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RESEARCH ARTICLE

Fiat categorification of the symmetric inverse semigroup IS_n and the semigroup F_n^*

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Abstract Starting from the symmetric group S_n, we construct two fiat 2-categories. One of them can be viewed as the fiat "extension" of the natural 2-category associated with the symmetric inverse semigroup (considered as an ordered semigroup with respect to the natural order). This 2-category provides a fiat categorification for the integral semigroup algebra of the symmetric inverse semigroup. The other 2-category can be viewed as the fiat "extension" of the 2-category associated with the maximal factorizable subsemigroup of the dual symmetric inverse semigroup (again, considered as an ordered semigroup with respect to the natural order). This 2-category provides a fiat categorification for the integral semigroup algebra of the maximal factorizable subsemigroup of the dual symmetric inverse semigroup.

Keywords Categorification · 2-category · Symmetric inverse semigroup · Dual symmetric inverse semigroup

1 Introduction and description of the results

Abstract higher representation theory has its origins in the papers [2,4,40,41] with principal motivation coming from [15,43]. For finitary 2-categories, basics

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of 2-representation theory were developed in [29–34] and further investigated in [3,11,12,16,23,36,44–47], see also [17] for applications. For different ideas on higher representation theory, see also [1,6,8,14,37] and references therein.

The major emphasis in [29–34] is on the study of so-called *fiat* 2-categories, which are 2-categorical analogues of finite dimensional algebras with involution. Fiat 2-categories appear naturally both in topology and representation theory. They have many nice properties and the series of papers mentioned above develops an essential starting part of 2-representation theory for fiat categories.

Many examples of 2-categories appear naturally in semigroup theory, see [9,11, 12,21]. The easiest example is the 2-category associated to a monoid with a fixed admissible partial order, see Sect. 4.1 for details. Linear analogues of these 2-categories show up naturally in representation theory, see [11,12]. A classical example of an ordered monoid is an inverse monoid with respect to the natural partial order. There is a standard linearization procedure, which allows one to turn a 2-category of a finite ordered monoid into a finitary 2-category, see Sect. 3.2 for details.

One serious disadvantage with linearizations of 2-categories associated to finite ordered monoids is the fact that they are almost never fiat. The main reason for that is lack of 2-morphisms which start from the identity 1-morphism. In the present paper we construct two natural "extensions" of the symmetric group to 2-categories whose linearizations are fiat. One of them becomes a nice 2-categorical analogue (categorification) for the symmetric inverse semigroup IS_n. The other one becomes a nice 2-categorical analogue for the maximal factorizable subsemigroup F_n^* in the dual symmetric inverse semigroup I_n^*.

The main novel component of the present paper is in the definitions and constructions of the main objects. To construct our 2-categories, we, essentially, have to define three things:

- sets of 2-morphisms between elements of S_n;
- horizontal composition of 2-morphisms;
- vertical composition of 2-morphisms.

In the case which eventually leads to IS_n, we view elements of S_n as binary relations in the obvious way and define 2-morphisms between two elements of S_n as the set of all binary relations contained in both these elements. We chose vertical composition to be given by intersection of relations and horizontal composition to be given by the usual composition of relations. Although all these choices are rather natural, none of them seems to be totally obvious. Verification that this indeed defines a 2-category requires some technical work. In the case which eventually leads to F_n^*, we do a similar thing, but instead of binary relations, we realize S_n inside the partition monoid. For 2-morphisms between elements σ and τ in S_n, we use those partitions which contain both σ and τ. All details on both constructions and all verifications can be found in Sect. 2.

Section 3 recalls the theory of $\Bbbk$-linear 2-categories and gives explicit constructions for a finitary $\Bbbk$-linear 2-category starting from a finite 2-category. In Sect. 4 we establish that our constructions lead to fiat 2-categories. We also recall, in more details, the standard constructions of finitary 2-categories, starting from IS_n and F_n^*, considered as ordered monoids, and show that the 2-categories obtained in this way are not fiat. In Sect. 5 we make the relation between our constructions and IS_n and F_n^* precise.

In fact, we show that the decategorification of our first construction is isomorphic to the semigroup algebra $\mathbb{Z}[IS_n]$, with respect to the so-called Möbius basis in $\mathbb{Z}[IS_n]$, cf. [42, Theorem 4.4]. Similarly, we show that the decategorification of our second construction is isomorphic to the semigroup algebra $\mathbb{Z}[F_n^*]$, with respect to a similarly defined basis. We complete the paper with two explicit examples in Sect. 6.

2 Two 2-categorical "extensions" of S_n

2.1 2-Categories

A 2-*category* is a category enriched over the monoidal category **Cat** of small categories. This means that a 2-category $\mathscr{C}$ consists of

- objects $\mathtt{i}, \mathtt{j}, \ldots$;
- small morphism categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$;
- bifunctorial compositions;
- identity objects $\mathbb{1}_{\mathtt{i}} \in \mathscr{C}(\mathtt{i}, \mathtt{i})$;

which satisfy the obvious collection of (strict) axioms. Objects in morphism categories are usually called 1-*morphisms* (for example, all $\mathbb{1}_{\mathtt{i}}$ are 1-morphisms) while morphisms in morphism categories are usually called 2-*morphisms*. Composition of 2-morphisms inside a fixed $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called *vertical* and denoted $\circ_1$. Composition of 2-morphisms coming from the bifunctorial composition in $\mathscr{C}$ is called *horizontal* and denoted $\circ_0$. We refer the reader to [22,24] for more details on 2-categories.

The main example of a 2-category is **Cat** itself, where

- objects are small categories;
- 1-morphisms are functors;
- 2-morphisms are natural transformations;
- composition is the usual composition;
- identity 1-morphisms are identity functors.

2.2 First 2-category extending S_n

For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, consider the set $\mathbf{n} = \{1, 2, \ldots, n\}$ and let S_n denote the *symmetric group* of all bijective transformations of $\mathbf{n}$ under composition. We consider also the monoid $\mathbf{B}_n = 2^{\mathbf{n}\times\mathbf{n}}$ of all *binary relations* on $\mathbf{n}$ which is identified with the monoid of $n \times n$-*matrices* over the Boolean semiring $\mathbf{B} := \{0, 1\}$ by taking a relation to its adjacency matrix. Note that $\mathbf{B}_n$ is an ordered monoid with respect to usual inclusions of binary relations. We identify S_n with the group of invertible elements in $\mathbf{B}_n$ in the obvious way.

We now define a 2-category $\mathscr{A} = \mathscr{A}_n$. To start with, we declare that

- $\mathscr{A}$ has one object $\mathtt{i}$;
- 1-morphisms in $\mathscr{A}$ are elements in S_n;
- composition $\cdot$ of 1-morphisms is induced from S_n;
- the identity 1-morphism is the identity transformation $\mathrm{id}_{\mathbf{n}} \in S_n$.

