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Discrete-Time Dynamical Networks with Diagonal Controllability Gramian ⋆

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Abstract: The controllability Gramian of a dynamical network carries rich information of the fundamental properties of the network. How to identify the connections from these fundamental properties to the network topology and weights is of great interest. It is, however, very challenging to do that because the Gramian is an extremely complicated function of the network topology and weights. In this paper, we consider the simplest case where the Gramian is diagonal. One of the main contributions of this paper is to prove the necessary and sufficient graphical conditions for a discrete-time dynamical network to feature a diagonal Gramian. The explicit relations between the values of the diagonal entries of the Gramian and the network weights are also established. The proposed results may be used to design networks with desired control energy and robustness performance.

1. INTRODUCTION

The controllability Gramian of a dynamical network carries rich information of the fundamental properties of the network. For example, its nonsingularity indicates that the network is controllable; its eigenvalues quantify the minimum control energy required to steer the network state along the eigenvectors [Yan et al. 2012, Pasqualetti et al. 2014, Cortesi et al. 2014, Kumar et al. 2015, Yan et al. 2015, Bof et al. 2016, Zhao and Cortes 2016, Tzoumas et al. 2016]; and its trace measures how robust the network state is against external disturbance [Summers et al. 2016, Zhou et al. 1995]. It is of great interest to identify the connections from these properties to the network structure and weights. These connections, which are generally difficult to characterize, can be used to design networks to achieve desired properties such as control energy and robustness performance.

In this paper, we study the simplest case of when a network features a *diagonal* Gramian. Since the Gramian is a solution to the Lyapunov equation, our approach is to study when the Lyapunov equation has a unique positive definite diagonal solution. While continuous-time Lyapunov equations with diagonal solutions have been studied in the context of D-stability [Kaszkurewicz and Hsu 1984, Geromel 1985, Hershkowitz 1992], in our work we focus on the discrete-time case and explore its application to network design.

The main contribution of this paper is to prove the necessary and sufficient graphical conditions for a discrete-time network to feature a diagonal controllability Gramian. In particular, we prove that the Gramian of a network with a single control input is diagonal if and only if the network is a stem or a bud (see Theorem 1). When there are multiple control inputs, we show that the Gramian is diagonal if and only if the network is a combination of stem and bud networks (see Theorem 2). Additionally, we also derive the expression of the diagonal entries of the Gramian in terms of the network weights. With the proposed results, we are able to design networks to feature any desired control energy or robust performance.

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2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Network Dynamics

Consider a network with n nodes and n_c independent control inputs. The control inputs are injected into the network through n_c distinct *control nodes*. The network interaction is described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $\mathcal{V}_c = \{k_1, \dots, k_{n_c}\} \subseteq \mathcal{V}$ be the set of control nodes. The network dynamics are described by the linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the network state and $u(t) \in \mathbb{R}^{n_c}$ is the input vector. The matrix $A = [a_{ij}]$ is the weighted adjacency matrix of the graph $\mathcal{G}$, where $a_{ij} \neq 0$ when there is a directed edge from node j to node i. Two nodes are called adjacent if either $a_{ij} \neq 0$ or $a_{ji} \neq 0$. The input matrix is

$$B = [\mathrm{e}_{k_1}, \mathrm{e}_{k_2}, \dots, \mathrm{e}_{k_{n_c}}] \in \mathbb{R}^{n \times n_c}, \tag{2}$$

where e_{k_i} is the k_ith canonical vector of dimension n.

In this paper, we always assume that the network is connected. If the network consists of disconnected components, the results presented in this paper are applicable to each disconnected component. Finally, let d_i^{in} and d_i^{out} be the in-degree and out-degree of node i, respectively. The value of d_i^{in} (d_i^{out}) equals the number of edges entering (leaving) node i, that is, the number of nonzero entries in the ith row (column) of A.

