal{X}\} \leq \infty$  and let  $V(X(t)) = \text{dist}(X(t), \mathcal{X})$ . For all  $t \in [0, t]$

$$\begin{aligned} V(X(t+dt)) - V(X(t)) &= \|X(t+dt) - \Pi_{\mathcal{X}}(X(t))\| \\ &\quad - \|X(t) - \Pi_{\mathcal{X}}(X(t))\| \\ &= \frac{1}{\|X(t)+dX(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) + dX(t) - \Pi_{\mathcal{X}}(X(t))\|^2 - \\ &\quad \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \end{aligned}$$

From the definition of infinitesimal generator

$$\begin{aligned} \mathcal{L}V(X(t)) &= \lim_{dt \rightarrow 0} \frac{\mathbb{E}V(X(t+dt)) - V(X(t))}{dt} \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} \left[ \mathbb{E} \left( \frac{1}{\|X(t)+dX(t)-\Pi_{\mathcal{X}}(X(t))\|} \right. \right. \\ &\quad \left. \left. \|X(t) + dX(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \right) \right. \\ &\quad \left. - \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \right] \\ &\leq \frac{1}{\|X(t)-\Pi_{\mathcal{X}}(X(t))\|} \left[ \partial_X V(X, t)^T \left( [A - BR^{-1}B^T P]X(t) \right. \right. \\ &\quad \left. \left. - BR^{-1}B^T \Psi + C \right) \right. \\ &\quad \left. + \frac{1}{2} (\hat{\sigma}_{11}^2 \partial_{xx} V(X, t) + \hat{\sigma}_{22}^2 \partial_{yy} V(X, t)) \right]. \end{aligned}$$

From (39) the above implies that  $\mathcal{L}V(X(t)) < 0$ , for all  $X(t) \notin \mathcal{M}$  and this concludes our proof.

*Proof of Theorem 4*

Isolating the HJI equation in (13), we have

$$\begin{cases} -\partial_t \mathcal{V}_t(X) - \sup_u \inf_w \left\{ \partial_X \mathcal{V}_t(X)^T (AX + Bu \right. \\ \quad \left. + C + Dw) + \frac{1}{2} (X(t)^T Q X(t) \right. \\ \quad \left. + u(t)^T Ru(t) - \gamma^2 w(t)^T w(t)) + L^T X(t) \right\} = 0, \\ \text{in } \mathcal{S} \times [0, T[, \\ \mathcal{V}_T(X) = g(X) \text{ in } \mathcal{S}. \end{cases} \quad (58)$$

Let us consider the following value function

$$v(X, t) = \frac{1}{2} X^T P(t) X + \Psi(t)^T X + \chi(t),$$

and the corresponding mean-field equilibrium control and worst-case disturbance

$$\begin{aligned} u^* &= -R^{-1}B^T[PX + \Psi], \\ w^* &= \frac{1}{\gamma^2} D^T[PX + \Psi]. \end{aligned}$$

Then (58) can be rewritten as

$$\begin{cases} \frac{1}{2} X^T \dot{P}(t) X + \dot{\Psi}(t)^T X + \dot{\chi}(t) \\ \quad + (P(t)X + \Psi(t))^T \left[ -BR^{-1}B^T + \frac{1}{\gamma^2} DD^T \right] \\ \quad \cdot (P(t)x + \Psi(t)) + (P(t)x + \Psi(t))^T (AX + C) \\ \quad + \frac{1}{2} (X(t)^T Q X(t) + u(t)^T Ru(t) - \gamma^2 w(t)^T w(t)) \\ \quad + L^T X(t) + \frac{1}{2} \sum_{i=1}^2 \sigma_{ii}(\cdot)^2 P_{ii}(t) = 0 \\ \quad \text{in } \mathbb{R}^2 \times [0, T[, \\ P(T) = \phi, \quad \Psi(T) = 0, \quad \chi(T) = 0. \end{cases}$$

The boundary conditions are obtained by imposing that

$$v(X, T) = \frac{1}{2} X^T P(T) X + \Psi(T)^T X + \chi(T) = \frac{1}{2} X^T \phi X =: g(X).$$

The above set of identities in  $x$  yields the following three equations in the variable  $P(t)$ ,  $\Psi(t)$ , and  $\chi(t)$ :

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P + P(-BR^{-1}B^T \\ \quad + \frac{1}{\gamma^2} DD^T)P + Q = 0 \text{ in } [0, T[, \quad P(T) = \phi, \\ \dot{\Psi}(t) + A^T \Psi + PC + (-BR^{-1}B^T \\ \quad + \frac{1}{\gamma^2} DD^T)\Psi + L = 0 \text{ in } [0, T[, \quad \Psi(T) = 0, \\ \dot{\chi}(t) + \Psi(t)^T C + \frac{1}{2} \Psi^T (-BR^{-1}B^T \\ \quad + \frac{1}{\gamma^2} DD^T)\Psi = 0 \text{ in } [0, T[, \quad \chi(T) = 0. \end{cases} \quad (59)$$

Substituting the expressions of the mean-field equilibrium strategies  $u^* = -R^{-1}B^T[PX + \Psi]$  and  $w^* = \frac{1}{\gamma^2}D^T[PX + \Psi]$  as in (43) in the open-loop microscopic dynamics  $\dot{X}(t) = AX(t) + Bu(t) + C$  introduced in (40), and averaging both LHS and RHS we obtain the following closed-loop macroscopic dynamics

$$\begin{aligned} \dot{\bar{X}}(t) &= [A + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)P]\bar{X}(t) \\ &+ (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)\Psi(\bar{X}(t)) + C, \end{aligned} \quad (60)$$

and this concludes our proof.

#### Proof of Corollary 4

Let  $X(t)$  be a solution of dynamics (44) with initial value  $X(0) \neq \mathcal{X}$ . Set  $t = \{\inf t > 0 | X(t) \in \mathcal{X}\} \leq \infty$  and let  $V(X(t)) = \text{dist}(X(t), \mathcal{X})$ . For all  $t \in [0, t]$

$$\begin{aligned} V(X(t+dt)) - V(X(t)) &= \|X(t+dt) - \Pi_{\mathcal{X}}(X(t))\| \\ &- \|X(t) - \Pi_{\mathcal{X}}(X(t))\| \\ &= \frac{1}{\|X(t) - \Pi_{\mathcal{X}}(X(t))\|} \|X(t) + dX(t) - \Pi_{\mathcal{X}}(X(t))\|^2 \\ &- \frac{1}{\|X(t) - \Pi_{\mathcal{X}}(X(t))\|} \|X(t) - \Pi_{\mathcal{X}}(X(t))\|^2. \end{aligned}$$

Using the asymptotic limit to differentiate the distance we have

$$\begin{aligned} \dot{V}(X(t)) &= \lim_{dt \rightarrow 0} \frac{V(X(t+dt)) - V(X(t))}{dt} \\ &\leq \frac{1}{\|X(t) - \Pi_{\mathcal{X}}(X(t))\|} \left[ \partial_X V(X, t)^T \left( [A + (-BR^{-1}B^T \right. \right. \\ &+ \frac{1}{\gamma^2}DD^T)P]X(t) \\ &\left. \left. + (-BR^{-1}B^T + \frac{1}{\gamma^2}DD^T)\Psi + C \right) \right] \leq 0 \end{aligned}$$

which implies  $\dot{V}(X(t)) < 0$ , for all  $X(t) \neq \mathcal{X}$  and this concludes our proof.

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