It remains to define 2-morphisms in $\mathscr{A}$ and their compositions.

- For $\pi, \sigma \in S_n$, we define $\mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ as the set of all $\alpha \in \mathbf{B}_n$ such that $\alpha \subseteq \pi \cap \sigma$.
- For $\pi, \sigma, \tau \in S_n$, and also for $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$, we define $\beta \circ_1 \alpha := \beta \cap \alpha$.
- For $\pi \in S_n$, we define the identity element in $\mathrm{Hom}_{\mathscr{A}}(\pi, \pi)$ to be π.
- For $\pi, \sigma, \tau, \rho \in S_n$, and also for $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{A}}(\tau, \rho)$, we define $\beta \circ_0 \alpha := \beta\alpha$, the usual composition of binary relations.

Proposition 1 *The construct $\mathscr{A}$ above is a 2-category.*

Proof Composition $\cdot$ of 1-morphisms is associative as S_n is a group. The vertical composition $\circ_1$ is clearly well-defined. It is associative as $\cap$ is associative. If we have $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ or $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \pi)$, then $\alpha \subseteq \pi$ and thus $\alpha \cap \pi = \alpha$. Therefore $\pi \in \mathrm{Hom}_{\mathscr{A}}(\pi, \pi)$ is the identity element.

Let us check that the horizontal composition $\circ_0$ is well-defined. From $\alpha \subseteq \pi$ and $\beta \subseteq \tau$ and the fact that $\mathbf{B}_n$ is ordered, we have $\beta\alpha \subseteq \tau\alpha \subseteq \tau\pi$. Similarly, from $\alpha \subseteq \sigma$ and $\beta \subseteq \rho$ and the fact that $\mathbf{B}_n$ is ordered, we have $\beta\alpha \subseteq \rho\alpha \subseteq \rho\sigma$. It follows that $\beta\alpha \in \mathrm{Hom}_{\mathscr{A}}(\tau\pi, \rho\sigma)$ and thus $\circ_0$ is well-defined. Its associativity follows from the fact that usual composition of binary relations is associative.

It remains to check the *interchange law*, that is the fact that, for any 1-morphisms $\pi, \sigma, \rho, \tau, \mu, \nu$ and for any $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$, $\beta \in \mathrm{Hom}_{\mathscr{A}}(\tau, \mu)$, $\gamma \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \rho)$ and $\delta \in \mathrm{Hom}_{\mathscr{A}}(\mu, \nu)$, we have

$$(\delta \circ_0 \gamma) \circ_1 (\beta \circ_0 \alpha) = (\delta \circ_1 \beta) \circ_0 (\gamma \circ_1 \alpha). \tag{2.1}$$

Assume first that $\sigma = \mu = \mathrm{id}_{\mathbf{n}}$. In this case both α, β, γ and δ are subrelations of the identity relation $\mathrm{id}_{\mathbf{n}}$. Note that, given two subrelations x and y of the identity relation $\mathrm{id}_{\mathbf{n}}$, their product xy as binary relations equals $x \cap y$. Hence, in this particular case, both sides of (2.1) are equal to $\alpha \cap \beta \cap \gamma \cap \delta$.

Before proving the general case, we will need the following two lemmata:

Lemma 2 *Let $\pi, \sigma, \tau, \rho \in S_n$.*

(i) Left composition with π induces a bijection from $\mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$ to $\mathrm{Hom}_{\mathscr{A}}(\pi\sigma, \pi\tau)$.
(ii) For any $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$ and $\beta \in \mathrm{Hom}_{\mathscr{A}}(\tau, \rho)$, we have

$$\pi \circ_0 (\beta \circ_1 \alpha) = (\pi \circ_0 \beta) \circ_1 (\pi \circ_0 \alpha).$$

Proof Left composition with π maps an element (y, x) of $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$ to $(\pi(y), x) \in \mathrm{Hom}_{\mathscr{A}}(\pi\sigma, \pi\tau)$. As π is an invertible transformation of $\mathbf{n}$, multiplying with π^{-1} returns $(\pi(y), x)$ to (y, x). This implies claim (i). Claim (ii) follows from claim (i) and the observation that composition with invertible maps commutes with taking intersections. □

Lemma 3 *Let $\pi, \sigma, \tau, \rho \in S_n$.*

(i) Right composition with π induces a bijection from $\mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$ to $\mathrm{Hom}_{\mathscr{A}}(\sigma\pi, \tau\pi)$.

(*ii*) *For any* $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$ *and* $\beta \in \mathrm{Hom}_{\mathscr{A}}(\tau, \rho)$*, we have*

$$(\beta \circ_1 \alpha) \circ_0 \pi = (\beta \circ_0 \pi) \circ_1 (\alpha \circ_0 \pi).$$

Proof Analogous to the proof of Lemma 2. ☐

Using Lemmata 2 and 3 together with associativity of $\circ_0$, right multiplication with σ^{-1} and left multiplication with μ^{-1} reduces the general case of (2.1) to the case $\sigma = \mu = \mathrm{id}_{\mathbf{n}}$ considered above. This completes the proof. ☐

2.3 Second 2-category extending S_n

For $n \in \mathbb{N}$, consider the corresponding *partition semigroup* $\mathbf{P}_n$, see [13,25–27]. Elements of $\mathbf{P}_n$ are partitions of the set

$$\bar{\mathbf{n}} := \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$$

into disjoint unions of non-empty subsets, called *parts*. Alternatively, one can view elements of $\mathbf{P}_n$ as equivalence relations on $\bar{\mathbf{n}}$. Multiplication $(\rho, \pi) \mapsto \rho\pi$ in $\mathbf{P}_n$ is given by the following *mini-max algorithm*, see [13,25–27] for details:

- Consider ρ as a partition of $\{1', 2', \ldots, n', 1'', 2'', \ldots, n''\}$ using the map $x \mapsto x'$ and $x' \mapsto x''$, for $x \in \mathbf{n}$.
- Let τ be the minimum, with respect to inclusions, partition of

$$\{1, 2, \ldots, n, 1', 2', \ldots, n', 1'', 2'', \ldots, n''\},$$

  such that each part of both ρ and π is a subset of a part of τ.
- Let σ be the maximum, with respect to inclusions, partition of

$$\{1, 2, \ldots, n, 1'', 2'', \ldots, n''\},$$

  such that each part of σ is a subset of a part of τ.
- Define the product $\rho\pi$ as the partition of $\mathbf{n}$ induced from σ via the map $x'' \mapsto x'$, for $x \in \mathbf{n}$.

Note that S_n is, naturally, a submonoid of $\mathbf{P}_n$. Moreover, S_n is the maximal subgroup of all invertible elements in $\mathbf{P}_n$.

