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A HAUSDORFF MEASURE VERSION OF THE JARNÍK–SCHMIDT THEOREM IN DIOPHANTINE APPROXIMATION

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Abstract. We solve the problem of giving sharp asymptotic bounds on the Hausdorff dimensions of certain sets of badly approximable matrices, thus improving results of Broderick and Kleinbock (preprint 2013) as well as Weil (preprint 2013), and generalizing to higher dimensions those of Kurzweil ('51) and Hensley ('92). In addition we use our technique to compute the Hausdorff f-measure of the set of matrices which are not ψ-approximable, given a dimension function f and a function $\psi : (0,\infty) \to (0,\infty)$. This complements earlier work by Dickinson and Velani ('97) who found the Hausdorff f-measure of the set of matrices which are ψ-approximable.

1. Introduction

1.1. **Notation.** Fix $m, n \in \mathbb{N}$, and let $\mathcal{M}$ denote the set of $m \times n$ matrices. Given a function $\psi : (0,\infty) \to (0,\infty)$ and two norms $\|\cdot\|_\mu$, $\|\cdot\|_\nu$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, a matrix $A \in \mathcal{M}$ is called *ψ-approximable* (with respect to the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$) if there exist infinitely many vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that

$$\|A\mathbf{q} - \mathbf{p}\|_\mu \leq \psi(\|\mathbf{q}\|_\nu).$$

If $\|\cdot\|_\nu$ is the max norm, then we may allow $\psi : \mathbb{N} \to (0,\infty)$ in place of $\psi : (0,\infty) \to (0,\infty)$. We denote the set of ψ-approximable matrices by $W(\psi)$, or by $W_{\mu,\nu}(\psi)$ when we wish to emphasize the role of the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$.

Remark. When $n = 1$, then $W(\psi)$ is the set of simultaneously ψ-approximable points in $\mathbb{R}^m$, and when $m = 1$, then $W(\psi)$ is the set of dually ψ-approximable points in $\mathbb{R}^n$. This convention agrees with most sources, but some swap the roles of m and n (e.g. [4, 11, 30]).

1.2. **A brief history of Diophantine approximation.** Diophantine approximation is traditionally considered to have begun in 1842, when Dirichlet proved the following theorem:

Theorem (Dirichlet's Theorem)**.** *For any $A \in \mathcal{M}$ and for any $Q > 1$, there exist $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that*

$$\|\mathbf{q}\|_\infty \leq Q \text{ and } \|A\mathbf{q} - \mathbf{p}\|_\infty \leq Q^{-n/m},$$

where $\|\cdot\|_\infty$ denotes the max norm. In particular, if

$$\psi_*(q) = q^{-n/m}$$

then

$$W_{\infty,\infty}(\psi_*) = \mathcal{M}. \tag{1.1}$$

We will refer to the equation (1.1) as *the corollary to Dirichlet's theorem.*

A natural question is whether Dirichlet's theorem or its corollary can be improved. The first result in this direction is due to Liouville, who showed in 1844 that the set of *badly approximable matrices*

$$\mathrm{BA} \stackrel{\text{def}}{=} \bigcup_{\gamma>0} \mathcal{M} \setminus W(\gamma\psi_*)$$

is nonempty when $m = n = 1$. The assumption $m = n = 1$ was removed by O. Perron in 1921 [27].

Liouville and Perron's result shows that for any function ψ satisfying $\psi/\psi_* \to 0$, we have $W(\psi) \neq \mathcal{M}$ (cf. [14, Section 2] for an extended discussion of this point). Nevertheless, for such a ψ the set $W(\psi)$ may

still be "large" in a number of senses. For example, it is trivial to show that for any function ψ, $W(\psi)$ is comeager.[1] The first nontrivial result concerning the size of $W(\psi)$ came in 1924, when A. Y. Khinchin proved the case $m = n = 1$ of the theorem below [20]. Two years later Khinchin extended his result to include the case $n = 1$, m arbitrary [21]; the general case was proven by A. V. Groshev in 1938 [15].

Theorem (The Khinchin–Groshev theorem)**.** *Fix $\psi : \mathbb{N} \to (0, \infty)$, and consider the series*

$$\sum_{q=1}^{\infty} q^{n-1} \psi^m(q). \tag{1.2}$$

(i) *If* (1.2) *converges, then $W_{\infty,\infty}(\psi)$ is of Lebesgue measure zero.*
(ii) *If* (1.2) *diverges and the function $q \mapsto q^n \psi^m(q)$ is nonincreasing,*[2] *then $W_{\infty,\infty}(\psi)$ is of full Lebesgue measure (i.e. its complement has measure zero).*

Remark. Case (i) (the "convergence case") of the Khinchin–Groshev theorem is a standard application of the Borel–Cantelli lemma; case (ii) (the "divergence case") is the difficult direction.

Fix $c > 0$, and let $\psi_c(q) = q^{-c}$. One naturally observes that (1.2) converges if $c > n/m$, and thus in this case the sets $W(\psi_c)$ are of Lebesgue measure zero. Nevertheless, the sets $W(\psi_c)$ still turn out to have positive Hausdorff dimension. The following result was proven for $n = 1$, m arbitary by V. Jarník in 1929 [19], and in general by J. D. Bovey and M. M. Dodson in 1986 [4]. Moreover, the case $m = n = 1$ was proven independently by A. S. Besicovitch in 1934 [3].

Theorem 1.1 (The Jarník–Besicovitch–Bovey–Dodson Theorem)**.** *For $c > n/m$,*

$$\dim_H(W(\psi_c)) = (n-1)m + \frac{m+n}{1+c}.$$

Here and throughout, $\dim_H(S)$ denotes the Hausdorff dimension of a set S.

Remark. In Theorem 1.1, the limiting dimension $\lim_{c\to\infty} \dim_H(W(\psi_c)) = (n-1)m$ is not zero unless $n = 1$. The reason for this is that for every function ψ, $W(\psi)$ contains the set $\{A \in \mathcal{M} : A\mathbf{q} \in \mathbb{Z}^m \text{ for some } \mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\}$, and it can be computed that the dimension of this set is $(n-1)m$. Thus by monotonicity of Hausdorff dimension, we have $\dim_H(W(\psi)) \geq (n-1)m$ for every function ψ.

In fact, Jarník proved a stronger statement than the above. Given a dimension function f,[3] let $\mathcal{H}^f$ denote the Hausdorff f-measure (see Section 2). Let $f_* : (0, \infty) \to (0, \infty)$ be the function defined by the formula

$$f_*(\rho) = \rho^{mn}. \tag{1.3}$$

The following result was proven for $n = 1$, m arbitrary by Jarník in 1929 [19], and in general by D. Dickinson and S. L. Velani in 1997 [11]:

Theorem 1.2 (The Jarník–Dickinson–Velani Theorem)**.** *Let $\psi : \mathbb{N} \to (0, \infty)$, let f be a dimension function, and assume that*

- *the functions ψ/ψ_* and f/f_* are nonincreasing and satisfy*

$$\frac{\psi(q)}{\psi_*(q)} \xrightarrow[q\to\infty]{} 0, \qquad\qquad \frac{f(\rho)}{f_*(\rho)} \xrightarrow[\rho\to 0]{} \infty;$$

[1] We recall that a subset of a Baire space is called *comeager* or *residual* if it contains a dense G_δ set. By the Baire category theory, the class of comeager sets is closed under countable intersections. The complement of a comeager set is called *meager* or *of the first category*.

[2] Weakening this assumption has been the motivation for much further research in this area, culminating in its complete elimination in the case $(m, n) \neq (1, 1)$ [1]. When $m = n = 1$, the theorem is false without the assumption, and the expected measure of $W(\psi)$ for arbitrary ψ is described by the Catlin conjecture [16, p.28].

[3] We recall that a *dimension function* is a nondecreasing function $f : (0, \infty) \to (0, \infty)$ such that $\lim_{\rho\to 0} f(\rho) = 0$.

- *the function*

$$q \mapsto \left(\frac{\psi(q)}{\psi_*(q)}\right)^m \frac{f(\Psi(q))}{f_*(\Psi(q))}$$

is nonincreasing, where

$$\Psi(q) = \psi(q)/q.$$

Consider the series

$$\sum_{q=1}^{\infty} \frac{1}{q} \left(\frac{\psi(q)}{\psi_*(q)}\right)^m \frac{f(\Psi(q))}{f_*(\Psi(q))}. \tag{1.4}$$

Then

$$\mathcal{H}^f(W_{\infty,\infty}(\psi)) = \begin{cases} 0 & \text{if (1.4) converges} \\ \infty & \text{if (1.4) diverges} \end{cases}.$$

Note that Theorem 1.2 includes Theorem 1.1 as a corollary. Indeed, setting $f(\rho) = \rho^s$ and $\psi = \psi_c$ yields Theorem 1.1.

Remark. As with the Khinchin–Groshev theorem, the convergence case of Theorem 1.2 is a straightforward application of the Hausdorff–Cantelli lemma [2, Lemma 3.10], while the divergence case is the difficult direction.

One may also ask about the characteristics of the sets $B(\psi) \stackrel{\text{def}}{=} \mathcal{M} \setminus W(\psi)$. From the measure-theoretic and topological points of view, there is no new information here; $B(\psi)$ is meager, and it is of measure zero if and only if $W(\psi)$ is of full measure. However, from the dimension point of view there is more to ask. In this context, the main result was proven by Jarník in 1928 for the case $m = n = 1$. The general case was proven by W. M. Schmidt in 1969 [29], using the game he had introduced three years earlier [28] which is now known as Schmidt's game.

Theorem (The Jarník–Schmidt Theorem)**.** *The set* $\mathrm{BA} = \bigcup_{\gamma>0} B(\gamma\psi_*)$ *has Hausdorff dimension* mn*.*

Since mn is the largest possible dimension, this closes the question of the dimension of $B(\psi)$ for functions ψ such that $\psi/\psi_* \to 0$, since such functions satisfy $\mathrm{BA} \subseteq B(\psi)$. On the other hand, for functions ψ such that $\psi \geq \psi_*$, we have $B_{\infty,\infty}(\psi) = \varnothing$ by the corollary to Dirichlet's theorem. To summarize,

$$\dim_H(B_{\infty,\infty}(\psi)) = \begin{cases} mn & \text{if } \psi/\psi_* \to 0 \\ 0 & \text{if } \psi \geq \psi_* \\ \text{unknown} & \text{otherwise} \end{cases}.$$

1.3. **Main results.** Although the Jarník–Schmidt theorem tells us the Hausdorff dimension of $B(\psi)$ for any function ψ satisfying $\psi/\psi_* \to 0$, it leaves open the following two natural questions:

Question 1. What are the dimensions of the sets $B(\kappa)$ $(\kappa > 0)$, where for each $\kappa > 0$

$$B(\kappa) = B_{\mu,\nu}(\kappa) \stackrel{\text{def}}{=} B_{\mu,\nu}(\kappa^{1/m}\psi_*) = \left\{ A \in \mathcal{M} : \begin{array}{c} \|A\mathbf{q} - \mathbf{p}\|_\mu^m \|\mathbf{q}\|_\nu^n > \kappa \text{ for all but} \\ \text{finitely many } (\mathbf{p},\mathbf{q}) \in \mathbb{Z}^{m+n} \end{array} \right\} \quad ?$$

Question 2. Given a dimension function f and a function ψ such that $\psi/\psi_* \to 0$, what is $\mathcal{H}^f(B(\psi))$?

In the present paper, we give fairly complete answers to both of these questions. To Question 1, we give the following asymptotic answer:

Theorem 1.3. *Given any two norms* $\|\cdot\|_\mu$ *and* $\|\cdot\|_\nu$ *on* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *respectively, we have*

$$\lim_{\kappa\to 0} \frac{mn - \dim_H(B_{\mu,\nu}(\kappa))}{\kappa} = \theta_{\mu,\nu} \stackrel{\text{def}}{=} \frac{V_\mu V_\nu}{2\zeta(m+n)} \frac{mn}{m+n}$$

or equivalently

$$\dim_H(B_{\mu,\nu}(\kappa)) = mn - \theta_{\mu,\nu}\kappa + o(\kappa). \tag{1.5}$$

Here V_μ (resp. V_ν) denotes the volume of the unit ball in $\mathbb{R}^m$ (resp. $\mathbb{R}^n$) with respect to the $\|\cdot\|_\mu$ (resp. $\|\cdot\|_\nu$) norm, ζ denotes the Riemann zeta function, and $\dim_H$ denotes Hausdorff dimension.

This theorem has been preceded by many partial results. When $m = n = 1$, J. Kurzweil [24, Theorem VII] proved that

$$1 - .99\kappa \leq \dim_H(B_{1,1}(\kappa)) \leq 1 - .25\kappa \tag{1.6}$$

for all sufficiently small κ. Here $\|\cdot\|_1$ denotes the absolute value function on $\mathbb{R}$. Later D. Hensley [17] (cf. [8, Theorem 1.9]) improved this result by showing that

$$\dim_H(B_{1,1}(\kappa)) = 1 - \frac{6}{\pi^2}\kappa - \frac{72}{\pi^4}\kappa^2|\log(\kappa)| + O(\kappa^2). \tag{1.7}$$

Of course, (1.7) is stronger than (1.6) since $.25 < 6/\pi^2 < .99$. (1.7) is also slightly stronger than (1.5), since it has an estimate on the error term $o(\kappa)$. Here is the calculation which checks that (1.5) and (1.7) agree on the second term:

$$\theta_{1,1} = \frac{2 \cdot 2}{2\zeta(2)} \frac{1 \cdot 1}{1+1} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

When $n = 1$ and m is arbitrary, S. Weil [30] showed that

$$m - k_1 \frac{\kappa^{1/(2m)}}{|\log(\kappa)|} \leq \dim_H(B(\kappa)) \leq m - k_2 \frac{\kappa^{m+1}}{|\log(\kappa)|}$$

for some constants $k_1, k_2 > 0$ depending on m. This was improved by R. Broderick and D. Y. Kleinbock [7] who showed that for m, n both arbitrary,

$$mn - k_3 \frac{\kappa^{1/p(m,n)}}{|\log(\kappa)|} \leq \dim_H(B(\kappa)) \leq mn - k_4 \frac{\kappa}{|\log(\kappa)|}. \tag{1.8}$$

for some constants $k_3, k_4, p(m,n) > 0$ depending on m and n, with the property that $p(m,1) = 2m \ \ \forall m$. Note that the results of Kurzweil, Weil, and Broderick–Kleinbock all follow immediately from (1.5), while the result of Hensley does not. The conjecture of Broderick and Kleinbock made on [7, last para. of p.3] also follows immediately from (1.5).

Remark. Although the result (1.8) is superceded by (1.5), the left-hand inequality of (1.8) is still interesting in that it was proven using Schmidt games [28], and appears to be the optimal result that one can prove using this technique. Thus the left-hand inequality of (1.8) illustrates both the strengths and weaknesses of the Schmidt games technique when applied to problems more precise than simply asking for full dimension of BA.

