STABLE POSTNIKOV DATA OF PICARD 2-CATEGORIES

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ABSTRACT. Picard 2-categories are symmetric monoidal 2-categories with invertible 0-, 1-, and 2-cells. The classifying space of a Picard 2-category $\mathcal{D}$ is an infinite loop space, the zeroth space of the K-theory spectrum $K\mathcal{D}$. This spectrum has stable homotopy groups concentrated in levels 0, 1, and 2. In this paper, we describe part of the Postnikov data of $K\mathcal{D}$ in terms of categorical structure. We use this to show that there is no strict skeletal Picard 2-category whose K-theory realizes the 2-truncation of the sphere spectrum. As part of the proof, we construct a categorical suspension, producing a Picard 2-category ΣC from a Picard 1-category C, and show that it commutes with K-theory in that $K\Sigma C$ is stably equivalent to ΣKC.

1. INTRODUCTION

This paper is part of a larger effort to refine and expand the theory of algebraic models for homotopical data, especially that of *stable* homotopy theory. Such modeling has been of interest since [May72, Seg74] gave K-theory functors which build connective spectra from symmetric monoidal categories. Moreover, Thomason [Tho95] proved that symmetric monoidal categories have a homotopy theory which is equivalent to that of all connective spectra.

Our current work is concerned with constructing models for stable homotopy 2-types using symmetric monoidal 2-categories. Preliminary foundations for this appear, for example, in [GO13, GJO15, JO12, SP11]. In forthcoming work [GJO16] we prove that all stable homotopy 2-types are modeled by a special kinds of symmetric monoidal 2-categories which we describe below and call *strict Picard 2-categories*.

This paper has three essential goals. First, we explicitly describe part of the Postnikov tower for strict Picard 2-categories. Second, and of independent interest, we show that the K-theory functor commutes with suspension up to stable equivalence. This allows us to bootstrap previous results on Picard 1-categories to give algebraic formulas for the two nontrivial Postnikov layers of a Picard 2-category. Third, we combine these to show that, while strict Picard 2-categories are expected to model all stable homotopy 2-types, strict and *skeletal* Picard 2-categories cannot. We prove that there is no strict and skeletal Picard 2-category modeling the truncation of the sphere spectrum.

1.1. **Background and motivation.** Homotopical invariants, and therefore homotopy *types*, often have a natural interpretation as categorical structures. The fundamental groupoid is a complete invariant for homotopy 1-types, while pointed connected homotopy 2-types are characterized by their associated crossed module or $\mathcal{C}at^1$-group structure [Whi49, MW50, BS76, Lod82, Con84]. Such characterizations provide the low-dimensional cases of Grothendieck's *Homotopy Hypothesis* [Gro83].

Homotopy Hypothesis. *There is an equivalence of homotopy theories between $\mathcal{G}pd^n$, weak n-groupoids equipped with categorical equivalences, and $\mathcal{T}op^n$, homotopy n-types equipped with weak homotopy equivalences.*

Restricting attention to stable phenomena, we replace homotopy n-types with stable homotopy n-types: spectra X such that $\pi_i X = 0$ unless $0 \leq i \leq n$. On the categorical side, we take a cue from [May72, Tho95] and replace n-groupoids with a grouplike, symmetric monoidal version that we call Picard n-categories. The stable version of the Homotopy Hypothesis is then the following.

Stable Homotopy Hypothesis. *There is an equivalence of homotopy theories between $\mathcal{P}ic^n$, Picard n-categories equipped with categorical equivalences, and $\mathcal{S}p_0^n$, stable homotopy n-types equipped with stable equivalences.*

For $n = 0$, $\mathcal{P}ic^0$ is the category of abelian groups $\mathcal{A}b$ with weak equivalences given by group isomorphisms. It is equivalent to the homotopy theory of Eilenberg-Mac Lane spectra. For $n = 1$, a proof of the Stable Homotopy Hypothesis appears in [JO12], and a proof for $n = 2$ will appear in the forthcoming [GJO16].

More than a proof of the Stable Homotopy Hypothesis, we seek a complete dictionary translating between stable homotopical invariants and the algebra of Picard n-categories. The search for such a dictionary motivated three questions that lie at the heart of this paper. First, how can we express invariants of stable homotopy types in algebraic terms? Second, how can we construct stable homotopy types of interest, such as Postnikov truncations of the sphere spectrum, from a collection of invariants? Third, can we make simplifying assumptions, such as strict inverses, about Picard n-categories without losing homotopical information?

1.2. **Postnikov invariants and strict skeletalization.** It has long been folklore that the symmetry in a Picard 1-category should model the bottom k-invariant, k_0. Along with a proof of the Stable Homotopy Hypothesis in dimension 1, this folklore result was established in [JO12]. This shows that a Picard 1-category is characterized by exactly three pieces of data: an abelian group of isomorphism classes of objects (π_0), an abelian group of automorphisms of the unit object (π_1), and a group homomorphism $k_0 \colon \pi_0 \otimes \mathbb{Z}/2 \to \pi_1$ (i.e., a stable quadratic map from π_0 to π_1) corresponding to the symmetry. Such a characterization is implied by the following result.

Theorem 1.1 ([JO12, Theorem 2.2])**.** *Every Picard category is equivalent to one which is both strict and skeletal.*

We call this phenomenon *strict skeletalization*. This theorem is quite surprising given that it is false without the symmetry. Indeed, Baez and Lauda [BL04] give a good account of the failure of strict skeletalization for 2-groups (the non-symmetric version of Picard 1-categories), and how it leads to a cohomological classification for 2-groups. Johnson and Osorno [JO12] show, in effect, that the relevant obstructions are *unstable* phenomena which become trivial upon stabilization.

When we turn to the question of building models for specific homotopy types, the strict and skeletal ones are the simplest: given a stable 1-type X, a strict and skeletal model will have objects equal to the elements of $\pi_0 X$ and automorphisms of the every object equal to the elements of $\pi_1 X$, with no morphisms between distinct objects. All that then remains is to define the correct symmetry isomorphisms, and these are determined entirely by the map k_0.

As an example, a strict and skeletal model for the 1-truncation of the sphere spectrum has objects the integers, each hom-set of automorphisms the integers mod 2, and k_0 given by the identity map on $\mathbb{Z}/2$ corresponding to the fact that the generating object 1 has a nontrivial symmetry with itself. One might be tempted to build a strict and skeletal model for the 2-type of the sphere spectrum (the authors here certainly were, and such an idea appears for example in [Bar14, 5.2]). But here we prove that this is not possible for the sphere spectrum, and in fact a large class of stable 2-types.

Theorem 1.2 (Theorem 3.13). *Let $\mathcal{D}$ be a strict skeletal Picard 2-category with k_0 surjective. Then the 0-connected cover of $K\mathcal{D}$ splits as a product of Eilenberg-Mac Lane spectra. In particular, there is no strict and skeletal model of the 2-truncation of the sphere spectrum.*

Our proof of this theorem identifies both the bottom k-invariant k_0 and the first Postnikov layer $k_1 i_1$ (see Section 3) explicitly using the symmetric monoidal structure. In order to give a formula for the first Postnikov layer, we must show that K-theory functors are compatible with suspension. More precisely, given a strict monoidal category C, one can construct a one-object 2-category ΣC, where the category of morphisms is given by C, with composition defined using the monoidal structure. Further, if C is a permutative category then ΣC is naturally a symmetric monoidal 2-category, with the monoidal structure also defined using the structure of C. Unstably, it is known that this process produces a categorical delooping: if C is a strict monoidal category with invertible objects, the classifying space $B(\Sigma C)$ is a delooping of BC [Jar91, CCG10]. We prove the stable analogue.

Theorem (Theorem 3.10). *For any permutative category C, the spectra $K(\Sigma C)$ and $\Sigma(KC)$ are stably equivalent.*

Here $K(-)$ denotes both the K-theory spectrum associated to a symmetric monoidal category [May72, Seg74] and the K-theory spectrum associated to a symmetric monoidal 2-category [GO13, GJO15]. In order to produce the comparison we work with the corresponding Γ-spaces and make substantial use of the theory of 2-monads.

Outline. In Section 2 we sketch the basic theory of Picard categories and Picard 2-categories. This includes some background to fix notation and some recent results about symmetric monoidal 2-categories [GJO15]. In Section 3 we develop algebraic models for some of the Postnikov data of the spectrum associated to a Picard 2-category, giving formulas for the two nontrivial layers in terms of the symmetric monoidal structure. This section closes with applications showing that strict skeletal Picard 2-categories cannot model all stable 2-types. Section 4 establishes formal strictification results for 2-categorical diagrams using 2-monad theory. We use those results in Section 5 to prove that the K-theory functor commutes with suspension.

2. PICARD CATEGORIES AND PICARD 2-CATEGORIES

This section introduces the primary categorical structures of interest which we call Picard 2-categories, as well as the particularly relevant variant of strict skeletal Picard 2-categories. Note that we use the term 2-category in its standard sense [KS74], and in particular all composition laws are strictly associative and unital.

Notation 2.1. We let $\mathcal{Cat}$ denote the category of categories and functors, and let $2\mathcal{Cat}$ denote the category of 2-categories and 2-functors. Note that these are both 1-categories.

Notation 2.2. We let Cat_2 denote the 2-cateogry of categories, functors, and natural transformations. This can be thought of as the 2-category of categories enriched in Set. Similarly, we let $2Cat_2$ denote the 2-category of 2-categories, 2-functors and 2-natural transformations; the 2-category of categories enriched in Cat.

2.1. **Picard categories.** We will begin by introducing all of the 1-categorical notions before going on to discuss their 2-categorical analogues. First we recall the notion of a permutative category (i.e., symmetric strict monoidal category); the particular form of this definition allows an easy generalization to structures on 2-categories.

Definition 2.3. A *permutative category C* consists of a strict monoidal category $(C, \oplus, e)$ together with a natural isomorphism,

$$C \times C \xrightarrow{\ \tau\ } C \times C$$
$$\begin{array}{ccc} & \nearrow \beta & \\ \oplus \searrow & & \swarrow \oplus \\ & C & \end{array}$$

where $\tau \colon C \times C \to C \times C$ is the symmetry isomorphism in Cat, such that the following axioms hold for all objects x, y, z of C.

- $\beta_{y,x}\beta_{x,y} = \mathrm{id}_{x \oplus y}$
- $\beta_{e,x} = \mathrm{id}_x = \beta_{x,e}$
- $\beta_{x,y \oplus z} = (y \oplus \beta_{x,z}) \circ (\beta_{x,y} \oplus z)$

Remark 2.4. We will sometimes say that a symmetric monoidal structure on a category is strict if its underlying monoidal structure is. Note that this does not imply that the symmetry is the identity, even though the other coherence isomorphisms are. Thus a permutative category is nothing more than a strict symmetric monoidal category.

Notation 2.5. Let $PermCat$ denote the category of permutative categories and symmetric, strict monoidal functors between them.

Next we require a notion of invertibility for the objects in a symmetric monoidal category.

Definition 2.6. Let $(C, \oplus, e)$ be a monoidal category. An object x is *invertible* if there exists an object y together with isomorphisms $x \oplus y \cong e$, $y \oplus x \cong e$.

Definition 2.7. A *Picard category* is a symmetric monoidal category in which all of the objects and morphisms are invertible.

The terminology comes from the following example.

Example 2.8. Let R be a commutative ring, and consider the symmetric monoidal category of R-modules. We have the subcategory $PicR$ of invertible R-modules and isomorphisms between them. The set of isomorphism classes of objects of $PicR$ is the classical Picard group of R.

Remark 2.9. If we drop the symmetric structure in Definition 2.7 above, we get the notion of what is both called a categorical group [JS93] or a 2-group [BL04]. These are equivalent to crossed modules [Whi49, Lod82], and hence are a model for pointed connected homotopy 2-types (i.e., spaces X for which $\pi_i(X) = 0$ unless $i = 1, 2$).

One should consider Picard categories as a categorified version of abelian groups. Just as abelian groups model the homotopy theory of spectra with trivial homotopy groups aside from π_0, Picard categories do the same for spectra with trivial homotopy groups aside from π_0 and π_1.

Theorem 2.10 ([JO12, Theorem 1.5])**.** *There is an equivalence of homotopy theories between the category of Picard categories, Pic^1, equipped with categorical equivalences, and the category of stable 1-types, Sp_0^1, equipped with stable equivalences.*

Forthcoming work [GJO16] proves the 2-dimensional analogue of Theorem 2.10. This requires a theory of Picard 2-categories which began in [GJO15] and motivated the work of the current paper. We now turn to such theory.