A part of $\rho \in \mathbf{P}_n$ is called a *propagating* part provided that it intersects both sets $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots, n'\}$. Partitions in which all parts are propagating are called *propagating partitions*. The set of all propagating partitions in $\mathbf{P}_n$ is denoted by $\mathbf{PP}_n$, it is a submonoid of $\mathbf{P}_n$.

The monoid $\mathbf{P}_n$ is naturally ordered with respect to refinement: $\rho \leq \tau$ provided that each part of ρ is a subset of a part in τ. With respect to this order, the partition of $\mathbf{n}$ with just one part is the maximum element, while the partition of $\bar{\mathbf{n}}$ into singletons is the minimum element. This order restricts to $\mathbf{PP}_n$. As elements of $\mathbf{P}_n$ are just equivalence relations, the poset $\mathbf{P}_n$ is a lattice and we denote by $\wedge$ and $\vee$ the corresponding meet and join operations, respectively. The poset $\mathbf{PP}_n$ is a sublattice in $\mathbf{P}_n$ with the same meet and join. As $S_n \subset \mathbf{PP}_n$, all meets and joins in $\mathbf{P}_n$ of elements from S_n belong to $\mathbf{PP}_n$.

We now define a 2-category $\mathscr{B} = \mathscr{B}_n$. Similarly to Sect. 2.2, we start by declaring that

- $\mathscr{B}$ has one object $\mathtt{i}$;
- 1-morphisms in $\mathscr{B}$ are elements in S_n;
- composition of 1-morphisms is induced from S_n;
- the identity 1-morphism is the identity transformation $\mathrm{id}_{\mathbf{n}}$.

It remains to define 2-morphisms in $\mathscr{B}$ and their compositions.

- For $\pi, \sigma \in S_n$, we define $\mathrm{Hom}_{\mathscr{B}}(\pi, \sigma)$ as the set of all $\alpha \in \mathbf{PP}_n$ such that we have both, $\pi \leq \alpha$ and $\sigma \leq \alpha$.
- For $\pi, \sigma, \tau \in S_n$, and also any $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{B}}(\sigma, \tau)$, we define $\beta \circ_1 \alpha := \beta \vee \alpha$.
- For $\pi \in S_n$, we define the identity element in $\mathrm{Hom}_{\mathscr{B}}(\pi, \pi)$ to be π.
- For $\pi, \sigma, \tau, \rho \in S_n$, and also any $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{B}}(\tau, \rho)$, we define $\beta \circ_0 \alpha := \beta\alpha$, the usual composition of partitions.

Proposition 4 *The construct $\mathscr{B}$ above is a 2-category.*

Proof The vertical composition $\circ_1$ is clearly well-defined. It is associative as $\vee$ is associative. If $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\pi, \sigma)$ or $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\sigma, \pi)$, then $\pi \leq \alpha$ and thus $\alpha \vee \pi = \alpha$. Therefore $\pi \in \mathrm{Hom}_{\mathscr{B}}(\pi, \pi)$ is the identity element.

Let us check that the horizontal composition $\circ_0$ is well-defined. From $\pi \leq \alpha$ and $\tau \leq \beta$ and the fact that $\mathbf{P}_n$ is ordered, we have $\tau\pi \leq \tau\alpha \leq \beta\alpha$. Similarly, from $\sigma \leq \alpha$ and $\rho \leq \beta$ and the fact that $\mathbf{P}_n$ is ordered, we have $\rho\sigma \leq \rho\alpha \leq \beta\alpha$. It follows that $\beta\alpha \in \mathrm{Hom}_{\mathscr{B}}(\tau\pi, \rho\sigma)$ and thus $\circ_0$ is well-defined. Its associativity follows from the fact that usual composition of partitions is associative.

It remains to check the interchange law (2.1). For this we fix any 1-morphisms π, σ, ρ, τ, μ, ν and any $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\pi, \sigma)$, $\beta \in \mathrm{Hom}_{\mathscr{B}}(\tau, \mu)$, $\gamma \in \mathrm{Hom}_{\mathscr{B}}(\sigma, \rho)$ and $\delta \in \mathrm{Hom}_{\mathscr{B}}(\mu, \nu)$. Assume first that $\sigma = \mu = \mathrm{id}_{\mathbf{n}}$. In this case both α, β, γ and δ are partitions containing the identity relation $\mathrm{id}_{\mathbf{n}}$. Note that, given two partitions x and y containing the identity relation $\mathrm{id}_{\mathbf{n}}$, their product xy as partitions equals $x \vee y$. Hence, in this particular case, both sides of (2.1) are equal to $\alpha \vee \beta \vee \gamma \vee \delta$.

Before proving the general case, we will need the following two lemmata:

Lemma 5 *Let $\pi, \sigma, \tau, \rho \in S_n$.*

(*i*) *Left composition with π induces a bijection between the sets* $\mathrm{Hom}_{\mathscr{B}}(\sigma, \tau)$ *and* $\mathrm{Hom}_{\mathscr{B}}(\pi\sigma, \pi\tau)$.
(*ii*) *For any $\alpha \in$* $\mathrm{Hom}_{\mathscr{B}}(\sigma, \tau)$ *and $\beta \in$* $\mathrm{Hom}_{\mathscr{B}}(\tau, \rho)$*, we have*

$$\pi \circ_0 (\beta \circ_1 \alpha) = (\pi \circ_0 \beta) \circ_1 (\pi \circ_0 \alpha).$$

Proof Left composition with π simply renames elements of $\{1', 2', \ldots, n'\}$ in an invertible way. This implies claim (i). Claim (ii) follows from claim (i) and the observation that composition with invertible maps commutes with taking unions. □

Lemma 6 *Let* $\pi, \sigma, \tau, \rho \in S_n$.

(*i*) *Right composition with* π *induces a bijection between the sets* $\mathrm{Hom}_{\mathscr{B}}(\sigma, \tau)$ *and* $\mathrm{Hom}_{\mathscr{B}}(\sigma\pi, \tau\pi)$.

(*ii*) *For any* $\alpha \in \mathrm{Hom}_{\mathscr{B}}(\sigma, \tau)$ *and* $\beta \in \mathrm{Hom}_{\mathscr{B}}(\tau, \rho)$, *we have*

$$(\beta \circ_1 \alpha) \circ_0 \pi = (\beta \circ_0 \pi) \circ_1 (\alpha \circ_0 \pi).$$

Proof Analogous to the proof of Lemma 5. □

Using Lemmata 5 and 6 together with associativity of $\circ_0$, right multiplication with σ^{-1} and left multiplication with μ^{-1} reduces the general case of (2.1) to the case $\sigma = \mu = \mathrm{id}_{\mathbf{n}}$ considered above. This completes the proof. □

3 2-Categories in the linear world

For more details on all the definitions in Sect. 3, we refer to [12].

