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Multiobjective Optimization for Interwoven Systems

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Abstract

In practical situations, complex systems are often composed of subsystems or subproblems with single or multiple objectives. These subsystems focus on different aspects of the overall system, but they often have strong interactions with each other and they are usually not sequentially ordered or obviously decomposable. Thus, the individual solutions of subproblems do not generally induce a solution for the overall system. Here, we strive to identify "re-composition architectures" of such "interwoven" systems. Our intention is to connect the subsystems adequately, analyze the resulting performance, model/solve the overall system, and improve the overall solution instead of just solving each subsystem separately. We review recent developments in this field and discuss modelling and solution paradigms in a general and unified framework using the example of an interwoven system consisting of two interacting subsystems.

1 Introduction

The optimization of complex systems is a consequence of the demand that computational sciences solve increasingly complex problems. For our purpose, we define a complex system as a natural or engineered system that is difficult to understand and analyze because it: (i) involves interactions among many phenomena; (ii) has multiple and dissimilar components or subsystems that may be connected in a variety of ways and as a whole exhibit one or more properties not obvious from the properties of the individual parts or; (iii) is characterized by non-comparable and conflicting criteria. For instance, many subjects of interest to humans are complex systems. In the literature, these systems are also referred to as "interwoven systems" or "systems of systems" [1]. Natural complex systems, such as the human body, oceans, climate, and many more, are present constantly and of great interest and significance to the public. Energy or telecommunication infrastructures, manufacturing systems, and service sector systems are examples of man-made and engineered complex systems.

For complex systems, the overall decision-making goal is to harmonize local requirements and goals to attain the objectives required of the entire system. The overall system performance depends on the interactions and synergy of all its parts, which make it particularly hard to model. Moreover, human preferences may exist that might not be captured in the mathematical model. In the presence of multiple components and criteria, a unique decision that is optimal for the whole system does usually not exist, but rather many or even infinitely many decisions are potentially suitable. Because of a correlation between the components, the overall system performance does not equal the simple sum of their performances but could be enriched by a synergy among them. Furthermore, the "optimal" solution of a mathematical model may not correspond to the actual optimal solution of the whole system due to possible inaccuracies in the model. Even if an adequate model does exist, it could result in prohibitively expensive computations. Without usable models, solutions to complex systems are achieved by optimizing only their components and coordinating their optimal solutions. In effect, decision-making for complex systems requires tools originated from multiobjective optimization that additionally account for the coupling of components and the coordination of subsystem optimal solutions.

The goal of this pilot study is to present a preliminary mathematical model of a complex or interwoven system and approaches to its optimization based on known principles of multiobjective optimization. As discussed in the next section, models and methods for dealing with multiobjective complex systems have been proposed in the literature, especially in the area of engineering optimization. However, they lack rigorous mathematical analyses and optimality proofs.

In this preliminary study, we consider interwoven systems that can be modeled as two interacting subsystems, each modeled as a multiobjective optimization problem (cf. Sec. 3). Several examples of such interwoven systems are presented in support of the proposed modeling paradigm (cf. Sec. 4). Notions of optimality that recognize the overall system optimality as well as local subsystem optimality are introduced, cf. Sec. 5. Different composition architectures allowing the computation of the optimal solutions are presented in Sec. 6. Finally, in Sec. 7, connections of the proposed approach to other areas of optimization and systems science are discussed.

2 Related Work

The term *Interwoven Systems* chosen to describe the systems under investigation here has infrequently been used in the past. We found one instance in Tomforde et al. [2], where systems are described that have several properties in common with the ones we address here. The main difference to the systems discussed is their magnitude. While we concentrate on smaller systems, where a chance of understanding the basic working principles exists, Tomforde et al. consider much larger and more complex systems. This different understanding is based on the intention of the publication. While Tomforde et al. introduce the term interwoven systems, we aim for the description and analysis of systems.

Literature on complex systems with multiple criteria is rather limited. The first studies on multiobjective complex problems are undertaken for hierarchical systems in [3, 4, 5, 6, 7, 8] and later continued in [9, 10]. Large-scale hierarchical multiobjective systems are studied in [4, 5, 6]. Other papers propose: (i) decomposition of the original problem into a collection of smaller-sized, better manageable subproblems; and (ii) coordination of the solutions of the subproblems to obtain the solution of the original problem. A large number of such approaches exists for specific applications in management sciences, engineering, and multidisciplinary optimization (see [3, 11] among many others). Other papers deal with decomposition and coordination due to a large number of criteria in the original problem [12, 6, 13, 9, 10]. Finally, some papers study objective decompositions from a predominantly mathematical perspective [14, 15, 16, 17].

Multidisciplinary Design Optimization (MDO) has been developed within the engineering community to coordinate results of various disciplines involved in system design. The MDO focus has been to either encapsulate disciplinary optimizations into subproblems that are coordinated by a super-optimizer or use sensitivity information to relate the effect of one disciplinary optimization on another. Multiobjective optimization has been introduced to strengthen MDO techniques attempting to deal with noncomparable and conflicting design objectives that are characteristic for each design discipline.

Numerous papers present applications of multiobjective MDO in various areas of engineering design, however, formal methodologies such as Multiobjective Collaborative Optimization (MOColO) [11], Multiobjective Concurrent Subspace Optimization (MOCSSO)[18, 19], Collaborative Optimization Strategy for Multiobjective Sytems (COSMOS) [20], and a bilevel method [21] making use of the weighted-sum scalarization are also proposed.

The MOColO, MOCSSO, and COSMOS methods support a multidisciplinary design environment by performing distributed optimization among different design disciplines. The original problem has three types of optimization variables: global variables that are shared between the disciplines, local variables that are particular to each discipline, and coupling variables accounting for the interaction between the disciplines. The overall objective function assessing the performance of the entire system is vector-valued, while each component function assesses the system performance with respect to a particular discipline. The original problem is reformulated into a bilevel problem with system optimization at the higher level and subsystem optimization for each discipline at the lower level. All three methods promote disciplinary autonomy while achieving interdisciplinary compatibility, but they differ in how the optimization is organized between the two levels.

In MOColO at the system-level, the overall objective function (or its scalarization) is optimized and the optimizations at the lower level are coordinated. The system-level optimization variables include global and coupling variables, and the optimization is performed subject to consistency constraints that match these variables with their lower-level counterparts. The subsystem-level optimization variables include the local variables and the counterparts to the system-level variables which serve as the system-level target values for the lower-level counterparts. At the subsystem-level, an auxiliary scalar objective function is minimized for each discipline. This function models the deviation between the system-level target values and the subsystem-level counterpart values.