2.2. **Picard 2-categories.** To give the correct 2-categorical version of Picard categories, we must first describe the analogue of a mere strict monoidal category: such a structure is called a Gray-monoid. It is most succinctly defined using the Gray tensor product of 2-categories, written $\mathcal{A} \otimes \mathcal{B}$ for a pair of 2-categories $\mathcal{A}, \mathcal{B}$. We will not give the full definition of $\otimes$ here (see [GJO15, Gur13a, BG15a, BG15b]) but instead give the reader the basic idea. The objects of $\mathcal{A} \otimes \mathcal{B}$ are tensors $a \otimes b$ for $a \in \mathcal{A}, b \in \mathcal{B}$, but the 1-cells are *not* tensors of 1-cells as one would find in the cartesian product. Instead they are generated under composition by 1-cells $f \otimes 1$ and $1 \otimes g$ for $f \colon a \to a'$ a 1-cell in $\mathcal{A}$ and $g \colon b \to b'$ a 1-cell in $\mathcal{B}$. These different kinds of generating 1-cells do not commute with each other strictly, but instead up to specified isomorphism 2-cells

$$\Sigma_{f,g} \colon (f \otimes 1) \circ (1 \otimes g) \cong (1 \otimes g) \circ (f \otimes 1)$$

which obey appropriate naturality and bilinearity axioms. We call these Σ the *Gray structure 2-cells*. The 2-cells of $\mathcal{A} \otimes \mathcal{B}$ are defined similarly, generated by $\alpha \otimes 1, 1 \otimes \beta$, and the $\Sigma_{f,g}$. The function $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \otimes \mathcal{B}$ is the object part of a functor of categories

$$\mathcal{2Cat} \times \mathcal{2Cat} \to \mathcal{2Cat}$$

which is the tensor product for a symmetric monoidal structure on $\mathcal{2Cat}$ with unit the terminal 2-category.

Definition 2.11. A *Gray-monoid* is a monoid object $(\mathcal{D}, \oplus, e)$ in the monoidal category $(\mathcal{2Cat}, \otimes)$.

Remark 2.12. By the coherence theorem for monoidal bicategories [GPS95, Gur13a], every monoidal bicategory is equivalent (in the appropriate sense) to a Gray-monoid. There is a stricter notion, namely that of a monoid object in $(\mathcal{2Cat}, \times)$, but a general monoidal bicategory will not be equivalent to one of these.

We now turn to the symmetry.

Definition 2.13. A *symmetric monoidal 2-category* is a symmetric monoidal bicategory (see [GJO15] for a sketch or [McC00] for full details) in which the underlying bicategory is a 2-category.

Definition 2.14. A *permutative Gray-monoid* $\mathcal{D}$ consists of a Gray-monoid $(\mathcal{D}, \oplus, e)$ together with a 2-natural isomorphism,

$$
\begin{array}{ccc}
\mathcal{D} \otimes \mathcal{D} & \xrightarrow{\ \tau\ } & \mathcal{D} \otimes \mathcal{D} \\
& \underset{\oplus}{\searrow} \ \nearrow \beta \ \underset{\oplus}{\swarrow} & \\
& \mathcal{D} &
\end{array}
$$

where $\tau \colon \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is the symmetry isomorphism in $\mathcal{2Cat}$ for the Gray tensor product, such that the following axioms hold.

- The following pasting diagram is equal to the identity 2-natural transformation for the 2-functor $\oplus$.

$$
\begin{array}{ccccc}
& & \xrightarrow{\quad 1 \quad} & & \\
\mathcal{D} \otimes \mathcal{D} & \xrightarrow{\ \tau\ } & \mathcal{D} \otimes \mathcal{D} & \xrightarrow{\ \tau\ } & \mathcal{D} \otimes \mathcal{D} \\
& \searrow \nearrow \beta \quad \downarrow{\oplus} & & \Rightarrow \beta \quad \swarrow & \\
\oplus & & & & \oplus \\
& & \mathcal{D} & &
\end{array}
$$

- The following pasting diagram is equal to the identity 2-natural transformation for the canonical isomorphism $1 \otimes \mathcal{D} \cong \mathcal{D}$.

$$
\begin{array}{ccccc}
1 \otimes \mathcal{D} & \xrightarrow{\;e \otimes \mathrm{id}\;} & \mathcal{D} \otimes \mathcal{D} & \xrightarrow{\;\tau\;} & \mathcal{D} \otimes \mathcal{D} \\
 & {\scriptstyle =} \searrow \quad {\scriptstyle \cong} & \quad{\scriptstyle \oplus}\downarrow \quad {\Rightarrow}\beta & & \downarrow{\scriptstyle \oplus} \\
 & & \mathcal{D} & \leftarrow &
\end{array}
$$

- The following equality of pasting diagrams holds where we have abbreviated the tensor product to concatenation when labeling 1- or 2-cells.

Remark 2.15. Every symmetric monoidal bicategory is equivalent as such to a symmetric monoidal 2-category by strictifying the underlying bicategory and transporting the structure as in [Gur12]. A deeper result is that every symmetric monoidal bicategory is equivalent as such to a permutative Gray-monoid; this is explained fully in [GJO15], making use of [SP11].

Notation 2.16. For convenience and readability, we use following notational conventions for cells in a Gray-monoid $\mathcal{D}$.

- For objects, we may use concatenation instead of explicitly indicating the monoidal product.
- For an object b and a 1-cell $f \colon a \to a'$, we denote by fb the 1-cell in $\mathcal{D}$ which is the image under $\oplus$ of $f \otimes 1 \colon a \otimes b \to a' \otimes b$ in $\mathcal{D} \otimes \mathcal{D}$. We use similar notation for multiplication on the other side, and for 2-cells.
- We let $\Sigma_{f,g}$ also denote the image in $\mathcal{D}$ of the Gray structure 2-cells under $\oplus$:

$$\Sigma_{f,g} \colon (fb') \circ (ag) \cong (a'g) \circ (fb).$$

Notation 2.17. Let $\mathcal{P}\!erm\mathcal{G}ray\mathcal{M}on$ denote the category of permutative Gray-monoids and strict symmetric monoidal 2-functors between them.

We are actually interested in permutative Gray-monoids which model stable homotopy 2-types, and we therefore restrict to those in which all the cells are invertible. We begin by defining invertibility in a Gray-monoid, then the notion of a Picard 2-category, and finish with that of a strict skeletal Picard 2-category.

Definition 2.18. Let $(\mathcal{D}, \oplus, e)$ be a Gray-monoid.

  i. A 2-cell of $\mathcal{D}$ is invertible if it has an inverse in the usual sense.
 ii. A 1-cell $f \colon x \to y$ is invertible if there exists a 1-cell $g \colon y \to x$ together with invertible 2-cells $g \circ f \cong \mathrm{id}_x$, $f \circ g \cong \mathrm{id}_y$. In other words, f is invertible if it is an internal equivalence (denoted with the $\simeq$ symbol) in $\mathcal{D}$.
iii. An object x of $\mathcal{D}$ is invertible if there exists another object y together with invertible 1-cells $x \oplus y \simeq e$, $y \oplus x \simeq e$.

Remark 2.19. The above definition actually used none of the special structure of a Gray-monoid that is not also present in a more general monoidal bicategory.

Definition 2.20. A *Picard 2-category* is a symmetric monoidal 2-category (see Definition 2.13) in which all of the objects, 1-cells, and 2-cells are invertible. A *strict Picard 2-category* is a permutative Gray-monoid which is a Picard 2-category.

Remark 2.21. Note that the definition of a strict Picard 2-category does not require that cells be invertible in the strict sense, i.e., having inverses on the nose rather than up to mediating higher cells. It only requires that the underlying symmetric monoidal structure is strict in the sense of being a permutative Gray-monoid.

Definition 2.22. A 2-category $\mathcal{A}$ is *skeletal* if the following condition holds: whenever there exists an invertible 1-cell $f : x \simeq y$, then $x = y$.

Remark 2.23. This definition might more accurately be named *skeletal on objects*, as one could impose a further condition of being skeletal on 1-cells as well. We have no need of this further condition, and so we work with this less restrictive notion of a skeletal 2-category. It is also important to remember that, in the definition above, the invertible 1-cell f need not be the identity 1-cell. The slogan is that "every equivalence is an autoequivalence": an object is allowed to have many non-identity autoequivalences, and there can be 1-cells between different objects as long as they are not equivalences.

Definition 2.24. A *strict skeletal Picard 2-category* is a strict Picard 2-category whose underlying 2-category is skeletal.

2.3. **Two adjunctions.** Our goal in this subsection is to present two different adjunctions between strict Picard categories and strict Picard 2-categories. While we focus on the categorical algebra here, later we will give each adjunction a homotopical interpretation. The unit of the first adjunction will categorically model Postnikov 1-truncation (Proposition 3.5), universally making π_2 zero, while the counit of the second will categorically model the 0-connected cover (Proposition 3.9).

Recall that for any category C, we have its set of path components denoted $\pi_0 C$; these are given by the path components of the nerve of C, or equivalently by quotienting the set of objects by the equivalence relation generated by $x \sim y$ if there exists an arrow $x \to y$. This is the object part of a functor $\pi_0 \colon \mathcal{C}at \to \mathcal{S}et$, and it is easy to verify that this functor preserves finite products. It is also left adjoint to the functor $d \colon \mathcal{S}et \to \mathcal{C}at$ which sends a set S to the discrete category with the same set of objects. Being a right adjoint, d preserves all products. The counit $\pi_0 \circ d \Rightarrow \mathrm{id}$ is the identity, and the unit $\mathrm{id} \Rightarrow d \circ \pi_0$ is the quotient functor $C \to d\pi_0 C$ sending every object to its path component and every morphism to the identity. Since d and π_0 preserve products, by applying them to hom-objects they induce change of enrichment functors d_* and $(\pi_0)_*$, respectively. We obtain the following result.

Lemma 2.25. *The adjunction* $\pi_0 \dashv d$ *lifts to a 2-adjunction*

$$2\mathcal{C}at_2 \underset{d_*}{\overset{(\pi_0)_*}{\rightleftarrows}} \bot \; \mathcal{C}at_2 .$$

Notation 2.26. We will write the functor $(\pi_0)_*$ as $\mathcal{D} \mapsto \mathcal{D}_1$ to lighten the notation. This anticipates the homotopical interpretation in Proposition 3.5. Furthermore, we will write d_* as d, it will be clear from context which functor we are using.

Lemma 2.27. *The functor* $\mathcal{D} \mapsto \mathcal{D}_1$ *is strong symmetric monoidal* $(2\mathcal{C}at, \otimes) \to (\mathcal{C}at, \times)$. *The functor* d *is lax symmetric monoidal* $(\mathcal{C}at, \times) \to (2\mathcal{C}at, \otimes)$.

Proof. The second statement follows from the first by doctrinal adjunction [Kel74]. For the first, one begins by checking that

$$\mathcal{D}_1 \times \mathcal{E}_1 \cong (\mathcal{D} \otimes \mathcal{E})_1;$$

this is a simple calculation using the definition of $\otimes$ that we leave to the reader. If we let I denote the terminal 2-category, the unit for $\otimes$, then I_1 is the terminal category, so $(-)_1$ preserves units up to (unique) isomorphism. It is then easy to check that these isomorphisms interact with the associativity, unit, and symmetry isomorphisms to give a strong symmetric monoidal functor. $\square$

Remark 2.28. It is useful to point out that if A, B are categories, then the comparison 2-functor

$$\chi_{A,B} \colon dA \otimes dB \to d(A \times B)$$

is the 2-functor which quotients all the 2-cells $\Sigma_{f,g}$ to be the identity. In view of the adjunction in Lemma 2.25, the 2-functor $\chi_{A,B}$ can be identified with the component of the unit at $dA \otimes dB$.

Our first adjunction between Picard 1- and 2-categories is contained in the following result.

Proposition 2.29. *The functors $\mathcal{D} \mapsto \mathcal{D}_1$ and d induce adjunctions between*

- *the categories PermGrayMon and PermCat, and*
- *the category of strict Picard 2-categories and the category of strict Picard categories.*

The counits of these adjunctions are both identities.

Proof. It is immediate from Lemma 2.27 and the definitions that applying $\mathcal{D} \mapsto \mathcal{D}_1$ to a permutative Gray-monoid gives a permutative category, and that the resulting permutative category is a strict Picard category if $\mathcal{D}$ is a strict Picard 2-category; this constructs both left adjoints. To construct the right adjoints, let $(C, \oplus, e)$ be a permutative category. We must equip dC with a permutative Gray-monoid structure. The tensor product is given by

$$dC \otimes dC \xrightarrow{\chi_{C,C}} d(C \times C) \xrightarrow{d \oplus} dC$$

using Lemma 2.27 or the explicit description in Remark 2.28. The 2-natural isomorphism β^{dC} is $d(\beta^C) * \chi_{C,C}$, using the fact that $d(\tau^\times) \circ \chi = \chi \circ \tau^\otimes$ by the second part of Lemma 2.27. The permutative Gray-monoid axioms for dC then reduce to the permutative category axioms for C and the lax symmetric monoidal functor axioms for d. Once again, dC is a strict Picard 2-category if C is a strict Picard category. The statement about counits follows from the corresponding statement about the counit for the adjunction $\pi_0 \dashv d$, and the unit is a strict symmetric monoidal 2-functor by inspection. The triangle identities then follow from those for $\pi_0 \dashv d$, concluding the construction of both adjunctions. $\square$

Remark 2.30. The proof above is simple, but not entirely formal: while symmetric monoidal categories are the symmetric pseudomonoids in the symmetric monoidal 2-category Cat, permutative Gray-monoids do not admit such a description due to the poor interaction between the Gray tensor product and 2-natural transformations.