Remark. Although Theorem 1.3 provides asymptotic information about the nonincreasing function $f_{\mu,\nu}(\kappa) = \dim_H(B_{\mu,\nu}(\kappa))$, the following questions concerning the behavior of $f_{\mu,\nu}$ are still open:

1. Is $f_{\mu,\nu}$ continuous? It was recently proven by C. G. T. de A. Moreira [26] that $f_{1,1}$ is continuous, but his techniques depend heavily on continued fractions and do not generalize to higher dimensions, where the question remains open.
2. What is the supremum of κ for which $f_{\mu,\nu}(\kappa) > 0$? Again, the answer is known only in dimension one, where Moreira showed that $f_{1,1}(\kappa) = 0$ if and only if $\kappa \geq 1/3$ [26, Theorem 1(ii)].

Remark. It is worthwhile to note that Theorem 1.3 provides an alternate proof of the Jarník–Schmidt Theorem. The existing proofs [29, 12, 6] have all been minor variations of each other, so it is nice to get an independent proof. (It should be noted here that when $m = 1$ or $n = 1$, there are easier proofs available [23, 13, 5] which rely on the so-called Simplex Lemma [23, Lemma 4].)

Regarding Question 2, we prove the following theorem, which should be thought of as an analogue of Theorem 1.2 for the sets $B(\psi)$ ($\psi : (0,\infty) \to (0,\infty)$):

Theorem 1.4. *Let f be a dimension function, and let $\psi : (0,\infty) \to (0,\infty)$ be a function such that* (1.2) *diverges but ψ/ψ_* is nonincreasing and tends to zero. Let*

$$L_{f,\psi} \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log\left(\frac{f(\rho)}{f_*(\rho)}\right)}{F_\psi(\rho^{-m/(m+n)})} \tag{1.9}$$

(cf. (1.3)*), where*

$$F_\psi(Q) = \sum_{q=1}^{Q} q^{n-1}\psi^m(q). \tag{1.10}$$

Then

$$\mathcal{H}^f(B_{\mu,\nu}(\psi)) = \begin{cases} 0 & \text{if } L_{f,\psi} < \eta_{\mu,\nu} \\ \infty & \text{if } L_{f,\psi} > \eta_{\mu,\nu} \\ \text{unknown} & \text{if } L_{f,\psi} = \eta_{\mu,\nu} \end{cases}, \tag{1.11}$$

where

$$\eta_{\mu,\nu} = \frac{nV_\mu V_\nu}{2\zeta(m+n)} = \frac{m+n}{m}\theta_{\mu,\nu}. \tag{1.12}$$

Remark. The divergence case of the Khinchin–Groshev theorem is a corollary of Theorem 1.4. Indeed, let $\psi : \mathbb{N} \to (0,\infty)$ be a function such that (1.2) diverges but ψ/ψ_* is nonincreasing and tends to zero. Since $\mathcal{H}^{f_*}$ is Lebesgue measure and $L_{f_*,\psi} = 0$, Theorem 1.4 says that the set $B_{\infty,\infty}(\psi)$ has Lebesgue measure zero, or equivalently that the set $W_{\infty,\infty}(\psi)$ has full Lebesgue measure. If ψ/ψ_* is nonincreasing but does not tend to zero, then $W_{\infty,\infty}(\psi)$ can be seen to have full measure via comparison with the function $\psi(q) = \psi_*(q)/\log(2 \vee q)$.[4]

Definition 1.5. In what follows, we shall call a function $\psi : (0,\infty) \to (0,\infty)$ such that (1.2) diverges but ψ/ψ_* is nonincreasing a *nice approximation function*.

To illustrate Theorem 1.4, we consider its special case for a certain family of nice approximation functions $(\psi_\gamma)_{\gamma>0}$, and we find a "dual" family of dimension functions $(f_s)_{s>0}$ which are useful for measuring the sets $(W(\psi_\gamma))_{\gamma>0}$:

Example 1.6. For each $\gamma, s > 0$, let

$$\psi_\gamma(q) = \gamma^{1/m} q^{-n/m} \log(2 \vee q)^{-1/m}, \qquad f_s(\rho) = \rho^{mn} |\log(\rho)|^s,$$

we have

$$F_\psi(Q) = \gamma \sum_{q=1}^{Q} \log(2 \vee q)^{-1} \frac{1}{q} \sim \gamma \log\log(Q)$$

$$\log\left(\frac{f_s(\rho)}{f_*(\rho)}\right) = \log\left(|\log(\rho)|^s\right) = s \log|\log(\rho)|$$

$$L_{f_s,\psi_\gamma} = \liminf_{\rho\to 0} \frac{s\log|\log(\rho)|}{\gamma \log\log(\rho^{-m/(m+n)})} = \frac{s}{\gamma}.$$

So by Theorem 1.4, we have

$$\mathcal{H}^{f_s}(B(\psi_\gamma)) = \begin{cases} 0 & \text{if } s < \gamma\eta_{\mu,\nu} \\ \infty & \text{if } s > \gamma\eta_{\mu,\nu} \\ \text{unknown} & \text{if } s = \gamma\eta_{\mu,\nu} \end{cases}.$$

[4]Here and hereafter $A \vee B$ denotes the maximum of two numbers A and B, and $A \wedge B$ denotes their minimum.

Example 1.6 reveals a surprising fact about the sets $B(\psi)$, namely that there exist pairs (f, ψ) and constants $0 < \gamma_- < \gamma_+$ such that $\mathcal{H}^f(B(\gamma_-\psi)) = \infty$ but $\mathcal{H}^f(B(\gamma_+\psi)) = 0$. This stands in marked contrast to Theorem 1.2, which demonstrates that $\mathcal{H}^f(W(\gamma\psi))$ is independent of γ for all functions f and ψ and for all $\gamma > 0$. Going further, we show that the phenomenon of the above example is in fact commonplace:

Corollary 1.7. *For any nice approximation function $\psi : (0,\infty) \to (0,\infty)$ such that $\psi/\psi_* \to 0$, there exists a dimension function f and constants $0 < \gamma_- < \gamma_+$ such that*

$$\mathcal{H}^f(B(\gamma_+\psi)) = 0 \text{ but } \mathcal{H}^f(B(\gamma_-\psi)) = \infty.$$

Proof. We define the dimension function f via the formula

$$f(\rho) = \begin{cases} \rho^{mn} \exp F_\psi(\rho^{-m/(m+n)}) & \rho \le \rho_0 \\ f(\rho_0) & \rho > \rho_0 \end{cases}$$

where $\rho_0 > 0$ is a small constant. We know that f is indeed a dimension function (i.e. is nondecreasing and satisfies $\lim_{\rho\to 0} f(\rho) = 0$) because $q^n\psi^m(q) \le 1 < n(m+n)$ for all q sufficiently large, so if ρ_0 is small enough then f is nondecreasing. This choice of f guarantees that $L_{f,\gamma\psi} = \gamma^{-m}$ for all $\gamma > 0$; choosing $0 < \gamma_- < \gamma_+$ so that $\gamma_+^{-m} < \eta_{\mu,\nu} < \gamma_-^{-m}$ completes the proof. □

Corollary 1.7 is less surprising than it first appears if we recall the contrast between the corollary of Dirichlet's theorem and the Jarník–Schmidt theorem: $W_{\infty,\infty}(\psi_*) = \mathcal{M}$ (so that $\dim_H(B_{\infty,\infty}(\psi_*)) = 0$), but $\dim_H(\mathrm{BA}) = mn$ (so that $\dim_H(B(\gamma\psi_*)) \to mn$ as $\gamma \to 0$, and in particular $\dim_H(B(\gamma\psi_*)) > 0$ for all sufficiently small $\gamma > 0$). Thus the conclusion of Corollary 1.7 was already known to hold for one function, namely $\psi = \psi_*$.

Another surprising fact about (1.11) is that it shows that $\mathcal{H}^f(B_{\mu,\nu}(\psi))$ depends on the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ used to define $B_{\mu,\nu}(\psi)$. Actually, this is not a different fact than our previous surprising fact, since multiplying the $\|\cdot\|_\mu$ norm by a constant factor is equivalent to dividing ψ by a constant factor; precisely,

$$B_{\mu,\nu}(\gamma\psi) = B_{\mu',\nu}(\psi) \text{ where } \|\cdot\|_{\mu'} = \|\cdot\|_\mu/\gamma.$$

We conclude this introduction by unifying Theorems 1.3 and 1.4 into a single theorem from which they can both be derived as corollaries.

Theorem 1.8. *Let f be a dimension function, and let $\psi : (0,\infty) \to (0,\infty)$ be a nice approximation function. Let $L_{f,\psi}$ be given by (1.9), and let $M_\psi = \psi^m(1)$. Then there exist continuous functions $c_{\mu,\nu}, C_{\mu,\nu} : [0,\infty) \to [0,\infty]$, depending only on m, n, μ, and ν, such that*

$$\mathcal{H}^f(B_{\mu,\nu}(\psi)) = \begin{cases} 0 & \text{if } L_{f,\psi} < c_{\mu,\nu}(M_\psi) \\ \infty & \text{if } L_{f,\psi} > C_{\mu,\nu}(M_\psi) \\ \text{unknown} & \text{if } c_{\mu,\nu}(M_\psi) \le L_{f,\psi} \le C_{\mu,\nu}(M_\psi) \end{cases} \tag{1.13}$$

AND

$$c_{\mu,\nu}(0) = C_{\mu,\nu}(0) = \eta_{\mu,\nu}.$$

Proof of Theorem 1.3 assuming Theorem 1.8. Fix $s, \kappa > 0$, and let $f_s(\rho) = \rho^{mn-s}$, $\psi_\kappa(q) = \kappa^{1/m}\psi_*(q)$. Then

$$M_\psi = \kappa \qquad F_\psi(Q) = \sum_{q=1}^{Q} \frac{\kappa}{q} \sim \kappa \log(Q)$$

$$\log\left(\frac{f(\rho)}{f_*(\rho)}\right) = s|\log(\rho)| \qquad L_{f,\psi} = \liminf_{\rho\to 0} \frac{s|\log(\rho)|}{\kappa \log(\rho^{-m/(m+n)})} = \frac{s}{\kappa}\frac{m+n}{m}$$

and thus

$$\mathcal{H}^{mn-s}(B_{\mu,\nu}(\kappa)) = \mathcal{H}^{f_s}(B_{\mu,\nu}(\psi_\kappa)) = \begin{cases} 0 & \text{if } s < \kappa\frac{m}{m+n} c_{\mu,\nu}(\kappa) \\ \infty & \text{if } s > \kappa\frac{m}{m+n} C_{\mu,\nu}(\kappa) \\ \text{unknown} & \text{if } \kappa\frac{m}{m+n} c_{\mu,\nu}(\kappa) \le s \le \kappa\frac{m}{m+n} C_{\mu,\nu}(\kappa) \end{cases}.$$

It follows that

$$\kappa\frac{m}{m+n}c_{\mu,\nu}(\kappa) \leq mn - \dim_H(B_{\mu,\nu}(\kappa)) \leq \kappa\frac{m}{m+n}C_{\mu,\nu}(\kappa)$$

and hence by the squeeze theorem,

$$\lim_{\kappa\to 0}\frac{mn-\dim_H(B_{\mu,\nu}(\kappa))}{\kappa} = \frac{m}{m+n}\eta_{\mu,\nu} = \theta_{\mu,\nu}. \qquad \square$$

Proof of Theorem 1.4 assuming Theorem 1.8. Suppose that $\psi/\psi_* \to 0$. Then for each $\gamma > 0$, the function $\psi_\gamma = \psi \wedge \gamma\psi_*$ agrees with ψ for all q sufficiently large. It follows that $B(\psi_\gamma) = B(\psi)$. Taking the limit of $(1.13)_{\psi=\psi_\gamma}$ as $\gamma \to 0$ yields (1.11). $\square$

The remainder of the paper is devoted to giving first a heuristic argument for Theorem 1.8, and then a rigorous proof motivated by the heuristic argument.

Note. To avoid needlessly cluttering the notation, in what follows we usually omit the subscripts of norms, writing $\|\mathbf{p}\|$ for $\|\mathbf{p}\|_\mu$, $\|\mathbf{q}\|$ for $\|\mathbf{q}\|_\nu$, etc. When it is important to clarify or emphasize which norm is being used, we will do so.

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2. Hausdorff measure and dimension

A function $f : (0,\infty) \to (0,\infty)$ is called a *dimension function* if f is nondecreasing and $\lim_{\rho\to 0} f(\rho) = 0$. In this section, we fix a dimension function f and a set $S \subseteq \mathbb{R}^d$. The *Hausdorff f-measure* of a set $S \subseteq \mathbb{R}^d$ is defined by the formula

$$\mathcal{H}^f(S) = \lim_{\varepsilon\to 0}\inf\left\{\sum_{i=1}^{\infty} f(\rho_i) : \big(B_{\rho_i}(x_i)\big)_1^\infty \text{ is a countable cover of } S \text{ with } \rho_i \leq \varepsilon \ \ \forall i\right\},$$

where here and hereafter, $B_\rho(x) = B(x,\rho) \stackrel{\text{def}}{=} \{y : d(x,y) \leq \rho\}$. Finally, the *Hausdorff dimension* of S is given by the formula

$$\dim_H(S) = \inf\{s \geq 0 : \mathcal{H}^s(S) = 0\},$$

where $\mathcal{H}^s$ is the Hausdorff measure with respect to the dimension function

$$f_s(\rho) = \rho^s.$$

We recall some well-known upper and lower bounds on the quantities $\mathcal{H}^f(S)$ and $\dim_H(S)$. The upper bounds come from considering only covers whose balls all have the same radius. Such a cover is represented

by two quantities: the set F of centers of balls in the cover, and the common radius ρ. The cost of such a cover is equal to $\#(F)f(\rho)$. It follows that

$$\mathcal{H}^f(S) \leq \underline{\mathcal{B}}^f(S) \stackrel{\text{def}}{=} \liminf_{\rho \to 0} \mathrm{N}_\rho(S) f(\rho),$$

where

$$\mathrm{N}_\rho(S) = \min \left\{ \#(F) : F \subseteq \mathbb{R}^d,\ S \subseteq \bigcup_{x \in F} B_\rho(x) \right\}. \tag{2.1}$$

We call the number $\underline{\mathcal{B}}^f(S)$ the *lower box-counting f-measure*[5] of S, in analogy with the *lower box-counting dimension*, which is given by the formula

$$\underline{\dim_B}(S) = \liminf_{\rho \to 0} \frac{\log \mathrm{N}_\rho(S)}{-\log(\rho)} = \inf\{s \geq 0 : \underline{\mathcal{B}}^s(S) = 0\}.$$

Note that

$$\dim_H(S) \leq \underline{\dim_B}(S).$$

To get a lower bound on $\mathcal{H}^f(S)$ and $\dim_H(S)$, a more sophisticated argument is needed.