In MOCSSO, the system level does not perform optimization of the original vector-valued objective function but only coordinates the lower-level optimizations in each discipline. The vector-valued objective is distributed among the disciplines so that each of them is responsible for optimizing their own scalar-valued objective function. The critical assumption in MOCSSO is that in the course of the procedure, the variables associated with a discipline based on initial problem modeling may end up being allocated to some other discipline for the optimization purposes. This reallocation of variables to the disciplines where they have the greatest impact on both the objective and constraint functions yields the greatest improvement possible in the objective while maintaining discipline's feasibility. Subsystem-level optimization is conducted for each discipline with respect to the variables currently allocated to that discipline and subject to the feasibility constraints of that discipline but also subject to the constraint and objective functions of the other disciplines. In this way, the optimization follows the rules of the epsilon-constraint method and hence ensures the generation of Pareto points. The system-level optimization is performed with respect to auxiliary variables modeling the influence of the variables allocated to a discipline on other disciplines.

In COSMOS, the CO method is integrated with a multiobjective optimization genetic algorithm. The disciplines are allowed to have their own vector-valued objective functions. At the system and subsystem levels the optimizations are performed with respect to the global and local variables respectively. The linking variable values evolve within each discipline optimization and are sent up to the system level that passes them to the other disciplines. The use of a genetic algorithm ensures the Pareto set to the original problem may be computed or approximated.

In all multiobjective MDO models methods reviewed above, the proposed algorithms are not supported with proofs of correctness or optimality.

The discipline-based decomposition of a system, the driving force for MDO, has also been replaced with other types of decomposition such as scenario-based or object-based decomposition, each leading to studying a collection of multiobjective problems. If a system performs in multiple scenarios and each of them is driven by different objective functions, the resulting collection represents a set of multiobjective problems where each of them models the performance of the system in a certain scenario. Refer to [22, 23, 24] for multiscenario multiobjective optimization in engineering design. An effort to quantify trade-offs between disciplines or scenarios is undertaken in [25, 26]. Physical or object-based decomposition leads to studying a system composed of subsystems and components that can interact with each other in various ways, which additionally increases the complexity of the overall problem. A collection of multiobjective problems naturally emerges because each of the elements may perform according to multiple criteria. Calculation of the Pareto sets of such complex systems is studied in [27, 28, 29].

3 Model

In this section, we suggest a general mathematical model for complex or interwoven systems that relies on known principles of multiobjective optimization. To keep the model simple while capturing the characteristics of interwoven systems, we consider systems that can be modeled as two interacting subsystems, each modeled as a multiobjective optimization problem. Such a simple yet non-trivial setup of an interwoven system consists of three parts: two subsystems and the interaction between them. The subsystems come in the form of the following optimization subproblems.

Subproblem 1:

$$\begin{aligned} &\min\ f_1(x_0, x_1, y_{21}) \\ &\text{s.t.}\ \ g_1(x_0, x_1, y_{12}) \leqq 0 \\ &\qquad x_0 \in X_0,\ x_1 \in X_1 \end{aligned}$$

and Subproblem 2:

$$\begin{aligned} &\min\ f_2(x_0, x_2, y_{12}) \\ &\text{s.t.}\ \ g_2(x_0, x_2, y_{21}) \leqq 0 \\ &\qquad x_0 \in X_0,\ x_2 \in X_2 \end{aligned}$$

where $X_i \subseteq \mathbb{R}^{n_i}$ for $i = 0, 1, 2$ and $y_{21} \in \mathbb{R}^{n_{21}}$, $y_{12} \in \mathbb{R}^{n_{12}}$ for some $n_0, n_1, n_2, n_{21}, n_{12} \in \mathbb{N}$. Each subproblem has objective functions $f_i : \mathbb{R}^{n_0} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_{ji}} \to \mathbb{R}^{p_i}$, and constraint functions $g_i : \mathbb{R}^{n_0} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_{ij}} \to \mathbb{R}^{q_i}$, $i, j = 1, 2$, $i \neq j$, for some $p_i, q_i \in \mathbb{N}$. When comparing two vectors $u, v \in \mathbb{R}^n$, we write $u \leqq v$ when $u_i \leq v_i$ for all $i = 1, \ldots, n$, $u \leq v$ when $u \leqq v$ and $u \neq v$, and $u < v$ when $u_i < v_i$ for all $i = 1, \ldots, n$. Note that Subproblems 1 and 2 share some common (global) decision variables $x_0 \in X_0$ while they also comprise model-specific (local) decision variables $x_1 \in X_1$ and $x_2 \in X_2$, respectively.

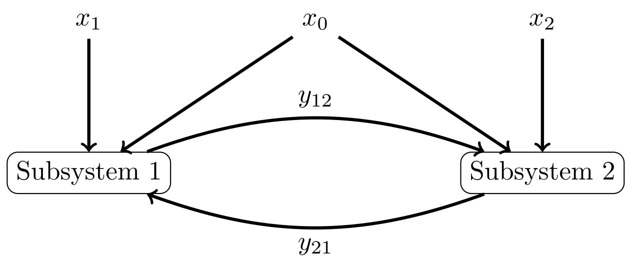
The interaction between the subsystems is modeled with *linking functions* ℓ_1 and ℓ_2 that yield the values of the *linking variables* y_{21} and y_{12}. The interaction is then typically expressed by a system of *interaction equations*:

$$y_{12} = \ell_1(x_0, x_1, y_{21}) \qquad \text{and} \qquad y_{21} = \ell_2(x_0, x_2, y_{12})$$

where $\ell_i : \mathbb{R}^{n_0} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_{ji}} \to \mathbb{R}^{n_{ij}}$ for $i, j = 1, 2, i \neq j$. This system of interaction equations has the form of *implicit representation* of linking variables y_{12} and y_{21} by means of linking functions ℓ_1 and ℓ_2. However, an explicit representation of y_{12} and y_{21} of the following form may also exist:

$$\begin{aligned} y_{12} &= y_{12}(x_0, x_1, x_2) \\ y_{21} &= y_{21}(x_0, x_1, x_2). \end{aligned}$$

A graphical exemplification of this setup is given in the following figure:



Feasibility of decision variables may refer to either of the two subsystems or to the interwoven system. This observation motivates the following definitions.