3.1 Finitary 2-categories

Let $\Bbbk$ be a field. A $\Bbbk$-linear category $\mathcal{C}$ is called *finitary* provided that it is additive, idempotent split and Krull–Schmidt (cf. [38, Section 2.2]) with finitely many isomorphism classes of indecomposable objects and finite dimensional homomorphism spaces.

A 2-category $\mathscr{C}$ is called *prefinitary (over* $\Bbbk$) provided that

(I) $\mathscr{C}$ has finitely many objects;
(II) each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is a finitary $\Bbbk$-linear category;
(III) all compositions are biadditive and $\Bbbk$-linear whenever the notion makes sense.

Following [29], a prefinitary 2-category $\mathscr{C}$ is called *finitary* provided that

(IV) all identity 1-morphisms are indecomposable.

3.2 $\Bbbk$-Linearization of finite categories

For any set X, let us denote by $\Bbbk[X]$ the vector space (over $\Bbbk$) of all formal linear combinations of elements in X with coefficients in $\Bbbk$. Then we can view X as the *standard basis* in $\Bbbk[X]$. By convention, $\Bbbk[X] = \{0\}$ if $X = \varnothing$.

Let $\mathcal{C}$ be a finite category, that is a category with a finite number of morphisms. Define the $\Bbbk$-*linearization* $\mathcal{C}_{\Bbbk}$ of $\mathcal{C}$ as follows:

- the objects in $\mathcal{C}_{\Bbbk}$ and $\mathcal{C}$ are the same;
- we have $\mathcal{C}_{\Bbbk}(\mathtt{i}, \mathtt{j}) := \Bbbk[\mathcal{C}(\mathtt{i}, \mathtt{j})]$;
- composition in $\mathcal{C}_{\Bbbk}$ is induced from that in $\mathcal{C}$ by $\Bbbk$-bilinearity.

3.3 $\Bbbk$-additivization of finite categories

Assume that objects of the category $\mathcal{C}$ are $\mathtt{1}, \mathtt{2}, \dots, \mathtt{k}$. For $\mathcal{C}$ as in Sect. 3.2, define the *additive* $\Bbbk$*-linearization* $\mathcal{C}_{\Bbbk}^{\oplus}$ of $\mathcal{C}$ in the following way:

- objects in $\mathcal{C}_{\Bbbk}^{\oplus}$ are elements in $\mathbb{Z}_{\geq 0}^{k}$, we identify $(m_1, m_2, \dots, m_k) \in \mathbb{Z}_{\geq 0}^{k}$ with the symbol

$$\underbrace{\mathtt{1} \oplus \dots \oplus \mathtt{1}}_{m_1 \text{ times}} \oplus \underbrace{\mathtt{2} \oplus \dots \oplus \mathtt{2}}_{m_2 \text{ times}} \oplus \dots \oplus \underbrace{\mathtt{k} \oplus \dots \oplus \mathtt{k}}_{m_k \text{ times}};$$

- the set $\mathcal{C}_{\Bbbk}^{\oplus}(\mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \dots \oplus \mathtt{i}_l, \mathtt{j}_1 \oplus \mathtt{j}_2 \oplus \dots \oplus \mathtt{j}_m)$ is given by the set of all matrices of the form

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{ml} \end{pmatrix}$$

  where $f_{st} \in \mathcal{C}_{\Bbbk}(\mathtt{i}_t, \mathtt{j}_s)$;
- composition in $\mathcal{C}_{\Bbbk}^{\oplus}$ is given by the usual matrix multiplication;
- the additive structure is given by addition in $\mathbb{Z}_{\geq 0}^{k}$.

One should think of $\mathcal{C}_{\Bbbk}^{\oplus}$ as the additive category generated by $\mathcal{C}_{\Bbbk}$.

3.4 $\Bbbk$-Linearization of finite 2-categories

Let now $\mathscr{C}$ be a finite 2-category. We define the $\Bbbk$*-linearization* $\mathscr{C}_{\Bbbk}$ of $\mathscr{C}$ over $\Bbbk$ as follows:

- $\mathscr{C}_{\Bbbk}$ and $\mathscr{C}$ have the same objects;
- we have $\mathscr{C}_{\Bbbk}(\mathtt{i}, \mathtt{j}) := \mathscr{C}(\mathtt{i}, \mathtt{j})_{\Bbbk}^{\oplus}$;
- composition in $\mathscr{C}_{\Bbbk}$ is induced from composition in $\mathscr{C}$ by biadditivity and $\Bbbk$-bilinearity.

By construction, the 2-category $\mathscr{C}_{\Bbbk}$ satisfies conditions (I) and (III) from the definition of a finitary 2-category. A part of condition (II) related to additivity and finite dimensionality of morphism spaces is also satisfied. Therefore, the 2-category $\mathscr{C}_{\Bbbk}$ is finitary if and only if, the 2-endomorphism $\Bbbk$-algebra of every 1-morphism in $\mathscr{C}_{\Bbbk}$ is local.

3.5 $\Bbbk$-Finitarization of finite 2-categories

Let $\mathscr{C}$ be a finite 2-category. Consider the 2-category $\mathscr{C}_{\Bbbk}$. We define the *finitarization* $\Bbbk\mathscr{C}$ of $\mathscr{C}_{\Bbbk}$ as follows:

- $\Bbbk\mathscr{C}$ and $\mathscr{C}_\Bbbk$ have the same objects;
- $\Bbbk\mathscr{C}(\mathtt{i},\mathtt{j})$ is defined to be the idempotent completion of $\mathscr{C}_\Bbbk(\mathtt{i},\mathtt{j})$;
- composition in $\Bbbk\mathscr{C}$ is induced from composition in $\mathscr{C}$.

By construction, the 2-category $\Bbbk\mathscr{C}$ is prefinitary. Therefore, the 2-category $\Bbbk\mathscr{C}$ is finitary if and only if, the 2-endomorphism $\Bbbk$-algebra of every identity 1-morphism in $\Bbbk\mathscr{C}$ is local.

3.6 Idempotent splitting

Let $\mathscr{C}$ be a prefinitary 2-category. If $\mathscr{C}$ does not satisfy condition (IV), then there is an object $\mathtt{i}\in\mathscr{C}$ such that the endomorphism algebra $\mathrm{End}_{\Bbbk\mathscr{C}}(\mathbb{1}_\mathtt{i})$ is not local, that is, contains a non-trivial idempotent. In this subsection we describe a version of "idempotent splitting", for all $\mathrm{End}_{\Bbbk\mathscr{C}}(\mathbb{1}_\mathtt{i})$, to turn $\mathscr{C}$ into a finitary 2-category which we denote by $\overline{\mathscr{C}}$.