We now move on to our second adjunction between permutative categories and permutative Gray-monoids which restricts to one between strict Picard categories and strict Picard 2-categories. This adjunction models loop and suspension functors, and appears informally in work of Baez and Dolan [BD95] on stabilization phenomena in higher categories.

Lemma 2.31. *Let $(C, \oplus, e)$ be a permutative category with symmetry σ. Then the 2-category ΣC with one object $*$, hom-category $\Sigma C(*, *) = C$, and horizontal composition given by $\oplus$ admits the structure of a permutative Gray-monoid $(\Sigma C, \widetilde{\oplus})$. The assignment $(C, \oplus) \mapsto (\Sigma C, \widetilde{\oplus})$ is the function on objects of a functor*

$$\Sigma \colon \mathcal{P}\!\mathit{ermCat} \to \mathcal{P}\!\mathit{ermGrayMon}.$$

Proof. Since C is a strict monoidal category, ΣC is a strict 2-category when horizontal composition is given by $\oplus$. We can define a 2-functor $\widetilde{\oplus} \colon \Sigma C \otimes \Sigma C \to \Sigma C$ as the unique function on 0-cells, by sending any cell of the form $a \otimes 1$ to a, any cell of the form $1 \otimes b$ to b, and $\Sigma_{a,b}$ to the symmetry $\sigma_{a,b} \colon a \oplus b \cong b \oplus a$. With the unique object as the unit, it is simple to check that this 2-functor makes ΣC into a Gray-monoid. All that remains is to define β and check the three axioms. Since there is only one object and it is the unit, the second axiom shows that the unique component of β must be the identity 1-cell. Then naturality on 1-cells is immediate, and the only two-dimensional naturality that is not obvious is for the cells $\Sigma_{a,b}$. This axiom becomes the equation

$$\beta \oplus \Sigma_{a,b} = \Sigma_{b,a}^{-1} \oplus \beta$$

which is merely the claim that $\sigma_{a,b}$ is a symmetry rather than a braid. It is then obvious that this assignment defines a functor as stated. $\qquad\square$

Example 2.32. The permutative Gray-monoid constructed in [SP11, Example 2.30] is a suspension ΣC for the following permutative category C.

- The objects of C are the elements of $\mathbb{Z}/2$ with the monoidal structure given by addition.
- Each endomorphism monoid of C is $\mathbb{Z}/2$ and there are no morphisms between distinct objects.
- The symmetry of the non-unit object with itself is the nontrivial morphism.

Remark 2.33. It is natural to expect that the permutative Gray-monoid ΣC in the previous example models the 0-connected cover of the 2-type of the sphere spectrum, and indeed this will follow from Theorem 3.10. One might also hope that a skeletal model for the sphere spectrum can be constructed as a "many-object" version of ΣC together with an appropriate symmetry. However Theorem 3.13 will prove that this is not possible.

Lemma 2.34. *Let $(\mathcal{D}, \oplus, e)$ be a permutative Gray-monoid. Then the category $\mathcal{D}(e, e)$ is a permutative category, with tensor product given by composition. The assignment $\mathcal{D} \mapsto \mathcal{D}(e, e)$ is the function on objects of a functor*

$$\Omega \colon \mathcal{P}\!\mathit{ermGrayMon} \to \mathcal{P}\!\mathit{ermCat}.$$

Proof. For a Gray-monoid $\mathcal{D}$, the hom-category $\mathcal{D}(e, e)$ is a braided, strict monoidal category [GPS95, CG11] in which the tensor product is given by composition and the braid $f \circ g \cong g \circ f$ is the morphism $\Sigma_{f,g}$ in $\mathcal{D}(e, e)$; we note that $fe = f$ and $eg = g$ since all the 1-cells involved are endomorphisms of the unit object, and the unit object in a Gray-monoid is a strict two-sided unit. The component $\beta_{e,e}$ is necessarily the identity, and the calculations in the proof of Lemma 2.31 show that $\Sigma_{f,g} = \Sigma_{g,f}^{-1}$, so we have a permutative structure on $\mathcal{D}(e, e)$. $\qquad\square$

Proposition 2.35. *The functor $\Sigma \colon \mathcal{P}\!\mathit{ermCat} \to \mathcal{P}\!\mathit{ermGrayMon}$ is left adjoint to the functor $\Omega \colon \mathcal{P}\!\mathit{ermGrayMon} \to \mathcal{P}\!\mathit{ermCat}$.*

Proof. It is easy to check that the composite $\Omega\Sigma$ is the identity functor on $\mathcal{P}\!\mathit{ermCat}$, and we take this equality to be the unit of the adjunction. The counit would be a functor $\Sigma(\mathcal{D}(e, e)) \to \mathcal{D}$ which we must define to send the single object of $\Sigma(\mathcal{D}(e, e))$ to the unit

object e of $\mathcal{D}$ and then to be the obvious inclusion on the single hom-category. This is clearly a 2-functor, and the arguments in the proofs of the previous two lemmas show that this is a strict map of permutative Gray-monoids.

The counit is then obviously the identity on the only hom-category when $\mathcal{D}$ has a single object, and this statement is in fact the commutativity of one of the triangle identities for the adjunction. It is simple to check that Ω applied to the counit is the identity as well since the counit is the identity functor when restricted to the hom-category of the unit objects, and this is the other triangle identity, completing the verification of the adjunction. $\square$

Since the unit $1 \Rightarrow \Omega\Sigma$ is the identity, and the counit is an isomorphism on permutative Gray-monoids with one object, we have the following corollary.

Corollary 2.36. *The adjunction $\Sigma \dashv \Omega$ in Proposition 2.35 restricts to the categories of strict Picard categories and strict Picard 2-categories. Moreover, this adjunction gives equivalences between*

- *the category of permutative categories and the category of one-object permutative Gray-monoids, and*
- *the category of strict Picard categories and the category of one-object strict Picard 2-categories.*

Proof. The first statement follows from the definitions, since both Σ and Ω send strict Picard objects in one category to strict Picard objects in the other. The other two statements are obvious from the proof above. $\square$

3. Stable homotopy theory of Picard 2-categories

In this section we describe how to use the algebra of Picard 2-categories to express homotopical features of their corresponding connective spectra categorically. We begin with a brief review of stable Postnikov towers, mainly for the purpose of fixing notation. Subsequently, we identify algebraic models for this homotopical data in terms of the categorical structure present in a Picard 2-category.

For an abelian group π, the Eilenberg-Mac Lane spectrum of π is denoted $H\pi$. Its nth suspension is denoted $\Sigma^n H\pi$, and has zeroth space given by the Eilenberg-Mac Lane space $K(\pi, n)$. With this notation, the stable Postnikov tower of a connective spectrum X is given as follows.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
\Sigma^2 H(\pi_2 X) & \xrightarrow{i_2} & X_2 & \xrightarrow{k_2} & \Sigma^4 H(\pi_3 X) \\
 & & \downarrow & & \\
\Sigma^1 H(\pi_1 X) & \xrightarrow{i_1} & X_1 & \xrightarrow{k_1} & \Sigma^3 H(\pi_2 X) \\
 & & \downarrow & & \\
 & & X_0 & \xrightarrow{k_0} & \Sigma^2 H(\pi_1 X)
\end{array}
$$

Since X is connective, it follows that $X_0 = H(\pi_0 X)$ and k_0 is therefore a stable map from $H(\pi_0 X)$ to $\Sigma^2 H(\pi_1 X)$. When X is the K-theory spectrum of a strict Picard 2-category, we will model k_0 and $k_1 i_1$ algebraically via stable quadratic maps. A *stable quadratic map* is a homomorphism from an abelian group A to the 2-torsion of an abelian group B. The abelian group of stable homotopy classes $[HA, \Sigma^2 HB]$ is naturally isomorphic to the abelian group of stable quadratic maps $A \to B$ by [EM54a, Equation (27.1)]. Moreover [EM54b, Theorem 20.1] implies that under this identification

$k_0 \colon H(\pi_0 X) \to \Sigma^2 H(\pi_1 X)$ corresponds to the stable quadratic map $\pi_0 X \to \pi_1 X$ given by precomposition with the Hopf map $\eta \colon \Sigma \mathbb{S} \to \mathbb{S}$ where $\mathbb{S}$ denotes the sphere spectrum.

The stable Postnikov tower can be constructed naturally in X, so that if

$$X' \to X$$

is a map of spectra, we have the following commuting naturality diagram of stable Postnikov layers.

(3.1)

$$
\begin{array}{ccc}
\Sigma^n H(\pi_n X') \xrightarrow{\ i'_n\ } X'_n \xrightarrow{\ k'_n\ } \Sigma^{n+2} H(\pi_{n+1} X') \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
\Sigma^n H(\pi_n X) \xrightarrow{\ i_n\ } X_n \xrightarrow{\ k_n\ } \Sigma^{n+2} H(\pi_{n+1} X)
\end{array}
$$

Picard 2-categories model stable 2-types via K-theory. The K-theory functors for symmetric monoidal n-categories, constructed in [Seg74, Tho95, Man10] for $n = 1$ and [GJO15] for $n = 2$, give faithful embeddings of Picard n-categories into stable homotopy. For the purposes of this section we can take K-theory largely as a black box; in Section 5 we give necessary definitions and properties.

3.1. **Modeling stable Postnikov data.** For a Picard category $(C, \oplus, e)$, the two possibly nontrivial stable homotopy groups of its K-theory spectrum $K(C)$ are given by

$$
\begin{array}{rcl}
\pi_0 K(C) & \cong & \mathrm{ob}\,C/\{x \sim y \text{ if there exists a 1-cell } f \colon x \to y\} \\
\pi_1 K(C) & \cong & C(e, e).
\end{array}
$$

The stable homotopy groups of the K-theory spectrum of a strict Picard 2-category can be calculated similarly. We denote the classifying space of a 2-category $\mathcal{D}$ by $B\mathcal{D}$ [CCG10].

Lemma 3.2. *Let $\mathcal{D}$ be a strict Picard 2-category. The classifying space $B\mathcal{D}$ is equivalent to $\Omega^\infty K(\mathcal{D})$. The stable homotopy groups $\pi_i K(\mathcal{D})$ are zero except when $0 \le i \le 2$, in which case they are given by the formulas below.*

$$
\begin{array}{rcl}
\pi_0 K(\mathcal{D}) & \cong & \mathrm{ob}\,\mathcal{D}/\{x \sim y \text{ if there exists a 1-cell } f \colon x \to y\} \\
\pi_1 K(\mathcal{D}) & \cong & \mathrm{ob}\,\mathcal{D}(e, e)/\{f \sim g \text{ if there exists a 2-cell } \alpha \colon f \Rightarrow g\} \\
\pi_2 K(\mathcal{D}) & \cong & \mathcal{D}(e, e)(\mathrm{id}_e, \mathrm{id}_e)
\end{array}
$$

Proof. First, note that $\mathcal{D}$ has underlying 2-category a bigroupoid, and the above are the unstable homotopy groups of the pointed space $(B\mathcal{D}, e)$ by [CCG10, Remark 4.4]. Since the objects of $\mathcal{D}$ are invertible, the space $B\mathcal{D}$ is group-complete, and hence it is the zeroth space of the spectrum $K(\mathcal{D})$. Thus the stable homotopy groups of $K(\mathcal{D})$ agree with the unstable ones for $B\mathcal{D}$. $\qquad\square$

We will use the adjunctions from Section 2.3 to reduce the calculation of the stable quadratic maps corresponding to k_0 and $k_1 i_1$ of $K(\mathcal{D})$ to two instances of the calculation of k_0 in the 1-dimensional case.

Lemma 3.3 ([JO12])**.** *Let C be a strict Picard category with unit e and symmetry β. Then the bottom stable Postnikov invariant $k_0 \colon H\pi_0 K(C) \to \Sigma^2 H\pi_1 K(C)$ is modeled by the stable quadratic map $k_0 \colon \pi_0 K(C) \to \pi_1 K(C)$,*

$$[x] \mapsto (e \xrightarrow{\ \cong\ } x\,x\,x^*\,x^* \xrightarrow{\ \beta_{x,x}\,x^*\,x^*\ } x\,x\,x^*\,x^* \xrightarrow{\ \cong\ } e),$$

where x is an object in C and x^* denotes an inverse of x.

Remark 3.4. The middle term of the composite $k_0(x)$ was studied in [Sín75, JS93] and is called the *signature* of x.

Proof of Lemma 3.3. Note that $k_0 \colon \pi_0 K(C) \to \pi_1 K(C)$ is a well-defined function (does not depend on the choices of x, x^*, and $xx^* \cong e$). Indeed, given isomorphisms $x \cong y$, $xx^* \cong e$ and $yy^* \cong e$, there is a unique isomorphism $j \colon x^* \cong y^*$ such that

$$
\begin{array}{ccc}
yx^* & \longrightarrow & xx^* \\
{\scriptstyle yj}\big\downarrow & & \big\downarrow \\
yy^* & \longrightarrow & e
\end{array}
$$

commutes.