Definition 1 *A solution (x_0, x_i, y_{ji}) $(i \neq j)$ is said to be i-*subsystem feasible *if $x_0 \in X_0$, $x_i \in X_i$, $g_i(x_0, x_i, y_{ij}) \leqq 0$ and y_{ji} satisfies the interaction equations $y_{ji} = \ell_j(x_0, x_j, y_{ij})$ for some $x_j \in X_j$, where $y_{ij} = \ell_i(x_0, x_i, y_{ji})$.*

Definition 2 *A solution $(x_0, x_1, x_2, y_{12}, y_{21})$ is called* multisystem feasible *if $x_0 \in X_0$, $x_1 \in X_1$, $x_2 \in X_2$, and if all constraints and interaction equations are satisfied, i.e., $g_1(x_0, x_1, y_{12}) \leqq 0$, $g_2(x_0, x_2, y_{21}) \leqq 0$, $y_{12} = \ell_1(x_0, x_1, y_{21})$, and $y_{21} = \ell_2(x_0, x_2, y_{12})$. A pair of solutions (x_0, x_1, y_{21}) and (x_0, x_2, y_{12}) is said to be* multisystem feasible *if $(x_0, x_1, x_2, y_{12}, y_{21})$ is multisystem feasible.*

4 Examples

An interwoven system consists of interacting subsystems. In some areas of human activity, the susbsystems are developed independently from each other. For example, in engineering design the subsystems of an automotive vehicle such as an engine or a suspension are designed by different groups within a company, or even by different companies. Whilst the designers within each group/company anticipate that these subsystems will work together within one system (a vehicle), the subsystem designs are carried out with limited or even absent information about the future interaction between the subsystems. In other applications, such as location of facilities, subsystems were not even meant to work together when they were being developed but later, due to new circumstances, they necessarily start to interact with each other as an interwoven system.

A countless number of interwoven systems are encountered in daily life and numerous examples can be identified e. g., in traffic systems, multidisciplinary design, or evacuation planning to name just some areas. Nonetheless, some comprehensible examples shall be listed in the following for the sake of intended exemplification of the proposed model.

4.1 An Academic Example

Let $X_i = [0,1] \subset \mathbb{R}$, $i = 0,1,2$ and, let $x_i, y_{12}, y_{21} \in \mathbb{R}$, $i = 0,1,2$. The scalar-valued objective functions f_1 and f_2 of the subproblems are defined as

$$\begin{aligned} f_1(x_0, x_1, y_{21}) &= x_0^2 - x_1 + x_1^2 y_{21} \\ f_2(x_0, x_2, y_{12}) &= (x_0 - 5)^2 + x_2^2 y_{12}. \end{aligned}$$

The values of the linking variables y_{21} and y_{12} are specified by the following linking functions ℓ_1 and ℓ_2:

$$\begin{aligned} y_{12} &= \ell_1(x_0, x_1, y_{21}) = 2x_0 - 3x_1 + y_{21} \\ y_{21} &= \ell_2(x_0, x_2, y_{12}) = -x_0 + 4x_2 - y_{12}. \end{aligned}$$

Formally, these linear linkage relations can be written in matrix form: $y = Q\,x + C\,y$ where

$$Q = \begin{pmatrix} q_{11} & q_{12} & 0 \\ q_{21} & 0 & q_{23} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & c_{12} \\ c_{21} & 0 \end{pmatrix}.$$

If $c_{12} \cdot c_{21} \neq 1$ there is a unique solution $y = (I - C)^{-1}\,Q\,x$, where I is the unit matrix of dimension 2. In the example above we obtain:

$$\begin{aligned} y_{12} &= \frac{1}{2}x_0 - \frac{3}{2}x_1 + 2x_2 \\ y_{21} &= -\frac{3}{2}x_0 + \frac{3}{2}x_1 + 2x_2. \end{aligned}$$

4.2 Integrated Location Problem

Let a finite set of customer locations $A = \{a_1, \ldots, a_M\}$ be given in the plane $\mathbb{R}^2$. Suppose that some group of decision makers, referred to as DM 1, wants to locate an airport at a location $x_1 \in X_1 \subseteq \mathbb{R}^2$. Suppose that for some given weights $w_m^1 \geq 0$, $m = 1, \ldots, M$, the sum of weighted distances between the customers and the airport is to be minimized. Another group of decision makers, say DM 2, wants to locate a hospital at a location $x_2 \in X_2 \subseteq \mathbb{R}^2$ which should (among others) also serve the same set of customers. Given some weights $w_m^2 \geq 0$, $m = 1, \ldots, M$, the maximum weighted distance between the customers and the hospital is to be minimized. The hospital acts as a repulsive facility for the airport (due to noise) which is expressed by some weight $-\lambda_2 < 0$. The airport acts as an attractive facility for the hospital (in the sense that a maximum acceptable travel time should not be exceeded) since emergencies occurring at the airport have to reach the hospital quickly. This aspect is modeled by a weight $\lambda_1 > 0$. Staff of the airport and of the hospital will jointly use a service facility, e. g., providing childcare, which has to be located at a location $x_0 \in X_0 \subseteq \mathbb{R}^2$.

The resulting interwoven system can again be specified by identifying the two subproblems corresponding to the two subsystems and by expressing the linking functions.

Subproblem 1: Location of the Airport

$$\begin{aligned}
&\min\ f_{11}(x_0, x_1, y_{21}) = \sum_{m=1}^{M} w_m^1 d(x_1, a_m) - \lambda_2 d(x_1, y_{21}) \\
&\min\ f_{12}(x_0, x_1, y_{21}) = d(x_0, x_1) \\
&\text{s.t. } \ x_0 \in X_0,\ x_1 \in X_1
\end{aligned}$$

Subproblem 2: Location of the Hospital

$$\begin{aligned}
&\min\ f_{21}(x_0, x_2, y_{12}) = \max\left\{ \max_{m=1,\ldots,M} w_m^2 d(x_2, a_m);\ \lambda_1 d(x_2, y_{12}) \right\} \\
&\min\ f_{22}(x_0, x_2, y_{12}) = d(x_0, x_2) \\
&\text{s.t. } x_0 \in X_0,\ x_2 \in X_2
\end{aligned}$$

Interaction Equations The interaction equations are given by the linking functions ℓ_1 and ℓ_2 as:

$$\begin{aligned}
y_{12} &= \ell_1(x_0, x_1, y_{21}) := x_1 \\
y_{21} &= \ell_2(x_0, x_2, y_{12}) := x_2,
\end{aligned}$$

which is again an explicit representation of the linking variables.

4.3 Traveling Thief Problem

The Traveling Thief Problem (TTP) [30] consists of two well-known interacting combinatorial subproblems: the Traveling Salesman Problem (TSP) and the Knapsack Problem (KP). In original KP, there are m items each with a weight w_k, $k = 1, \ldots, m$, and value v_k, $k = 1, \ldots, m$. A subset of these items has to be packed into a knapsack with limited capacity W so as to maximize the total value of items chosen. In original TSP, a salesman must visit each one of n cities exactly once and return to his starting position in a way to minimize total traveled distance. In TTP, there are objects located in the cities that need to be packed into a knapsack while the thief performs a visit to every city by building a complete tour. In other words, TTP seeks a tour of n cities and a subset of items located in the cities to be packed in a way to maximize total value and minimize total travel time. In TTP, subproblems KP and TSP are interlinked because the traveling speed of the thief between two cities depends on the weight of items he has placed in the knapsack. Moreover, the value associated with a picked item deteriorates as a function of travel time and what contributes to the objective function is the value of the item at the end of the tour.