For $\mathtt{i}\in\mathscr{C}$, the 2-endomorphism algebra of $\mathbb{1}_\mathtt{i}$ is equipped with two unital associative operations, namely, $\circ_0$ and $\circ_1$. These two operations satisfy the interchange law. By the classical Eckmann–Hilton argument (see, for example, [5] or [19, Subsection 1.1]), both these operations, when restricted to the 2-endomorphism algebra of $\mathbb{1}_\mathtt{i}$, must be commutative and, in fact, coincide. Therefore we can unambiguously speak about the commutative 2-endomorphism algebra $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})$. Let $\varepsilon_\mathtt{i}^{(j)}$, where $j=1,2,\dots,k_\mathtt{i}$, be a complete list of primitive idempotents in $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})$. Note that the elements $\varepsilon_\mathtt{i}^{(j)}$ are identities in the minimal ideals of $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})$ and hence are canonically determined (up to permutation).

We now define a new 2-category, which we denote by $\overline{\mathscr{C}}$, in the following way:

- Objects in $\overline{\mathscr{C}}$ are $\mathtt{i}^{(s)}$, where $\mathtt{i}\in\mathscr{C}$ and $s=1,2,\dots,k_\mathtt{i}$.
- 1-morphisms in $\overline{\mathscr{C}}(\mathtt{i}^{(s)},\mathtt{j}^{(t)})$ are the same as 1-morphisms in $\mathscr{C}(\mathtt{i},\mathtt{j})$.
- for 1-morphisms $\mathrm{F},\mathrm{G}\in\overline{\mathscr{C}}(\mathtt{i}^{(s)},\mathtt{j}^{(t)})$, the set $\mathrm{Hom}_{\overline{\mathscr{C}}}(\mathrm{F},\mathrm{G})$ equals

$$\varepsilon_\mathtt{j}^{(t)}\circ_0\mathrm{Hom}_{\mathscr{C}}(\mathrm{F},\mathrm{G})\circ_0\varepsilon_\mathtt{i}^{(s)}.$$

- The identity 1-morphism in $\overline{\mathscr{C}}(\mathtt{i}^{(s)},\mathtt{i}^{(s)})$ is $\mathbb{1}_\mathtt{i}$.
- All compositions are induced from $\mathscr{C}$.

Lemma 7 *Let $\mathscr{C}$ be a prefinitary 2-category. Then the construct $\overline{\mathscr{C}}$ is a finitary 2-category.*

Proof The fact that $\overline{\mathscr{C}}$ is a 2-category follows from the fact that $\mathscr{C}$ is a 2-category, by construction. For $\overline{\mathscr{C}}$, conditions (I), (II) and (III) from the definition of a prefinitary 2-category, follow from the corresponding conditions for the original category $\mathscr{C}$.

It remains to show that $\overline{\mathscr{C}}$ satisfies (IV). By construction, the endomorphism algebra of the identity 1-morphism $\mathbb{1}_\mathtt{i}$ in $\overline{\mathscr{C}}(\mathtt{i}^{(s)},\mathtt{i}^{(s)})$ is

$$\varepsilon_\mathtt{i}^{(s)}\circ_0\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})\circ_0\varepsilon_\mathtt{i}^{(s)}.$$

The latter algebra is local as $\varepsilon_{\mathtt{i}}^{(s)}$ is a minimal idempotent. This means that condition (IV) is satisfied and completes the proof. □

Starting from $\overline{\mathscr{C}}$ and taking, for each $\mathtt{i} \in \mathscr{C}$, a direct sum of $\mathtt{i}^{(s)}$, where $s = 1, 2, \dots, k_{\mathtt{i}}$, one obtains a 2-category biequivalent to the original 2-category $\mathscr{C}$. The 2-categories $\mathscr{C}$ and $\overline{\mathscr{C}}$ are, clearly, Morita equivalent in the sense of [32].

Warning: Despite of the fact that $\overline{\mathscr{C}}(\mathtt{i}^{(s)}, \mathtt{j}^{(t)})$ and $\mathscr{C}(\mathtt{i}, \mathtt{j})$ have *the same* 1-morphisms, these two categories, in general, have *different* indecomposable 1-morphisms as the sets of 2-morphisms are different. In particular, indecomposable 1-morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ may become isomorphic to zero in $\overline{\mathscr{C}}(\mathtt{i}^{(s)}, \mathtt{j}^{(t)})$.

We note that the operation of idempotent splitting is also known as taking *Cauchy completion* or *Karoubi envelope*.

4 Comparison of $\overline{\Bbbk\mathscr{A}_n}$ and $\overline{\Bbbk\mathscr{B}_n}$ to 2-categories associated with ordered monoids IS_n and F_n^*

4.1 2-Categories and ordered monoids

Let $(S, \cdot, 1)$ be a monoid and $\leq$ be an *admissible order* on S, that is a partial (reflexive) order such that $s \leq t$ implies both $sx \leq tx$ and $xs \leq xt$, for all $x, s, t \in S$. Then we can associate with S a 2-category $\mathscr{S} = \mathscr{S}_S = \mathscr{S}_{(S,\cdot,1,\leq)}$ defined as follows:

- $\mathscr{S}$ has one object $\mathtt{i}$;
- 1-morphisms are elements in S;
- for $s, t \in S$, the set $\mathrm{Hom}_{\mathscr{S}}(s, t)$ is empty if $s \not\leq t$ and contains one element (s, t) otherwise;
- composition of 1-morphisms is given by $\cdot$;
- both horizontal and vertical compositions of 2-morphism are the only possible compositions (as sets of 2-morphisms are either empty or singletons);
- the identity 1-morphism is 1.

Admissibility of $\leq$ makes the above well-defined and ensures that $\mathscr{S}$ becomes a 2-category.

A canonical example of the above is when S is an inverse monoid and $\leq$ is the *natural partial order* on S defined as follows: $s \leq t$ if and only if $s = et$ for some idempotent $e \in S$.

4.2 (Co)ideals of ordered semigroups

Let S be a semigroup equipped with an admissible order $\leq$. For a non-empty subset $X \subset S$, let

$$X^{\downarrow} := \{s \in S : \quad \text{there is}\, x \in X \text{ such that}\, s \leq x\}$$

denote the *lower set* or *ideal* generated by X. Let

$$X^{\uparrow} := \{s \in S : \quad \text{there is } x \in X \text{ such that } x \leq s\}$$

denote the *upper set* or *coideal* generated by X.

Lemma 8 *For any subsemigroup $T \subset S$, both $T^{\downarrow}$ and $T^{\uparrow}$ are subsemigroups of S.*

Proof We prove the claim for $T^{\downarrow}$, for $T^{\uparrow}$ the arguments are similar. Let $a, b \in T^{\downarrow}$. Then there exist $s, t \in T$ such that $a \leq s$ and $b \leq t$. As $\leq$ is admissible, we have $ab \leq sb \leq st$. Now, $st \in T$ as T is a subsemigroup, and thus $ab \in T^{\downarrow}$. □

4.3 The symmetric inverse monoid

For $n \in \mathbb{N}$, we denote by IS_n the *symmetric inverse monoid* on $\mathbf{n}$, see [10]. It consists of all bijections between subsets of $\mathbf{n}$. Alternatively, we can identify IS_n with $S_n^{\downarrow}$ inside the ordered monoid $\mathbf{B}_n$. The monoid IS_n is an inverse monoid. The natural partial order on the inverse monoid IS_n coincides with the inclusion order inherited from $\mathbf{B}_n$. The group S_n is the group of invertible elements in IS_n.