Lemma 2.1 (Generalized mass distribution principle)**.** *Let μ be a probability measure on S such that for all $\mathbf{x} \in \mathbb{R}^d$ and $\rho > 0$,*

$$\mu(B_\rho(\mathbf{x})) \leq g(\rho),$$

where g is a dimension function. Then

$$\mathcal{H}^f(S) \geq \liminf_{\rho \to 0} \frac{f(\rho)}{g(\rho)}.$$

Proof. This can be deduced as a corollary of [25, Theorem 8.2], but the proof is much easier. If $(B_{\rho_i}(\mathbf{x}_i))_1^\infty$ is a cover of S such that $\rho_i \leq \varepsilon\ \ \forall i$, then

$$1 = \mu(S) \leq \sum_{i=1}^\infty \mu(B_{\rho_i}(\mathbf{x}_i)) \leq \sum_{i=1}^\infty g(\rho_i) \leq \left(\sup_{0 < \rho \leq \varepsilon} \frac{g(\rho)}{f(\rho)} \right) \sum_{i=1}^\infty f(\rho_i)$$

and thus

$$\mathcal{H}^f(S) \geq \lim_{\varepsilon \to 0} \inf_{0 < \rho \leq \varepsilon} \frac{f(\rho)}{g(\rho)} = \liminf_{\rho \to 0} \frac{f(\rho)}{g(\rho)}. \qquad \square$$

3. Heuristic argument

In this section we give a heuristic argument explaining where the formulas (1.5) and (1.11) come from. Let f be a dimension function, and let ψ be a nice approximation function (cf. Definition 1.5). The reader can keep in mind the special case $f(\rho) = \rho^s$, $\psi(q) = \kappa^{1/m}\psi_*(q) = (\kappa q^{-n})^{1/m}$, where $0 < s < mn$ and $\kappa > 0$, i.e. the special case needed to prove Theorem 1.3.

In this section, the notation $A \sim B$ indicates that the ratio A/B is asymptotically close to 1, but we do not specify the circumstances under which this asymptotic holds. Similarly, $A \approx B$ indicates that the ratio A/B is bounded from above and below, but again the contextual requirements are unspecified. A more rigorous way of denoting asymptotic relations will be described at the beginning of Section 4.

The first step in our calculation is to rewrite the set $B(\psi)$ as the union of countably many closed sets. For each $Q_0 \in \mathbb{N}$, let

$$W_{\psi,Q_0} \stackrel{\text{def}}{=} \bigcup_{\substack{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^{m+n} \\ \|\mathbf{q}\| \geq Q_0}} \Delta_\psi(\mathbf{p},\mathbf{q}), \qquad\qquad B_{\psi,Q_0} \stackrel{\text{def}}{=} \mathcal{M} \setminus W_{\psi,Q_0}, \tag{3.1}$$

[5]Warning: Although we are calling $\underline{\mathcal{B}}^f$ a "measure", it is not countably additive on any reasonable collection of sets, although it is finitely additive on sets A, B such that $d(A, B) > 0$.

where for each $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{q} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we have

$$\Delta_\psi(\mathbf{p}, \mathbf{q}) = \{A \in \mathcal{M} : \|A\mathbf{q} - \mathbf{p}\| \le \psi(\|\mathbf{q}\|)\}. \tag{3.2}$$

Then we have

$$B(\psi) = \bigcup_{Q_0=1}^{\infty} B_{\psi,Q_0},$$

so

$$\mathcal{H}^f(B(\psi)) = \sup_{Q_0 \ge 1} \mathcal{H}^f(B_{\psi,Q_0}) = \infty \cdot \sup_{Q_0 \ge 1} \mathcal{H}^f(B_{\psi,Q_0} \cap \mathcal{K}),^{6} \tag{3.3}$$

where $\mathcal{K}$ denotes the set of $m \times n$ matrices whose entries are all in $[0, 1]$, i.e. the "unit cube" of $\mathcal{M}$.

Fix $Q_0 \in \mathbb{N}$, and we will estimate $\mathcal{H}^f(B_{\psi,Q_0} \cap \mathcal{K})$. We will use the lower box-counting measure $\underline{\mathcal{B}}^f(B_{\psi,Q_0} \cap \mathcal{K})$ (cf. Section 2) as a proxy for $\mathcal{H}^f(B_{\psi,Q_0} \cap \mathcal{K})$. Intuitively, $\underline{\mathcal{B}}^f(B_{\psi,Q_0} \cap \mathcal{K})$ is a good proxy for $\mathcal{H}^f(B_{\psi,Q_0} \cap \mathcal{K})$ as long as the closed set $B_{\psi,Q_0} \cap \mathcal{K}$ is sufficiently homogeneous.

Heuristic assumption 1. For all sufficiently large $Q_0 \in \mathbb{N}$, we have $\mathcal{H}^f(B_{\psi,Q_0} \cap \mathcal{K}) \approx \underline{\mathcal{B}}^f(B_{\psi,Q_0} \cap \mathcal{K})$.

Now to compute $\underline{\mathcal{B}}^f(B_{\psi,Q_0} \cap \mathcal{K})$, we should fix $\rho > 0$ and estimate the number $\mathrm{N}_\rho(B_{\psi,Q_0} \cap \mathcal{K})$ defined in (2.1). To do this, we will estimate the measure of the set $\mathcal{N}(B_{\psi,Q_0}, \rho) \cap \mathcal{K}$ in order to apply the formula

$$\mathrm{N}_\rho(B_{\psi,Q_0} \cap \mathcal{K}) \approx \rho^{-mn}\, \lambda_{\mathcal{K}}\big(\mathcal{N}(B_{\psi,Q_0}, \rho)\big).$$

Here $\lambda_{\mathcal{K}}$ denotes Lebesgue measure on $\mathcal{K}$, normalized to be a probability measure, and

$$\mathcal{N}(S, \rho) \stackrel{\text{def}}{=} \{A \in \mathcal{M} : d(A, S) \le \rho\}.$$

(The distance on $\mathcal{M}$ comes from the operator norm $\|A\| = \max_{\|\mathbf{q}\|_\nu = 1} \|A\mathbf{q}\|_\mu$.) Fix $Q \ge Q_0$ to be determined, and let

$$W_\psi(Q_0, Q) = \bigcup_{\substack{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^{m+n} \\ Q_0 \le \|\mathbf{q}\| \le Q}} \Delta_\psi(\mathbf{p}, \mathbf{q}), \qquad B_\psi(Q_0, Q) = \mathcal{M} \setminus W_\psi(Q_0, Q). \tag{3.4}$$

We will estimate the measure of $B_\psi(Q_0, Q)$ using an independence assumption, and then we will compare the measure of $B_\psi(Q_0, Q)$ with the measure of $\mathcal{N}(B_{\psi,Q_0}, \rho)$, assuming an appropriate relation between Q and ρ. However, we must deal with the fact that if $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ and $k \in \mathbb{Z} \setminus \{0\}$, then $\Delta_\psi(k\mathbf{p}, k\mathbf{q}) \subseteq \Delta_\psi(\mathbf{p}, \mathbf{q})$, and in particular the sets $\Delta_\psi(\mathbf{p}, \mathbf{q})$ and $\Delta_\psi(k\mathbf{p}, k\mathbf{q})$ are not independent. Let $\mathcal{P} \subseteq \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$ be a set consisting of primitive integer vectors such that for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$, we have $(\mathbf{p}, \mathbf{q}) \in k\mathcal{P}$ for exactly one value $k \in \mathbb{Z} \setminus \{0\}$.

Heuristic assumption 2. The sets $\Delta_\psi(\mathbf{p}, \mathbf{q})$ ($(\mathbf{p}, \mathbf{q}) \in \mathcal{P}$, $Q_0 \le \|\mathbf{q}\| \le Q$) are approximately independent with respect to the probability measure $\lambda_{\mathcal{K}}$.

Using this assumption, we make the following calculation:

$$\begin{aligned} -\log \lambda_{\mathcal{K}}\big(B_\psi(Q_0, Q)\big) &\sim \sum_{\substack{(\mathbf{p},\mathbf{q}) \in \mathcal{P} \\ Q_0 \le \|\mathbf{q}\| \le Q}} -\log\big(1 - \lambda_{\mathcal{K}}(\Delta_\psi(\mathbf{p}, \mathbf{q}))\big) \\ &\sim \sum_{\substack{(\mathbf{p},\mathbf{q}) \in \mathcal{P} \\ Q_0 \le \|\mathbf{q}\| \le Q}} \lambda_{\mathcal{K}}\big(\Delta_\psi(\mathbf{p}, \mathbf{q})\big). \end{aligned}$$

Heuristic assumption 3. The set $\mathcal{P}$ is "homogeneous" in the sense that sums over subsets of $\mathcal{P}$ are asymptotic to a fixed constant, say $\gamma_{\mathcal{P}}$, times integrals over the same regions.

[6] Here we use the convention that $\infty \cdot 0 = 0$.

To compute $\gamma_{\mathcal{P}}$, we observe that for R large,

$$\begin{aligned} V_\mu V_\nu R^{m+n} &\sim \#\{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\mathbf{0}\}) : \|\mathbf{p}\| \vee \|\mathbf{q}\| \leq R\} \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \#\{(\mathbf{p},\mathbf{q}) \in \mathcal{P} : \|k\mathbf{p}\| \vee \|k\mathbf{q}\| \leq R\} \\ &\sim \gamma_{\mathcal{P}} \sum_{k \in \mathbb{Z} \setminus \{0\}} V_\mu V_\nu (R/|k|)^{m+n} = 2\zeta(m+n)\gamma_{\mathcal{P}} V_\mu V_\nu R^{m+n}, \end{aligned}$$

so $\gamma_{\mathcal{P}} = 1/(2\zeta(m+n))$. Thus,

$$\begin{aligned} &-\log \lambda_{\mathcal{K}}\left(B_\psi(Q_0,Q)\right) \\ &\sim \frac{1}{2\zeta(m+n)} \int_{\substack{(\mathbf{p},\mathbf{q}) \in \mathbb{R}^{m+n} \\ Q_0 \leq \|\mathbf{q}\| \leq Q}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{p},\mathbf{q})\right) \, \mathrm{d}(\mathbf{p},\mathbf{q}) \\ &= \frac{1}{2\zeta(m+n)} \int_{Q_0 \leq \|\mathbf{q}\| \leq Q} \int_{\mathcal{K}} \lambda_{\mathbb{R}^m}\left(\left\{\mathbf{p} \in \mathbb{R}^m : \|A\mathbf{q} - \mathbf{p}\| \leq \psi(\|\mathbf{q}\|)\right\}\right) \, \mathrm{d}A \, \mathrm{d}\mathbf{q} \\ &= \frac{1}{2\zeta(m+n)} \int_{Q_0 \leq \|\mathbf{q}\| \leq Q} V_\mu \psi^m(\|\mathbf{q}\|) \, \mathrm{d}\mathbf{q}. \end{aligned}$$

Lemma 3.1. *If $f : [0,\infty) \to [0,\infty)$ is locally integrable, then*

$$\int f(\|\mathbf{q}\|) \, \mathrm{d}\mathbf{q} = nV_\nu \int_0^\infty q^{n-1} f(q) \, \mathrm{d}q.$$

Proof. It suffices to prove the equality for functions of the form $f(q) = \left[q \leq Q\right]$ (cf. Convention 5 below), and for these,

$$\int \left[\|\mathbf{q}\| \leq Q\right] \, \mathrm{d}\mathbf{q} = \lambda_{\mathbb{R}^n}\left(B_\nu(\mathbf{0},Q)\right) = V_\nu Q^n = V_\nu \int_0^Q nq^{n-1} \, \mathrm{d}q. \qquad \square$$

Using Lemma 3.1, we continue the calculation:

$$\begin{aligned} -\log \lambda_{\mathcal{K}}\left(B_\psi(Q_0,Q)\right) &\sim \frac{V_\mu}{2\zeta(m+n)} \int_{Q_0 \leq \|\mathbf{q}\| \leq Q} \psi^m(\|\mathbf{q}\|) \, \mathrm{d}\mathbf{q} \\ &= \frac{nV_\mu V_\nu}{2\zeta(m+n)} \int_{Q_0}^Q q^{n-1} \psi^m(q) \, \mathrm{d}q \\ &\sim \eta_{\mu,\nu} F_\psi(Q), \end{aligned} \tag{3.5}$$

where F_ψ is as in (1.10).

Now that we have estimated the measure of $B_\psi(Q_0,Q)$, we want to compare it to the measure of $\mathcal{N}(B_{\psi,Q_0},\rho)$. To do this, we compute the "thickness" of the sets $\Delta_\psi(\mathbf{p},\mathbf{q})$ ($(\mathbf{p},\mathbf{q}) \in \mathcal{P}$). The following lemma (which will be used in the real proof of Theorem 1.8) suggests that the thickness should be interpreted as being $\Psi(\|\mathbf{q}\|) \stackrel{\text{def}}{=} \psi(\|\mathbf{q}\|)/\|\mathbf{q}\|$:

Lemma 3.2. *Let $\psi_1, \psi_2 : \mathbb{N} \to (0,\infty)$. Then for all $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{q} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\mathcal{N}\left(\Delta_{\psi_1}(\mathbf{p},\mathbf{q}), \frac{\psi_2(\|\mathbf{q}\|)}{\|\mathbf{q}\|}\right) \subseteq \Delta_{\psi_1+\psi_2}(\mathbf{p},\mathbf{q}).$$

For example, $\Delta_0(\mathbf{p},\mathbf{q})$ is an affine linear subspace of $\mathcal{M}$, and $\mathcal{N}(\Delta_0(\mathbf{p},\mathbf{q}), \Psi(\|\mathbf{q}\|)) \subseteq \Delta_\psi(\mathbf{p},\mathbf{q})$.

Proof. Fix $A \in \Delta_{\psi_1}(\mathbf{p},\mathbf{q})$ and $B \in \mathcal{M}$ with $\|B - A\| \leq \psi_2(\|\mathbf{q}\|)/\|\mathbf{q}\|$. Then

$$\begin{aligned} \|B\mathbf{q} - \mathbf{p}\| &\leq \|A\mathbf{q} - \mathbf{p}\| + \|B - A\| \cdot \|\mathbf{q}\| \\ &\leq \psi_1(\|\mathbf{q}\|) + \frac{\psi_2(\|\mathbf{q}\|)}{\|\mathbf{q}\|} \cdot \|\mathbf{q}\| = (\psi_1 + \psi_2)(\|\mathbf{q}\|). \end{aligned}$$

$\square$

Heuristic assumption 4. The measure of $B_\psi(Q_0, Q)$ is approximately the same as the measure of $\mathcal{N}(B_{\psi,Q_0}, \rho)$, where $\rho = \Psi(Q)$ is the minimum "thickness" of the sets $\Delta_\psi(\mathbf{p}, \mathbf{q})$ $(Q_0 \le \|\mathbf{q}\| \le Q)$.

