**Article:**

Booton, R.D., Iwasa, Y., Marshall, J.A.R. et al. (1 more author) (2017) Stress-mediated allee effects can cause the sudden collapse of honey bee colonies. Journal of Theoretical Biology, 420. pp. 213-219. ISSN 0022-5193

https://doi.org/10.1016/j.jtbi.2017.03.009

# Supplementary Information
# Stress-Mediated Allee Effects Can Cause the Sudden Collapse of Honey Bee Colonies

Ross D. Booton[a,*], Yoh Iwasa[b], James A. R. Marshall[c], Dylan Z. Childs[a]

[a] *Department of Animal and Plant Sciences, University of Sheffield, Sheffield, S10 2TN, United Kingdom*
[b] *Department of Biology, Faculty of Science, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan*
[c] *Department of Computer Science, University of Sheffield, Sheffield, S10 2TN, United Kingdom*

---

*Linearisation Around The Zero Equilibrium*

Clearly, $(H, F) = (0, 0)$ is an equilibrium of the system. We can perform a linearisation around the zero equilibrium to determine the stability of this fixed point. Let us define the following functions

$$\begin{aligned} g_1(H,F) &= \frac{dH}{dt} \\ &= L\frac{H+F}{\omega+H+F} - H\left(\alpha - \sigma\frac{F}{k+F+H}\right) - \frac{\mu H}{\phi+H+F} - \gamma(H+F)H \\ g_2(H,F) &= \frac{dF}{dt} \\ &= H\left(\alpha - \sigma\frac{F}{k+F+H}\right) - mF - \frac{\mu F}{\phi+H+F} \end{aligned}$$

*Corresponding author.
*E-mail address:* r.booton@sheffield.ac.uk.

Now, since $\exists$ $(H^*, F^*)$, steady state of

$$\frac{dH}{dt} = \frac{dF}{dt} = 0$$

Then

$$\frac{dH^*}{dt} = g_1(H^*, F^*)$$
$$\frac{dF^*}{dt} = g_2(H^*, F^*)$$

We can calculate the Jacobian matrix for $(H^*, F^*)$ as

$$J = \begin{pmatrix} \left(\frac{dg_1}{dH}\right)_* & \left(\frac{dg_1}{dF}\right)_* \\ \left(\frac{dg_2}{dH}\right)_* & \left(\frac{dg_2}{dF}\right)_* \end{pmatrix}$$

$$
\begin{aligned}
\frac{dg_1}{dH} =& -\alpha - F\gamma - 2H\gamma + \frac{F^2\sigma}{(F+H+k)^2} + \frac{Fk\sigma}{(F+H+k)^2} + \\
& \frac{H\mu}{(F+H+\phi)^2} - \frac{\mu}{F+H+\phi} - \frac{(F+H)L}{(F+H+\omega)^2} + \frac{L}{F+H+\omega} \\
\frac{dg_1}{dF} =& -H\gamma + \frac{H(H+k)\sigma}{(F+H+k)^2} + \frac{H\mu}{(F+H+\phi)^2} - \frac{(F+H)L}{(F+H+\omega)^2} + \frac{L}{F+H+\omega} \\
\frac{dg_2}{dH} =& \alpha - \frac{F(F+k)\sigma}{(F+H+k)^2} + \frac{F\mu}{(F+H+\phi)^2} \\
\frac{dg_2}{dF} =& -m - \frac{H(H+k)\sigma}{(F+H+k)^2} - \frac{\mu(H+\phi)}{(F+H+\phi)^2}
\end{aligned}
$$

Evaluating the Jacobian at the equilbrium point $(H_0^*, F_0^*) = (0, 0)$,

$$
J^* = \begin{pmatrix} -\alpha - \frac{\mu}{\phi} + \frac{L}{\omega} & \frac{L}{\omega} \\ \alpha & -m - \frac{\mu}{\phi} \end{pmatrix}
$$