2.2 Controllability Gramian

The dynamical system (1) or the pair (A, B) is controllable if and only if the *controllability matrix*

$$K := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times(nn_c)} \tag{3}$$

has full row rank. Controllability can also be evaluated based on the *controllability Gramian*, which is defined as

$$W = \sum_{k=0}^{\infty} A^k BB^{\mathrm{T}}(A^k)^{\mathrm{T}}. \tag{4}$$

The controllability Gramian is an n by n positive semi-definite matrix, and it becomes positive definite (i.e., nonsingular) if and only if the system is controllable [Zhou et al. 1995, Lemma 21.2]. We write $W > 0$ ($W \geq 0$) when W is positive definite (positive semi-definite).

For unstable systems, the calculation of W by (4) may diverge and hence W may not be well defined. For stable systems, W is well defined and it equals the unique solution to the Lyapunov equation

$$AWA^T - W = -BB^T. \tag{5}$$

More information on equation (5) can be found in [Zhou et al. 1995, Lemma 21.2].

2.3 Nodal Energy

The minimum energy required to control a network is usually of great theoretical and practical interest. This minimum energy can be calculated from the Gramian. In particular, if x_f is the desired final state, the minimum energy required to drive the state from the origin to x_f over the infinite time horizon is $x_f^{\mathrm{T}} W^{-1} x_f$ [Pasqualetti et al. 2014]. If x_f is a unit-norm eigenvector of W, then the minimum energy equals $x_f^{\mathrm{T}} W^{-1} x_f = 1/\lambda$, where λ is the eigenvalue associated with x_f. The value of $1/\lambda$ is referred to as *eigen-energy* in [Yan et al. 2015]. Clearly a small eigenvalue corresponds to large eigen-energy. In the special yet important case of $x_f = \mathrm{e}_i$, we have

$$\varepsilon_i := \mathrm{e}_i^{\mathrm{T}} W^{-1} \mathrm{e}_i = [W^{-1}]_{ii},$$

where $[W^{-1}]_{ii}$ denotes the ith diagonal entry of W^{-1}. The value of ε_i is referred to as ith *nodal energy* in this paper, The nodal energy is of particular interest because it has a clear intuitive interpretation: the ith nodal energy is the energy required to drive the state of node i from 0 to 1, while leaving the final states of the other nodes to 0.

Nodal and eigen energies are usually different for general networks. They, however, coincide with each other when W is diagonal because the canonical vectors $\mathrm{e}_1, \ldots, \mathrm{e}_n$ are eigenvectors of W in this case. In particular, if $W = \mathrm{diag}(w_1, \ldots, w_n) > 0$, both the ith nodal energy and eigen-energy equal to

$$\varepsilon_i = \frac{1}{w_i}.$$

When W is diagonal, nodal energies also indicate the robustness of the states against input disturbance. In particular, in addition to the input dynamics (1), consider the output $y(t) = Cx(t)$ with C as a given output matrix. Let $G(z)$ be the transfer function of the discrete-time system (A, B, C). The $\mathcal{H}_2$ norm of $G(z)$ can be computed as

$$\|G\|_2^2 = \mathrm{tr}(CWC^{\mathrm{T}}). \tag{6}$$

The derivation of (6) can be found in [Zhou et al. 1995, Remark 21.6]. The $\mathcal{H}_2$ norm can be interpreted as the expected root mean square value of the output in response to white noise excitation or, equivalently, the energy of the output response to unit impulse inputs [Zhou et al. 1995]. If $C = \mathrm{e}_i^{\mathrm{T}}$, then $y(t) = x_i(t)$. Substituting $C = \mathrm{e}_i^{\mathrm{T}}$ into (6) gives

$$\|G\|_2^2 = \frac{1}{\varepsilon_i}, \tag{7}$$

which indicates that the inverse of the nodal energy, $1/\varepsilon_i$, equals the $\mathcal{H}_2$ norm of the network when the output is $y(t) = x_i(t)$. As a result, a larger nodal energy of a node leads to less sensitivity or stronger robustness of the state against input disturbances.

2.4 Problem Statement

Nodal energies quantify both network controllability and state robustness when the Gramian is diagonal. It is important to study how to design networks that feature specified nodal energies. This problem, which is solved in this paper, is formally stated below.