So

$$-\log \lambda_{\mathcal{K}} \left(\mathcal{N}(B_{\psi,Q_0}, \rho)\right) \sim \eta_{\mu,\nu} F_\psi(\Psi^{-1}(\rho)).$$

Since ψ is assumed to be a nice approximation function, we have $\log \psi(q) \sim -\frac{n}{m}\log(q)$ and thus $\log \Psi(q) \sim -\frac{m+n}{m}\log(q)$ and $\log \Psi^{-1}(\rho) \sim -\frac{m}{m+n}\log(\rho)$. Since F_ψ is either logarithmic or sublogarithmic, this gives

$$-\log \lambda_{\mathcal{K}} \left(\mathcal{N}(B_{\psi,Q_0}, \rho)\right) \sim \eta_{\mu,\nu} F_\psi(\rho^{-m/(m+n)}).$$

We now have all the ingredients needed to finish the calculation. In what follows, the symbol $\sim$ just means intuitively that "the quantities are close" and we do not specify a more precise relation.

$$\begin{aligned}
\log \underline{\mathcal{B}}^f(B_{\psi,Q_0} \cap \mathcal{K}) &= \liminf_{\rho\to 0} \log\left(\mathrm{N}_\rho(B_{\psi,Q_0}) f(\rho)\right) \\
&\sim \liminf_{\rho\to 0} \left[\log \lambda_{\mathcal{K}} \left(\mathcal{N}(B_{\psi,Q_0}, \rho)\right) - mn\log(\rho) + \log f(\rho)\right] \\
&\sim \liminf_{\rho\to 0} \left[-\eta_{\mu,\nu} F_\psi(\rho^{-m/(m+n)}) + \log \frac{f(\rho)}{f_*(\rho)}\right] \\
&= \begin{cases} -\infty & \text{if } L_{f,\psi} < \eta_{\mu,\nu} \\ \infty & \text{if } L_{f,\psi} > \eta_{\mu,\nu} \\ \text{unknown} & \text{if } L_{f,\psi} = \eta_{\mu,\nu} \end{cases}.
\end{aligned}$$

Using heuristic assumption 1 and applying (3.3) gives

$$\mathcal{H}^f(B(\psi)) = \begin{cases} 0 & \text{if } L_{f,\psi} < \eta_{\mu,\nu} \\ \infty & \text{if } L_{f,\psi} > \eta_{\mu,\nu} \\ \text{unknown} & \text{if } L_{f,\psi} = \eta_{\mu,\nu} \end{cases}.$$

4. Preliminaries and notation

The notation introduced in this section will be used throughout the paper. We provide a summary in Appendix A for convenience. Recall that $m, n \in \mathbb{N}$ are fixed, and that $\mathcal{M}$ denotes the set of $m \times n$ matrices.

Convention 1. The symbols $\lessapprox$, $\gtrapprox$, and $\approx$ will denote coarse multiplicative asymptotics, with a subscript of $+$ indicating that the asymptotic is additive instead of multiplicative, and other subscripts indicating variables that the implied constant is allowed to depend on. If there are no variables indicated, then the implied constant only depends on the parameters m, n, μ, and ν. The implied constant is understood to depend continuously on all parameters. We emphasize that all parameters the implied constant depends on other than m, n, μ, and ν will be explicitly notated, even if they appear to be clear from context.

For example, $A \lessapprox_K B$ means that there exists a constant $k \ge 1$ that depends continuously on the parameters m, n, μ, ν, and K, and on no other parameters, such that $A \le kB$. Similarly, $A \approx_+ B$ means that there exists a constant $k \ge 0$, depending only on m, n, μ, and ν, such that $B - k \le A \le B + k$. Note that in the first example, if we have an upper bound and a lower bound on K then we can use continuity to get an upper bound on the implied constant k.

Convention 2. When limits are written using arrow notation, they are assumed to be uniform with respect to all parameters except for those mentioned in parentheses. For example, the notation

$$A \xrightarrow[\substack{Q\to\infty \\ \delta\to 0\ (K)}]{} 0$$

means: For all $\varepsilon > 0$, ther exist $Q_0, \delta_0 > 0$, with Q_0 depending only on ε and δ_0 depending on both K and ε (and on m, n, μ, ν), such that if $Q \ge Q_0$ and $0 < \delta \le \delta_0$, then $|A| \le \varepsilon$.

Convention 3. The symbols $\sim$, $\lesssim$, and $\gtrsim$ will denote asymptotics whose implied constants tend to 1 (or 0 for additive asymptotics) as certain quantities approach their limits. The convergence of the implied constant is notated using Convention 2. For example, the formula

$$A \underset{\delta \to 0 \ (Q)}{\sim} B$$

means: For all $\varepsilon > 0$, there exists $\delta_0 > 0$ depending only on ε, Q, and m, n, μ, ν such that if $0 < \delta \leq \delta_0$, then $(1-\varepsilon)B \leq A \leq (1+\varepsilon)B$.

Similarly, the notation $A \ll B$ means that $A/B \to 0$; again the convergence is notated underneath the $\ll$ sign using Convention 2. (This differs from the common usage of $\ll$ as the Vinogradov symbol; in this paper, $\lessapprox$ plays the function of the Vinogradov symbol.)

Convention 4. Whenever a vector $\mathbf{r} \in \mathbb{R}^{m+n}$ is fixed, we denote its components by $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{q} \in \mathbb{R}^n$, so that $\mathbf{r} = (\mathbf{p}, \mathbf{q})$. Similarly, whenever $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{q} \in \mathbb{R}^n$ are fixed, we use the shorthand $\mathbf{r} = (\mathbf{p}, \mathbf{q})$. The same convention applies to $\mathbf{r}' = (\mathbf{p}', \mathbf{q}')$.

Convention 5. We use the Iverson bracket notation $[\text{statement}] = \begin{cases} 1 & \text{statement true} \\ 0 & \text{statement false} \end{cases}$.

Convention 6. The symbol $\lhd$ will be used to indicate the end of a nested proof.

4.1. **Cantor series expansion.** Let $D = mn$, so that $\mathbb{R}^D$ is isomorphic to $\mathcal{M}$ as a vector space. In what follows, we identify an element of $\mathcal{M}$ with its image in $\mathbb{R}^D$ under the map which sends a matrix to the list of its entries, and we take the norm on $\mathcal{M}$ to be the operator norm $\|A\| = \max_{\|\mathbf{q}\|=1} \|A\mathbf{q}\|$. We write $\mathcal{K} = [0,1]^D \subseteq \mathcal{M}$. Note that this subsection and the next one (i.e. §4.1-4.2) make sense in any finite-dimensional normed space $(\mathbb{R}^D, \|\cdot\|)$, without any reference to Diophantine approximation.

During the proof of Theorem 1.8, we will specify a sequence of integers $(N_k)_1^\infty$, depending on the approximation function ψ and on an auxiliary parameter β. We will describe how this sequence is chosen later, but for now we assume that we are given a sequence $(N_k)_1^\infty$ satisfying $N_k \geq 2 \ \ \forall k$. We introduce the following notations:

$$E_k \stackrel{\text{def}}{=} \{0, \ldots, N_k - 1\}^D \subseteq \mathcal{M} \qquad (1 \leq k < \infty)$$

$$E^k \stackrel{\text{def}}{=} \prod_{j=1}^{k} E_j \quad \text{(Cartesian product)} \qquad (0 \leq k \leq \infty)$$

$$N^k \stackrel{\text{def}}{=} \prod_{j=1}^{k} N_j = \#(E^k)^{1/D}. \qquad (0 \leq k < \infty)$$

Warning. The notations E^k and N^k should not be confused with the Cartesian and arithmetic powers of a set E and a number N, respectively. In this paper, there is no set E and there is no number N. As a plea to forgive this abuse of notation, we note that in the special case occurring in the proof of Theorem 1.3, the sequence $(N_k)_1^\infty$ is constant (cf. Remark 4.3 below), and so in that case the notations E^k and N^k really do denote Cartesian and arithmetic powers.

For each $\omega \in E^k$, we write $|\omega| = k$. Let $E^* \stackrel{\text{def}}{=} \bigcup_{k \geq 0} E^k$. ($E^\infty$ is not included in this union.) For all $\omega \in E^* \cup E^\infty$ and $k \leq |\omega|$, we denote the kth element of ω by ω_k, and we denote the initial segment of ω of length k by $\omega \upharpoonright k$. We define the *coding map* $\pi : E^* \cup E^\infty \to \mathcal{K}$ via the formula

$$\pi(\omega) = \sum_{k=1}^{|\omega|} \frac{\omega_k}{N^k}.$$

This series is called the *Cantor series expansion* of $\pi(\omega)$. Note that every element of $\mathcal{K}$ has at least one Cantor series expansion, and almost every element of $\mathcal{K}$ (with respect to Lebesgue measure) has exactly one Cantor series expansion. Moreover, for each $\omega \in E^*$, if

$$[\omega] \stackrel{\text{def}}{=} \{\tau \in E^\infty : \tau \upharpoonright |\omega| = \omega\}, \qquad \mathcal{K}_\omega \stackrel{\text{def}}{=} \pi([\omega]),$$

then

$$\mathcal{K}_\omega = \pi(\omega) + \frac{1}{N^{|\omega|}}\mathcal{K}. \tag{4.1}$$

In particular, $\mathcal{K}_\omega$ is a cube of side length $1/N^{|\omega|}$. We observe that if $\omega, \tau \in E^*$ are incomparable,[7] then $\mathcal{K}_\omega$ and $\mathcal{K}_\tau$ have disjoint interiors.

The following notation will sometimes be convenient: Given $\omega \in E^*$ and $A \in \mathcal{K}$, let

$$\Phi_\omega(A) = \pi(\omega) + \frac{A}{N^{|\omega|}}.$$

By (4.1), $\Phi_\omega : \mathcal{K} \to \mathcal{K}_\omega$ is a bijection.

4.2. **Evaporation rates.** A *tree* in E^* is a subset of E^* which is closed under taking initial segments. Let $T^* \subseteq E^*$ be a tree. For each $k \in \mathbb{N}$, let $T^k \stackrel{\text{def}}{=} E^k \cap T^*$, and for each $\omega \in T^*$, let

$$T_\omega = \{a \in E_{|\omega|+1} : \omega a \in T^{|\omega|+1}\}.$$

Here and hereafter, ωa denotes the concatenation of ω and a. For each $k \geq 1$, let

$$P_k^+ \stackrel{\text{def}}{=} \frac{1}{(N_k)^D} \max_{\omega \in T^{k-1}} \#(E_k \setminus T_\omega)$$
$$P_k^- \stackrel{\text{def}}{=} \frac{1}{(N_k)^D} \min_{\omega \in T^{k-1}} \#(E_k \setminus T_\omega).$$

In other words, P_k^+ (resp. P_k^-) is the maximum (resp. minimum) proportion of children that get removed from the tree T at stage k. We call the sequences $(P_k^+)_1^\infty$ and $(P_k^-)_1^\infty$ the *upper evaporation rate* and the *lower evaporation rate* of T^*, respectively.

Let T^∞ denote the set of infinite branches through T^*, i.e. the set $\{\omega \in E^\infty : \omega \restriction k \in T^k \ \ \forall k\}$. The goal of this subsection is to relate the Hausdorff f-measure of $\pi(T^\infty)$ to the evaporation rate of T^*, where f is a dimension function. To do this, we introduce some more notation. Namely, let

$$M_k^- \stackrel{\text{def}}{=} (N_k)^D(1 - P_k^+) = \min_{\omega \in T^{k-1}} \#(T_\omega)$$
$$M_k^+ \stackrel{\text{def}}{=} (N_k)^D(1 - P_k^-) = \max_{\omega \in T^{k-1}} \#(T_\omega)$$
$$M_\pm^k \stackrel{\text{def}}{=} \prod_{j=1}^k M_j^\pm.$$

In other words, M_k^- (resp. M_k^+) is the minimum (resp. maximum) number of children of any stage $k-1$ node that remain in T. For each $0 < \rho \leq 1$, let $k(\rho) \in \mathbb{N}$ be the unique integer satisfying $1/N^{k(\rho)+1} < \rho \leq 1/N^{k(\rho)}$, and let

$$f_\pm(\rho) \stackrel{\text{def}}{=} \frac{(N^{k(\rho)}\rho)^D}{M_\mp^{k(\rho)}} = \left(\prod_{j=1}^{k(\rho)} \frac{1}{1 - P_j^\pm}\right)\rho^D.$$

Remark. The subscripts and superscripts above have been chosen so that $\square_- \leq \square_+$ for all relevant objects.

Proposition 4.1. *Let f be a dimension function. With notation as above,*

(i) *If $T^* \subseteq E^*$ is a tree such that $\varnothing \in T^*$ and $\sup_{k\geq 1} P_k^+ < 1$, then*

$$\mathcal{H}^f(\pi(T^\infty)) \gtrapprox \liminf_{\rho \to 0} \frac{f(\rho)}{f_+(\rho)}. \tag{4.2}$$

[7] Two strings are called *incomparable* if neither is an initial segment of the other.