4.4 The dual symmetric inverse monoid

For $n \in \mathbb{N}$, we denote by I_n^* the *dual symmetric inverse monoid* on $\mathbf{n}$, see [7]. It consists of all bijections between quotients of $\mathbf{n}$. Alternatively, we can identity I_n^* with $\mathbf{PP}_n$ in the obvious way. The monoid I_n^* is an inverse monoid. The natural partial order on the inverse monoid I_n^* coincides with the order inherited from $\mathbf{P}_n$. The group S_n is the group of invertible elements in I_n^*.

We also consider the *maximal factorizable submonoid* F_n^* of I_n^*, that is the submonoid of all elements which can be written in the form $\sigma\varepsilon$, where $\sigma \in S_n$ and ε is an idempotent in I_n^*. Idempotents in I_n^* are exactly the identity transformations of quotient sets of $\mathbf{n}$, equivalently, idempotents in I_n^* coincide with the principal coideal in $\mathbf{P}_n$ generated by the identity element.

Lemma 9 *The monoid F_n^* coincides with the subsemigroup $S_n^{\uparrow}$ of $\mathbf{PP}_n$.*

Proof As $S_n^{\uparrow}$ contains both S_n and all idempotents of I_n^*, we have $F_n^* \subset S_n^{\uparrow}$. On the other hand, let $\rho \in S_n^{\uparrow}$. Then $\sigma \leq \rho$ for some $\sigma \in S_n$. This means that $\mathrm{id}_{\mathbf{n}} \leq \rho\sigma^{-1}$. Hence $\rho\sigma^{-1}$ is an idempotent and $\rho = (\rho\sigma^{-1})\sigma \in F_n^*$. □

4.5 Fiat 2-categories

Following [29], we say that a finitary 2-category $\mathscr{C}$ is *fiat* provided that there exists a weak anti-involution $\star : \mathscr{C} \to \mathscr{C}^{\mathrm{co,op}}$, such that, for any objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and any 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, there are 2-morphisms $\eta : \mathbb{1}_{\mathtt{i}} \to \mathrm{F}^{\star}\mathrm{F}$ and $\varepsilon : \mathrm{F}\mathrm{F}^{\star} \to \mathbb{1}_{\mathtt{j}}$ such that

$$(\varepsilon \circ_0 \mathrm{id}_{\mathrm{F}}) \circ_1 (\mathrm{id}_{\mathrm{F}} \circ_0 \eta) = \mathrm{id}_{\mathrm{F}} \quad \text{and} \quad (\mathrm{id}_{\mathrm{F}^{\star}} \circ_0 \varepsilon) \circ_1 (\eta \circ_0 \mathrm{id}_{\mathrm{F}^{\star}}) = \mathrm{id}_{\mathrm{F}^{\star}}.$$

This means that F and $\mathrm{F}^{\star}$ are biadjoint in $\mathscr{C}$ and hence also in any 2-representation of $\mathscr{C}$. The above property is usually called *existence of adjunction 2-morphisms*.

There are various classes of 2-categories whose axiomatization covers some parts of the axiomatization of fiat 2-categories, see, for example, *compact categories*, *rigid categories*, *monoidal categories with duals* and *2-categories with adjoints*.

4.6 Comparison of fiatness

Theorem 10 *Let* $n \in \mathbb{N}$.

(*i*) *Both* 2*-categories,* $\overline{\Bbbk\mathscr{A}_n}$ *and* $\overline{\Bbbk\mathscr{B}_n}$*, are fiat.*
(*ii*) *Both* 2*-categories,* $\Bbbk\mathscr{S}_{IS_n}$ *and* $\Bbbk\mathscr{S}_{F_n^*}$*, are finitary but not fiat.*

Proof The endomorphism algebra of any 1-morphism in $\Bbbk\mathscr{S}_{IS_n}$ is $\Bbbk$, by definition. Therefore $\Bbbk\mathscr{S}_{IS_n}$ is finitary by construction. The category $\Bbbk\mathscr{S}_{IS_n}$ cannot be fiat as it contains non-invertible indecomposable 1-morphisms but it does not contain any non-zero 2-morphisms from the identity 1-morphism to any non-invertible indecomposable 1-morphism. Therefore adjunction 2-morphisms for non-invertible indecomposable 1-morphisms cannot exist. The same argument also applies to $\Bbbk\mathscr{S}_{F_n^*}$, proving claim (ii).

By construction, the 2-category $\Bbbk\mathscr{A}_n$ satisfies conditions (I), (II) and (III) from the definition of a finitary 2-category. Therefore the 2-category $\overline{\Bbbk\mathscr{A}_n}$ is a finitary 2-category by Lemma 7. Let us now check existence of adjunction 2-morphisms.

Recall that an adjoint to a direct sum of functors is a direct sum of adjoints to components. Therefore, as $\overline{\Bbbk\mathscr{A}_n}$ is obtained from $(\mathscr{A}_n)_\Bbbk$ by splitting idempotents in 2-endomorphism rings, it is enough to check that adjunction 2-morphisms exist in $(\mathscr{A}_n)_\Bbbk$. Any 1-morphism in $(\mathscr{A}_n)_\Bbbk$ is, by construction, a direct sum of $\sigma \in S_n$. Therefore it is enough to check that adjunction 2-morphisms exist in $\mathscr{A}_n$. In the latter category, each 1-morphism $\sigma \in S_n$ is invertible and hence both left and right adjoint to σ^{-1}. This implies existence of adjunction 2-morphisms in $\mathscr{A}_n$.

The above shows that the 2-category $\overline{\Bbbk\mathscr{A}_n}$ is fiat. Similarly one shows that the 2-category $\overline{\Bbbk\mathscr{B}_n}$ is fiat. This completes the proof. □

5 Decategorification

5.1 Decategorification via Grothendieck group

Let $\mathscr{C}$ be a finitary 2-category. A *Grothendieck decategorification* $[\mathscr{C}]$ of $\mathscr{C}$ is a category defined as follows:

- $[\mathscr{C}]$ has the same objects as $\mathscr{C}$.
- For $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, the set $[\mathscr{C}](\mathtt{i}, \mathtt{j})$ coincides with the split Grothendieck group $[\mathscr{C}(\mathtt{i}, \mathtt{j})]_\oplus$ of the additive category $\mathscr{C}(\mathtt{i}, \mathtt{j})$.
- The identity morphism in $[\mathscr{C}](\mathtt{i}, \mathtt{i})$ is the class of $\mathbb{1}_\mathtt{i}$.
- Composition in $[\mathscr{C}]$ is induced from composition of 1-morphisms in $\mathscr{C}$.