Problem 1. (**Network nodal energy design**). Given a network with node set $\mathcal{V}$, control node set $\mathcal{V}_c \subseteq \mathcal{V}$, input matrix B as in (2), and desired nodal energies $\{\varepsilon_i\}_{i=1}^n$ with $\varepsilon_i > 0$, the task is to design the network adjacency matrix A such that the following three conditions hold:

(a) A is stable,
(b) (A, B) is controllable, and
(c) $W = \mathrm{diag}(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1})$ is the controllability Gramian.

Problem 1 is to design a stable and controllable network that features a specified positive definite diagonal Gramian. The nodal energies $\{\varepsilon_i\}_{i=1}^n$ can be set according to practical requirement. For example, if we wish to render node i very robust against any external disturbance, then we can set ε_i to be large.

Our approach to solve Problem 1 is to first identify the graphical conditions for the controllability Gramian to be diagonal. Considering that the Gramian solves the Lyapunov equation, we next define a notion that will be used throughout the paper.

Definition 1. (**Diagonally admissible networks**). Given a positive definite diagonal Gramian W and an input matrix B as in (2), a network with the adjacency matrix A is *diagonally admissible* for W if $AWA^T - W = -BB^{\mathrm{T}}$.

A network that solves Problem 1 must be diagonally admissible. The converse is, however, not true because a diagonally admissible network may not be stable or controllable. Thus, we must study when a diagonally admissible network is both stable and controllable.

We next derive algebraic conditions for diagonally admissible networks. Let $W = \mathrm{diag}(w_1, \ldots, w_n)$ with $w_i > 0$ for all i. Comparing the diagonal entries of the both sides of the Lyapunov equation gives

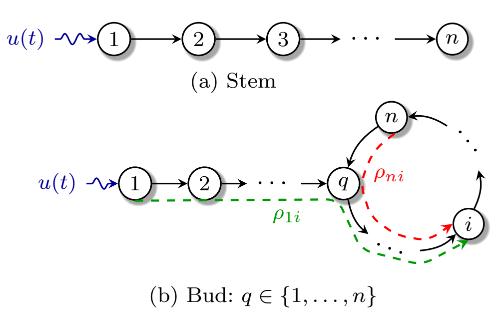


Fig. 1. An illustration of stem and bud networks.

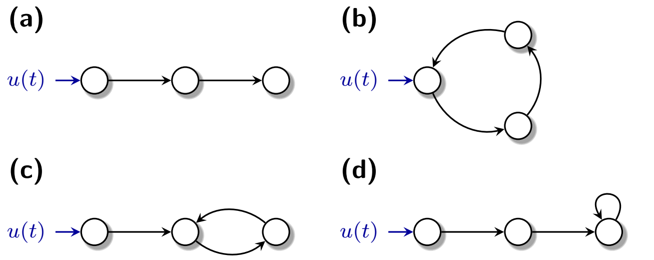


Fig. 2. All the possible stem and bud networks in the case of $n = 3$.

$$[AWA^{\mathrm{T}} - W]_{ii} = \sum_{k=1}^{n} a_{ik}^2 w_k - w_i = \begin{cases} -1, & i \in \mathcal{V}_c, \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Comparing the off-diagonal entries of both sides gives

$$[AWA^{\mathrm{T}} - W]_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} w_k = 0, \quad i, j \in \mathcal{V}, i \neq j. \tag{9}$$

Equations (8) and (9) are necessary and sufficient conditions for the network to be diagonally admissible, but they need to be further explored to reveil their graphical interpretation. Since (9) is difficult to analyze when the edge weights may assume arbitrary values, we make the following assumption.

Assumption 1. (**Positive Edge Weights**). All nonzero entries of A are positive.

In the rest of the paper, we first consider the case where the network has a single input and then analyze the multi-input case.

3. NETWORKS WITH SINGLE INPUTS

In this section, we consider discrete-time networks with single control inputs, and derive conditions for stability and controllability of diagonally admissible networks. We start with some important definitions.

Definition 2. (**Stem and Bud Networks**).

(a) A *stem* network is of the form as shown in Figure 1(a), where the network is a path starting from a control node.
(b) A *bud* network is of the form as shown in Figure 1(b), where the network is a stem combined with the edge pointing from the ending node to an arbitrary node in the stem including the ending node itself.