Calculating the eigenvalues,

$$
\begin{aligned}
\lambda_1 =& \frac{L\phi - 2\mu\omega - m\phi\omega - \alpha\phi\omega - \phi\sqrt{L^2 + 2Lm\omega + 2L\alpha\omega + m^2\omega^2 - 2m\alpha\omega^2 + \alpha^2\omega^2}}{2\phi\omega} \\
\lambda_2 =& \frac{L\phi - 2\mu\omega - m\phi\omega - \alpha\phi\omega + \phi\sqrt{L^2 + 2Lm\omega + 2L\alpha\omega + m^2\omega^2 - 2m\alpha\omega^2 + \alpha^2\omega^2}}{2\phi\omega}
\end{aligned}
$$

In order for the origin to be stable, we require all eigenvalues to be real and of negative sign. If we assume that all other parameters are nonzero, $\omega > 0, \alpha > 0, \phi > 0, L > 0, m > 0$ and that stress is also nonzero $\mu > 0$ then this happens either when (1)

$$0 < \omega < \frac{L(m+\alpha)}{m\alpha}$$

and when

$$\mu > \frac{L\phi + \omega\Big( - \phi(m+\alpha) + \sqrt{\frac{\phi^2(L^2+2L\omega(m+\alpha)+(m-\alpha^2)\omega^2)}{\omega^2}}\Big)}{2\omega}$$

or when (2),

$$\omega \geq \frac{L(m+\alpha)}{m\alpha}$$

So in order for the system to have a stable equilibrium at $(0,0)$, we require either (1) or (2) to hold true. If stress is present $\mu > 0$, then we have obtained the conditions for the stability of the origin.

*Bifurcations of Critical Parameters*

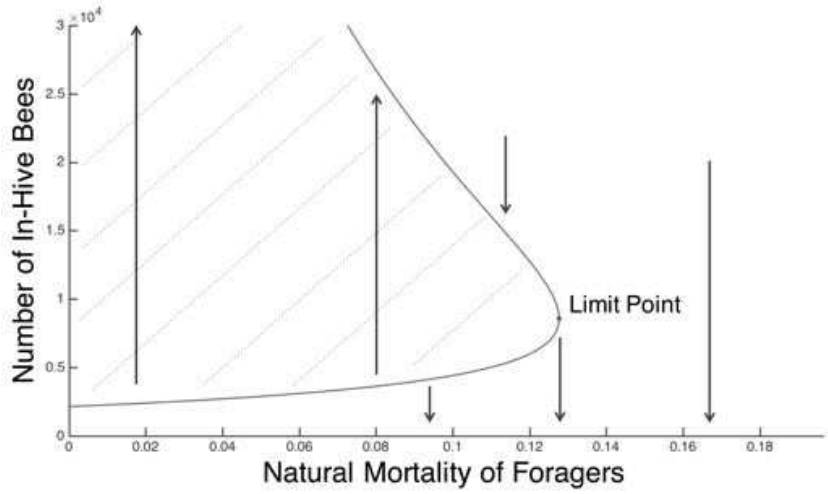



Figure S1: The saddle-node bifurcation through the parameter of natural mortality rate, $m$, with parameters taken from Table 1. The location of the limit point represents the critical death rate after which the total number of in-hive bees will become 0. For low initial values of in-hive bees, the colony can fail with no natural mortality present.