Moreover, it is clear that k_0 is compatible with equivalences of Picard categories. By [JO12, Theorem 2.2], we can thus replace C by a strict skeletal Picard category. In [*loc. cit.*, Section 3], a natural action $S \times C \to C$ is defined, where S is a strict skeletal model for the 1-truncation of the sphere spectrum. It follows from the definition of the action that

$$
\pi_1(BS) \times \pi_1(BC, x) \to \pi_1(BC, e)
$$

sends (η, id_x) to $\beta_{x,x} x^* x^*$, where η denotes the generator of $\pi_1(BS) \cong \mathbb{Z}/2$. Finally it follows from [*loc. cit.*, Proposition 3.4] that the action $S \times C \to C$ models the truncation of the action of the sphere spectrum on KC, thus the image under the action of (η, id_x) agrees with the image of $[x]$ under the stable quadratic map associated to the bottom stable Postnikov invariant. $\square$

Proposition 3.5. *Let $\mathcal{D}$ be a strict Picard 2-category and let $\mathcal{D} \to d(\mathcal{D}_1)$ be the unit of the adjunction in Proposition 2.29. Then*

$$
K(\mathcal{D}) \to K\big(d(\mathcal{D}_1)\big)
$$

is the 1-truncation of $K(\mathcal{D})$.

Proof. Using the formulas in Lemma 3.2, it is clear that $\mathcal{D} \to d(\mathcal{D}_1)$ induces an isomorphism on π_0 and π_1, and that $\pi_2 K\big(d(\mathcal{D}_1)\big) = 0$. Moreover, both K-theory spectra have $\pi_i = 0$ for $i > 2$, so $\mathcal{D}_1$ models the 1-truncation of $\mathcal{D}$. $\square$

Lemma 3.6. *For any permutative category C, the K-theory spectrum of C is stably equivalent to the K-theory spectrum of the corresponding permutative Gray-monoid, dC.*

Proof. This follows directly from the formulas in [GJO15], and in particular Remark 6.32. $\square$

For any connective spectrum X, the bottom stable Postnikov invariant of X and its 1-truncation X_1 agree. Thus combining Lemma 3.3, Proposition 3.5 and Lemma 3.6 yields the following result.

Corollary 3.7. *Let $\mathcal{D}$ be a strict Picard 2-category with unit e and symmetry β. Then the bottom stable Postnikov invariant $k_0 \colon H\pi_0 K(\mathcal{D}) \to \Sigma^2 H\pi_1 K(\mathcal{D})$ is modeled by the stable quadratic map $k_0 \colon \pi_0 K(\mathcal{D}) \to \pi_1 K(\mathcal{D})$,*

$$
[x] \mapsto [e \xrightarrow{\simeq} x x x^* x^* \xrightarrow{\beta_{x,x} x^* x^*} x x x^* x^* \xrightarrow{\simeq} e],
$$

where x is an object in $\mathcal{D}$ and x^* denotes an inverse of x.

Remark 3.8. It can be checked directly that the function $k_0 \colon \mathrm{ob}(\mathcal{D}) \to \pi_1(K\mathcal{D})$ is well-defined using the essential uniqueness of the inverse: given another object $\overline{x}$ together with an equivalence $e \simeq x\overline{x}$, there is an equivalence $x^* \simeq \overline{x}$ and an isomorphism 2-cell in the obvious triangle which is unique up to unique isomorphism. This follows from the techniques in [Gur12], and many of the details are explained there in Section 6.

In order to identify the composite $k_1 i_1$ categorically, we analyze the relationship between Postnikov layers and categorical suspension.

Proposition 3.9. *Let $\mathcal{D}$ be a strict Picard 2-category and let $\Sigma\Omega\mathcal{D} \to \mathcal{D}$ be the counit of the adjunction in Proposition 2.35. Then*

$$K(\Sigma\Omega\mathcal{D}) \to K(\mathcal{D})$$

is a 0-connected cover of $K(\mathcal{D})$.

Proof. It is clear from the formulas in Lemma 3.2 that $\Sigma\Omega\mathcal{D} \to \mathcal{D}$ induces an isomorphism on π_1 and π_2, and moreover, the corresponding K-theory spectra have $\pi_i = 0$ for $i > 2$. Since $\Sigma\Omega\mathcal{D}$ has only one object, we have $\pi_0 K(\Sigma\Omega\mathcal{D}) = 0$, so $\Sigma\Omega\mathcal{D}$ models the 0-connected cover of $\mathcal{D}$. $\qquad\square$

In addition to the elementary algebra and homotopy theory of Picard 2-categories discussed above, we require the following result.

Theorem 3.10. *Let C be a permutative category. Then $\Sigma K(C)$ and $K(\Sigma C)$ are stably equivalent.*

The proof of Theorem 3.10 requires a nontrivial application of 2-monad theory. We develop the relevant 2-monadic techniques in Section 4 and give the proof in Section 5. These two sections are independent of the preceding sections.

Lemma 3.11. *Let $(\mathcal{D}, \oplus, e)$ be a strict Picard 2-category. Then the composite*

$$k_1 i_1 \colon \Sigma H \pi_1 K(\mathcal{D}) \to \Sigma^3 H \pi_2 K(\mathcal{D})$$

is modeled by the stable quadratic map $\pi_1 K(\mathcal{D}) \to \pi_2 K(\mathcal{D})$,

$$[f] \mapsto (\mathrm{id}_e \xrightarrow{\cong} f \circ f \circ f^* \circ f^* \xrightarrow{\Sigma_{f,f} f^* \circ f^*} f \circ f \circ f^* \circ f^* \xrightarrow{\cong} \mathrm{id}_e),$$

where $f \colon e \to e$ is a 1-cell in $\mathcal{D}$ and f^* denotes an inverse of f.

Proof. We use superscripts to distinguish Postnikov data of different spectra. The composite $k_1^{\mathcal{D}} i_1^{\mathcal{D}}$ in the first Postnikov layer of the spectrum $K(\mathcal{D})$ identifies with the composite $k_1^{\Sigma\Omega\mathcal{D}} i_1^{\Sigma\Omega\mathcal{D}}$ since $K(\Sigma\Omega\mathcal{D})$ is the 0-connected cover of $K(\mathcal{D})$ by Proposition 3.9 and the Postnikov tower can be constructed naturally (Display (3.1)).

Since $K(\Sigma\Omega\mathcal{D}) \simeq \Sigma K(\Omega\mathcal{D})$ by Theorem 3.10 and $K(\Omega\mathcal{D})$ is connective, it follows that

$$k_1^{\Sigma\Omega\mathcal{D}} i_1^{\Sigma\Omega\mathcal{D}} = \Sigma(k_0^{\Omega\mathcal{D}} i_0^{\Omega\mathcal{D}}) = \Sigma(k_0^{\Omega\mathcal{D}})$$

in the stable homotopy category.

Finally, we deduce from Lemma 3.3 that the map $\Sigma(k_0^{\Omega\mathcal{D}})$ is represented by the desired group homomorphism. $\qquad\square$

3.2. **Application to strict skeletal Picard 2-categories.**

Now we make an observation about the structure 2-cells $\Sigma_{f,g}$ in a strict Picard 2-category. This algebra will be a key input for our main application, Theorem 3.13.

Lemma 3.12. *Let $(\mathcal{D}, \oplus, e)$ be a strict Picard 2-category. Let $g \colon e \to e$ be any 1-cell and let $s = \beta_{x,x} x^* x^*$ be a representative of the signature of some object x with inverse x^*. Then $\Sigma_{s,g}$ and $\Sigma_{g,s}$ are identity 2-cells in $\mathcal{D}$.*

Proof. By naturality of the symmetry and interchange, $\Sigma_{\beta_{y,z},h}$ and $\Sigma_{h,\beta_{y,z}}$ are identity 2-cells for any 1-cell h [GJO15, Prop. 3.41]. The result for $\Sigma_{g,s}$ follows by noting that $\Sigma_{g,f} w = \Sigma_{g,fw}$ for any 1-cells f, g and object w by the associativity axiom for a Gray-monoid. Hence $\Sigma_{g,s} = \Sigma_{g,\beta_{x,x} x^* x^*} = \Sigma_{g,\beta_{x,x}} x^* x^*$, which is the identity 2-cell.

For the other equality, we note the final axiom of [Gur13a, Prop. 3.3] reduces to the following equality of pasting diagrams for objects y, z, w with endomorphisms t_y, t_z, t_w respectively.

Thus the result for $\Sigma_{s,g}$ follows by taking $(y,z,w) = (xx, x^* x^*, e)$, $t_y = \beta_{x,x}$, $t_z = \text{id}$, and $t_w = g$. $\qquad\square$

We are now ready to give our main application regarding stable Postnikov data of strict skeletal Picard 2-categories.

Theorem 3.13. *Let $\mathcal{D}$ be a strict skeletal Picard 2-category and assume that*

$$k_0 \colon \pi_0 K(\mathcal{D}) \to \pi_1 K(\mathcal{D})$$

is surjective. Then $k_1 i_1$ is trivial.

Proof. We prove that the stable quadratic map $\pi_1 K(\mathcal{D}) \to \pi_2 K(\mathcal{D})$ from Lemma 3.11 that models the composite $k_1 i_1$ is trivial. Since k_0 is surjective by assumption, it suffices to consider $k_1 i_1(f)$ for f of the form

(3.14)
$$e \xrightarrow{w} xxx^* x^* \xrightarrow{\beta_{x,x} x^* x^*} xxx^* x^* \xrightarrow{w^*} e$$

for some object x with inverse x^*. Here w denotes the composite

$$e \xrightarrow{u} xx^* \xrightarrow{xux^*} xxx^* x^*$$

for a chosen equivalence $u \colon e \simeq xx^*$ and w^* denotes the corresponding reverse composite for a chosen $u^* \colon xx^* \simeq e$ inverse to u. Note that the isomorphism class of f is independent of the choices of the inverse object x^* and the equivalences u and u^* (see Remark 3.8). Since $\mathcal{D}$ is skeletal, it must be that $xx^* = e$. Therefore we can choose the equivalence $u \colon e \simeq xx^*$ to be id_e and then choose u^* to be id_e as well. With these choices, the composite f is actually equal to $\beta_{x,x} x^* x^*$. By Lemma 3.12 the Gray structure 2-cell $\Sigma_{f,f}$ is the identity 2-cell $\text{id}_{f \circ f}$. This implies that $k_1 i_1(f) = \text{id}_{\text{id}_e}$. $\qquad\square$

Remark 3.15. The result of Theorem 3.13 may be viewed as the computation of a differential in the spectral sequence arising from mapping into the stable Postnikov tower of $K\mathcal{D}$. This spectral sequence appears, for example, in [Kah66] and is a cocellular construction of the Atiyah-Hirzebruch spectral sequence (see [GM95, Appendix B]).

Our most important application concerns the sphere spectrum.

Corollary 3.16. *Let $\mathcal{D}$ be a strict skeletal Picard 2-category. Then $\mathcal{D}$ cannot be a model for the 2-truncation of the sphere spectrum.*

Proof. The nontrivial element in π_1 of the sphere spectrum is given by $k_0(1)$, so k_0 is surjective and therefore Theorem 3.13 applies. But $k_1 i_1$ is Sq^2, which is the nontrivial element of $H^2(\mathbb{Z}/2; \mathbb{Z}/2)$ [MT68, pp. 117–118]. $\qquad\square$

Remark 3.17. To understand the meaning of this result, recall that one can specify a unique Picard category by choosing two abelian groups for π_0 and π_1 together with a stable quadratic map k_0 for the symmetry. This is the content of Theorem 1.1. However, one does not specify a Picard 2-category by simply choosing three abelian groups and two group homomorphisms. This is tantamount to specifying a stable 2-type by choosing the bottom Postnikov invariant k_0 and the composite $k_1 i_1$. Theorem 3.13 shows that such data do not always assemble to form a strict Picard 2-category. For example, the construction of [Bar14, 5.2] does not satisfy the axioms of a permutative Gray-monoid.

4. STRICTIFICATION VIA 2-MONADS

In this section we develop the 2-monadic tools used in the proof of Theorem 3.10. In Section 4.1 we recall some basic definitions as well as abstract coherence theory from the perspective of 2-monads. Our focus is on various strictification results for algebras and pseudoalgebras over 2-monads, and how strictification can often be expressed as a 2-adjunction with good properties. In Section 4.2 we apply this to construct a strictification of pseudodiagrams as a left 2-adjoint.

The formalism of this section aids the proof of Theorem 3.10 in two ways. First, it allows us to produce strict diagrams of 2-categories by working with diagrams which are weaker (e.g., whose arrows take values in pseudofunctors) but more straightforward to define. This occurs in Section 5.1. Second, it allows us to construct strict equivalences of strict diagrams by working instead with pseudonatural equivalences between them. This occurs in Section 5.2.

4.1. **Review of 2-monad theory.** We recall relevant aspects of 2-monad theory and fix notation. These include maps of monads and abstract coherence theory [KS74, BKP89, Lac02]. Let $\mathcal{A}$ be a 2-category, and $(T\colon \mathcal{A} \to \mathcal{A}, \eta, \mu)$ be a 2-monad on $\mathcal{A}$. We then have the following 2-categories of algebras and morphisms with varying levels of strictness.

   i. T-**Alg**$_s$ is the 2-category of strict T-algebras, strict morphisms, and algebra 2-cells. Its underlying category is just the usual category of algebras for the underlying monad of T on the underlying category of $\mathcal{A}$.
   ii. T-**Alg** is the 2-category of strict T-algebras, pseudo-T-morphisms, and algebra 2-cells.
   iii. **Ps**-T-**Alg** is the 2-category of pseudo-T-algebras, pseudo-T-morphisms, and algebra 2-cells.