All the edges in a stem or bud network are directed. To illustrate, Figure 2 shows all the possible stem and bud networks with 3 nodes. The definition of stem and bud networks in our work is different from [Lin 1974] because (i) the location of the control node is specified and (ii) the joint node q in a bud network may be any node in the network.

For bud networks, two useful weight products are defined below. For the directed path $(1, 2, \ldots, i)$ from control node 1 to node i, define the weight product ρ_{1i} as

$$\rho_{1i} = \begin{cases} 1, & i = 1, \\ a_{21}^2 a_{32}^2 \ldots a_{i(i-1)}^2, & i \geq 2. \end{cases}$$

See Figure 2(b) for an illustration of ρ_{1i}. For the directed path $(n, q, q+1, \ldots, i)$ from node n to node i, define the weight product ρ_{ni} as

$$\rho_{ni} = \begin{cases} 0, & i < q, \\ a_{qn}^2, & i = q, \\ a_{qn}^2 a_{(q+1)q}^2 \ldots a_{i(i-1)}^2, & i > q. \end{cases}$$

See Figure 2(b) for an illustration of ρ_{ni}. Since a stem network can be viewed as a special case of a bud network with $a_{qn} = 0$, the weight products defined above are also applicable to stem networks.

With the above definitions, we are able to give necessary and sufficient graphical condition for a diagonally admissible network.

Theorem 1. (**Graphical Condition**). Under Assumption 1, a discrete-time dynamical network with a single control input is controllable and diagonally admissible if and only if it is a stem or bud. Moreover, the network is admissible for the Gramian $W = \mathrm{diag}(w_1, \ldots, w_n)$ with

$$w_i = \rho_{1i} + \rho_{ni} w_n, \quad i \in \mathcal{V}, \tag{10}$$

where

$$w_n = \frac{\rho_{1n}}{1 - \rho_{nn}}. \tag{11}$$

Proof. The proof consists of two parts. In the first part, we determine the topology of the network by analyzing (9). In the second part, we determine the expression of the diagonal Gramian by analyzing (8).

Part 1: Network topology (Necessity) Suppose the network is controllable and diagonally admissible (i.e., satisfying (8) and (9)). Since all edge weights are assumed to be positive, equation (9) indicates that

$$a_{ik} a_{jk} = 0, \quad \forall i, j, k \in \mathcal{V}, i \neq j,$$

which means each column of A has at most one nonzero entry. As a result, there are at most n directed edges in the network. On the other hand, since we assume the network is connected, it must have at least $n-1$ directed edges. Hence the number of edges n_e in the network satisfies

$$n - 1 \leq n_e \leq n. \tag{12}$$

We next determine the topology of the network by studying the in- and out-degrees of each node. In the sequel, we call the nodes that are not control nodes as *follower nodes*. Since each column of A has at most one nonzero entry, we have $d_i^{\mathrm{out}} \leq 1$ for all i. For a control node, we must have $d_i^{\mathrm{out}} \geq 1$; otherwise, the follower nodes would not be reachable from the control input and hence the network would not be controllable. As a result, we have

$$d_i^{\mathrm{out}} = \begin{cases} 1, & \text{node } i \text{ is a control node}, \\ 0 \text{ or } 1, & \text{node } i \text{ is a follower node}. \end{cases} \tag{13}$$

Since the network is controllable, every follower node must have at least one in-degree and hence

$$d_i^{\text{in}} \geq \begin{cases} 0 & \text{node } i \text{ is a control node,} \\ 1 & \text{node } i \text{ is a follower node.} \end{cases} \quad (14)$$

Moreover, note $\sum_{i=1}^n d_i^{\text{in}} = \sum_{i=1}^n d_i^{\text{out}} = n_e$. Since n_e equals either $n-1$ or n by (12), we study the two cases respectively.

(a) *Case 1:* $n_e = n-1$. Due to $\sum_{i=1}^n d_i^{\text{out}} = n-1$ and (13), we know $d_i^{\text{out}} = 1$ for $n-1$ nodes and $d_i^{\text{out}} = 0$ for one (follower) node. Due to $\sum_{i=1}^n d_i^{\text{in}} = n-1$ and (14), we know $d_i^{\text{in}} = 1$ for $n-1$ follower nodes and $d_i^{\text{in}} = 0$ for the control node. Therefore, for the control node we have $d^{\text{in}} = 0$ and $d^{\text{out}} = 1$; for $n-2$ follower nodes we have $d^{\text{in}} = d^{\text{out}} = 1$; and for the remaining follower node we have $d^{\text{in}} = 1$ and $d^{\text{out}} = 0$. With these in- and out-degrees, the topology of the network must be a stem (see Figure 2(a) for illustration).