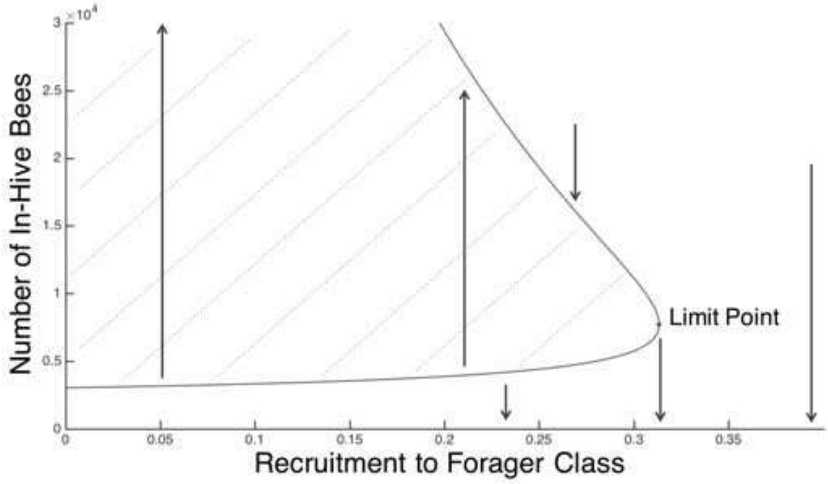



Figure S2: The saddle-node bifurcation through the parameter of recruitment $\alpha$, with parameters taken from Table 1. The bifurcation is similar to the natural mortality bifurcation, with the limit point being approached from low parameter values. If recruitment is critically high, then the colony will fail.

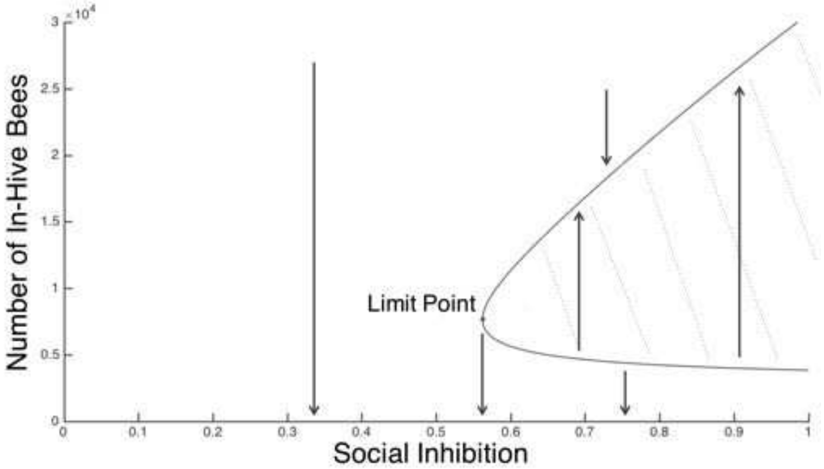



Figure S3: The reversed direction saddle-node bifurcation through the parameter of social inhibition $\sigma$, with parameters taken from Table 1. For high values of social inhibition task switching, the colony can sustain if the initial number of bees is greater than the critically low amount. If the social inhibition decreases below the limit point, then the colony will fail.

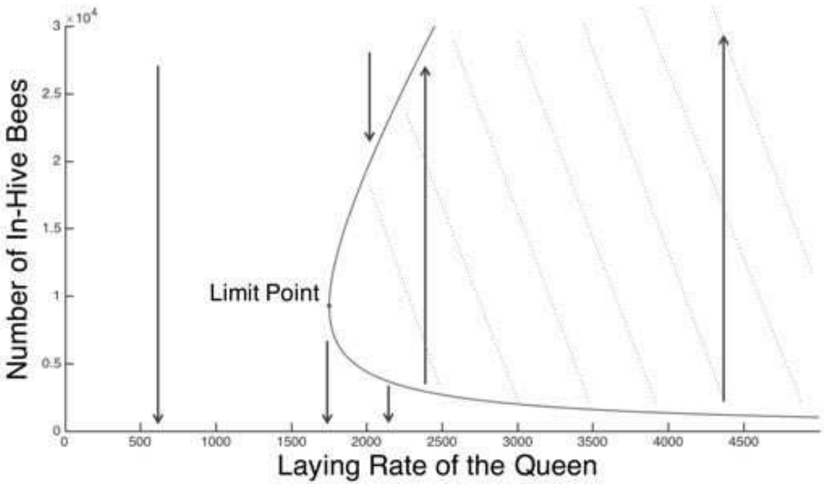
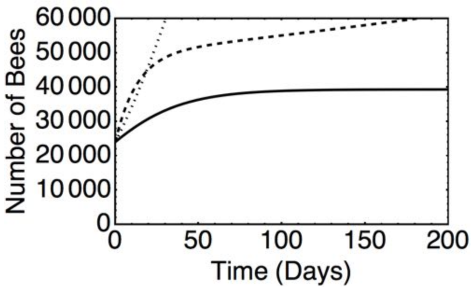
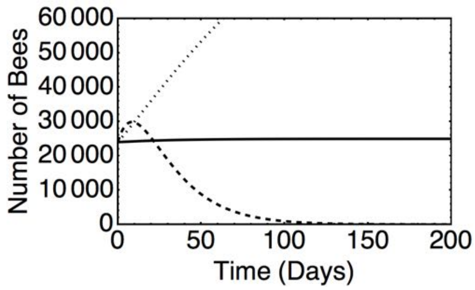