We refer to [28, Lecture 1] for more details.

For a finitary 2-category $\mathscr{C}$, the above allows us to define the *decategorification* of $\mathscr{C}$ as the $\mathbb{Z}$-algebra

$$A_{\mathscr{C}} := \bigoplus_{\mathtt{i},\mathtt{j}\in\mathscr{C}} [\mathscr{C}](\mathtt{i},\mathtt{j})$$

with the induced composition. The algebra $A_{\mathscr{C}}$ is positively based in the sense of [18] with respect to the basis corresponding to indecomposable 1-morphisms in $\mathscr{C}$.

5.2 Decategorifications of $\overline{\Bbbk\mathscr{A}_n}$ and $\Bbbk\mathscr{S}_{IS_n}$

Theorem 11 *We have* $A_{\overline{\Bbbk\mathscr{A}_n}} \cong A_{\Bbbk\mathscr{S}_{IS_n}} \cong \mathbb{Z}[IS_n]$.

Proof Indecomposable 1-morphisms in $\Bbbk\mathscr{S}_{IS_n}$ correspond exactly to elements of IS_n, by construction. This implies that $A_{\Bbbk\mathscr{S}_{IS_n}} \cong \mathbb{Z}[IS_n]$ where an indecomposable 1-morphism σ on the left hand side is mapped to itself on the right hand side. So, we only need to prove that $A_{\overline{\Bbbk\mathscr{A}_n}} \cong \mathbb{Z}[IS_n]$.

For $\sigma \in IS_n$, set

$$\underline{\sigma} := \sum_{\rho\subset\sigma} (-1)^{|\sigma\setminus\rho|}\rho \in \mathbb{Z}[IS_n]. \tag{5.1}$$

Then $\{\underline{\sigma} : \sigma \in IS_n\}$ is a basis in $\mathbb{Z}[IS_n]$ which we call the *Möbius basis*, see, for example, [42, Theorem 4.4].

The endomorphism monoid $\mathrm{End}_{\mathscr{A}_n}(\mathrm{id}_{\mathbf{n}})$ is, by construction, canonically isomorphic to the Boolean $2^{\mathbf{n}}$ of $\mathbf{n}$ with both $\circ_0$ and $\circ_1$ being equal to the operation on $2^{\mathbf{n}}$ of taking the intersection. We identify elements in $\mathrm{End}_{\mathscr{A}_n}(\mathrm{id}_{\mathbf{n}})$ and in $2^{\mathbf{n}}$ in the obvious way. With this identification, in the construction of $\overline{\Bbbk\mathscr{A}_n}$, we can take, for $X \subset \mathbf{n}$,

$$\varepsilon_{\mathtt{i}}^{(X)} = \sum_{Y\subseteq X} (-1)^{|X|-|Y|} Y. \tag{5.2}$$

For $\sigma \in S_n$ and $X, Y \subset \mathbf{n}$, consider the element

$$\varepsilon_{\mathtt{i}}^{(Y)} \circ_0 \sigma \circ_0 \varepsilon_{\mathtt{i}}^{(X)} \in \mathrm{End}_{\Bbbk\mathscr{A}_n}(\sigma) \tag{5.3}$$

and write it as a linear combination of subrelations of σ [this is the standard basis in $\mathrm{End}_{\Bbbk\mathscr{A}_n}(\sigma)$]. A subrelation $\rho \subset \sigma$ may appear in this linear combination with a non-zero coefficient only if ρ consist of pairs of the form (y, x), where $x \in X$ and $y \in Y$.

Assume that $\sigma(X) = Y$. Then the relation

$$\rho_\sigma = \bigcup_{x\in X} \{(\sigma(x), x)\},$$

clearly, appears in the linear combination above with coefficient one. Moreover, the idempotent properties of $\varepsilon_{\mathtt{i}}^{(X)}$ and $\varepsilon_{\mathtt{i}}^{(Y)}$ imply that the element in (5.3) is exactly $\underline{\rho_\sigma}$.

Assume that $\sigma(X) \neq Y$. Then the inclusion-exclusion formula implies that any subrelation of σ appears in the linear combination above with coefficient zero. This means that the 1-morphism $\sigma \in \overline{\Bbbk\mathscr{A}_n}(\mathtt{i}^{(Y)}, \mathtt{i}^{(X)})$ is zero if and only if $\sigma(X) \neq Y$.

If $|X| = |Y|$ and $\sigma, \pi \in S_n$ are such that $\sigma(x) = \pi(x) \in Y$, for all $x \in X$, then

$$\underline{\rho_\sigma} = \underline{\rho_\pi} \in \mathrm{Hom}_{\Bbbk\mathscr{A}_n}(\sigma, \pi) \cap \mathrm{Hom}_{\Bbbk\mathscr{A}_n}(\pi, \sigma)$$

gives rise to an isomorphism between σ and π in $\overline{\Bbbk\mathscr{A}_n}(\mathtt{i}^{(Y)}, \mathtt{i}^{(X)})$. If $\sigma(x) \neq \pi(x)$, for some $x \in X$, then any morphism in $\mathrm{Hom}_{\Bbbk\mathscr{A}_n}(\sigma, \pi)$ is a linear combination of relations which are properly contained in both ρ_σ and ρ_π. Therefore σ and π are not isomorphic in $\Bbbk\mathscr{A}_n$.

Consequently, isomorphism classes of indecomposable 1-morphisms in the category $\overline{\Bbbk\mathscr{A}_n}(\mathtt{i}^{(Y)}, \mathtt{i}^{(X)})$ correspond precisely to elements in IS_n with domain X and image Y. Composition of these indecomposable 1-morphisms is inherited from S_n. By comparing formulae (5.1) and (5.2), we see that composition of 1-morphisms in $\overline{\Bbbk\mathscr{A}_n}$ corresponds to multiplication of the Möbius basis elements in $\mathbb{Z}[IS_n]$. This completes the proof of the theorem. □

Theorem 11 allows us to consider $\overline{\Bbbk\mathscr{A}_n}$ and $\mathscr{S}_{IS_n}$ as two different categorifications of IS_n. The advantage of $\overline{\Bbbk\mathscr{A}_n}$ is that this 2-category is fiat.

The construction we use in our proof of Theorem 11 resembles the partialization construction from [20].

5.3 Decategorifications of $\overline{\Bbbk\mathscr{B}_n}$ and $\Bbbk\mathscr{S}_{F_n^*}$

Theorem 12 *We have* $A_{\overline{\Bbbk\mathscr{B}_n}} \cong A_{\Bbbk\mathscr{S}_{F_n^*}} \cong \mathbb{Z}[F_n^*]$.