(b) *Case 2:* $n_e = n$. Due to $\sum_{i=1}^n d_i^{\text{out}} = n$ and (13), we know $d_i^{\text{out}} = 1$ for all i. Due to $\sum_{i=1}^n d_i^{\text{in}} = n$ and (14), we have (i) $d_i^{\text{in}} = 1$ for all nodes, or (ii) $d_i^{\text{in}} = 0$ for the control node, $d_i^{\text{in}} = 1$ for $n-2$ follower nodes, and $d_i^{\text{in}} = 2$ for one follower node. For the subcase (i), we have $d_i^{\text{out}} = d_i^{\text{in}} = 1$ for all nodes and consequently the network is a circle (see Figure 2(b) for illustration). For the subcase (ii), due to the in- and out-degrees of the nodes, the network must be a stem network together with a directed edge pointing from the rightmost node to any other node except the control node (see Figure 2(c)-(d) for illustration).

To sum up, the network has one of the topologies as shown in Figure 1, which is either a stem or bud.

(Sufficiency) If the network has the topology as shown in Figure 1, it is obvious that (9) is satisfied. Moreover, by indexing the nodes properly, we have the adjacency and input matrix as

$$A = \begin{bmatrix} 0 & & & & 0 \\ & & & & \vdots \\ a_{21} & 0 & & & \vdots \\ & a_{32} & 0 & & a_{qn} \\ & & \ddots & \ddots & \vdots \\ & & & a_{n(n-1)} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad (15)$$

where a_{qn} can be either zero or positive. It can be easily calculated that the controllability matrix is expressed as

$$\begin{aligned} K &= [B, AB, A^2B, \ldots, A^{n-1}B] \\ &= \text{diag}\left(1, a_{21}, a_{21}a_{32}, \ldots, \prod_{i=2}^n a_{i(i-1)}\right) > 0, \end{aligned}$$

which indicates the network is always controllable.

Part 2: Expression of the diagonal Gramian If the network is a stem or bud, substituting (15) into (8) gives

$$\begin{bmatrix} -1 & & & & 0 \\ a_{21}^2 & -1 & & & \\ & a_{32}^2 & -1 & & a_{qn}^2 \\ & & \ddots & \ddots & \\ & & & a_{n(n-1)}^2 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \quad (16)$$

In order to solve w_i from (16), we consider three cases: (i) $a_{qn} = 0$; (ii) $a_{qn} \neq 0$ and $q = 1$; and (iii) $a_{qn} \neq 0$ and $q \geq 2$. We first solve case (iii) which is the most general one. In case (iii), equation (16) can be rewritten as

$$\begin{aligned} -w_1 &= -1, \\ a_{i(i-1)}^2 w_{i-1} - w_i &= 0, \quad 1 < i < q \\ a_{i(i-1)}^2 w_{i-1} - w_i + a_{qn}^2 w_n &= 0, \quad i = q \\ a_{i(i-1)}^2 w_{i-1} - w_i &= 0, \quad i > q \end{aligned}$$

which implies that

$$\begin{aligned} w_1 &= 1, \\ w_i &= a_{21}a_{32}\ldots a_{i(i-1)}w_1 = \rho_{1i}w_1, \quad 1 < i < q \\ w_q &= \rho_{1q}w_1 + a_{qn}^2 w_n, \quad i = q \\ w_i &= \rho_{1i}w_1 + a_{qn}^2 a_{(q+1)q}^2 \cdots a_{i(i-1)}^2 w_n, \quad i > q \\ &= \rho_{1i}w_1 + \rho_{ni}w_n. \end{aligned}$$