Figure S4: The reversed direction saddle-node bifurcation through the parameter of laying rate of the queen $L$, with parameters taken from Table 1. The bifurcation is similar to the bifurcation for social inhibition. If the queen lays more than the critically low value around the limit point, then the colony will sustain given that the initial number of bees is greater than the unstable branch.

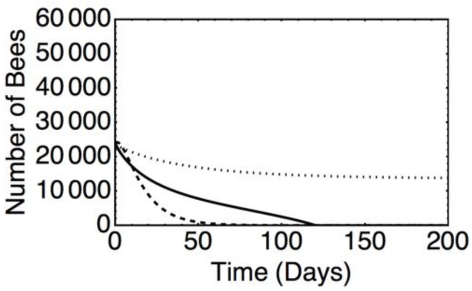
*Model Comparison*



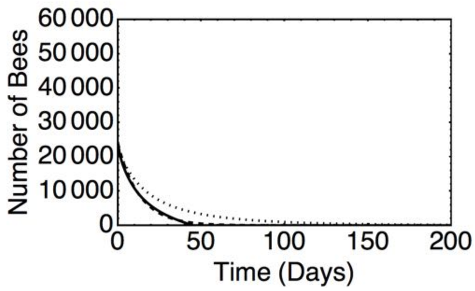

(a) Death Rate $m = 0$




(b) Death Rate $m = 0.05$

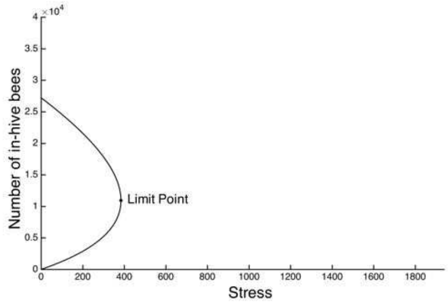



(c) Death Rate $m = 0.2$

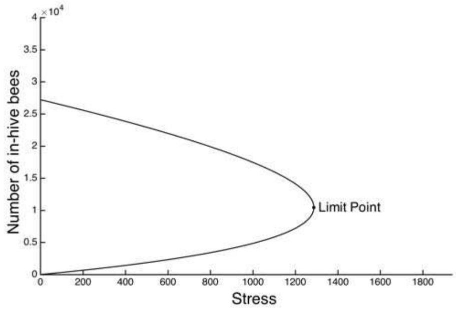



(d) Death Rate $m = 0.5$

Figure S5: The comparison between our simulated model, and other author's models, with parameters taken from Table 1. The black curve represents our adjusted model including the limiting $\gamma$ function. The dotted curve represents the Khoury et al. [2011] model and dashed curve represents the Bryden et al. [2013] model. As we increase each model through the death rate parameter, the sudden switch between indefinite growth and collapse exhibited by the Bryden model, and the very gradual decrease in the population exhibited by the Khoury model can be compared to the stabilisation of the population at low mortality and sudden collapse at higher mortality exhibited by our model.




(a) Stress function only in-hive bees




(b) Stress function only forager bees

Figure S6: The bifurcation through stress when acting only on the in-hive population (a) and only on the forager population (b), with parameters taken from Table 1. In (a) we see similar dynamics to the original model, however in (b) the limit point occurs at a higher stress level. The same qualitative dynamics are present (saddle-node bifurcation) in both cases.