(ii) *For any tree* $T^* \subseteq E^*$,

$$\mathcal{H}^f(\pi(T^\infty)) \leq \underline{\mathcal{B}}^f(\pi(T^\infty)) \lessapprox \liminf_{\rho \to 0} \frac{f(\rho)}{f_-(\rho)}. \tag{4.3}$$

Proof of (i). By pruning the tree T^*, we may without loss of generality assume that $\#(T_\omega) = M^-_{|\omega|+1}$ for all $\omega \in T^*$. Let ν be the unique measure on T^∞ such that

$$\nu([\omega]) = 1/M_-^{|\omega|} \ \ \forall \omega \in T^*.$$

Let μ be the image of ν under the coding map π. Now fix $A \in \mathcal{K}$ and $0 < \rho \leq 1$, and let $k = k(\rho)$. Let $\|\mathcal{K}\| = \max_{B \in \mathcal{K}} \|B\|$. Then

$$\begin{aligned}
\mu(B_\rho(A)) &= \sum_{\omega \in E^{k+1}} \nu\big([\omega] \cap \pi^{-1}(B_\rho(A))\big) \\
&\leq \frac{1}{M_-^{k+1}} \#\{\omega \in E^{k+1} : \mathcal{K}_\omega \cap B_\rho(A) \neq \varnothing\} \\
&\leq \frac{1}{M_-^{k+1}} \#\{\omega \in E^{k+1} : \mathcal{K}_\omega \subseteq B(A, (1 + \|\mathcal{K}\|)\rho)\} \\
&\text{(since } \operatorname{diam}(\mathcal{K}_\omega) \leq \|\mathcal{K}\|/N^{k+1} < \|\mathcal{K}\|\rho \text{ for all } \omega \in E^{k+1}\text{)} \\
&\leq \frac{1}{M_-^{k+1}} \frac{\lambda_{\mathcal{K}}\left(B(A, (1 + \|\mathcal{K}\|)\rho)\right)}{(1/N^{k+1})^D} \\
&\text{(since } \lambda_{\mathcal{K}}(\mathcal{K}_\omega) = (1/N^{k+1})^D \text{ for all } \omega \in E^{k+1}\text{)} \\
&\approx \frac{(N^{k+1})^D}{M_-^{k+1}} \rho^D \\
&\approx \frac{(N^k)^D}{M_-^k} \rho^D \\
&\text{(since } \sup_{k \geq 1} P_k^+ < 1\text{)} \\
&= f_+(\rho).
\end{aligned}$$

Applying Lemma 2.1 demonstrates (4.2). □

Proof of (ii). Fix $0 < \rho \leq 1$, and let $k = k(\rho)$. For each $\omega \in T^k$, we have $\mathrm{N}_\rho(\mathcal{K}_\omega) = \mathrm{N}_{N^k \rho}(\mathcal{K}) \approx 1/(N^k \rho)^D$. Since $\#(T^k) \leq M_+^k$, it follows that

$$\mathrm{N}_\rho(\pi(T^\infty)) \leq \sum_{\omega \in T^k} \mathrm{N}_\rho(\mathcal{K}_\omega) \lessapprox \frac{M_+^k}{(N^k \rho)^D} = \frac{1}{f_-(\rho)}.$$

Multiplying by $f(\rho)$ and taking the liminf as $\rho \to 0$ finishes the proof. □

4.3. **Choice of the sequence** $(N_k)_1^\infty$**.** In this subsection we describe how the sequence $(N_k)_1^\infty$ to be used in the proof of Theorem 1.8 will be chosen, depending on a nice approximation function ψ and a parameter $\beta > 0$. To motivate this choice, we recall that in the heuristic argument of Section 3, we estimated (cf. (3.5)) that for all $1 \leq Q_1 \leq Q_2$,

$$-\log \lambda_{\mathcal{K}}\left(B_\psi(Q_1, Q_2)\right) \sim \eta_{\mu,\nu} F_\psi(Q_1, Q_2), \tag{4.4}$$

where $B_\psi(Q_1, Q_2)$ is defined by (3.4), and

$$F_\psi(Q_1, Q_2) \stackrel{\text{def}}{=} \int_{Q_1}^{Q_2} q^{n-1} \psi^m(q) \, \mathrm{d}q. \tag{4.5}$$

Note that the two-input function F_ψ defined here is related to the one-input function F_ψ defined in (1.10) via the asymptotic

$$F_\psi(Q) \underset{Q\to\infty\ (Q_0)}{\sim} F_\psi(Q_0, Q).$$

We have switched from a sum to an integral for ease of calculations later.

Coming back to the motivation of the choice of $(N_k)_1^\infty$, we want to choose it so that the companion sequence

$$Q^k \stackrel{\text{def}}{=} (N^k)^{m/(m+n)} \quad (k \in \mathbb{N}) \tag{4.6}$$

satisfies $F_\psi(Q^k, Q^{k+1}) \sim \beta$ for all k, for an appropriate limiting process. The idea is to make sure that Q^k and Q^{k+1} are "far enough apart" that the heuristic formula (4.4) is true for $Q_1 = Q^k$ and $Q_2 = Q^{k+1}$, but "close enough together" that the left-hand side of (4.4) is essentially the same as $\lambda_{\mathcal{K}}\left(W_\psi(Q^k, Q^{k+1})\right)$. The significance of the sequence (4.6) is that the "minimum thickness" of $B_\psi(Q^k, Q^{k+1})$ is approximately $1/N^{k+1}$ (cf. Lemma 3.2 and Heuristic assumption 4) and thus heuristically, we expect that if $\omega \in E^\infty$ is the Cantor series expansion of a matrix $A = \pi(\omega) \in \mathcal{K}$, then it should be possible to control whether or not $A \in B_\psi(Q^k, Q^{k+1})$ by controlling the coordinate ω_{k+1}, given information about the initial segment $\omega \restriction k$. This will allow us to create a tree contained in B_{ψ,Q_0} that we can control the evaporation rate of.

Now we need to show that it is possible to choose the sequence $(N_k)_1^\infty$ in such a way so as to satisfy $F_\psi(Q^k, Q^{k+1}) \sim \beta\ \ \forall k$, or more precisely,

$$F_\psi(Q^k, Q^{k+1}) \underset{M_\psi/\beta\to 0}{\sim} \beta. \tag{4.7}$$

Before we do, we collect some facts about the function F_ψ. Since ψ is a nice approximation function, the function

$$\phi(q) \stackrel{\text{def}}{=} \left(\frac{\psi(q)}{\psi_*(q)}\right)^m = q^n\psi^m(q) \tag{4.8}$$

is nonincreasing, and thus $\phi(Q_2) \le \phi(q) \le \phi(Q_1)$ for all $q \in [Q_1, Q_2]$. Since

$$F_\psi(Q_1, Q_2) = \int_{Q_1}^{Q_2} \phi(q)\frac{\mathrm{d}q}{q},$$

we get

$$\phi(Q_2)\log(Q_2/Q_1) \le F_\psi(Q_1, Q_2) \le \phi(Q_1)\log(Q_2/Q_1). \tag{4.9}$$

In particular, since $\phi(Q_1) \le \phi(1) = M_\psi$ we have

$$F_\psi(Q_1, Q_2) \le M_\psi \log(Q_2/Q_1). \tag{4.10}$$

Lemma 4.2. *Fix $\alpha, \beta > 0$ and a nice approximation function ψ. There exists a sequence of integers $(N_k)_1^\infty$ such that for all $k \ge 1$,*

$$\beta \le F_\psi(Q^k, Q^{k+1}) \le \beta + M_\psi\alpha\log(2), \tag{4.11}$$

where

$$N^k \stackrel{\text{def}}{=} \prod_{j=1}^{k} N_j, \qquad\qquad Q^k \stackrel{\text{def}}{=} (N^k)^\alpha.$$

In fact, the sequence $(N_k)_1^\infty$ can be chosen to consist of powers of 2.

In particular, letting $\alpha = m/(m+n)$ shows that we can choose $(N_k)_1^\infty$ so as to satisfy (4.7).

Remark 4.3. For the case occurring in the proof of Theorem 1.3, namely $\psi(q) = \kappa^{1/m}\psi_*(q)$ for some $\kappa > 0$, the proof of this lemma can be accomplished by letting $(N_k)_1^\infty$ be the constant sequence whose terms are all equal to $2^{\lceil \beta/(\kappa\alpha\log(2))\rceil}$.

Proof of Lemma 4.2. Given $k \geq 0$ and $N_1, \ldots, N_k$, we find $N_{k+1} \geq 2$ such that (4.11) holds. Let $\ell \geq 0$ be the smallest integer for which $G_\psi(N^k, 2^\ell N^k) \geq \beta$, where

$$G_\psi(M_1, M_2) \stackrel{\text{def}}{=} F_\psi(M_1^\alpha, M_2^\alpha).$$

Such an integer exists since

$$\lim_{\ell \to \infty} G_\psi(N^k, 2^\ell N^k) = \int_{Q^k}^\infty q^{n-1} \psi^m(q) \, \mathrm{d}q = \infty,$$

where the last equality holds because ψ is a nice approximation function (cf. Definition 1.5). Since $\beta > 0$, $\ell \geq 1$. So

$$\begin{aligned} G_\psi(N^k, 2^{\ell-1} N^k) &\leq \beta \\ G_\psi(2^{\ell-1} N^k, 2^\ell N^k) &\leq M_\psi \alpha \log(2), \qquad \text{(by (4.10))} \end{aligned}$$

and adding these inequalities yields

$$\beta \leq G_\psi(N^k, 2^\ell N^k) \leq \beta + M_\psi \alpha \log(2).$$

Letting $N_{k+1} = 2^\ell$ finishes the proof. □

4.4. **Homogeneous dynamics.** Although we do not use it directly, it is worth recalling the connection between Diophantine approximation and dynamics in homogeneous spaces first discovered by S. G. Dani [10] and subsequently generalized by D. Y. Kleinbock and G. A. Margulis [22]. To state their result precisely, we need some notation. This notation will end up being convenient for our purposes as well.

Let $d = m + n$, let Ω_d denote the set of unimodular (i.e. covolume one) lattices in $\mathbb{R}^d$, and let $\Lambda_* = \mathbb{Z}^d \in \Omega_d$. Let $G = \mathrm{SL}_d(\mathbb{R})$ and $\Gamma = \mathrm{SL}_d(\mathbb{Z})$. Note that Ω_d is isomorphic to the homogeneous space G/Γ via the mapping $g\Gamma \mapsto g(\Lambda_*)$. Let the function $\Delta : \Omega_d \to \mathbb{R}$ be defined by the formula

$$\Delta(\Lambda) = -\log \min_{\substack{\mathbf{r} \in \Lambda \\ \mathbf{r} \neq \mathbf{0}}} \|\mathbf{r}\|,$$

where

$$\|(\mathbf{p}, \mathbf{q})\| \stackrel{\text{def}}{=} \|\mathbf{p}\| \vee \|\mathbf{q}\| = \|\mathbf{p}\|_\mu \vee \|\mathbf{q}\|_\nu. \tag{4.12}$$

For each $A \in \mathcal{M}$ and $t \in \mathbb{R}$, let

$$u_A \stackrel{\text{def}}{=} \begin{bmatrix} I_m & -A \\ & I_n \end{bmatrix}, \qquad g_t \stackrel{\text{def}}{=} \begin{bmatrix} e^{t/m} I_m & \\ & e^{-t/n} I_n \end{bmatrix},$$

and note that $u_A, g_t \in G$. Here I_k denotes the k-dimensional identity matrix.

Remark. The maps $A \mapsto u_A$ and $t \mapsto g_t$ are exponential homomorphisms (in the sense that their domains are additive groups but their common codomain is the multiplicative group G). Dynamically, the subgroup (u_A) is a multiparameter unipotent flow on Ω_d, while the subgroup (g_t) is a one-parameter diagonal flow on Ω_d.

Let $\delta = D/d = mn/(m+n)$. Then the conjugation relation between the (u_A) flow and the (g_t) flow can be expressed as follows:

$$g_{-\delta t} u_A g_{\delta t} = u_{e^{-t} A}. \tag{4.13}$$

We are now ready to state the Kleinbock–Margulis correspondence principle:

Theorem 4.4 ([22, Theorem 8.5])**.** *Let $\psi : [1, \infty) \to (0, \infty)$ be a nonincreasing continuous function, and let $r_\psi : [t_0, \infty) \to \mathbb{R}$ be defined by the formula*

$$\psi(e^{t/n - r_\psi(t)}) = e^{-t/m - r_\psi(t)} \tag{4.14}$$

as well as the requirement that both sides should be decreasing with respect to t. (The existence and uniqueness of such a function is proven in [22, Lemma 8.3]*). Given $A \in \mathcal{M}$, we have $A \in W(\psi)$ if and only if there exist arbitrarily large $t \geq 0$ such that*

$$\Delta(g_t u_A \Lambda_*) \geq r_\psi(t). \tag{4.15}$$

Equivalently, if for each $t \geq 0$ we let

$$K(\psi, t) = \{\Lambda \in \Omega_d : \Delta(\Lambda) < r_\psi(t)\}, \tag{4.16}$$

then $A \in W(\psi)$ if and only if there exist arbitrarily large $t \geq 0$ such that $g_t u_A \Lambda_* \notin K(\psi, t)$.

Remark. Theorem 4.4 was originally stated for the case where $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are both the max norm, but the only place this is needed in the proof is to ensure that the relation (4.12) holds.

5. Lattice estimates

The goal of this section is to justify the heuristic calculation (3.5), but in a "local" way that will later allow us to deduce estimates on the evaporation rates of certain trees. This "localness" will be represented by the fact that we consider an arbitrary unimodular lattice $\Lambda \in \Omega_d$ rather than just the lattice $\Lambda_* = \mathbb{Z}^d$. Specifically, let us fix:

- a lattice $\Lambda \in \Omega_d$,
- numbers Q_1, Q_2 with $1 \leq Q_1 \leq Q_2$,
- a function $\psi : [Q_1, Q_2] \to (0, \infty)$ such that ψ/ψ_* is nonincreasing.

Then the goal of this section is to prove the following theorem:

Theorem 5.1. *The set*

$$W_{\psi,\Lambda}(Q_1, Q_2) \stackrel{\text{def}}{=} \bigcup_{\substack{\mathbf{r} \in \Lambda \\ Q_1 \leq \|\mathbf{q}\| \leq Q_2}} \Delta_\psi(\mathbf{r})$$

(cf. (3.2)*) satisfies*

$$\lambda_{\mathcal{K}}\left(W_{\psi,\Lambda}(Q_1, Q_2)\right) \underset{\substack{Q_1/\operatorname{Irr}(\Lambda)\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0 \\ F_\psi(Q_1,Q_2)\to 0}}{\sim} \eta_{\mu,\nu} F_\psi(Q_1, Q_2), \tag{5.1}$$

where

$$\operatorname{Irr}(\Lambda) \stackrel{\text{def}}{=} \lambda_1^{-(2d-1)}(\Lambda) \tag{5.2}$$

and ϕ is defined by (4.8)*. Here $\lambda_1(\Lambda)$ denotes the first Minkowski minimum of Λ (with respect to the unit ball), i.e.*

$$\lambda_1(\Lambda) \stackrel{\text{def}}{=} \min_{\mathbf{r} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{r}\|.$$

Remark. The "irregularity" $\operatorname{Irr}(\Lambda)$ should be thought of as a way to measure how far away Λ is from the "standard lattice" Λ_*. The precise formula for $\operatorname{Irr}(\Lambda)$ is used in only two places: the proof of Lemma 5.8 and the proof of Claim 7.2.

The following corollary is what we will actually use in the proof of Theorem 1.8:

Corollary 5.2. *Let $\psi : (0, \infty) \to (0, \infty)$ be a nice approximation function and let the sequence $(N_k)_1^\infty$ be as in Lemma 4.2. Fix $\omega \in E^*$, and let*

$$k = |\omega|, \qquad g_\omega = g_{\delta \log(N^k)} u_{\pi(\omega)}, \qquad \Lambda_\omega = g_\omega \Lambda_*.$$

Then for all $Q^k \leq Q_1 \leq Q_2 \leq Q^{k+1}$,

$$\lambda_{\mathcal{K}}\left(\mathcal{K}_\omega \cap W_\psi(Q_1, Q_2)\right) \underset{\substack{Q_1/(Q^k \operatorname{Irr}(\Lambda_\omega))\to\infty \\ \beta\to 0 \\ M_\psi/\beta\to 0 \\ F_\psi(Q^k,Q_1)/\beta\to 0 \\ F_\psi(Q_2,Q^{k+1})/\beta\to 0}}{\sim} (N^k)^{-D} \eta_{\mu,\nu} \beta.$$

Proof. This follows from applying Theorem 5.1 to the lattice $\Lambda' = \Lambda_\omega$, the numbers $Q_i' = Q_i/Q^k$ $(i = 1, 2)$, and the function $\psi'(q) = (Q^k)^{n/m}\psi(Q^k q)$, and then using (4.11) and the formulas

$$\begin{aligned}\mathcal{K}_\omega \cap W_{\psi,\Lambda_*}(Q_1, Q_2) &= \Phi_\omega\big(\mathcal{K} \cap W_{\psi',\Lambda'}(Q_1', Q_2')\big)\\ F_{\psi'}(Q_1', Q_2') &= F_\psi(Q_1, Q_2) = F_\psi(Q^k, Q^{k+1}) - [F_\psi(Q^k, Q_1) + F_\psi(Q_2, Q^{k+1})]\\ \phi(Q_1) &\le M_\psi.\end{aligned}$$

□

This corollary is the only part of this section which is used in the proof of Theorem 1.8, so Sections 6 and 7 can be read independently of this section.