Due to the definition of ρ_{1i} and ρ_{ni}, the expression of w_i can be written in a unified way as (10). In case (i) where $a_{qn} = 0$, it is easy to see that (10) still holds since $\rho_{ni} = 0$ for all i. In case (ii) where $a_{qn} \neq 0$ and $q = 1$, we have $w_1 = 1 + a_{1n}^2 w_n$ and $w_i = a_{i(i-1)}^2 w_{i-1}$, which can also be expressed in (10). In order to calculate w_n, we substitute $i = n$ into (10) and obtain $w_n = \rho_{1n} + \rho_{nn}w_n$, which implies (11). □

The expression of w_i in equation (10) has a clear graphical meaning. In the case of stem, we have $w_i = \rho_{1i}$ for all i. In the case of bud, we have

$$w_i = \begin{cases} \rho_{1i} & i < q. \\ \rho_{1i} + \rho_{ni}w_n & i \geq q. \end{cases}$$

It is obvious that the control energy of node i ($i \geq q$) is influenced jointly by nodes 1 and n. The expression of w_i suggests that larger edge weights would yield larger w_i and consequently less control energy.

The converse problem, which is important for network design, is to determine the edge weights given desired w_i or ε_i. In the simplest case where the network is a stem as in Figure 1(a), if the specified nodal energies are $\{\varepsilon_i\}_{i=1}^n$ where $\varepsilon_1 = 1$, then the nodal energies can be achieved by setting the edge weights as

$$a_{i(i-1)} = \sqrt{\frac{\varepsilon_{i-1}}{\varepsilon_i}}, \quad i = 2, \ldots, n.$$

That is because in this case we have $\rho_{1i} = a_{21}^2 \ldots a_{i(i-1)}^2 = \varepsilon_1/\varepsilon_i = 1/\varepsilon_i$ for all i and, consequently, $W = \text{diag}(\rho_{11}, \ldots, \rho_{1n}) = \text{diag}(\varepsilon_1^{-1}, \ldots, \varepsilon_n^{-1})$ according to Theorem 1. It is worth mentioning that if the network is a stem then ε_1 can only be selected as 1 because $\varepsilon_1 = 1/\rho_{11}$ where $\rho_{11} = 1$.

Finally, the conditions in Theorem 1 may lead to unstable networks. In order to ensure the network stability, we need an additional condition.

Proposition 1. (**Stability Condition**). A controllable and diagonally admissible discrete-time network is stable if and only if $\rho_{nn} < 1$.

Proof. The adjacency matrix A of a controllable and diagonally admissible network can be written as the form in (15). It can be verified that $\det(\lambda I - A) = \lambda^{q-1}(\lambda^{n-q+1} - a_{qn}a_{(q+1)q}\cdots a_{n(n-1)}) = \lambda^{q-1}(\lambda^{n-q+1} - \sqrt{\rho_{nn}})$. Therefore, the spectral radius of A is less than 1 if and only if $\rho_{nn} < 1$. □

The intuition behind Proposition 1 is clear: since ρ_{nn} is the gain for a signal propagating along the cycle, if $\rho_{nn} > 1$, any perturbation of the state away from the equilibrium would be amplified while propagating along the cycle and hence cause network instability.

4. NETWORKS WITH MULTIPLE INPUTS

In this section we consider discrete-time networks with multiple control inputs. We show that the multiple-input case can be converted to a set of single-input cases.

When there are multiple inputs, the input matrix has the form of $B = [\cdots, e_i, \cdots]$ where $i \in \mathcal{V}_c$. Then, we have $BB^{\mathrm{T}} = \sum_{i\in\mathcal{V}_c} e_i e_i^{\mathrm{T}}$ and consequently the Gramian is

$$W = \sum_{k=0}^{\infty} A^k BB^{\mathrm{T}}(A^k)^{\mathrm{T}} = \sum_{k=0}^{\infty} A^k \left(\sum_{i\in\mathcal{V}_c} e_i e_i^{\mathrm{T}}\right)(A^k)^{\mathrm{T}}$$
$$= \sum_{i\in\mathcal{V}_c} \underbrace{\sum_{k=0}^{\infty} A^k e_i e_i^{\mathrm{T}} (A^k)^{\mathrm{T}}}_{W_i}. \quad (17)$$

The matrix W_i is the Gramian of the network with control input i. It is obvious that if all W_i are diagonal, then W is also diagonal. The converse is also true because all entries of W_i are nonnegative due to that all entries of A are nonnegative. We have the following result.