We have inclusions and forgetful functors as below.

$$T\text{-}\mathbf{Alg}_s \xrightarrow{\ i\ } T\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$$

A map of 2-monads is precisely the data necessary to provide a 2-functor between 2-categories of strict algebras.

Definition 4.1. Let S be a 2-monad on $\mathcal{A}$ and T a 2-monad on $\mathcal{B}$. A *strict map of 2-monads* $S \to T$ consists of a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ and a 2-natural transformation $\lambda\colon TF \Rightarrow FS$ satisfying two compatibility axioms [Bec69]:

$$\lambda \circ \mu F = F\mu \circ \lambda S \circ T\lambda$$
$$\lambda \circ \eta F = F\eta.$$

Proposition 4.2. *If $F\colon S \to T$ is a strict map of 2-monads, then F lifts to the indicated 2-functors in the following diagram.*

$$
\begin{array}{ccc}
S\text{-}\mathcal{A}lg_s & \xrightarrow{\;\;F\;\;} & T\text{-}\mathcal{A}lg_s \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
S\text{-}\mathcal{A}lg & \xrightarrow{\;\;F\;\;} & T\text{-}\mathcal{A}lg \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathcal{A} & \xrightarrow[\;\;F\;\;]{} & \mathcal{B}
\end{array}
$$

Abstract coherence theory provides left 2-adjoints to $T\text{-}\mathbf{Alg}_s \hookrightarrow T\text{-}\mathbf{Alg}$ and the composite $T\text{-}\mathbf{Alg}_s \hookrightarrow \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$. Lack discusses possible hypotheses in [Lac02, §3], so we give the following theorem in outline form.

Theorem 4.3. [Lac02, §3] *Under some assumptions on $\mathcal{A}$ and T, the inclusions*

$$
i\colon T\text{-}\mathbf{Alg}_s \hookrightarrow T\text{-}\mathbf{Alg}, \quad j\colon T\text{-}\mathbf{Alg}_s \hookrightarrow \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}
$$

have left 2-adjoints generically denoted Q. Under even further assumptions, the units $1 \Rightarrow iQ, 1 \Rightarrow jQ$ and the counits $Qi \Rightarrow 1, Qj \Rightarrow 1$ of these 2-adjunctions have components which are internal equivalences in $T\text{-}\mathbf{Alg}$ for $Q \dashv i$ and $\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$ for $Q \dashv j$, respectively.

Remark 4.4. The proofs in [Lac02] only concern the units, but the statement about counits follows immediately from the 2-out-of-3 property for equivalences and one of the triangle identities. We note that the components of the counits are actually always 1-cells in $T\text{-}\mathbf{Alg}_s$, so saying they are equivalences in $T\text{-}\mathbf{Alg}$ or $\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$ requires implicitly applying i or j, respectively.

Notation 4.5. We will always denote inclusions of the form $T\text{-}\mathbf{Alg}_s \hookrightarrow T\text{-}\mathbf{Alg}$ by i, and inclusions of the form $T\text{-}\mathbf{Alg}_s \hookrightarrow \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$ by j. If we need to distinguish between the left adjoints for i and j, we will denote them Q_i and Q_j, respectively.

4.2. **Two applications of 2-monads.** We are interested in two applications of Theorem 4.3: one which gives 2-categories as the strict algebras (Proposition 4.12), and one which gives 2-functors with fixed domain and codomain as the strict algebras (Proposition 4.16). Combining these in Theorem 4.19 we obtain the main strictification result used in our analysis of K-theory and suspension in Section 5.

We begin with the 2-monad for 2-categories and refer the interested reader to [Lac10b] and [LP08] for further details.

Definition 4.6.

  i. A *category-enriched graph* or $\mathcal{C}at$-*graph* $\big(S, S(x,y)\big)$ consists of a set of objects S and for each pair of objects $x, y \in S$, a category $S(x,y)$.

  ii. A *map of $\mathcal{C}at$-graphs* $(F, F_{x,y})\colon \big(S, S(x,y)\big) \to \big(T, T(w,z)\big)$ consists of a function $F\colon S \to T$ and a functor $F_{x,y}\colon S(x,y) \to T(Fx, Fy)$ for each pair of objects $x, y \in S$.

  iii. A $\mathcal{C}at$-*graph 2-cell* $\alpha\colon (F, F_{x,y}) \Rightarrow (G, G_{x,y})$ only exists when $F = G$ as functions $S \to T$, and then consists of a natural transformation $\alpha_{x,y}\colon F_{x,y} \Rightarrow G_{x,y}$ for each pair of objects $x, y \in S$.

Notation 4.7. $\mathcal{C}at$-graphs, their maps, and 2-cells form a 2-category, $\mathcal{C}at\text{-}\mathcal{G}rph$, with the obvious composition and unit structures.

Definition 4.8. Let $\mathcal{A}, \mathcal{B}$ be 2-categories, and $F, G\colon \mathcal{A} \to \mathcal{B}$ be a pair of 2-functors between them. An *icon* $\alpha\colon F \Rightarrow G$ exists only when $Fa = Ga$ for all objects $a \in \mathcal{A}$, and then consists of natural transformations

$$
\alpha_{a,b}\colon F_{a,b} \Rightarrow G_{a,b}\colon \mathcal{A}(a,b) \to \mathcal{B}(Fa, Fb)
$$

for all pairs of objects a, b, such that the following diagrams commute. (Note that we suppress the 0-cell source and target subscripts for components of the transformations $\alpha_{a,b}$ and instead only list the 1-cell for which a given 2-cell is the component.)

$$
\begin{array}{ccc}
\mathrm{id}_{Fa} \xrightarrow{\;=\;} F\mathrm{id}_a & \qquad Ff \circ Fg \xrightarrow{\;=\;} F(f \circ g) \\
\end{array}
$$

Remark 4.9. We can define icons between pseudofunctors or lax functors with only minor modifications, replacing some equalities above with the appropriate coherence cell; see [LP08, Lac10b].

Notation 4.10. 2-categories, 2-functors, and icons form a 2-category which we denote $\mathit{2Cat}_{2,\mathrm{i}}$. 2-categories, pseudofunctors, and icons form a 2-category which we denote $\mathit{2Cat}_{\mathrm{p,i}}$. Bicategories, pseudofunctors, and icons also form a 2-category which we denote $\mathit{Bicat}_{\mathrm{p,i}}$.

Recall that a 2-functor $U \colon \mathcal{A} \to \mathcal{K}$ is 2-monadic if it has a left 2-adjoint F and $\mathcal{A}$ is 2-equivalent to the 2-category of algebras $(UF)\text{-}\mathbf{Alg}_s$ via the canonical comparison map.

Proposition 4.11 ([LP08, Lac10b])**.** *The 2-functor $\mathit{2Cat}_{2,\mathrm{i}} \to \mathit{Cat\text{-}Grph}$ is 2-monadic, and the left 2-adjoint is given by the Cat-enriched version of the free category functor.*

The following is our first application of Theorem 4.3.

Proposition 4.12. *The two inclusions,*

$$
i \colon \mathit{2Cat}_{2,\mathrm{i}} \hookrightarrow \mathit{2Cat}_{\mathrm{p,i}}, \quad j \colon \mathit{2Cat}_{2,\mathrm{i}} \hookrightarrow \mathit{Bicat}_{p,i}
$$

have left 2-adjoints, and the components of the units and counits of both adjunctions are internal equivalences in $\mathit{2Cat}_{\mathrm{p,i}}$ for $Q_i \dashv i$ and $\mathit{Bicat}_{p,i}$ for $Q_j \dashv j$, respectively.

Proof. The induced monad T on $\mathit{Cat\text{-}Grph}$ satisfies a version of the hypotheses for Theorem 4.3 (for example, it is a finitary monad) so we get left 2-adjoints to both inclusions

$$
i \colon T\text{-}\mathbf{Alg}_s \to T\text{-}\mathbf{Alg}, \quad j \colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps\text{-}}T\text{-}\mathbf{Alg}.
$$

Now $T\text{-}\mathbf{Alg}$ can be identified with $\mathit{2Cat}_{\mathrm{p,i}}$, and one can check that $\mathbf{Ps\text{-}}T\text{-}\mathbf{Alg}$ can be identified with $\mathit{Bicat}_{\mathrm{p,i}}$, and using these the two left 2-adjoints above are both given by the standard functorial strictification functor, often denoted st (see [JS93] for the version with only a single object, i.e., monoidal categories). The objects of $\mathrm{st}(X)$ are the same as X, while the 1-cells are formal strings of composable 1-cells (including the empty string at each object). Internal equivalences in either $T\text{-}\mathbf{Alg}$ or $\mathbf{Ps\text{-}}T\text{-}\mathbf{Alg}$ for the 2-monad T are bijective-on-objects biequivalences, and it is easy to check that the unit is such; see [LP08, Gur13b] for further details. $\qquad\square$

Remark 4.13. We should note that $\mathit{2Cat}_{2,\mathrm{i}}$ is complete and cocomplete as a 2-category, since it is the 2-category of algebras for a finitary 2-monad on a complete and cocomplete 2-category. This will be necessary for later constructions. On the other hand, $\mathit{2Cat}_{\mathrm{p,i}}$ is not cocomplete as a 2-category, but is as a bicategory: coequalizers of pseudofunctors rarely exist in the strict, 2-categorical sense, but all bicategorical colimits do exist.

Our second application of Theorem 4.3 deals with functor 2-categories. Here we fix a small 2-category $\mathcal{A}$ and a complete and cocomplete 2-category $\mathcal{K}$.

Notation 4.14. Let $[\mathcal{A}, \mathcal{K}]$ denote the 2-category of 2-functors, 2-natural transformations, and modifications from $\mathcal{A}$ to $\mathcal{K}$. Let $\mathit{Bicat}(\mathcal{A}, \mathcal{K})$ denote the 2-category of pseudofunctors, pseudonatural transformations, and modifications from $\mathcal{A}$ to $\mathcal{K}$. Let $\mathit{Gray}(\mathcal{A}, \mathcal{K})$ denote the 2-category of 2-functors, pseudonatural transformations, and modifications

from $\mathcal{A}$ to $\mathcal{K}$. This is the internal hom-object corresponding to the Gray tensor product on *2Cat* [GPS95].

Remark 4.15. $\mathcal{B}icat(\mathcal{A},\mathcal{K})$ inherits its compositional and unit structures from the target 2-category $\mathcal{K}$ and is therefore a 2-category rather than a bicategory even though all of its cells are of the weaker, bicategorical variety.

Let ob$\mathcal{A}$ denote the discrete 2-category with the same set of objects as $\mathcal{A}$. We have an inclusion ob$\mathcal{A} \hookrightarrow \mathcal{A}$ which induces a 2-functor $U \colon [\mathcal{A},\mathcal{K}] \to [\text{ob}\mathcal{A},\mathcal{K}]$.

Proposition 4.16. *The forgetful 2-functor $U \colon [\mathcal{A},\mathcal{K}] \to [\text{ob}\mathcal{A},\mathcal{K}]$ is 2-monadic, and the left 2-adjoint is given by enriched left Kan extension. The induced 2-monad preserves all colimits, and so the inclusions*

$$i \colon [\mathcal{A},\mathcal{K}] \hookrightarrow \mathcal{G}ray(\mathcal{A},\mathcal{K}), \quad j \colon [\mathcal{A},\mathcal{K}] \hookrightarrow \mathcal{B}icat(\mathcal{A},\mathcal{K})$$

have left 2-adjoints. The units and counits of these adjunctions have components which are internal equivalences in $\mathcal{G}ray(\mathcal{A},\mathcal{K})$ for $\mathbf{Q}_i \dashv i$ and $\mathcal{B}icat(\mathcal{A},\mathcal{K})$ for $\mathbf{Q}_j \dashv j$, respectively.

Proof. That U is 2-monadic follows because it has a left 2-adjoint given by enriched left Kan extension and is furthermore conservative. Thus $[\mathcal{A},\mathcal{K}]$ is 2-equivalent to the 2-category of strict algebras for $U \circ \text{Lan}$. The 2-functor U also has a right adjoint given by right Kan extension since $\mathcal{K}$ is complete, so $U \circ \text{Lan}$ preserves all colimits as it is a composite of two left 2-adjoints. The 2-category $[\text{ob}\mathcal{A},\mathcal{K}]$ is cocomplete since $\mathcal{K}$ is, hence $T = U \circ \text{Lan}$ satisfies the strongest version of the hypotheses for Theorem 4.3. One can check that T-**Alg** is 2-equivalent to $\mathcal{G}ray(\mathcal{A},\mathcal{K})$ and **Ps**-T-**Alg** is 2-equivalent to $\mathcal{B}icat(\mathcal{A},\mathcal{K})$ [Lac10a]. This proves that the inclusions i,j in the statement have left 2-adjoints. The version of Theorem 4.3 which applies in this case proves, moreover, that the components of the units are internal equivalences in $\mathcal{G}ray(\mathcal{A},\mathcal{K})$ and $\mathcal{B}icat(\mathcal{A},\mathcal{K})$, respectively, and hence the claim about counits follows (see Remark 4.4). $\qquad\square$

We require one further lemma before stating the main result of this section.