Let $x_1 \in X_1 \subseteq \mathbb{B}^m \subseteq \mathbb{R}^m$ be a binary vector whose k^{th} entry indicates item k is picked if it has a value of 1 and is not picked if it has a value of 0. $x_2 \in \mathbb{R}^n$ describes the tour of the thief where x_{2i} is the index of the city that is visited i^{th} in the tour. Let $X_2 = \{x \in \mathbb{R}^n | x_i \in \{1, \ldots, n\}, i = 1, \ldots, n, x_i \neq x_j \forall i \neq j\}$. Then X_2 captures all possible tours.

For $k = 1, \ldots, m$ and $i = 1, \ldots, n$, let parameter $a_k(i)$ indicate that item k is located at city i if it is equal to 1, and not if it is 0. $s_{\max}$ and $s_{\min}$ are the maximum and minimum speed of the thief, respectively, and W is the capacity of the knapsack. $d(i, j)$ is the distance between city i and city j, $i, j = 1, \ldots, n$.

Subproblem 1: Knapsack Problem

$$\begin{aligned}\min\ f_1(x_1, v) = -\sum_{k=1}^{m} v_k x_{1k} \\ \text{s.t. } \sum_{k=1}^{m} w_k x_{1k} \le W \\ x_1 \in X_1\end{aligned}$$

Subproblem 2: Traveling Salesman Problem

$$\begin{aligned}&\min\ f_2(x_2, s) = \sum_{i=1}^{n-1} \frac{d(x_{2i}, x_{2(i+1)})}{s(x_{2i})} + \frac{d(x_{2n}, x_{21})}{s(x_{2n})} \\ &\text{s.t. } x_2 \in X_2\end{aligned}$$

The subproblems above have separate objective functions. While the first one aims to maximize the total value of items picked along the tour, the second subproblem minimizes total travel time. We note that the objectives can be combined into one by assuming a rent R per unit time of travel (see, for instance, the setup in [31]). We prefer to keep these objectives separate and explain the interrelationships between the two subproblems below.

Interaction Equations In subproblem 1, v, the vector that denotes the values of items, is indeed linked to the decisions made in the second subproblem because the value v_k of picked item k, $k = 1, \ldots, m$, declines over time. This would correspond to the variable denoted as y_{21} in our earlier presentation. Therefore, y_{21} could replace v in the subproblem formulation; however, we have opted not to do so to keep the notation more legible. Likewise, the speed at which the thief travels, s is determined by the decision variables in the first subproblem and s can be considered as a function of y_{12}.

The speed $s(x_{2i})$ of travel when leaving city x_{2i} is related to the knapsack's current weight at city x_{2i}, $i = 1, \ldots, n$. It can be expressed as follows:

$$s(x_{2i}) = \ell_{1,x_{2i}}(x_1, x_2) := s_{\max} - \left(\frac{s_{\max} - s_{\min}}{W}\right) \sum_{j=1}^{i} \sum_{k=1}^{m} a_k(x_{2j}) w_k x_{1k}.$$

According to this formulation, the speed of the thief decreases when the weight of the knapsack increases, i.e., the speed is one of the factors that captures the interaction between the two subproblems of the interwoven system.

The final value of the item at the end of the travel is not the same as its initial value. This value is dependent on travel time and here it is modeled as:

$$v_k = \ell_{2,k}(x_2, s) := b_k^{\text{init}} - \rho_k T_k(x_2, s)$$

where $T_k(x_2, s)$ is the time between the moment when item k is picked and the end of the tour, which depends on the location of item k and the order at which it was visited, say i^{th} in the tour, as well as the cities visited afterwards as in the following equation:

$$T_k(x_2, s) = \sum_{j=i}^{n} \frac{d(x_{2j}, x_{2(j+1)})}{s(x_{2j})} + \frac{d(x_{2n}, x_{21})}{s(x_{2n})}.$$

We assume that ρ_k is a rate of decline in the value of v_k so that $v_k \ge 0$ for all possible values of T_k. There might be alternative ways of modeling the value of an item k in a way to depend on the duration of the tour from the time the item is picked until the end. As such, this becomes another factor that captures the interaction between the two subproblems of the interwoven system.

5 Notions of Optimality

It is of interest to establish a concept of optimality for the interwoven systems presented above. Note that such a concept could recognize all three parts of the system or just a subset of them. We propose three notions of optimality depending on the level of engagement of each subsystem in the overall system. The three concepts can be used to model different levels of involvement of decision makers of the individual subsystems (bottom-up decisions) and/or of decision makers for the overall system (top-down decisions). Corresponding solution concepts may reflect an individual (local) or corporate (global) decision makers and a spectrum of their attitudes from ego-centric to teamwork.

We first take the perspective of a decision maker of one of the subsystems who assumes the highest priority for this subsystem and respects solely feasibility requirements of the other subsystem. Under the scenario that each subsystem would like to operate at its best for itself regardless of the values of the linking variables passed from the other system, we define individually Pareto optimal solutions for each system.

Definition 3 *A solution* (x_0, x_1, y_{21}) *is said to be* individually Pareto optimal for Subsystem 1 *if it can be extended to a multisystem feasible solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *and if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that*

$$f_1(x'_0, x'_1, y'_{21}) \quad \leq \quad f_1(x_0, x_1, y_{21}). \tag{1}$$

Similarly, a solution (x_0, x_2, y_{12}) *is said to be* individually Pareto optimal for Subsystem 2 *if it can be extended to a multisystem feasible solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *and if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that*

$$f_2(x'_0, x'_2, y'_{12}) \quad \leq \quad f_2(x_0, x_2, y_{12}). \tag{2}$$

Note that an individually Pareto optimal solution for Subsystem i must be i-subsystem feasible, $i \in \{1, 2\}$.

Taking the perspective of a decision maker for the overall system, we distinguish between two closely related concepts that both reflect the trade-off between the performances of the individual subsystems. While in both cases candidate solutions must be "stable" in the sense that no (strict) improvement should be possible for both subsystems simultaneously, (weak) Pareto optimality of a subsystem is considered at different levels, giving a slightly higher priority to the decision makers of the individual subsystems in the second notion of optimality than in the third.

Accordingly, we assume in the second notion of optimality that each subsystem would like to perform best to the common good of both subsystems, we define cooperative Pareto solutions that are feasible for both systems.

Definition 4 *A multisystem feasible solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is said to be* cooperatively Pareto optimal *if it is Pareto optimal with respect to all objective functions, i.e., if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that* $f_1(x'_0, x'_1, y'_{21}) \leq f_1(x_0, x_1, y_{21})$ *and* $f_2(x'_0, x'_2, y'_{12}) \leq f_2(x_0, x_2, y_{12})$. *Similarly, a multisystem feasible solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is said to be* cooperatively weakly Pareto optimal*, if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that* $f_1(x'_0, x'_1, y'_{21}) < f_1(x_0, x_1, y_{21})$ *and* $f_2(x'_0, x'_2, y'_{12}) < f_2(x_0, x_2, y_{12})$.