Lemma 1. Under Assumption 1, the Gramian W in (17) is diagonal if and only if W_i is diagonal for all $i \in \mathcal{V}_c$.

It is notable that W_i may be singular because there may exist some nodes unreachable from control node i (see Figure 3 for illustration). When W_i is singular, it is important to study under what conditions $W = \sum_{i\in\mathcal{V}_c} W_i$ is nonsingular. In order to solve this problem, we introduce the following definitions. If there is a directed path from a control input to a given node, then the given node is called *accessible* by the control input; otherwise, it is called *unaccessible*. The accessible nodes for a control input compose a subnetwork as defined below.

Definition 3. (**Accessible Subnetwork**). The *accessible subnetwork* of a control input is the network obtained by deleting all the unaccessible nodes and the associated edges from the original network.

An illustration of accessible subnetworks is given in Figure 3, where the accessible subnetworks for each input are highlighted.

With the above preparation, we are ready to present the necessary and sufficient graphical condition for multiple-input diagonally admissible networks.

Theorem 2. (**Graphical Condition**). Under Assumption 1, a discrete-time dynamical network with multiple control inputs is controllable and diagonally admissible if and only if the following conditions hold:

(a) For each control input, the accessible subnetwork is a stem or bud;
(b) Each node is accessible by at least one control input.

Proof. According to Lemma 1, W is diagonal if and only if W_i is diagonal for all $i \in \mathcal{V}_c$. We next analyze the graphical conditions for W_i to be diagonal. Consider the case where control input i is the only control input and all the other inputs are removed. Without loss of generality, we can permute the states such that the state vector $x(t)$ can be expressed as $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, where $x_1(t)$ and $x_2(t)$ are the states corresponding to the accessible and unaccessible nodes, respectively. The network dynamics can be expressed as

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t),$$

where the adjacency matrix and input matrix are partitioned into block matrices according to the accessible and unaccessible nodes. Consequently, the Gramian W_i is expressed by

$$W_i = \sum_{k=0}^{\infty} \begin{bmatrix} A_{11}^k & * \\ 0 & A_{22}^k \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} B_1^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} (A_{11}^k)^{\mathrm{T}} & 0 \\ * & (A_{22}^k)^{\mathrm{T}} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=0}^{\infty} A_{11}^k B_1 B_1^{\mathrm{T}} (A_{11}^k)^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{W}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad (18)$$

where $*$ denotes matrix entries that do not contribute to derivation. Equation (18) indicates that W_i is diagonal if and only if the matrix $\tilde{W}_i$ is diagonal. The matrix $\tilde{W}_i$ is the Gramian of the accessible subnetwork with the control input on node i. According to Theorem 1, the Gramian $\tilde{W}_i$ is diagonal if and only if the accessible subnetwork is a stem or bud network, which proves condition (a) in the theorem.

We next analyze when $W = \sum_{i\in\mathcal{V}_c} W_i$ is nonsingular. Equation (18) indicates that the diagonal entries of W_i that correspond to the unaccessible nodes of control input i are zero. Since $W = \sum_{i\in\mathcal{V}_c} W_i$, there are no zero diagonal entries in W if and only if there are no unaccessible nodes for any control input, which proves condition (b) in the theorem. □

In Theorem 2, condition (a) ensures that W is diagonal and condition (b) guarantees that W is nonsingular (i.e., the entire network is controllable). The two conditions are illustrated by an example in Figure 3.

Finally, a simulation example is shown in Figure 4 to illustrate Theorem 2. In this example, the controllability Gramian is diagonal because the accessible subnetwork of either input is a bud. When the inputs are discrete white noises, the states of the nodes have a response with very different magnitude. More specifically, if the nodal energy of a node is large (small), the state response of the node has a small (large) magnitude (see Figure 4(c)). This simulation result verifies the implication of equation (7). In practice, we may assign a node with a large nodal energy if we would like to protect its state against input disturbance. This phenomenon also reveal a tradeoff between controllability and robustness; that is when the nodal energy of a node is small, the node can be easily controlled, but its state is also vulnerable to input disturbances.