Lemma 4.17. *For a fixed 2-category $\mathcal{A}$, $\mathcal{B}icat(\mathcal{A},-)$ is an endo-2-functor of the 2-category of 2-categories, 2-functors, and 2-natural transformations.*

Proof. For any 2-category $\mathcal{B}$, we know that $\mathcal{B}icat(\mathcal{A},\mathcal{B})$ is a 2-category. Furthermore, if $F \colon \mathcal{B} \to \mathcal{C}$ is a 2-functor, it is straightforward to check that $F_* \colon \mathcal{B}icat(\mathcal{A},\mathcal{B}) \to \mathcal{B}icat(\mathcal{A},\mathcal{C})$ is also a 2-functor. The only interesting detail to check is on the level of 2-cells where we must show that if $\sigma \colon F \Rightarrow G$ is 2-natural, then so is σ_*. The component of σ_* at $H \colon \mathcal{A} \to \mathcal{B}$ is the pseudonatural transformation $\sigma H \colon FH \Rightarrow GH$ with $(\sigma H)_a = \sigma_{Ha}$ and similarly for pseudonaturality isomorphisms. We must verify that σ_* is 2-natural in H. Thus for any $\alpha \colon H \Rightarrow K$, we must check that $G\alpha \circ \sigma H = \sigma K \circ F\alpha$ as pseudonatural transformations and then similarly for modifications. At an object a, we have components

$$(G\alpha \circ \sigma H)_a = G(\alpha_a) \circ \sigma_{Ha} = \sigma_{Ka} \circ F(\alpha_a) = (\sigma K \circ F\alpha)_a$$

by the 2-naturality of σ in Ha. A short and simple pasting diagram argument that we leave to the reader also shows that the pseudonaturality isomorphisms for $G\alpha \circ \sigma H$ and $\sigma K \circ F\alpha$ are the same, once again relying on the 2-naturality of σ in its argument. This completes the 1-dimensional part of 2-naturality, and the 2-dimensional part is a direct consequence of the 2-naturality of σ when written out on components. $\qquad\square$

Remark 4.18. While the argument above is simple, it is not entirely formal. The "dual" version for $\mathcal{B}icat(-,\mathcal{A})$ does not hold due to an asymmetry in the definition of the pseudonaturality isomorphisms for a horizontal composite of pseudonatural transformations.

We are now ready to prove the main result of this section, namely that we can replace pseudofunctors $\mathcal{A} \to \mathcal{2Cat}_{p,i}$ with equivalent 2-functors $\mathcal{A} \to \mathcal{2Cat}_{2,i}$.

Theorem 4.19. *The inclusion $J : [\mathcal{A}, 2Cat_{2,\mathrm{i}}] \hookrightarrow \mathcal{B}icat(\mathcal{A}, 2Cat_{\mathrm{p,i}})$ has a left 2-adjoint Q. The unit and counit of this adjunction have components which are internal equivalences in $\mathcal{B}icat(\mathcal{A}, 2Cat_{\mathrm{p,i}})$.*

Proof. We will combine Propositions 4.12 and 4.16. The inclusion J factors into the two inclusions

$$[\mathcal{A}, 2Cat_{2,\mathrm{i}}] \xrightarrow{j} \mathcal{B}icat(\mathcal{A}, 2Cat_{2,\mathrm{i}}) \xrightarrow{i_*} \mathcal{B}icat(\mathcal{A}, 2Cat_{\mathrm{p,i}}).$$

Since $2Cat_{2,\mathrm{i}}$ is cocomplete, j has a left 2-adjoint Q_j by Proposition 4.16. The inclusion i has a left 2-adjoint Q_i by Proposition 4.12, so i_* has a left 2-adjoint $(Q_i)_*$ by Lemma 4.17. Both of these 2-adjunctions have units whose components are equivalences, so the composite $Q = Q_j(Q_i)_*$ does as well, from which the claim about counits follows. $\qquad\square$

5. Categorical suspension models stable suspension

The purpose of this section is to prove Theorem 3.10, which states that K-theory commutes with suspension, in the appropriate sense. More precisely, we show that for any permutative category C, the K-theory spectrum of the one-object permutative Gray-monoid ΣC is stably equivalent to the suspension of the K-theory spectrum of C.

This entails a comparison between constructions of K-theory for categories and 2-categories. Both constructions use the theory of Γ-spaces developed by Segal [Seg74]. We recall this theory in Section 5.1. Our interest in Γ-spaces arises from the fact that they model the homotopy theory of connective spectra, as developed by Bousfield and Friedlander [BF78] in the simplicial setting. Thus, in what follows, we will work with Γ-simplicial sets to prove Theorem 3.10.

We model the spectra $K(\Sigma C)$ and ΣKC with Γ-simplicial sets which are constructed from certain Γ-objects in simplicial categories. These Γ-objects in simplicial categories are two different strictifications of the same pseudofunctor $\mathscr{F} \to \mathcal{B}icat(\Delta^{\mathrm{op}}, Cat_2)$, where $\mathscr{F}$ is the category of finite pointed sets and pointed maps. In Section 5.2 we use the formalism of Section 4 to compare the two strictifications via a zigzag of levelwise equivalences.

5.1. **Constructions of K-theory spectra and suspension.** Let $\mathscr{F}$ denote the following skeletal model for the category of finite pointed sets and pointed maps. An object of $\mathscr{F}$ is determined by an integer $m \geq 0$, which represents the pointed set $\underline{m}_+ = \{0, 1, \dots, m\}$, where 0 is the basepoint. This category is isomorphic to the opposite of the category Γ defined by Segal [Seg74].

Definition 5.1. Let $\mathcal{C}$ be a category with a terminal object $*$. A Γ-*object* in $\mathcal{C}$ is a functor $X : \mathscr{F} \to \mathcal{C}$ such that $X(\underline{0}_+) = *$.

We give the above definition in full generality, but are only interested in the cases when $\mathcal{C}$ is one of Cat, $2Cat$, the category of simplicial sets $sSet$ or of topological spaces Top. In each of these cases, we have finite products and a notion of weak equivalence. In Top and $sSet$ this is the classical notion of weak homotopy equivalence, and in both Cat and $2Cat$ we define a functor or 2-functor to be a weak equivalence if it induces a weak homotopy equivalence in $sSet$ after applying the nerve [Gur09, CCG10].

Definition 5.2. Let X be a Γ-object in $\mathcal{C}$. We say X is *special* if the Segal maps

$$X(\underline{n}_+) \to X(\underline{1}_+)^n$$

are weak equivalences.

The main result of [Seg74] is that, given a Γ-space X, one can produce a connective spectrum $\widetilde{X}$. Moreover, if X is special then $\widetilde{X}$ is an almost Ω-spectrum such that $\Omega^\infty \widetilde{X}$ is a group completion of $X(\underline{1}_+)$. We recall how to express suspension of spectra in terms of

Γ-simplicial sets using the standard "inclusion" $\Delta^{\mathrm{op}} \to \mathscr{F}$ as specified in [MT78, Lemma 3.5] and the following smash product. Let $\wedge \colon \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ be the functor that sends $(\underline{n}_+, \underline{p}_+)$ to $\underline{(np)}_+ = \underline{n}_+ \vee \ldots \vee \underline{n}_+$. Our reverse lexicographic convention differs from the smash product in [MT78, Construction 3.4] which considers $\underline{(np)}_+$ as $\underline{p}_+ \vee \ldots \vee \underline{p}_+$.

Notation 5.3. Let
$$\Phi \colon \mathscr{B}icat(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \to \mathscr{B}icat(\mathcal{A}, \mathscr{B}icat(\mathcal{B}, \mathcal{C}))$$
denote the biequivalence of functor bicategories given in [Str80], sending a pseudofunctor $F \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ to the pseudofunctor
$$\Phi(F)(a)(b) = F(a, b).$$
We also let Φ denote the isomorphism of functor 2-categories
$$[\mathcal{A} \times \mathcal{B}, \mathcal{C}] \xrightarrow{\cong} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]].$$
In order to justify using the same notation Φ for both of these, we note that both versions (reading vertical arrows upwards or downwards) of the square below commute,

(5.4)
$$\begin{array}{ccc} [\mathcal{A} \times \mathcal{B}, \mathcal{C}] & \longrightarrow & \mathscr{B}icat(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \\ {\scriptstyle \cong} \Big\updownarrow & & \Big\updownarrow {\scriptstyle \simeq} \\ [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \longrightarrow & \mathscr{B}icat(\mathcal{A}, \mathscr{B}icat(\mathcal{B}, \mathcal{C})) \end{array}$$

with the downward direction being given by Φ on the vertical arrows.

Definition 5.5. Let $X \colon \mathscr{F} \to s\mathcal{S}et$ be a special Γ-simplicial set and let $X \circ \wedge$ denote the composite
$$\mathscr{F} \times \Delta^{\mathrm{op}} \xrightarrow{\wedge} \mathscr{F} \xrightarrow{X} s\mathcal{S}et.$$
Let $d \colon [\Delta^{\mathrm{op}}, s\mathcal{S}et] \to s\mathcal{S}et$ denote the diagonal functor. We define the *suspension*, ΣX, as the special Γ-simplicial set $d \circ \Phi(X \circ \wedge)$.

Proposition 5.6 ([Seg74, BF78])**.** *Let X be a special Γ-simplicial set and $\widetilde{X}$ its associated spectrum. Then the spectrum associated to ΣX is stably equivalent to $\Sigma \widetilde{X}$.*

Given a permutative category C, there are several equivalent ways of constructing a special Γ-category. The following was first constructed by Thomason [Tho79, Definition 4.1.2].

Construction 5.7. Let $(C, \oplus, e)$ be a permutative category. We can construct a pseudofunctor
$$C^{(-)} \colon \mathscr{F} \to \mathcal{C}at_2$$
which sends $\underline{m}_+$ to C^m. Given a morphism $\phi \colon \underline{m}_+ \to \underline{n}_+$, the corresponding functor $\phi_* \colon C^m \to C^n$ is defined uniquely by the requirement that the squares below commute for each projection $\pi_j \colon C^n \to C$.

$$\begin{array}{ccc} C^m & \longrightarrow & C^{\phi^{-1}(j)} \\ {\scriptstyle \phi_*} \Big\downarrow & & \Big\downarrow {\scriptstyle \oplus} \\ C^n & \xrightarrow{\ \pi_j\ } & C \end{array}$$

The top horizontal map is the projection onto the coordinates which appear in $\phi^{-1}(j)$. The $\oplus$ appearing on the right vertical map is the iterated application of the tensor product $\oplus$, with the convention that if $\phi^{-1}(j)$ is empty, then the map is the constant functor on the unit e. This assignment is not strictly functorial, but the permutative structure provides natural isomorphisms
$$\psi_* \circ \phi_* \cong (\psi \circ \phi)_*$$

which are uniquely determined by the symmetry. These isomorphisms assemble to make $C^{(-)}$ a pseudofunctor.

Definition 5.8. The *K-theory* of C is the functor

$$KC = N \circ Q_j(C^{(-)}) \colon \mathscr{F} \to s\mathcal{S}et,$$

where N is the usual nerve functor $\mathcal{C}at \to s\mathcal{S}et$ and Q_j is the left 2-adjoint from Proposition 4.16 when $\mathcal{K} = \mathcal{C}at_2$.

Remark 5.9. Although the pseudofunctor $C^{(-)}$ satisfies the property that it maps $\underline{0}_+$ to $*$, its strictification $Q_j(C^{(-)})$ does not. Thus $Q_j(C^{(-)})$ is a functor $\mathscr{F} \to \mathcal{C}at$, but it is not a Γ-category as in Definition 5.1. Since $Q_j(C^{(-)})$ is levelwise equivalent to $C^{(-)}$, and in particular, $Q_j(C^{(-)})(\underline{0}_+)$ is contractible, we can replace $N \circ Q_j(C^{(-)})$ by a levelwise equivalent Γ-simplicial set. This replacement is made implicitly here, and throughout the remainder of the paper.

Lemma 5.10. *Consider the composite*

$$[\mathscr{F} \times \Delta^{\mathrm{op}}, \mathcal{C}at] \xrightarrow{\Phi} [\mathscr{F}, [\Delta^{\mathrm{op}}, \mathcal{C}at]] \xrightarrow{N_* \circ -} [\mathscr{F}, [\Delta^{\mathrm{op}}, s\mathcal{S}et]] \xrightarrow{d \circ -} [\mathscr{F}, s\mathcal{S}et].$$

If F is a levelwise weak equivalence of diagrams $\mathscr{F} \times \Delta^{\mathrm{op}} \to \mathcal{C}at$, then $dN_*\Phi(F)$ is a levelwise weak equivalence of diagrams $\mathscr{F} \to s\mathcal{S}et$.

Proof. This follows from [BF78, Theorem B.2], which states that if $f \colon X \to Y$ is a map of bisimplicial sets such that $X_{n,\bullet} \to Y_{n,\bullet}$ is a weak equivalence of simplicial sets for all $n \geq 0$, then $d(f) \colon d(X) \to d(Y)$ is a weak equivalence. $\square$

To relate the Γ-simplicial set ΣKC to the K-theory of the permutative Gray-monoid ΣC, we provide a new construction of a special Γ-2-category $\underline{K}(\Sigma C)$ and show it is levelwise weakly equivalent to the K-theory defined in [GJO15].