Proof Using the Möbius function for the poset of all quotients of $\mathbf{n}$ with respect to $\leq$ (see, for example, [39, Example 1]), Theorem 12 is proved mutatis mutandis Theorem 11. □

Theorem 12 allows us to consider $\overline{\Bbbk\mathscr{B}_n}$ and $\mathscr{S}_{F_n^*}$ as two different categorifications of F_n^*. The advantage of $\overline{\Bbbk\mathscr{B}_n}$ is that this 2-category is fiat.

The immediately following examples are in low rank, but show that these constructions can be worked with at the concrete as well as the abstract level. In particular, they illustrate the difference between the two constructions.

6 Examples for $n = 2$

6.1 Example of F_2^*

The monoid F_2^* consists of three elements which we write as follows:

$$\epsilon := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \sigma := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \tau := \begin{pmatrix} \{1,2\} \\ \{1,2\} \end{pmatrix}.$$

These are identified with the following partitions of $\{1, 2, 1', 2'\}$:

$$\epsilon \leftrightarrow \big\{\{1, 1'\}, \{2, 2'\}\big\}, \quad \sigma \leftrightarrow \big\{\{1, 2'\}, \{2, 1'\}\big\}, \quad \tau \leftrightarrow \big\{\{1, 2, 1', 2'\}\big\}.$$

The symmetric group S_2 consists of ϵ and σ.

Here is the table showing all 2-morphisms in $\mathscr{B}_2$ from x to y:

$y\backslash x$	ϵ	σ	τ
ϵ	ϵ, τ	τ	τ
σ	τ	σ, τ	τ
τ	τ	τ	τ

The 2-endomorphism algebra of both ϵ and σ in $(\mathscr{B}_2)_\Bbbk$ is isomorphic to $\Bbbk \oplus \Bbbk$ where the primitive idempotents are τ and $\epsilon - \tau$, in the case of ϵ, and τ and $\sigma - \tau$, in the case of σ.

The 2-category $\Bbbk\mathscr{B}_2$ has three isomorphism classes of indecomposable 1-morphisms, namely τ, $\epsilon - \tau$ and $\sigma - \tau$.

The 2-category $\overline{\Bbbk\mathscr{B}_2}$ has two objects, $\mathtt{i}_\tau$ and $\mathtt{i}_{\epsilon-\tau}$. The indecomposable 1-morphisms in $\Bbbk\mathscr{B}_2$ give indecomposable 1-morphisms in $\overline{\Bbbk\mathscr{B}_2}$ from x to y as follows:

$y\backslash x$	$\mathtt{i}_{\epsilon-\tau}$	$\mathtt{i}_\tau$
$\mathtt{i}_{\epsilon-\tau}$	$\epsilon - \tau, \sigma - \tau$	$\varnothing$
$\mathtt{i}_\tau$	$\varnothing$	τ

6.2 Example of IS_2

We write elements of IS_2 as follows:

$$\epsilon := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \sigma := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \tau := \begin{pmatrix} 1 & 2 \\ \varnothing & \varnothing \end{pmatrix},$$

$$\alpha := \begin{pmatrix} 1 & 2 \\ 1 & \varnothing \end{pmatrix}, \quad \beta := \begin{pmatrix} 1 & 2 \\ 2 & \varnothing \end{pmatrix}, \quad \gamma := \begin{pmatrix} 1 & 2 \\ \varnothing & 1 \end{pmatrix}, \quad \delta := \begin{pmatrix} 1 & 2 \\ \varnothing & 2 \end{pmatrix}.$$

The symmetric group S_2 consists of ϵ and σ.

Here is the table showing all 2-morphisms in $\mathscr{A}_2$ from x to y:

$y\backslash x$	ϵ	σ	τ	α	β	γ	δ
ϵ	$\epsilon, \alpha, \delta, \tau$	τ	τ	α, τ	τ	τ	δ, τ
σ	τ	$\sigma, \beta, \gamma, \tau$	τ	τ	β, τ	γ, τ	τ
τ	τ	τ	τ	τ	τ	τ	τ
α	α, τ	τ	τ	α, τ	τ	τ	τ
β	τ	β, τ	τ	τ	β, τ	τ	τ
γ	τ	β, τ	τ	τ	τ	γ, τ	τ
δ	α, τ	τ	τ	τ	τ	τ	δ, τ

The 2-endomorphism algebra of ϵ in $(\mathscr{A}_2)_{\Bbbk}$ is isomorphic to $\Bbbk \oplus \Bbbk \oplus \Bbbk \oplus \Bbbk$ where the primitive idempotents are τ, $\alpha - \tau$, $\delta - \tau$ and $\epsilon - \alpha - \delta + \tau$. Similarly one can describe the 2-endomorphism algebra of σ in $(\mathscr{A}_2)_{\Bbbk}$. The 2-endomorphism algebra of α in $(\mathscr{A}_2)_{\Bbbk}$ is isomorphic to $\Bbbk \oplus \Bbbk$ where the primitive idempotents are τ and $\alpha - \tau$. Similarly one can describe the 2-endomorphism algebras of β, γ and δ.

The 2-category $\Bbbk\mathscr{A}_2$ has seven isomorphism classes of indecomposable 1-morphisms, namely

$$\tau, \quad \alpha - \tau, \quad \beta - \tau, \quad \gamma - \tau, \quad \delta - \tau, \quad \epsilon - \alpha - \delta + \tau, \quad \sigma - \beta - \gamma + \tau.$$

The 2-category $\overline{\Bbbk\mathscr{A}_2}$ has four objects, $\mathtt{i}_{\tau}$, $\mathtt{i}_{\alpha-\tau}$, $\mathtt{i}_{\delta-\tau}$ and $\mathtt{i}_{\epsilon-\alpha-\delta+\tau}$. The indecomposable 1-morphisms in $\Bbbk\mathscr{A}_2$ give indecomposable 1-morphisms in $\overline{\Bbbk\mathscr{A}_2}$ from x to y as follows:

$y\backslash x$	$\mathtt{i}_{\epsilon-\alpha-\delta+\tau}$	$\mathtt{i}_{\alpha-\tau}$	$\mathtt{i}_{\delta-\tau}$	$\mathtt{i}_{\tau}$
$\mathtt{i}_{\epsilon-\alpha-\delta+\tau}$	$\epsilon - \alpha - \delta + \tau, \sigma - \beta - \gamma + \tau$	$\varnothing$	$\varnothing$	$\varnothing$
$\mathtt{i}_{\alpha-\tau}$	$\varnothing$	$\alpha - \tau$	$\gamma - \tau$	$\varnothing$
$\mathtt{i}_{\delta-\tau}$	$\varnothing$	$\beta - \tau$	$\delta - \tau$	$\varnothing$
$\mathtt{i}_{\tau}$	$\varnothing$	$\varnothing$	$\varnothing$	τ

This table can be compared with [35, Figure 1].

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