5. CONCLUSIONS

In this paper, we proved necessary and sufficient graphical conditions for discrete-time dynamical networks featuring diagonal controllability Gramians. With the graphical conditions, we are able to determine whether the Gramian of

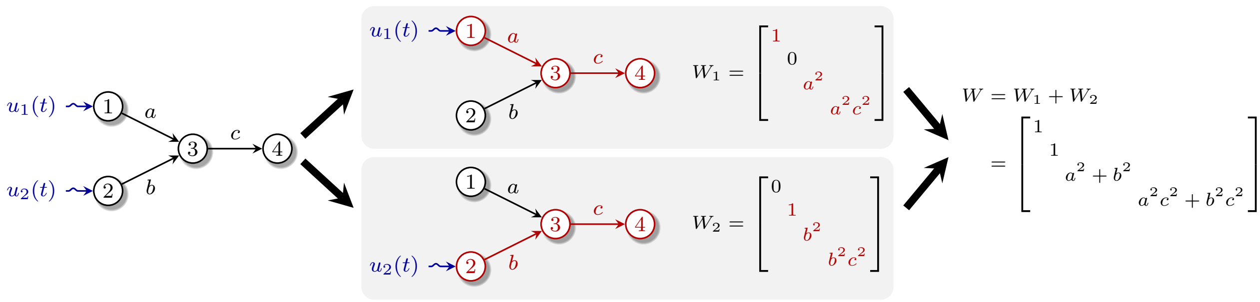
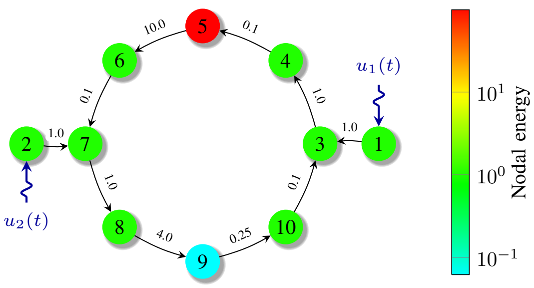
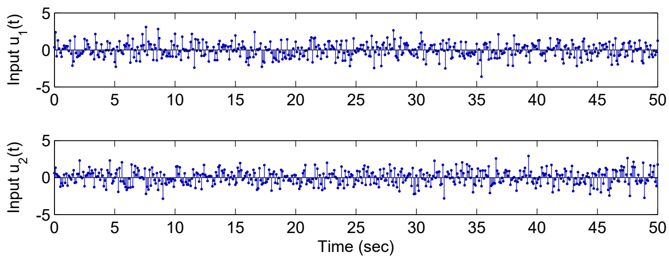


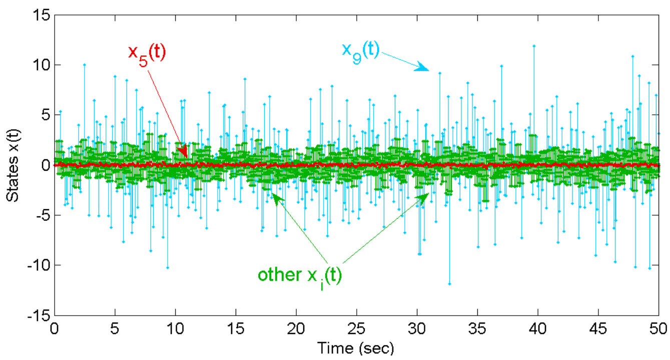
Fig. 3. An illustration of the graphical conditions in Theorem 2. The network is diagonally admissible because the accessible subnetwork for either input is a bud network and every node in the network is accessible by at least one control input.



(a) Color-coded nodal energies



(b) White-noise inputs



(c) State response

Fig. 4. An example of discrete-time networks with diagonal Gramians. There are 10 nodes and two inputs.

a network is diagonal by simply looking at its structure, and to determine the values of nodal energies by simply examining the edge weights. It has been shown by theoretical analysis and numerical simulation that nodes with high nodal energies are robust against input disturbance. This paper assumed that the edge weights are positive; in the future it is meaningful to study networks with both positive and negative weights.

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