The proof of Theorem 5.1 will consist of three main steps:

- (Lemma 5.12) estimating the measure of $W_{\psi,\Lambda}(Q_1, Q_2)$ "with multiplicity", i.e. estimating the sum of the measures of the sets $\Delta_\psi(\mathbf{r})$ $(\mathbf{r} \in \Lambda,\ Q_1 \le \|\mathbf{q}\| \le Q_2)$;
- (Lemma 5.13) proving a "quasi-independence on average" result for a certain subset of Λ, the "regular primitive" vectors;
- combining to estimate the measure of $W_{\psi,\Lambda}(Q_1, Q_2)$ "without multiplicity".

Before going into the proof of the main lemmas 5.12 and 5.13, we need some preliminaries. For these preliminaries, we forget about the numbers Q_1, Q_2 and the function ψ, and concentrate only on the lattice $\Lambda \in \Omega_d$.

5.1. **Preliminaries.** We collect here various facts needed in the proof of Theorem 5.1.

Lemma 5.3. *Let Λ be a unimodular lattice in $\mathbb{R}^d$, and let*

$$\Lambda^* \stackrel{\text{def}}{=} \{\mathbf{s} \in \mathbb{R}^d : \mathbf{r} \cdot \mathbf{s} \in \mathbb{Z} \ \ \forall \mathbf{r} \in \Lambda\}$$

be its dual lattice. Let $V \le \mathbb{R}^d$ be a subspace such that $\Lambda \cap V$ is a lattice in V, and let

$$V^\perp \stackrel{\text{def}}{=} \{\mathbf{s} \in \mathbb{R}^d : \mathbf{r} \cdot \mathbf{s} = 0 \ \ \forall \mathbf{r} \in V\}$$

be its dual subspace. Then

$$\operatorname{Covol}(\Lambda \cap V) \approx \operatorname{Covol}(\Lambda^* \cap V^\perp).$$

Here the *covolume* $\operatorname{Covol}(\Gamma)$ of a lattice Γ in a normed vector space $(V, \|\cdot\|)$ is the volume of a fundamental domain[8] of Γ with respect to λ_V, where λ_V denotes Lebesgue measure of V normalized so that the unit ball of $\|\cdot\|$ in V has measure 1.[9]

Proof. Without loss of generality suppose that the norm $\|\cdot\|$ is Euclidean, and we will show that in this case

$$V_{\dim(V)} \operatorname{Covol}(\Lambda \cap V) = V_{\dim(V^\perp)} \operatorname{Covol}(\Lambda^* \cap V^\perp),$$

where V_k denotes the volume of the Euclidean unit ball in $\mathbb{R}^k$.

Indeed, let $\mathbf{r}_1, \dots, \mathbf{r}_d$ be a basis of Λ such that $\mathbf{r}_1, \dots, \mathbf{r}_k$ is a basis of V, where $k = \dim(V)$. Let $\mathbf{s}_1, \dots, \mathbf{s}_d$ denote the dual basis, so that $\mathbf{s}_{k+1}, \dots, \mathbf{s}_d$ is a basis of $\Lambda^* \cap V^\perp$. Since Λ is unimodular,

$$(\mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_k) \wedge (\mathbf{r}_{k+1} \wedge \cdots \wedge \mathbf{r}_d) = 1 = (\mathbf{r}_{k+1} \wedge \cdots \wedge \mathbf{r}_d) \cdot (\mathbf{s}_{k+1} \wedge \cdots \wedge \mathbf{s}_d)$$

(see e.g. [31, §3.2 and §5.5.1]), and thus the equality

$$(\mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_k) \wedge \omega = \omega \cdot (\mathbf{s}_{k+1} \wedge \cdots \wedge \mathbf{s}_d) \qquad \left(\omega \in \bigwedge^{d-k} \mathbb{R}^d\right)$$

[8]We recall that a set $\mathcal{D} \subseteq \mathbb{R}^d$ is called a *fundamental domain* of Λ if $\mathcal{D} + \Lambda = \mathbb{R}^d$ and $\operatorname{Int}(\mathcal{D}) \cap (\mathbf{r} + \operatorname{Int}(\mathcal{D})) = \varnothing$ for all $\mathbf{r} \in \Lambda \setminus \{\mathbf{0}\}$. In this paper, we make the additional assumption that $\mathcal{D} \setminus \operatorname{Int}(\mathcal{D})$ has Lebesgue measure zero, so that $\lambda_{\mathbb{R}^d} = \sum_{\mathbf{r} \in \Lambda} \lambda_{\mathbf{r}+\mathcal{D}}$.

[9]Note that this definition disagrees slightly with the notion of covolume used in §4.4, in which Lebesgue measure was assumed to be normalized so that $\lambda([0,1]^d) = 1$. Hopefully this will not cause any confusion.

can be checked by verifying that it holds on a basis of $\bigwedge^{d-k} \mathbb{R}^d$. So

$$V_{\dim(V)} \operatorname{Covol}(\Lambda \cap V) = \|\mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_k\| = \max_{\|\omega\|=1} \|(\mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_k) \wedge \omega\|$$
$$= \max_{\|\omega\|=1} \|\omega \cdot (\mathbf{s}_{k+1} \wedge \cdots \wedge \mathbf{s}_d)\| = \|\mathbf{s}_{k+1} \wedge \cdots \wedge \mathbf{s}_d\| = V_{\dim(V^\perp)} \operatorname{Covol}(\Lambda^* \cap V^\perp). \qquad \square$$

Another important fact about dual lattices is that if λ_i denotes the ith successive Minkowski minimum, then

$$\lambda_i(\Lambda^*) \approx \lambda_{d+1-i}^{-1}(\Lambda); \tag{5.3}$$

see e.g. [9, Theorem VIII.5.VI].

Notation 5.4. If $\mathcal{D} \subseteq \mathbb{R}^d$ and $f : \mathbb{R}^d \to [0, \infty)$, we let

$$f^{(\mathcal{D})}(\mathbf{r}) = \sup_{\mathbf{x} \in \mathbf{r} + \mathcal{D}} f(\mathbf{x}), \qquad f_{(\mathcal{D})}(\mathbf{r}) = \inf_{\mathbf{x} \in \mathbf{r} + \mathcal{D}} f(\mathbf{x}),$$

and if $\rho > 0$ we let

$$f^{(\rho)} = f^{(B_\rho(\mathbf{0}))}, \qquad f_{(\rho)} = f_{(B_\rho(\mathbf{0}))}.$$

The following fundamental observation allows us to compare sums over the lattice Λ with integrals:

Observation 5.5. Let $f : \mathbb{R}^d \to [0, \infty)$, and let $\mathcal{D}$ be a fundamental domain of a unimodular lattice $\Lambda \in \Omega_d$ (cf. Footnote 8). Then

$$\int_{\mathbb{R}^d} f_{(\mathcal{D})}(\mathbf{r}) \, d\mathbf{r} \leq \sum_{\mathbf{r} \in \Lambda} f(\mathbf{r}) \leq \int_{\mathbb{R}^d} f^{(\mathcal{D})}(\mathbf{r}) \, d\mathbf{r}. \tag{5.4}$$

Furthermore, if $\|\Lambda\|$ denotes the codiameter of Λ, i.e.

$$\|\Lambda\| = \sup_{\mathbf{x} \in \mathbb{R}^d} d(\mathbf{x}, \Lambda),$$

then

$$\int_{\mathbb{R}^d} f_{(\|\Lambda\|)}(\mathbf{r}) \, d\mathbf{r} \leq \sum_{\mathbf{r} \in \Lambda} f(\mathbf{r}) \leq \int_{\mathbb{R}^d} f^{(\|\Lambda\|)}(\mathbf{r}) \, d\mathbf{r}. \tag{5.5}$$

Proof. For each $\mathbf{r} \in \Lambda$,

$$\int_{\mathbf{r} - \mathcal{D}} f_{(\mathcal{D})}(\mathbf{r}') \, d\mathbf{r}' \leq f(\mathbf{r}) \leq \int_{\mathbf{r} - \mathcal{D}} f^{(\mathcal{D})}(\mathbf{r}') \, d\mathbf{r}',$$

and summing over all $\mathbf{r} \in \Lambda$ demonstrates (5.4). Next, let $\mathcal{D}$ be the Dirichlet fundamental domain for Λ centered at $\mathbf{0}$, i.e. the set

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{r}, \mathbf{x}) \geq \|\mathbf{x}\| \ \ \forall \mathbf{r} \in \Lambda\}. \tag{5.6}$$

Then $\sup_{\mathbf{x} \in \mathcal{D}} \|\mathbf{x}\| = \|\Lambda\|$, so $\mathcal{D} \subseteq B_{\|\Lambda\|}(\mathbf{0})$ and thus

$$\int_{\mathbf{r} - \mathcal{D}} f_{(\|\Lambda\|)}(\mathbf{r}') \, d\mathbf{r}' \leq \int_{\mathbf{r} - \mathcal{D}} f_{(\mathcal{D})}(\mathbf{r}') \, d\mathbf{r}' \leq f(\mathbf{r}) \leq \int_{\mathbf{r} - \mathcal{D}} f^{(\mathcal{D})}(\mathbf{r}') \, d\mathbf{r}', \leq \int_{\mathbf{r} - \mathcal{D}} f^{(\|\Lambda\|)}(\mathbf{r}') \, d\mathbf{r}',$$

and summing over all $\mathbf{r} \in \Lambda$ demonstrates (5.5). $\square$

When $f(\mathbf{r})$ depends only on $\|\mathbf{r}\|$, a spherical coordinates calculation allows us to simplify the integrals of Observation 5.5, yielding the following upper bound:

Corollary 5.6. *Fix $\|\Lambda\| \leq Q_1 \leq Q_2$, and let $f : [Q_1, Q_2] \to [0, \infty)$ be a decreasing function. Then*

$$\sum_{\substack{\mathbf{r} \in \Lambda \\ Q_1 \leq \|\mathbf{r}\| \leq Q_2}} f(\|\mathbf{r}\|) \lessapprox Q_1^d f(Q_1) + \int_{Q_1}^{Q_2} f(r) r^{d-1} \, dr.$$

Proof. Write

$$F(q) = \begin{cases} f(Q_1 \vee q) & \text{if } q \leq Q_2 \\ 0 & \text{if } q > Q_2 \end{cases}.$$

Then

$$\begin{aligned} \sum_{\substack{\mathbf{r}\in\Lambda \\ Q_1 \leq \|\mathbf{r}\| \leq Q_2}} f(\|\mathbf{r}\|) &\leq \int_{\mathbb{R}^d} F(\|\mathbf{r}\| - \|\Lambda\|) \, \mathrm{d}\mathbf{r} && \text{(by (5.5))} \\ &\lessapprox \int_0^\infty r^{d-1} F(r - \|\Lambda\|) \, \mathrm{d}r && \text{(Lemma 3.1)} \\ &= \int_{-\|\Lambda\|}^\infty (r + \|\Lambda\|)^{d-1} F(r) \, \mathrm{d}r \\ &= \frac{1}{d}(Q_1 + \|\Lambda\|)^d f(Q_1) + \int_{Q_1}^{Q_2} (r + \|\Lambda\|)^{d-1} f(r) \, \mathrm{d}r \\ &\approx Q_1^d f(Q_1) + \int_{Q_1}^{Q_2} r^{d-1} f(r) \, \mathrm{d}r. && \text{(since } \|\Lambda\| \leq Q_1) \end{aligned}$$

□

5.2. **Regular lattice vectors.** The goal of this subsection is to find a "large" subset of Λ along which quasi-independence holds. As noted in Section 3, we will need to restrict at least to the set of primitive vectors. But actually we may need to restrict further; some primitive vectors may still be "bad" in the sense that including them in our set would cause problems for quasi-independence. It turns out that the right way to measure how "bad" a vector is in this sense is given by the following definition:

Definition 5.7. The *irregularity* of a vector $\mathbf{r} \in \Lambda \setminus \{\mathbf{0}\}$ (with respect to Λ) is the number

$$\mathrm{Irr}(\mathbf{r}) \stackrel{\mathrm{def}}{=} \frac{\|\mathbf{r}\|}{\left(\lambda_1(\Lambda^* \cap \mathbf{r}^\perp)\right)^{d-1}}. \tag{5.7}$$

As before, λ_1 denotes the first Minkowski minimum and Λ^* denotes the dual lattice. Note that by Lemma 5.3, we have

$$\|\mathbf{r}\| \geq \mathrm{Covol}(\Lambda \cap \mathbb{R}\mathbf{r}) \approx \mathrm{Covol}(\Lambda^* \cap \mathbf{r}^\perp),$$

and thus Minkowski's theorem implies that the irregularity of $\mathbf{r}$ is bounded from below (by a constant depending on $\|\cdot\|$). Intuitively, the irregularity of $\mathbf{r}$ tells us about the "worst hyperplane" that $\mathbf{r}$ is contained in, where "worst" means "smallest covolume". The advantage of vectors with low irregularity is that they can be used to construct fundamental domains of Λ which are approximately cylindrical; cf. the proof of Lemma 5.13 below. The following lemma shows that there is a sense in which the irregularity is bounded from above "on average":

Lemma 5.8 (Almost all lattice vectors are regular)**.** *If $d \geq 2$, then*

$$\lim_{K\to\infty} \sup_{\Lambda\in\Omega_d} \sup_{Q \geq K \,\mathrm{Irr}(\Lambda)} \frac{\#\{\mathbf{r} \in \Lambda : \mathrm{Irr}(\mathbf{r}) \geq K, \|\mathbf{r}\| \leq Q\}}{Q^d} = 0,$$

where $\mathrm{Irr}(\Lambda) \stackrel{\mathrm{def}}{=} \lambda_1^{-(2d-1)}(\Lambda)$ *as in* (5.2)*.*