Notation 5.11. Let $2\mathcal{C}at_{\mathrm{p,p,m}}$ denote the tricategory whose objects are 2-categories, and whose higher cells are pseudofunctors, pseudonatural transformations, and modifications [Gur13a].

Lemma 5.12. *Let $(\mathcal{D}, \oplus, e)$ be a permutative Gray-monoid. Then there is a pseudofunctor of tricategories $\mathcal{D}^{(-)} \colon \mathscr{F} \to 2\mathcal{C}at_{\mathrm{p,p,m}}$ with value at $\underline{m}_+$ given by $\mathcal{D}^m$. If $\mathcal{D}$ has a single object, then this becomes a pseudofunctor of 2-categories $\mathcal{D}^{(-)} \colon \mathscr{F} \to 2\mathcal{C}at_{\mathrm{p,i}}$.*

Proof. The first claim is a special case of [GO13, Theorem 2.5]. For the second claim, by Corollary 2.36, it suffices to work with ΣD for a permutative category D. Recall from Construction 5.7 that we have the pseudofunctor

$$D^{(-)} \colon \mathscr{F} \to \mathcal{C}at_2.$$

The permutative structure on D in fact makes each D^m a strict monoidal category with pointwise tensor product and unit, and each functor $\phi_* \colon D^m \to D^n$ for $\phi \colon \underline{m}_+ \to \underline{n}_+$ a strong monoidal functor. One can verify that the isomorphisms $\psi_* \circ \phi_* \cong (\psi \circ \phi)_*$ are themselves monoidal, so we get a pseudofunctor

$$D^{(-)} \colon \mathscr{F} \to \mathcal{S}t\mathcal{M}on\mathcal{C}at_{\mathrm{p}}$$

from $\mathscr{F}$ to the 2-category $\mathcal{S}t\mathcal{M}on\mathcal{C}at_{\mathrm{p}}$ of strict monoidal categories, strong monoidal functors, and monoidal natural transformations. Note that $(\Sigma D)^m \cong \Sigma(D^m)$, so we define

$$(\Sigma D)^{(-)} = \Sigma \circ D^{(-)}$$

where Σ is now the 2-functor $\mathcal{S}t\mathcal{M}on\mathcal{C}at_{\mathrm{p}} \to 2\mathcal{C}at_{\mathrm{p,i}}$ which views each strict monoidal category as the hom-category of a 2-category with a single object. This composite is the desired pseudofunctor. $\square$

Definition 5.13 ([LP08])**.** Let $\mathcal{A}$ be a 2-category. The *nerve* of $\mathcal{A}$ is the simplicial category $N\mathcal{A}\colon \Delta^{\mathrm{op}} \to \mathit{Cat}$ defined by

$$N\mathcal{A}_n = \mathit{2Cat}_{2,\mathrm{i}}([n], \mathcal{A})$$

where $[n]$ is the standard category $0 \to 1 \to \cdots \to n$ treated as a discrete 2-category. This is the function on objects of a 2-functor from $\mathit{2Cat}_{2,\mathrm{i}}$ to $[\Delta^{\mathrm{op}}, \mathit{Cat}_2]$.

Remark 5.14. We note that this is called the *2-nerve* by Lack and Paoli. It is related but not equal to the general bicategorical nerve of [Gur09, CCG10]. Detailed comparisons are given in [CCG10].

Unpacking this definition, $N\mathcal{A}_0 = \mathrm{ob}\mathcal{A}$ as a discrete category. When $n \geq 1$,

$$N\mathcal{A}_n = \coprod_{a_0,\ldots,a_n \in \mathrm{ob}\mathcal{A}} \mathcal{A}(a_{n-1}, a_n) \times \cdots \times \mathcal{A}(a_0, a_1).$$

Using this same formula, we define the nerve on $\mathit{Cat}\text{-}\mathit{Grph}$ which fits in the following commuting diagram.

$$
\begin{array}{ccc}
\mathit{2Cat}_{2,\mathrm{i}} & \xrightarrow{\ N\ } & [\Delta^{\mathrm{op}}, \mathit{Cat}_2] \\
\downarrow & & \downarrow \\
\mathit{Cat}\text{-}\mathit{Grph} & \xrightarrow{\ N\ } & [\mathrm{ob}\,\Delta^{\mathrm{op}}, \mathit{Cat}_2]
\end{array}
$$

Let S be the 2-monad on $\mathit{Cat}\text{-}\mathit{Grph}$ whose algebra 2-category is $\mathit{2Cat}_{2,\mathrm{i}}$ (Proposition 4.11). Let T be the 2-monad on $[\mathrm{ob}\,\Delta^{\mathrm{op}}, \mathit{Cat}_2]$ whose algebra 2-category is $[\Delta^{\mathrm{op}}, \mathit{Cat}_2]$ (Proposition 4.16). We now apply Proposition 4.2 to show that the nerve extends to $\mathit{2Cat}_{\mathrm{p,i}}$.

Lemma 5.15. *The nerve N is a strict map of 2-monads $S \to T$ and therefore provides the middle map in the commutative diagram below.*

$$
\begin{array}{ccc}
\mathit{2Cat}_{2,\mathrm{i}} & \xrightarrow{\ N\ } & [\Delta^{\mathrm{op}}, \mathit{Cat}_2] \\
i\downarrow & & \downarrow i \\
\mathit{2Cat}_{\mathrm{p,i}} & \xrightarrow{\ N\ } & \mathit{Gray}(\Delta^{\mathrm{op}}, \mathit{Cat}_2) \\
U\downarrow & & \downarrow U \\
\mathit{Cat}\text{-}\mathit{Grph} & \xrightarrow{\ N\ } & [\mathrm{ob}\,\Delta^{\mathrm{op}}, \mathit{Cat}_2]
\end{array}
$$

We now define the Γ-objects we will use to understand K-theory of a suspension.

Definition 5.16. Let C be a permutative category with ΣC its suspension permutative Gray-monoid. Let $Q = Q_j(Q_i)_*$ denote the left 2-adjoint of the inclusion $J \colon [\mathscr{F}, \mathit{2Cat}_{2,\mathrm{i}}] \hookrightarrow \mathit{Bicat}(\mathscr{F}, \mathit{2Cat}_{\mathrm{p,i}})$ constructed in Theorem 4.19.

  i. Define $\underline{K}(\Sigma C)$ to be $Q\big((\Sigma C)^{(-)}\big)$. This is a functor $\mathscr{F} \to \mathit{2Cat}$.

  ii. The composite $N \circ \underline{K}(\Sigma C)$ is a functor $\mathscr{F} \to [\Delta^{\mathrm{op}}, \mathit{Cat}]$. Define $K_{adj}(\Sigma C)$ to be $\Phi^{-1}\big(N \circ \underline{K}(\Sigma C)\big)$.

The composite

$$\mathit{2Cat} \xrightarrow{\ N\ } [\Delta^{\mathrm{op}}, \mathit{Cat}] \xrightarrow{\ N_*\ } [\Delta^{\mathrm{op}}, \mathit{sSet}] \xrightarrow{\ d\ } \mathit{sSet}$$

is one of the versions of the nerve for 2-categories in [CCG10]. Post-composing $\underline{K}(\Sigma C)$ with this functor (and, as noted in Remark 5.9, implicitly replacing with a reduced diagram) yields a Γ-simplicial set which is a model of the K-theory of ΣC. We make this rigorous in the following lemma.

Lemma 5.17. *Let $(C, \oplus, e)$ be a permutative category and let $\widetilde{K}$ be the K-theory functor of [GJO15]. There is a levelwise weak equivalence between the Γ-2-categories $\underline{K}(\Sigma C)$ and $\widetilde{K}(\Sigma C)$, hence a stable equivalence between the spectra these represent.*

Proof. We shall prove that there is a levelwise weak equivalence $\underline{K}(\Sigma C) \to \widetilde{K}(\Sigma C)$ of Γ-2-categories. Since both of these are special, it suffices to construct such a map and check that it is a weak equivalence when evaluated at $\underline{1}_+$. The functor Q is a left adjoint, so strict maps $Z \colon \underline{K}(\Sigma C) = Q\big((\Sigma C)^{(-)}\big) \to \widetilde{K}(\Sigma C)$ are in bijection with pseudonatural transformations

$$\check{Z} \colon (\Sigma C)^{(-)} \to \widetilde{K}(\Sigma C)$$

in $\mathcal{B}icat(\mathscr{F}, 2\mathcal{C}at_{\mathrm{p,i}})$. This bijection is induced by composition with a universal pseudonatural transformation $\eta \colon (\Sigma C)^{(-)} \to Q\big((\Sigma C)^{(-)}\big)$, so we have the commutative triangle shown below.

$$
\begin{array}{ccc}
(\Sigma C)^{(-)} & \xrightarrow{\ \eta\ } & Q\big((\Sigma C)^{(-)}\big) \\
& {\scriptstyle \check{Z}} \searrow & \ \ \downarrow {\scriptstyle Z} \\
& & \widetilde{K}(\Sigma C)
\end{array}
$$

We know that η is a levelwise weak equivalence by Theorem 4.19, so the component of Z at $\underline{1}_+$ is a weak equivalence if and only if the same holds for $\check{Z}$.

We will construct the pseudonatural transformation $\check{Z}$. In order to do so, we briefly review the data which define the cells of $\widetilde{K}(\Sigma C)(\underline{n}_+)$; we omit the axioms these data must satisfy and refer the reader to [GJO15]. Because ΣC has a single object, an object of $\widetilde{K}(\Sigma C)(\underline{n}_+)$ consists of objects $c_{s,t}$ of the permutative category C for s, t disjoint subsets of $\underline{n} = \{1, \ldots, n\}$. We denote such an object as $\{c_{s,t}\}$ or, when more detail is useful, a function

$$\{s, t \mapsto c_{s,t}\}.$$

A 1-cell $\{c_{s,t}\} \to \{d_{s,t}\}$ consists of objects x_s of C for $s \subset \underline{n}$ together with isomorphisms

$$\gamma_{s,t} \colon x_t \oplus x_s \oplus c_{s,t} \cong d_{s,t} \oplus x_{s \cup t}.$$

We denote this as $\{x_s, \gamma_{s,t}\}$ or, in functional notation,

$$\left\{ \begin{array}{l} s \mapsto x_s \\ s, t \mapsto \gamma_{s,t} \end{array} \right\}.$$

A 2-cell $\{x_s, \gamma_{s,t}\} \Rightarrow \{y_s, \delta_{s,t}\}$ consists of morphisms $\alpha_s \colon x_s \to y_s$ in C. We denote this $\{\alpha_s\}$ or with a corresponding functional notation.

Now $(\Sigma C)^{\underline{n}_+}$ is $(\Sigma C)^n \cong \Sigma(C^n)$ by definition. We define $\check{Z}$ on cells as follows.

- The unique 0-cell of $\Sigma(C^n)$ maps to the object of $\widetilde{K}(\Sigma C)(\underline{n}_+)$ with $c_{s,t} = e$ for all s, t.
- A 1-cell $(x_1, \ldots, x_n)$ maps to the 1-cell

$$\left\{ \begin{array}{l} s \mapsto \oplus_{i \in s} x_i \\ s, t \mapsto \lambda_{s,t} \end{array} \right\}.$$

  where $\lambda_{s,t}$ denotes the unique interleaving symmetry isomorphism

$$(\oplus_{i \in s} x_i) \oplus (\oplus_{j \in t} x_j) \cong \oplus_{k \in s \cup t} x_k.$$

- A 2-cell $(f_1, \ldots, f_n)$ maps to the 2-cell

$$\{s \mapsto \oplus_{i \in s} f_i\}.$$

Using the permutative structure of C, it is straightforward to verify that the formulas above satisfy the axioms of [GJO15, §6.1] and therefore define valid cells. Clearly $\check{Z}$ sends the identity 1-cell of $\Sigma(C^n)$, namely $(e, \ldots, e)$, to the identity 1-cell in $\widetilde{K}(\Sigma C)(\underline{n}_+)$. Now composition of 1-cells in $\Sigma(C^n)$ is given by the monoidal structure, so

$$(x_1, \ldots, x_n) \circ (y_1, \ldots, y_n) = (x_1 \oplus y_1, \ldots, x_n \oplus y_n).$$

We have a similar formula for composition in $\widetilde{K}(\Sigma C)(\underline{n}_+)$, with the object part of $\{x_s, \gamma_{s,t}\} \circ \{y_s, \delta_{s,t}\}$ being given on s by $x_s \oplus y_s$. From these formulas, we see that $\check{Z}$ does not strictly preserve 1-cell composition since

$$\check{Z}(x_1,\ldots,x_n) \circ \check{Z}(y_1,\ldots,y_n) = \left\{ \begin{array}{c} s \mapsto (\oplus_{i \in s} x_i) \oplus (\oplus_{i \in s} y_i) \\ s,t \mapsto \mu_{s,t} \end{array} \right\}$$

where μ denotes the unique interleaving symmetry isomorphism. On the other hand,

$$\check{Z}(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = \left\{ \begin{array}{c} s \mapsto \oplus_{i \in s}(x_i \oplus y_i) \\ s,t \mapsto \lambda_{s,t} \end{array} \right\}.$$

These are isomorphic by a unique symmetry, and that data equips

$$\check{Z}(\underline{n}_+) \colon (\Sigma C)^n \to \widetilde{K}(\Sigma C)(\underline{n}_+)$$

with the structure of a normal (i.e., strictly unit-preserving) pseudofunctor.