The third notion of optimality reflects the teamwork attitude of two decision makers whose systems perform at their best simultaneously even if one of them could perform better while the other system is ignored.

Definition 5 *A multisystem feasible solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is said to be* mutually Pareto optimal *if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that*

$$\begin{pmatrix} f_1(x'_0, x'_1, y'_{21}) \\ f_2(x'_0, x'_2, y'_{12}) \end{pmatrix} \leq \begin{pmatrix} f_1(x_0, x_1, y_{21}) \\ f_2(x_0, x_2, y_{12}) \end{pmatrix}. \tag{3}$$

Similarly, a multisystem feasible solution $(x_0, x_1, x_2, y_{12}, y_{21})$ *is called* mutually weakly Pareto optimal *if there is no other multisystem feasible solution* $(x'_0, x'_1, x'_2, y'_{12}, y'_{21})$ *such that*

$$\begin{pmatrix} f_1(x'_0 x'_1, y'_{21}) \\ f_2(x'_0, x'_2, y'_{12}) \end{pmatrix} < \begin{pmatrix} f_1(x_0, x_1, y_{21}) \\ f_2(x_0, x_2, y_{12}) \end{pmatrix}. \tag{4}$$

The following relations can be derived.

Proposition 1 *Let* $(x_0, x_1, x_2, y_{12}, y_{21})$ *be multisystem feasible. Then it holds:*

(1a) If (x_0, x_1, y_{21}) *is individually Pareto optimal for Subsystem 1, then* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually weakly Pareto optimal.*

(1b) If (x_0, x_2, y_{12}) *is individually Pareto optimal for Subsystem 2, then* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually weakly Pareto optimal.*

(2) If $(x_0, x_1, x_2, y_{12}, y_{21})$ *is cooperatively Pareto optimal, then it is mutually weakly Pareto optimal.*

(3) If $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually Pareto optimal, then it is cooperatively weakly Pareto optimal.*

(4) $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually weakly Pareto optimal if and only if it is cooperatively weakly Pareto optimal.*

Note that mutual Pareto optimality does not in general imply individual Pareto optimality for the respective subsystems. Similarly, cooperative Pareto optimality does not in general imply individual Pareto optimality.

6 Composition Approaches

We discuss some possible ways of composing the interwoven subsystems. While some of these composition approaches reveal a close relation to the notions of optimality discussed in Section 5 above, others give rise to alternative interpretations of interwoven systems, both with respect to modelling assumptions and optimality concepts.

6.1 Biobjective All-in-One System

This approach imposes the least additional structure upon the interwoven system while composing it by bringing together the two subsystems in a natural biobjective way as follows.

$$
\begin{aligned}
\min \ & \begin{pmatrix} f_1(x_0, x_1, y_{21}) \\ f_2(x_0, x_2, y_{12}) \end{pmatrix} \\
\text{s.t.} \ & g_1(x_0, x_1, y_{12}) \leqq 0 \\
& g_2(x_0, x_2, y_{21}) \leqq 0 \\
& \ell_1(x_0, x_1, y_{21}) = y_{12} \\
& \ell_2(x_0, x_2, y_{12}) = y_{21} \\
& x_0 \in X_0, \ x_1 \in X_1, \ x_2 \in X_2.
\end{aligned}
$$

The term biobjective is used in relation to the two subsystems involved. Note that if f_1 or f_2 is a vector-valued function, the number of objectives in the above formulation will be more than two. Therefore, in general, this is a multiobjective optimization formulation. A feasible solution of the biobjective all-in-one system is called *Pareto optimal* if there is no other feasible solution that performs at least as good in all objective functions, and strictly better in at least one objective function. A feasible solution is called *weakly Pareto optimal* if there is no other feasible solutions that performs strictly better in all objective functions. The Pareto optimal solutions to this multiobjective problem can be considered as the solutions to the interwoven system.

Proposition 2 *A solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually Pareto optimal for the interwoven system if and only if it is Pareto optimal for the biobjective all-in-one system. Similarly, a solution* $(x_0, x_1, x_2, y_{12}, y_{21})$ *is mutually weakly Pareto optimal for the interwoven system if and only if it is weakly Pareto optimal for the biobjective all-in-one system.*

In the spirit of Definition 5 and Proposition 1 above, the biobjective all-in-one system thus integrates the preferences of the decision makers of the individual subsystems who share an overall perspective and are willing to act as a team. This approach may serve, for example, as a source for solution alternatives and trade-off information in a top-down decision making processes.

As an example, consider again the academic example introduced in Section 4.1. The corresponding biobjective all-in-one system is in this case given by:

$$\begin{aligned}
&\min\ f_1(x_0, x_1, y_{21}) = x_0^2 - x_1 + x_1^2 y_{21}\\
&\min\ f_2(x_0, x_2, y_{12}) = (x_0 - 5)^2 + x_2^2 y_{12}\\
&\text{s.t. } y_{21} = -\frac{3}{2}x_0 + \frac{3}{2}x_1 + 2x_2\\
&\qquad y_{12} = \frac{1}{2}x_0 - \frac{3}{2}x_1 + x_2\\
&\qquad x_0, x_1, x_2 \in [0,1],\ y_{12}, y_{21} \in \mathbb{R}.
\end{aligned}$$

An approximation of the nondominated set of this all-in-one system is illustrated in Figure 1. The points shown are obtained by sampling feasible solutions and filtering for dominated points.

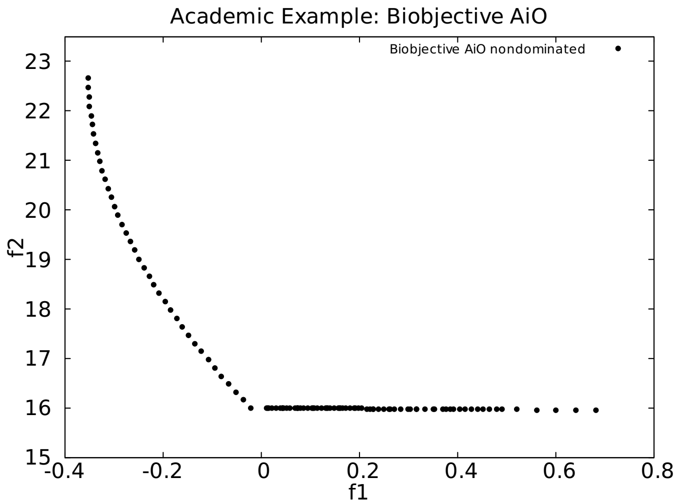


Figure 1: Approximation of the nondominated objective vectors, i.e., the images of the Pareto optimal solutions of the biobjective all-in-one system, for the academic example introduced in Section 4.1.