Proof. Fix $K \geq 1$, $\Lambda \in \Omega_d$, and $Q \geq K \,\mathrm{Irr}(\Lambda)$. Fix $\mathbf{r} \in \Lambda$ with $\mathrm{Irr}(\mathbf{r}) \geq K$ and $\|\mathbf{r}\| \leq Q$. Then

$$\lambda_1(\Lambda^* \cap \mathbf{r}^\perp) = \left(\frac{\|\mathbf{r}\|}{\mathrm{Irr}(\mathbf{r})}\right)^{1/(d-1)} \leq S \stackrel{\mathrm{def}}{=} \left(\frac{Q}{K}\right)^{1/(d-1)},$$

so there exists $\mathbf{s} \in \Lambda^*_{\mathrm{pr}} \cap \mathbf{r}^\perp$ such that $\|\mathbf{s}\| \leq S$, where Λ^*_{pr} denotes the set of primitive vectors of Λ^*. Equivalently,

$$\mathbf{r} \in \bigcup_{\substack{\mathbf{s}\in\Lambda^*_{\mathrm{pr}} \\ \|\mathbf{s}\| \leq S}} \Lambda \cap \mathbf{s}^\perp.$$

It follows that

$$\#\{\mathbf{r}\in\Lambda : \mathrm{Irr}(\mathbf{r})\geq K,\ \|\mathbf{r}\|\leq Q\} \leq \sum_{\substack{\mathbf{s}\in\Lambda^*_{\mathrm{pr}}\\ \|\mathbf{s}\|\leq S}} \#(\Lambda\cap\mathbf{s}^\perp\cap B_Q(\mathbf{0})). \tag{5.8}$$

Next, fix $\mathbf{s}\in\Lambda^*_{\mathrm{pr}}$ with $\|\mathbf{s}\|\leq S$, and let $\Gamma=\Lambda\cap\mathbf{s}^\perp$. Then

$$\#(\Gamma\cap B_Q(\mathbf{0}))\cdot\mathrm{Covol}(\Gamma) = \lambda_{\mathbf{s}^\perp}\left(\bigcup_{\mathbf{r}\in\Gamma\cap B_Q(\mathbf{0})}\mathbf{r}+\mathcal{D}_\Gamma\right) \leq \lambda_{\mathbf{s}^\perp}\Big(B(\mathbf{0},Q+\|\Gamma\|)\Big) = (Q+\|\Gamma\|)^{d-1}, \tag{5.9}$$

where $\mathcal{D}_\Gamma$ denotes the Dirichlet domain of Γ centered at 0, and $\lambda_{\mathbf{s}^\perp}$ denotes Lebesgue measure of $\mathbf{s}^\perp$, normalized so that $\lambda_{\mathbf{s}^\perp}(\mathbf{s}^\perp\cap B(\mathbf{0},1))=1$. Now by Lemma 5.3 we have

$$\mathrm{Covol}(\Gamma)\approx\|\mathbf{s}\| \tag{5.10}$$

and thus by Minkowski's Second Theorem [9, VIII.4.V],

$$\begin{aligned}\|\Gamma\|\approx\lambda_{d-1}(\Gamma) &\lessapprox \frac{\mathrm{Covol}(\Gamma)}{\lambda_1^{d-2}(\Gamma)} \leq \frac{\mathrm{Covol}(\Gamma)}{\lambda_1^{d-2}(\Lambda)} \approx \frac{\|\mathbf{s}\|}{\mathrm{Irr}^{-(d-2)/(2d-1)}(\Lambda)}\\ &\leq \frac{(Q/K)^{1/(d-1)}}{(Q/K)^{-(d-2)/(2d-1)}} \leq Q/K \leq Q.\end{aligned} \tag{5.11}$$

Plugging in (5.10) and (5.11) into (5.9) gives

$$\#(\Gamma\cap B_Q(\mathbf{0}))\lessapprox\frac{Q^{d-1}}{\|\mathbf{s}\|}.$$

Plugging this into (5.8), we get

$$\begin{aligned}&Q^{-d}\#\{\mathbf{r}\in\Lambda:\mathrm{Irr}(\mathbf{r})\geq K,\|\mathbf{r}\|\leq Q\}\\ &\lessapprox \sum_{\substack{\mathbf{s}\in\Lambda^*_{\mathrm{pr}}\\ \|\mathbf{s}\|\leq S}}\frac{1}{Q\|\mathbf{s}\|}\\ &\leq \frac{\#\{\mathbf{s}\in\Lambda^*:\|\mathbf{s}\|\leq\|\Lambda^*\|\}}{Q\lambda_1(\Lambda^*)} + \sum_{\substack{\mathbf{s}\in\Lambda^*\\ \|\Lambda^*\|\leq\|\mathbf{s}\|\leq S}}\frac{1}{Q\|\mathbf{s}\|}\\ &\lessapprox \frac{\|\Lambda^*\|^d}{Q\lambda_1(\Lambda^*)} + \int_0^S\frac{r^{d-2}}{Q}\,\mathrm{d}r && \text{(by Corollary 5.6)}\\ &\lessapprox \frac{\|\Lambda^*\|^{2d-1}}{Q} + \frac{r^{d-1}}{Q}\Big|_{r=0}^{S} && \text{(since } \lambda_1(\Lambda^*)\gtrapprox\|\Lambda^*\|^{-(d-1)} \text{ and } d\geq 2)\\ &\approx \frac{\mathrm{Irr}(\Lambda)}{Q}+\frac{1}{K}\leq\frac{2}{K}. && \text{(since } \|\Lambda^*\|\approx\lambda_1^{-1}(\Lambda) \text{ and } Q\geq K\,\mathrm{Irr}(\Lambda))\end{aligned}$$

Taking the appropriate suprema and letting $K\to\infty$ completes the proof. Note that the asymptotic inequality $\lambda_1(\Lambda^*)\gtrapprox\|\Lambda^*\|^{-(d-1)}$ follows from combining Minkowski's second theorem on successive minima with the asymptotic $\|\Lambda^*\|\approx\lambda_d(\Lambda^*)$, while the asymptotic $\|\Lambda^*\|\approx\lambda_1^{-1}(\Lambda)$ follows from combining (5.3) with the same asymptotic. □

For each $K\geq 1$, write

$$\varepsilon_K = \sup_{\Lambda\in\Omega_d}\sup_{Q\geq K\,\mathrm{Irr}(\Lambda)}\frac{\#\{\mathbf{r}\in\Lambda:\mathrm{Irr}(\mathbf{r})\geq K,\|\mathbf{r}\|\leq Q\}}{Q^d}. \tag{5.12}$$

Then Lemma 5.8 says that if $d\geq 2$, then $\varepsilon_K\to 0$ as $K\to\infty$. The notation ε_K allows us to write down the following analogue of Corollary 5.6 for sums over lattice vectors with large irrationality:

Observation 5.9. Fix $K \geq 1$ and $\Lambda \in \Omega_d$, and let $f : [0,\infty) \to [0,\infty)$ be a decreasing function which is constant on $[0, K \operatorname{Irr}(\Lambda)]$. Then

$$\sum_{\substack{\mathbf{r} \in \Lambda \\ \operatorname{Irr}(\mathbf{r}) \geq K}} f(\|\mathbf{r}\|) \leq d\varepsilon_K \int_0^\infty r^{d-1} f(r) \, \mathrm{d}r.$$

Proof. It suffices to check this for functions of the form $f(q) = \big[q \leq Q\big]$ $(Q \geq K \operatorname{Irr}(\Lambda))$, and for these functions the inequality follows from the definition of ε_K. □

5.3. **Proof of Theorem 5.1.** In the remainder of this section, we fix Q_1, Q_2, ψ as in the start of this section.

The following lemma allows us to simplify the notation somewhat:

Lemma 5.10 (Good lattice vectors satisfy $\|(\mathbf{p},\mathbf{q})\| = \|\mathbf{q}\|$)**.** *Suppose that $\psi(Q_1) \leq 1/2$ and $\|\mathcal{K}\| \leq 1/2$. Then for all $(\mathbf{p},\mathbf{q}) \in \mathbb{R}^d$ with $Q_1 \leq \|\mathbf{q}\| \leq Q_2$ such that $\mathcal{K} \cap \Delta_\psi(\mathbf{p},\mathbf{q}) \neq \varnothing$, we have*

$$\|(\mathbf{p},\mathbf{q})\| = \|\mathbf{q}\|. \tag{5.13}$$

Proof. If $A \in \mathcal{K} \cap \Delta_\psi(\mathbf{p},\mathbf{q})$, then

$$\|\mathbf{p}\| \leq \|A\mathbf{q} - \mathbf{p}\| + \|A\| \cdot \|\mathbf{q}\| \leq \psi(\|\mathbf{q}\|) + \|\mathcal{K}\| \cdot \|\mathbf{q}\|.$$

Since $\|\mathbf{q}\| \geq Q_1 \geq 1$ and ψ is nonincreasing, we have

$$\psi(\|\mathbf{q}\|) \leq \psi(Q_1) \leq \psi(Q_1)\|\mathbf{q}\|.$$

Thus $\|\mathbf{p}\| \leq (\psi(Q_1) + \|\mathcal{K}\|)\|\mathbf{q}\| \leq \|\mathbf{q}\|$, which implies (5.13). □

To make things easier, in this section we make the standing assumption that the hypotheses of Lemma 5.10 are satisfied:

Assumption 5.11. $\psi(Q_1) \leq 1/2$ and $\|\mathcal{K}\| \leq 1/2$.

In fact, Assumption 5.11 may be made without loss of generality: first of all, by replacing $\|\cdot\|_\mu$ by $\gamma\|\cdot\|_\mu$ for an appropriate $\gamma > 0$, we can assume that $\|\mathcal{K}\| \leq 1/2$; second of all, since in Theorem 5.1 we are trying to prove an asymptotic which includes the convergence assumption $\phi(Q_1) \to 0$, it follows that we can assume $\phi(Q_1)$ is as small as we want and in particular that $\psi(Q_1) \leq 1/2$.

For convenience of notation we introduce the function

$$\widetilde{\psi}(q) = \psi(Q_1 \vee q)$$

and we note that $\widetilde{\psi}$ is still nonincreasing (although the function $\widetilde{\phi}(q) = q^n \widetilde{\psi}^m(q)$ is not nonincreasing). We also let $\widetilde{\Psi}(q) = \Psi(Q_1 \vee q)$.

In the following lemma, Λ_{pr}^+ denotes a subset of Λ_{pr} (the set of primitive vectors of Λ) such that for all $\mathbf{r} \in \Lambda_{\mathrm{pr}}$, we have $\#(\{\mathbf{r}, -\mathbf{r}\} \cap \Lambda_{\mathrm{pr}}^+) = 1$.

Lemma 5.12 (Estimates of the measure of $W_{\psi,\Lambda}(Q_1,Q_2)$, with multiplicity)**.** *We have*

$$\sum_{\substack{\mathbf{r} \in \Lambda_{\mathrm{pr}}^+ \\ Q_1 \leq \|\mathbf{q}\| \leq Q_2}} \lambda_{\mathcal{K}}\big(\Delta_\psi(\mathbf{r})\big) \underset{\substack{Q_1/\operatorname{Irr}(\Lambda) \to \infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2) \to 0}}{\sim} \eta_{\mu,\nu} F_\psi(Q_1,Q_2). \tag{5.14}$$

Moreover, for all $K \geq 1$ such that $Q_1 \geq K \operatorname{Irr}(\Lambda)$ we have

$$\sum_{\substack{\mathbf{r} \in \Lambda \\ Q_1 \leq \|\mathbf{q}\| \leq Q_2 \\ \operatorname{Irr}(\mathbf{r}) \geq K}} \lambda_{\mathcal{K}}\big(\Delta_\psi(\mathbf{r})\big) \underset{\substack{K \to \infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2) \to 0}}{\ll} F_\psi(Q_1,Q_2) \tag{5.15}$$

and thus

$$\sum_{\substack{\mathbf{r} \in \Lambda_{\mathrm{pr}}^+ \\ Q_1 \leq \|\mathbf{q}\| \leq Q_2 \\ \operatorname{Irr}(\mathbf{r}) \leq K}} \lambda_{\mathcal{K}}\big(\Delta_\psi(\mathbf{r})\big) \underset{\substack{K \to \infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2) \to 0}}{\sim} \eta_{\mu,\nu} F_\psi(Q_1,Q_2). \tag{5.16}$$

Proof.

Step 1: Initial calculation. Consider the function $\eta : \mathbb{R}^d \setminus (\mathbb{R}^m \times \{\mathbf{0}\}) \to [0,\infty)$ defined by the formula

$$\eta(\mathbf{p},\mathbf{q}) \stackrel{\text{def}}{=} \prod_{i=1}^{m} \left(\mathop{\ast}_{\substack{j=1 \\ q_j \neq 0}}^{n} f_{q_j} \right)(p_i),$$

where

$$f_q(p) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{q}\big[p \in [0,q]\big] & \text{if } q > 0 \\ \frac{1}{|q|}\big[p \in [q,0]\big] & \text{if } q < 0 \end{cases} = \frac{\mathrm{d}[t \mapsto qt]_* \, \lambda_{[0,1]}}{\mathrm{d}\, \lambda_{\mathbb{R}}}$$

and $\ast$ denotes convolution. Here $f_*\mu$ denotes the pushforward of a measure μ under a function f, and $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ denotes the Radon–Nikodym derivative of μ with respect to ν. The significance of η is that for each $\mathbf{q} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\eta(\cdot,\mathbf{q})$ is a nonnegative Riemann integrable function which satisfies

$$\int_S \eta(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p} = \lambda_{\mathcal{K}}(\{A \in \mathcal{K} : A\mathbf{q} \in S\}) \ \ \forall S \subseteq \mathbb{R}^m. \tag{5.17}$$

Indeed, we have

$$\begin{aligned}
\eta \, \lambda_{\mathbb{R}^m} &= \prod_{i=1}^{m} \left(\mathop{\ast}_{\substack{j=1 \\ q_j \neq 0}}^{n} f_{q_j} \right) \lambda_{\mathbb{R}} \\
&= \prod_{i=1}^{m} \Big[(x_1,\ldots,x_n) \mapsto \Sigma_1^n x_j \Big]_* \prod_{j=1}^{n} \begin{cases} f_{q_j} \, \lambda_{\mathbb{R}} & q_j \neq 0 \\ \delta_0 & q_j = 0 \end{cases} \\
&= \prod_{i=1}^{m} \Big[(x_1,\ldots,x_n) \mapsto \Sigma_1^n x_j \Big]_* \prod_{j=1}^{n} \big[t \mapsto q_j t \big]_* \, \lambda_{[0,1]} \\
&= \prod_{i=1}^{m} \Big[(t_1,\ldots,t_n) \mapsto \Sigma_1^n q_j t_j \Big]_* \, \lambda_{[0,1]^n} \\
&= \Big[(t_{11},\ldots,t_{mn}) \mapsto (\Sigma_1^n t_{1j} q_j, \ldots, \Sigma_1^n t_{mj} q_j) \Big]_* \, \lambda_{[0,1]^{mn}} \\
&= \Big[A \mapsto A\mathbf{q} \Big]_* \, \lambda_{\mathcal{K}} \, .
\end{aligned}$$

Also, note that

$$\eta(\gamma \mathbf{r}) = \gamma^{-m} \eta(\mathbf{r}) \ \ \forall \gamma > 0 \ \ \forall \mathbf{r} \in \mathbb{R}^d \setminus (\mathbb{R}^m \times \{\mathbf{0}\}). \tag{5.18}$$