Now let $\phi \colon \underline{m}_+ \to \underline{n}_+$ in $\mathscr{F}$. We must construct an invertible icon in the square below.

$$
\begin{array}{ccc}
(\Sigma C)^m & \xrightarrow{\check{Z}} & \widetilde{K}(\Sigma C)(\underline{m}_+) \\
\phi_* \downarrow & & \downarrow \phi_* \\
(\Sigma C)^n & \xrightarrow[\check{Z}]{} & \widetilde{K}(\Sigma C)(\underline{n}_+)
\end{array}
$$

We begin by noting that this diagram obviously commutes on the unique object, so there can exist an icon (see Definition 4.8) between the two composite pseudofunctors. The top and right composite sends a 1-cell $(x_1,\ldots,x_n)$ to the 1-cell

$$\left\{ \begin{array}{c} u \mapsto \oplus_{i \in \phi^{-1}(u)} x_i \\ u,v \mapsto \lambda_{\phi^{-1}(u),\phi^{-1}(v)} \end{array} \right\}$$

The left and bottom composite then sends $(x_1,\ldots,x_n)$ to the 1-cell with

$$\left\{ \begin{array}{c} u \mapsto \oplus_{i \in u}\left(\oplus_{j \in \phi^{-1}(i)} x_j \right) \\ u,v \mapsto \kappa_{u,v} \end{array} \right\}$$

where $\kappa_{u,v}$ interleaves the blocks $\left(\oplus_{j \in \phi^{-1}(i)} x_j \right)$.

There is an invertible 2-cell between these 1-cells which is given by the symmetry isomorphism

$$\oplus_{i \in \phi^{-1}(u)} x_i \cong \oplus_{i \in u}\left(\oplus_{j \in \phi^{-1}(i)} x_j \right).$$

Coherence for symmetric monoidal categories, together with the naturality of symmetries, implies that the icon axioms hold. Further, the same coherence shows that these invertible icons are themselves the naturality isomorphisms which constitute a pseudonatural transformation between pseudofunctors $\mathscr{F} \to 2\mathcal{C}at_{\mathrm{p.i}}$.

Our final task is to verify that $\check{Z}(\underline{1}_+)$ is a weak equivalence. It is a simple calculation to check that in fact $\check{Z}(\underline{1}_+)$ induces an isomorphism of 2-categories $\widetilde{K}(\Sigma C)(\underline{1}_+) \cong \Sigma C$. □

Remark 5.18. One can check that the equivalence constructed in Lemma 5.17 is pseudonatural in the variable C.

5.2. **Proof of Theorem 3.10.** Given a permutative category C, we can construct two pseudofunctors from $\mathscr{F}$ to $\mathcal{B}icat(\Delta^{\mathrm{op}}, \mathcal{C}at_2)$. One is the composite

$$\mathscr{F} \xrightarrow{(\Sigma C)^{(-)}} 2\mathcal{C}at_{\mathrm{p.i}} \xrightarrow{N} \mathcal{G}ray(\Delta^{\mathrm{op}}, \mathcal{C}at_2) \hookrightarrow \mathcal{B}icat(\Delta^{\mathrm{op}}, \mathcal{C}at_2),$$

where N denotes the nerve functor of Lemma 5.15. The other is given by $\Phi(C^{(-)} \circ \wedge)$, where

$$\Phi : \mathcal{B}icat(\mathscr{F} \times \Delta^{\mathrm{op}}, \mathcal{C}at_2) \to \mathcal{B}icat(\mathscr{F}, \mathcal{B}icat(\Delta^{\mathrm{op}}, \mathcal{C}at_2))$$

is the 2-functor from Notation 5.3 and $C^{(-)} \circ \wedge$ is the composite

$$\mathscr{F} \times \Delta^{\mathrm{op}} \xrightarrow{\ \wedge\ } \mathscr{F} \xrightarrow{\ C^{(-)}\ } Cat_2.$$

Proposition 5.19. *With notation as above, $\Phi(C^{(-)} \circ \wedge) = N \circ (\Sigma C)^{(-)}$.*

Proof. This result follows from a direct comparison of $\Phi(C^{(-)} \circ \wedge)$ with $N \circ (\Sigma C)^{(-)}$. Both pseudofunctors send the object $\underline{m}_+$ in $\mathscr{F}$ to the 2-functor $\Delta^{\mathrm{op}} \to Cat_2$ given by

$$[p] \mapsto C^{m \cdot p} = (C^m)^p$$

$$([p] \xrightarrow{\alpha} [q]) \mapsto \big(C^{m \cdot p} \xrightarrow{(m \wedge \alpha)_*} C^{m \cdot q} \big).$$

For $\Phi(C^{(-)} \circ \wedge)$ this is immediate. For $N \circ (\Sigma C)^{(-)}$ this follows because ΣC has only one object and the horizontal composition of cells is given by the monoidal product in C.

Both pseudofunctors send a morphism $\phi \colon \underline{m}_+ \to \underline{n}_+$ in $\mathscr{F}$ to the pseudonatural transformation whose component at $[p] \in \Delta^{\mathrm{op}}$ is given by

$$C^{m \cdot p} \xrightarrow{(\phi \wedge p)_*} C^{n \cdot p}.$$

For $\Phi(C^{(-)} \circ \wedge)$ it is immediate that the pseudonaturality constraint has components given by

(5.20) $$(\underline{n}_+ \wedge \alpha)_* \circ (\phi \wedge [p])_* \cong (\phi \wedge \alpha)_* \cong (\phi \wedge [q])_* \circ (\underline{m}_+ \wedge \alpha)_*$$

at $\alpha \colon [p] \to [q]$. These isomorphisms are the pseudofunctoriality constraints of $C^{(-)}$ and are instances of the symmetry in C (see Construction 5.7). A straightforward check shows that the pseudofunctoriality constraint of $N \circ (\Sigma C)^{(-)}$ is given by the same instances of the symmetry of C.

For a composable pair $\phi \colon \underline{m}_+ \to \underline{n}_+$ and $\psi \colon \underline{n}_+ \to \underline{k}_+$, the symmetry of C provides

$$(\psi \wedge [p])_* \circ (\phi \wedge [p])_* \cong \big((\psi \circ \phi) \wedge [p] \big)_*$$

and these are the components of the pseudofunctoriality of $\Phi(C^{(-)} \circ \wedge)$. The same computation holds for $N \circ (\Sigma C)^{(-)}$. $\qquad\square$

We are now ready for the main theorem of this section, from which the proof of Theorem 3.10 follows. Let Q_j be as in Definition 5.8: the left 2-adjoint to the inclusion functor

$$j \colon [\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2] \hookrightarrow Bicat(\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2).$$

Theorem 5.21. *For any permutative category C, there is a zigzag of levelwise equivalences between $Q_j(C^{(-)}) \circ \wedge$ and $K_{adj}(\Sigma C)$.*

Proof. The components of the unit and counit of the 2-adjunction $Q_j \dashv j$ are internal equivalences in $Bicat(\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2)$ by Proposition 4.16. Assume that

$$\alpha \colon j\big(Q_j(C^{(-)}) \circ \wedge \big) \xrightarrow{\ \simeq\ } j\big(K_{adj}(\Sigma C) \big)$$

is a pseudonatural equivalence in $Bicat(\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2)$. Since a pseudonatural equivalence is an internal equivalence in $Bicat(\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2)$, we can apply Q_j and get an internal equivalence in $[\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2]$. This gives a zigzag

$$Q_j(C^{(-)}) \circ \wedge \xleftarrow{\ \varepsilon\ } Q_j j\big(Q_j(C^{(-)}) \circ \wedge \big) \xrightarrow{\ Q_j(\alpha)\ } Q_j j\big(K_{adj}(\Sigma C) \big) \xrightarrow{\ \varepsilon\ } K_{adj}(\Sigma C)$$

in $[\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2]$ in which the first and third arrows are levelwise equivalences as they are internal equivalences in $Bicat(\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2)$, and the second arrow is a levelwise equivalence as it is an internal equivalence (i.e., 2-equivalence) in $[\mathscr{F} \times \Delta^{\mathrm{op}}, Cat_2]$. It only remains to construct an equivalence α as above.

In order to construct the pseudonatural equivalence α, first recall from Definition 5.16 (*ii*) that

$$K_{adj}(\Sigma C) = \Phi^{-1}\Big(N \circ Q\big((\Sigma C)^{(-)}\big)\Big)$$

where Φ denotes the adjunction of Notation 5.3 and Q denotes the left adjoint constructed in Theorem 4.19. We define α as the composite below, which we explain afterwards.

$$
\begin{array}{rl}
j\big(Q_j(C^{(-)}) \circ \wedge\big) & \xrightarrow{=} \; jQ_j(C^{(-)}) \circ \wedge \\
& \xrightarrow{\simeq} \; C^{(-)} \circ \wedge \\
& \xrightarrow{\simeq} \; \Phi^{-1}\Big(N \circ (\Sigma C)^{(-)}\Big) \\
& \xrightarrow{\simeq} \; \Phi^{-1}\Big(N \circ JQ\big((\Sigma C)^{(-)}\big)\Big) \\
& \xrightarrow{=} \; j\Phi^{-1}\Big(N \circ Q\big((\Sigma C)^{(-)}\big)\Big) \\
& \xrightarrow{=} \; jK_{adj}(\Sigma C)
\end{array}
$$

The equality giving the first arrow is a simple calculation. The equivalence giving the second arrow is a pseudo-inverse of the unit for $Q_j \dashv j$, whiskered by $\wedge$ and hence still an equivalence. The equivalence giving the third arrow is the adjoint of the equality in Proposition 5.19. The equivalence giving the fourth arrow is derived from the unit of $Q \dashv J$ which is itself an equivalence, so whiskering with N and applying Φ^{-1} still yields an equivalence. The equality giving the fifth arrow follows from the commutativity of Display (5.4), and the equality giving the final arrow is Definition 5.16 (*ii*). □

Remark 5.22. The zigzag in Theorem 3.10 is natural up to homotopy. More precisely, this zigzag consists of three maps, two of which are counits for the 2-adjunction $Q_j \dashv j$. It is easy to see that $C \mapsto C^{(-)}$ sends symmetric, strong monoidal functors between permutative categories to pseudonatural transformations between their corresponding pseudofunctors $\mathscr{F} \to \mathcal{C}at_2$, so a symmetric, strong monoidal functor $F \colon C \to D$ will yield a 2-natural transformation

$$Q_j(C^{(-)}) \circ \wedge \to Q_j(D^{(-)}) \circ \wedge.$$

The counit ε is strictly natural with respect to such, so the first map in our zigzag is strictly natural in symmetric, strong monoidal functors. A similar argument holds for K_{adj}, so the third map in our zigzag is also strictly natural in symmetric, strong monoidal functors. The second map is what is called $Q_j(\alpha)$ in the proof above. It is more involved, but a careful check reveals that each of the maps of which it is a composite is pseudonatural in symmetric, strong monoidal functors, and so the same will be true after applying Q_j. Thus our zigzag is actually pseudonatural in the variable C, which in particular implies that it is natural up to homotopy when viewed as a zigzag of spectra.

Proof of Theorem 3.10. On one hand, the suspension of Γ-simplicial sets given in Definition 5.5 models the stable suspension by Proposition 5.6. Recalling [Tho79, MT78], the Γ-simplicial set $KC = N \circ Q_j(C^{(-)})$ from Definition 5.8 models the K-theory spectrum of C. Its suspension as a Γ-simplicial set, $\Sigma K(C)$, is given by composing the diagonal d with $\Phi(K(C) \circ \wedge)$. By naturality of Φ in its target 2-category, this is given by $dN_*\Phi(Q_j(C^{(-)}) \circ \wedge)$. By Lemma 5.10, a levelwise weak equivalence of functors $X, Y \colon \mathscr{F} \times \Delta^{\mathrm{op}} \to \mathcal{C}at_2$ induces a levelwise weak equivalence between $dN_*\Phi(X)$ and $dN_*\Phi(Y)$. Therefore it suffices to examine $Q_j(C^{(-)}) \circ \wedge$. On the other hand, in Definition 5.16 we have the Γ-2-category $\underline{K}(\Sigma C) = Q\big((\Sigma C)^{(-)}\big)$ and the related adjoint $K_{adj}(\Sigma C) = \Phi^{-1}\big(N \circ \underline{K}(\Sigma C)\big)$. Lemma 5.17 shows that $dN_*\Phi(K_{adj}(\Sigma C))$ models the K-theory spectrum of ΣC. Finally, the result follows by Theorem 5.21, which shows that there is a zigzag of levelwise equivalences between $Q_j(C^{(-)}) \circ \wedge$ and $K_{adj}(\Sigma C)$. □

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