6.2 Bilevel All-in-One System

In some situations, the interactions between the two subsystems may be modeled in a hierarchical way. In such cases, a bilevel programming framework may best describe the composed system. Such a composition does not need to utilize the variables x_0.

$$\begin{aligned}
&\min\ f_1(x_0, x_1, y_{21})\\
&\text{s.t. } g_1(x_0, x_1, y_{12}) \leqq 0\\
&\qquad y_{21} = \ell_2(x_0, x_2, y_{12})\\
&\qquad x_0 \in X_0,\ x_1 \in X_1\\
&\qquad x_2 \in \arg\min f_2(x_0, x_2, y_{12})\\
&\qquad\qquad \text{s.t. } g_2(x_0, x_2, y_{21}) \leqq 0\\
&\qquad\qquad\qquad y_{12} = \ell_1(x_0, x_1, y_{21})\\
&\qquad\qquad\qquad x_2 \in X_2.
\end{aligned}$$

In this bilevel problem the objective functions f_1 and f_2 can be scalar and/or vector-valued. We will usually assume an *optimistic* model in the sense that, if the set of optimal solutions to the lower-level problem is not a singleton, then the decision maker of the upper-level problem gets to select an optimal solution to the lower-level problem that is the most suitable to the upper level. Note that, even though the two subsystems are considered at different levels, it cannot in general be guaranteed that an optimal solution (or a Pareto optimal solution in the case that f_1 and f_2 are vector-valued) to the bilevel all-in-one system is individually Pareto optimal for Subsystem 1 or for Subsystem 2, or that it is Pareto optimal for the bi-objective all-in-one system, respectively. It should be noted that the bilevel formulation that is applicable to the interwoven system is not unique. In [32], different bilevel formulations are suggested and their relationships are analyzed. We refer to [33] for a discussion of the relationship between single-level and bilevel multiobjective optimization.

The Pareto optimal solutions to the bilevel all-in-one system can be considered as stable solutions or as equilibrium solutions to the interwoven system. The model best reflects hierarchical decision making processes where one subsystem is granted a higher priority (Subsystem 1 in this case), while the decision maker of the lower-level subsystem should be able to achieve a stable solution in the sense that no improvement is possible given the restrictions coming from the higher-level subsystem.

Considering again the academic example problem introduced in Section 4.1, we obtain

$$\begin{aligned}
\min\ & f_1(x_0, x_1, y_{21}) = x_0^2 - x_1 + x_1^2 y_{21}\\
\text{s.t. } & y_{21} = -\frac{3}{2}x_0 + \frac{3}{2}x_1 + 2x_2\\
& x_0, x_1 \in [0,1]\\
& x_2 \in \arg\min f_2(x_0, x_2, y_{12}) = (x_0-5)^2 + x_2^2 y_{12}\\
& \qquad \text{s.t. } y_{12} = \frac{1}{2}x_0 - \frac{3}{2}x_1 + 2x_2\\
& \qquad\qquad x_2 \in [0,1].
\end{aligned}$$

In this special case, the explicit representation of the linking variables can be used to rewrite the lower level problem as

$$\begin{aligned}
\min\ & f_2(x_0, x_1, x_2) = (x_0-5)^2 + \frac{1}{2}x_0 x_2^2 - \frac{3}{2}x_1 x_2^2 + 2x_2^3\\
\text{s.t. } & x_2 \in [0,1].
\end{aligned}$$

The values of x_0 and x_1 are passed from the upper level problem and are thus treated as constants. Since x_2 has to be chosen optimally for the lower level problem, we set $\frac{\partial f_2}{\partial x_2}(x_0, x_1, x_2) = 0$ and obtain as candidate solutions $x_2 = 0$ when $x_0 - 3x_1 \geq 0$, and $x_2 = \frac{1}{6}(-x_0 + 3x_1)$ when $x_0 - 3x_1 < 0$. Note that for all possible values of $x_0, x_1 \in [0,1]$, $x_2 = 1$ at the boundary of the feasible set is not optimal for the lower level problem. Passing this information back to the upper level problem and applying again optimality conditions, we obtain a minimum of the bilevel all-in-one system at $(x_0, x_1, x_2, y_{12}, y_{21}) = (0.1492, 0.4034, 0.1769, -0.1769, 0.7351)$ that is multisystem feasible and satisfies the lower level constraints (i.e., x_2 is optimal for the lower level problem with the given values of x_0 and x_1). The objective vector of this solution is obtained as $(f_1, f_2) = (-0.2615, 23.5248)$, which turns out to be dominated for the biobjective all-in-one problem, cf. Figure 2.

6.3 Individual Systems with Parameterized Interactions

The two subsystems may be decoupled by treating the linking variables in each subsystem as parameters. In this case, each subproblem becomes a parametric multiobjective optimization problem in which the set of unknowns is grouped into two subsets: decision variables that determine solutions and parameters that determine problem data. Such models may be used, for example, if decision makers of the individual subsystems have to suggest solutions independent

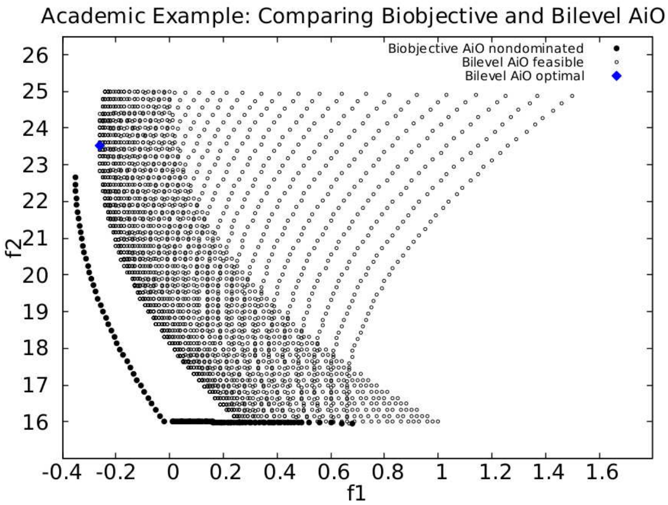


Figure 2: Comparison of biobjective and bilevel all-in-one solutions for the academic example problem introduced in Section 4.1. For the biobjective all-in-one problem, the nondominated objective vectors from Figure 1 are shown. For the bilevel all-in-one problem, objective vectors of multisystem feasible solutions that obey the optimality condition of the lower level problem were sampled in $X_i = [0, 1]$, $i = 0, 1$. The optimal solution for the bilevel all-in-one system was obtained analytically using optimality conditions.

from each other and without knowing the precise preferences from the respective other subsystem. Moreover, an extensive parametric analysis may provide a wide representation of reasonable solution alternatives of the individual subsystem, including trade-off information, and may thus be highly valuable for a decision maker for the overall system. However, note that the interrelation between the solution alternatives of the respective subsystems is not reflected in this model which may thus bear the risk of overestimating the potential performance of the overall system.

Whereas nonparametric multiobjective problems produce static Pareto optimal solutions, parametric optimization goes beyond traditional sensitivity analysis that is valid solely in a neighborhood of an optimal solution. In the parametric case, Pareto optimal solutions are functions from the parameter space to the decision space (and the objective space). Each solution maps an element of the parameter space to the decision that is obtained as the Pareto optimal decision when that parameter value is used for the problem data.

Specifically, for each subsystem having a vector-valued objective function, parametric optimization involves finding: (i) a representation (or approximation) of the Pareto optimal set as a function of the linking variables; (ii) a partition of the feasible space of the linking variables into critical regions for which a specific representation is valid. The virtue of parametric Pareto optimal solutions lies in mapping out the full range of potential outcomes prior to knowing the specific values of the linking variables. As these values change within their feasible sets, the Pareto optimal solutions can quickly be approximated using the obtained representation, bypassing the need for expensive or time consuming re-optimization of each subsystem. For decision makers, the linking variables acting as parameters offer a way to analyze and understand how changes in their values affect the Pareto optimal solutions to each subsystem.

In the proposed model, the common and local variables become decision variables while the linking variables are parameters. The overall problem is decoupled into two subproblems of the form:

Subproblem 1

$$\begin{aligned} \min_{x_0, x_1} \ & f_1(x_0, x_1; y_{21}) \\ \text{s.t. } & g_1(x_0, x_1; y_{12}) \leqq 0 \text{ where } y_{12} \in [t_L, t_R] \text{ is a parameter} \\ & x_0 \in X_0,\ x_1 \in X_1, \end{aligned}$$

and Subproblem 2

$$
\begin{aligned}
&\min_{x_0, x_2} \; f_2(x_0, x_2; y_{12}) \\
&\text{s.t. } g_2(x_0, x_2; y_{21}) \leqq 0 \text{ where } y_{21} \in [u_L, u_R] \text{ is a parameter} \\
&\qquad x_0 \in X_0, \; x_2 \in X_2.
\end{aligned}
$$

The intervals $[t_L, t_R]$ and $[u_L, u_R]$ are the feasible sets for the parameters y_{21} and y_{12}, respectively. Since $y_{ij} = \ell_i(x_0, x_i, y_{ji})$, $i, j = 1, 2$, $i \neq j$, the theoretically possible values for y_{ij} can be determined based on the linking functions ℓ_1 and ℓ_2 and the feasible sets X_0, X_1 and X_2. Since this may lead to large ranges for the values of y_{12} and y_{21}, it may be reasonable to consider smaller subsets of "likely" parameter values that can be determined based on the decision maker's experience. A reduced parameter set will, however, affect the ranges of the (approximated, parameter dependent) Pareto sets of the respective subproblems, and some individual Pareto optimal solutions, cooperative Pareto optimal solutions and mutually Pareto optimal solutions of the system may be missed (cf. Section 5).

It is anticipated that best solutions to the two subsystems will be found by solving the subproblems independently. However, since the two subsystems must agree on the common variable x_0 and the linking variables y_{12} and y_{21}, a solution mechanism that ensures such a consensus must be developed. In other words, it is of interest to compute parametric Pareto optimal sets which are related to each other by the commonality of certain variables.

We recognize that algorithms for parametric single-objective optimization have been developed [34, 35, 36, 37] but parametric multiobjective optimization is an open area of study in which just pioneering efforts have been undertaken [38, 39]. The parametric approach to interwoven systems clearly motivates further work in this area. Furthermore, the commonality requirement suggests other methods and implementations.

6.4 Distributed Approach

In [40, 41] two multiobjective decomposition algorithms are proposed to compute the Pareto optimal set of a multiobjective problem that is decomposable to multiobjective subproblems, each with different objective functions defined on their own feasible sets and subject to a common constraint allowing for passing information among the subproblems. Because the algorithms compute the Pareto optimal solutions to the original problem by working only with the subproblems, the approach is referred to as decentralized or distributed. We show that the case of interwoven systems lends itself to this method.

The interwoven system is decomposed into two subsystems by introducing copies of the common variables. Let x_0^{sub1} and x_0^{sub2} be copies of the common variable x_0 for subsystem 1 and 2, respectively. Let y_{12}^{sub1} and y_{12}^{sub2} be copies of the linking variable y_{12} for subsystem 1 and 2, respectively. Finally, let y_{21}^{sub1} and y_{21}^{sub2} be copies of the linking variable y_{21} for Subsystem 1 and 2, respectively. Given the copies, the overall problem is decomposed into the following two subproblems:

Subproblem 1

$$
\begin{aligned}
\min_{x_0^{sub1}, x_1, y_{12}^{sub1}, y_{21}^{sub1}} \quad & f_1(x_0^{sub1}, x_1, y_{21}^{sub1}) \\
\text{s.t.} \quad & g_1(x_0^{sub1}, x_1, y_{12}^{sub1}) \leqq 0 \\
& x_0^{sub1} \in X_0, \; x_1 \in X_1 \\
& x_0^{sub1} - x_0^{sub2} = 0 \\
& y_{21}^{sub1} - y_{21}^{sub2} = 0 \\
& y_{12}^{sub1} - y_{12}^{sub2} = 0,
\end{aligned}
$$

where $y_{12}^{sub1} = \ell_1(x_0^{sub1}, x_1, y_{21}^{sub2})$, and $x_0^{sub2}, y_{21}^{sub2}, y_{12}^{sub2}$ are constant.

Subproblem 2

$$\begin{aligned} \min_{x_0^{sub2}, x_2, y_{12}^{sub2}, y_{21}^{sub2}} \quad & f_2(x_0^{sub2}, x_2, y_{12}^{sub2}) \\ \text{s.t.} \quad & g_2(x_0^{sub2}, x_2, y_{21}^{sub2}) \leqq 0 \\ & x_0^{sub2} \in X_0, \; x_2 \in X_2 \\ & x_0^{sub2} - x_0^{sub1} = 0 \\ & y_{12}^{sub2} - y_{12}^{sub1} = 0 \\ & y_{21}^{sub2} - y_{21}^{sub1} = 0, \end{aligned}$$

where $y_{21}^{sub2} = \ell_2(x_0^{sub2}, x_2, y_{12}^{sub1})$, and $x_0^{sub1}, y_{12}^{sub1}, y_{21}^{sub1}$ are constant.

In each subproblem, there are now new equality constraints referred to as the consistency constraints because they ensure that the copies are identical to each other. In the distributed approaches we refer to, each subproblem is transformed into a single objective problem (SOP) by a suitable scalarization method using scalarizing parameters such as weights in the weighted-sum method or right-hand-side coefficients in the ε-constraint method. Once a subproblem has been solved, its optimal solutions are passed as constant targets to the other subproblem that is then solved. The new solutions are in turn passed back to the first subproblem. Lagrangian relaxation is applied to the consistency constraints of each SOP so that each of them is solved applying the block coordinate descent [40] or subgradient optimization [41]. The process continues iteratively until the solutions to every subproblem are within a tolerance level of or as close as possible to the desired targets. The convergence of the algorithms is claimed based on the Lagrangian duality theory integrated with either newly developed principles of distributed optimization [40] or subgradient optimization [41]. In each case the iterative process leads to the generation of a weakly Pareto solution to the biobjective all-in-one system which, by Proposition 2, is mutually weakly Pareto optimal to the interwoven system. The entire solution process can be repeated for a finite number of the scalarizing parameters to compute a representation of the mutually weakly Pareto optimal set for the interwoven system.

The solution method that follows on the algorithm in [41] is described in Algorithm 1 below in the form of a pseudocode. Again it is assumed that each subproblem has been scalarized. The variables $x_0^{sub1}, y_{12}^{sub1}, y_{21}^{sub1}$ are treated as targets being sent from Subproblem 1 to Subproblem 2, while the other variables $x_0^{sub2}, y_{21}^{sub2}, y_{12}^{sub2}$ act as targets being sent from Subproblem 2 to Subproblem 1. The additional superscript on the variables denotes the k-th iteration of the algorithm. In particular implementations, acceptable stopping criteria must be specified.

Algorithm 1 Distributed approach for scalarized subproblems

```
1: Initialize x_0^{sub2,0}, y_12^{sub2,0}, y_21^{sub2,0}
2: k ← 0
3: repeat
4:     k ← k + 1
5:     Solve scalarized Subproblem 1 for x_0^{sub1,k}, x_1^k, y_12^{sub1,k}, y_21^{sub1,k}
6:     Pass x_0^{sub1,k}, y_12^{sub1,k}, y_21^{sub1,k} to Subproblem 2
7:     Solve scalarized Subproblem 2 for x_0^{sub2,k}, x_2^k, y_12^{sub2,k}, y_21^{sub2,k}
8:     Pass x_0^{sub2,k}, y_12^{sub2,k}, y_21^{sub2,k} to Subproblem 1
9: until all consistency constraints are satisfied
```

As mentioned above, the distributed approach is particularly applicable to decision making situations in which the biobjective all-in-one system is not solvable but its weakly Pareto optimal solutions are computed in a distributed fashion by only working with the subsystems. The lack of solvability may result from lack of a model of the entire system, incompatibility of models of the subsystems, incompatibility of softwares optimizing the subsystems, or different geographical locations, backgrounds, and preferences of decision makers. Such situations may take place, for example, in big corporations in which every division is located on a different continent and is managed by different decision makers who never work together but, nevertheless, are supposed to make up a team working for the entire company. Another application refers to the design process of a complex engineering system, such as an automotive vehicle, which involves a number of design teams cooperating with each other rather than one team working on the entire vehicle. While each team works on a different component of the vehicle (e.g., engine, body, suspension, tires) and has access to limited information from the other teams, all teams work towards the same final design.

7 Connection to Other Disciplines

7.1 Game Theory

The structure of interwoven systems can be analysed using game theoretic approaches – methods that are based on one subsystem anticipating the response of another subsystem. The configuration of the interwoven system determines which type of game theory model is the most appropriate to use [42]. Where subsystems exhibit a hierarchical structure (see the bilevel formulation introduced in Section 6.2), a Stackelberg game (leader-follower) can be adopted; where the subsystems attempt to make decisions in parallel, a Nash game may be used [43]. Motivated by engineering design scenarios in which different design teams work in physical or organizational isolation, with only limited opportunity or incentives to share information, Xiao and colleagues [44] developed a framework based on a Nash model. In the approach, each subsystem constructs a surrogate model of its own rational reaction set, in anticipation of potential choices from other subsystems. To do this, a set of scenarios is defined for the interaction variables and an optimization run is performed by the subsystem for each scenario. A model is then estimated that generalises the relationship between the interaction variables and the optimized response. The Nash solutions for the overall system are identified by taking the intersection of the rational reaction sets for each subsystem. The approach was demonstrated on a simple pressure vessel design problem with two subsystems, where the underpinning physical models had low computational complexity – it remains an open question whether or not this approach could be scaled to larger classes of interwoven system and more computationally demanding evaluation functions.

7.2 Robust Design

During the process of solving its subproblem, each subsystem in an interwoven system is typically uncertain about the final values that will be chosen for the interaction variables that affect its objectives. This uncertainty can be re-framed within the context of robust optimization, enabling a search for subproblem solutions that are in some way robust to the remaining uncertainty over the choices that will be made for interaction variables. Such an idea was first proposed by Chang and Ward [45], who described the approach as *conceptual robustness*; the aim then being to find solutions that are robust to *conceptual noise*. A small number of robustness-inspired schemes have been proposed to deal with interaction variables that are presently undetermined [46]; some of the schemes are based on game theoretic models – see for example Chen and Lewis' Stackelberg-based approach [47]. Again, computational tractability remains a concern with these methods.

7.3 Co-Evolutionary Algorithms

Studies in co-evolutionary computation investigate how separate subpopulations solve their own subproblems as a means of solving the complete problem – an approach known as *cooperative co-evolution* [48]. The sub-populations exchange their information at certain intervals, e.g., at a certain number of generations in an evolutionary algorithm. Although subpopulations try to optimize their own objectives, they have to cooperate to solve the overall problem [49]. Mei and colleagues [50] used a co-evolutionary approach to tackle the TTP. In this formulation, the TTP was decomposed into a travelling salesman problem and a knapsack problem, with a separate sub-population devoted to solving each problem. At each iteration of the co-evolutionary algorithm, within each subproblem, the value of a candidate solution was evaluated on the full TTP, using collaborators from the previous iteration of the other subproblem. In empirical comparisons using instances of the TTP, the co-evolutionary approach was outpeformed by a memetic algorithm which applied perturbations to both the tour and picking plan simultaneously. Despite these discouraging initial findings, co-evolutionary strategies remain underexplored for interwoven systems, warranting further investigation.

8 Summary and Outlook

In this pilot study a mathematical model for an interwoven system consisting of two subproblems was introduced. Different concepts defining the optimal performance of such an interwoven system were proposed, and the relation to associated multiobjective and bilevel optimization models was discussed. Several existing optimization methodologies were suggested as tools for generating optimal solutions to interwoven systems.

This research raises a variety of challenging and interesting questions. This includes generalizations to interwoven systems with more than two subproblems, an in-depth analysis of the similarities and the differences between different notions of optimality and between the associated optimization models, and the development and critical evaluation of efficient solution methods. The example problems mentioned in this study may serve as a first benchmark for such approaches.

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