Now for $\rho > 0$ let

$$\tau^{(\rho)} = \max_{\|\mathbf{q}\|=1} \int_{\mathbb{R}^m} \eta^{(\rho)}(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p}, \qquad\qquad \tau_{(\rho)} = \min_{\|\mathbf{q}\|=1} \int_{\mathbb{R}^m} \eta_{(\rho)}(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p},$$

where $\eta^{(\rho)}$, $\eta_{(\rho)}$ are as in Notation 5.4. Note that $\tau^{(\rho)} \searrow 1$ and $\tau_{(\rho)} \nearrow 1$ as $\rho \to 0$. Also, by (5.18) we have

$$\begin{aligned}
&\int_{\mathbb{R}^m} \eta_{(\rho)}(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p} \geq \tau_{(\rho/\|\mathbf{q}\|)} \\
&\int_{\mathbb{R}^m} \eta^{(\rho)}(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p} \leq \tau^{(\rho/\|\mathbf{q}\|)}
\end{aligned} \qquad \forall \rho > 0 \ \ \forall \mathbf{q} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{5.19}$$

Next, for all $\mathbf{r} \in \Lambda_*$, by (5.17) we have

$$\lambda_{\mathcal{K}}\big(\Delta_\psi(\mathbf{r})\big) = \lambda_{\mathcal{K}}(\{A \in \mathcal{K} : A\mathbf{q} \in B_{\psi(\|\mathbf{q}\|)}(\mathbf{p})\}) = \int_{B_{\psi(\|\mathbf{q}\|)}(\mathbf{p})} \eta(\mathbf{p}',\mathbf{q}) \, \mathrm{d}\mathbf{p}' \tag{5.20}$$

and thus

$$\begin{aligned}
&\sum_{\substack{\mathbf{r}\in\Lambda\\ Q_1\le\|\mathbf{q}\|\le Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right)\\
&\le \sum_{\substack{\mathbf{r}\in\Lambda\\ Q_1\le\|\mathbf{q}\|\le Q_2}} \eta^{(\psi(Q_1))}(\mathbf{r}) V_\mu \psi^m(\|\mathbf{q}\|)\\
&\le V_\mu \int_{\mathbb{R}^d} \eta^{(\psi(Q_1)+\|\Lambda\|)}(\mathbf{r})\widetilde{\psi}^m(\|\mathbf{q}\|-\|\Lambda\|)\big[Q_1-\|\Lambda\|\le\|\mathbf{q}\|\le Q_2+\|\Lambda\|\big]\,\mathrm{d}\mathbf{r}\\
&\text{(by (5.5))}\\
&\le V_\mu \tau^{\left(\frac{\psi(Q_1)+\|\Lambda\|}{Q_1-\|\Lambda\|}\right)} \int_{\mathbb{R}^n} \widetilde{\psi}^m(\|\mathbf{q}\|-\|\Lambda\|)\big[Q_1-\|\Lambda\|\le\|\mathbf{q}\|\le Q_2+\|\Lambda\|\big]\,\mathrm{d}\mathbf{q}\\
&\text{(by (5.19))}\\
&= V_\mu \tau^{\left(\frac{\psi(Q_1)+\|\Lambda\|}{Q_1-\|\Lambda\|}\right)} n V_\nu \int_{Q_1-\|\Lambda\|}^{Q_2+\|\Lambda\|} \widetilde{\psi}^m(q-\|\Lambda\|) q^{n-1}\,\mathrm{d}q\\
&\text{(Lemma 3.1)}\\
&\le n V_\mu V_\nu \tau^{\left(\frac{\psi(Q_1)+\|\Lambda\|}{Q_1-\|\Lambda\|}\right)} \left(\frac{Q_1-\|\Lambda\|}{Q_1-2\|\Lambda\|}\right)^{n-1} \int_{Q_1-\|\Lambda\|}^{Q_2+\|\Lambda\|} \widetilde{\psi}^m(q-\|\Lambda\|)(q-\|\Lambda\|)^{n-1}\,\mathrm{d}q\\
&\le n V_\mu V_\nu \tau^{\left(\frac{\psi(Q_1)+\|\Lambda\|}{Q_1-\|\Lambda\|}\right)} \left(\frac{Q_1-\|\Lambda\|}{Q_1-2\|\Lambda\|}\right)^{n-1} \big[F_\psi(Q_1,Q_2)+\phi(Q_1)\big].
\end{aligned}$$

A similar argument (using (4.9)) shows that

$$\begin{aligned}
&\sum_{\substack{\mathbf{r}\in\Lambda\\ Q_1\le\|\mathbf{q}\|\le Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right)\\
&\ge n V_\mu V_\nu \tau^{\left(\frac{\psi(Q_1)+\|\Lambda\|}{Q_1+\|\Lambda\|}\right)} \left(\frac{Q_1+\|\Lambda\|}{Q_1+2\|\Lambda\|}\right)^{n-1} \left[F_\psi(Q_1,Q_2)-\phi(Q_1)\log\left(\frac{Q_1+2\|\Lambda\|}{Q_1}\right)\right].
\end{aligned}$$

Thus

$$\sum_{\substack{\mathbf{r}\in\Lambda\\ Q_1\le\|\mathbf{q}\|\le Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right) \underset{\substack{Q_1/\|\Lambda\|\to\infty\\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim} n V_\mu V_\nu F_\psi(Q_1,Q_2). \tag{5.21}$$

To get (5.14), we need to change (5.21) to a formula about primitive vectors. Before doing this, we first demonstrate (5.15).

Step 2: Ignoring irregular lattice vectors. Fix $\mathbf{r}\in\mathbb{R}^d$. Since $\eta(\mathbf{p}',\mathbf{q})\lessapprox\|\mathbf{q}\|^{-m}$ for all $\mathbf{p}'\in\mathbb{R}^m$, (5.20) implies that

$$\lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right)\lessapprox\|\mathbf{r}\|^{-m}\psi^m(\|\mathbf{r}\|), \tag{5.22}$$

where we have used Lemma 5.10 to get $\|\mathbf{q}\| = \|\mathbf{r}\|$ in the case where the left-hand side is nonzero. Now fix $K \geq 1$ such that $Q_1 \geq K \operatorname{Irr}(\Lambda)$, and let ε_K be as in (5.12). Then

$$\begin{aligned}
\sum_{\substack{\mathbf{r}\in\Lambda \\ Q_1\leq\|\mathbf{r}\|\leq Q_2 \\ \operatorname{Irr}(\mathbf{r})\geq K}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right) &\lessapprox \sum_{\substack{\mathbf{r}\in\Lambda \\ Q_1\leq\|\mathbf{r}\|\leq Q_2 \\ \operatorname{Irr}(\mathbf{r})\geq K}} \Psi^m(\|\mathbf{r}\|) && \text{(by (5.22))} \\
&\leq \sum_{\substack{\mathbf{r}\in\Lambda \\ \|\mathbf{r}\|\leq Q_2 \\ \operatorname{Irr}(\mathbf{r})\geq K}} \widetilde{\Psi}^m(\|\mathbf{r}\|) \\
&\leq \varepsilon_K d \int_0^{Q_2} q^{d-1}\widetilde{\Psi}^m(q)\,\mathrm{d}q && \text{(Observation 5.9)} \\
&= \varepsilon_K[dF_\psi(Q_1,Q_2) + \phi(Q_1)] \\
&\ll_{\substack{K\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}} F_\psi(Q_1,Q_2),
\end{aligned}$$

which completes the proof of (5.15).

Step 3: Accounting for repeated entries. Fix $k \in \mathbb{N}$ such that $k \leq Q_1$. The argument of Step 1 goes through just as well for the function $\psi_k : [Q_1/k, Q_2/k] \to [0,\infty)$ defined by the formula

$$\psi_k(q) = \psi(kq)/k,$$

which satisfies

$$\begin{aligned}
\Delta_{\psi'}(\mathbf{r}) &= \Delta_\psi(k\mathbf{r}) \ \ \forall \mathbf{r} \in \Lambda, & \phi_k(Q_1/k) &= k^{-d}\phi(Q_1), \\
F_{\psi_k}(Q_1/k, Q_2/k) &= k^{-d}F_\psi(Q_1,Q_2).
\end{aligned}$$

Thus, replacing ψ by ψ_k in (5.21) gives

$$\sum_{\substack{\mathbf{r}\in k\Lambda \\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right) \underset{\substack{Q_1/(k\|\Lambda\|)\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim} nV_\mu V_\nu k^{-d} F_\psi(Q_1,Q_2).$$

Now let

$$\begin{aligned}
A_k &\stackrel{\text{def}}{=} \frac{1}{nV_\mu V_\nu F_\psi(Q_1,Q_2)} \sum_{\substack{\mathbf{r}\in k\Lambda_{\mathrm{pr}} \\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right), \\
B_k &\stackrel{\text{def}}{=} \frac{1}{nV_\mu V_\nu F_\psi(Q_1,Q_2)} \sum_{\substack{\mathbf{r}\in k\Lambda \\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right) = \sum_{\ell\in\mathbb{N}} A_{k\ell},
\end{aligned}$$

and note that

$$B_k \underset{\substack{Q_1/\|\Lambda\|\to\infty\ (k) \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim_+} k^{-d}. \tag{5.23}$$

(To switch from a multiplicative asymptotic to an additive one, we have used the fact that $k^{-d} \leq 1$.) Recall that for all $\mathbf{r} \in \Lambda$, $\operatorname{Irr}(\mathbf{r})$ is bounded below by a constant depending only on m, n, μ, ν, say $\operatorname{Irr}(\mathbf{r}) \geq \delta_0 \ \ \forall \mathbf{r} \in \Lambda$. Now fix $K \in \mathbb{N}$ such that $Q_1 \geq (K/\delta_0)\operatorname{Irr}(\Lambda)$. Since $\operatorname{Irr}(k\mathbf{r}) = k\operatorname{Irr}(\mathbf{r}) \geq k\delta_0$ for all $\mathbf{r} \in \Lambda_{\mathrm{pr}}$ and $k \in \mathbb{N}$, by (5.15) we have

$$\sum_{|k|\geq K} A_k \lessapprox \frac{1}{F_\psi(Q_1,Q_2)} \sum_{\substack{\mathbf{r}\in\Lambda \\ Q_1\leq\|\mathbf{q}\|\leq Q_2 \\ \operatorname{Irr}(\mathbf{r})\geq K/\delta_0}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\right) \xrightarrow[\substack{K\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}]{} 0. \tag{5.24}$$

Now let P be the set of all primes strictly less than K, and let S be the set of all square-free integers whose prime factors are all in P (or equivalently, the set of products of distinct elements of P). Let μ denote the Möbius function.[10] Then

$$\begin{aligned} A_1 &\underset{\substack{K\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim_+} \sum_{\substack{k\in\mathbb{N} \\ k\notin \ell\mathbb{N}\ \forall \ell\in P}} A_k && \text{(by (5.24))} \\ &= \sum_{k\in S} \mu(k) B_k \\ &\underset{\substack{Q_1/\|\Lambda\|\to\infty\ (K) \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim_+} \sum_{k\in S} \mu(k) k^{-d} && \text{(by (5.23))} \\ &&& \text{(since } S \text{ is finite and depends only on } K\text{)} \\ &= \prod_{k\in P} \frac{1}{1-k^{-d}} \\ &\underset{K\to\infty}{\sim} \frac{1}{\zeta(d)}. \end{aligned}$$

Standard quantifier logic then gives

$$A_1 \underset{\substack{Q_1/\operatorname{Irr}(\Lambda)\to\infty \\ \phi(Q_1)/F_\psi(Q_1,Q_2)\to 0}}{\sim} \frac{1}{\zeta(d)}$$

and rearranging gives (5.14). Finally, combining (5.14) and (5.15) and using the assumption $Q_1 \geq K \operatorname{Irr}(\Lambda)$ to get $Q_1/\operatorname{Irr}(\Lambda) \xrightarrow[K\to\infty]{} \infty$ gives (5.16). □

Lemma 5.13 (Quasi-independence on average)**.** *Suppose that $Q_1 \geq \|\Lambda\|$. Then for all $\mathbf{r} = (\mathbf{p},\mathbf{q}) \in \Lambda_{\mathrm{pr}}$ such that $Q_1 \leq \|\mathbf{q}\| \leq Q_2$, we have*

$$\sum_{\substack{\mathbf{r}'\in\Lambda\setminus\mathbb{Z}\mathbf{r} \\ \|\mathbf{q}\|\leq\|\mathbf{q}'\|\leq Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}')\right) \lessapprox_{\operatorname{Irr}(\mathbf{r})} \Psi^m(\|\mathbf{r}\|) \overline{F}_\psi(Q_1,Q_2), \tag{5.25}$$

where

$$\overline{F}_\psi(Q_1,Q_2) \stackrel{\text{def}}{=} F_\psi(Q_1,Q_2) + \phi(Q_1).$$

Proof. We can assume that $\mathcal{K} \cap \Delta_\psi(\mathbf{r}) \neq \varnothing$, as otherwise (5.25) holds trivially. Thus by Lemma 5.10, we get $\|\mathbf{q}\| = \|\mathbf{r}\|$. Also, since $F_\psi(\|\mathbf{q}\|,Q_2) + \phi(\|\mathbf{q}\|) \leq F_\psi(Q_1,Q_2) + \phi(Q_1)$, we may without loss of generality assume that $Q_1 = \|\mathbf{q}\|$.

Let $\pi : \mathbb{R}^d \to \mathbf{r}^\perp$ be the orthogonal projection map, so that $\pi(\mathbf{r}) = 0$. Then $\pi(\Lambda)$ is a lattice in $\mathbf{r}^\perp$. Let $\mathcal{D}_1 \subseteq \mathbf{r}^\perp$ be the Dirichlet fundamental domain for $\pi(\Lambda)$ centered at $\mathbf{0}$ (cf. (5.6)). Let $\mathcal{D}_2 = [-1/2,1/2]\mathbf{r}$, and let $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$. Then $\mathcal{D}$ is a fundamental domain for Λ (though not necessarily a Dirichlet domain). Note that

$$\|\mathcal{D}\| \stackrel{\text{def}}{=} \max_{\mathbf{x}\in\mathcal{D}} \|\mathbf{x}\| \leq \|\pi(\Lambda)\| + \|\mathbf{r}\| \leq \|\Lambda\| + \|\mathbf{r}\| \leq \|\mathbf{q}\| + \|\mathbf{r}\| = 2Q_1. \tag{5.26}$$

Claim 5.14. *For all $\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r}$ and $\mathbf{r}'' \in \mathbf{r}' + \mathcal{D}$, we have*

$$d(\mathbf{r}', \mathbb{R}\mathbf{r}) \gtrapprox_{\operatorname{Irr}(\mathbf{r})} d(\mathbf{r}'', \mathbb{R}\mathbf{r}).$$

Proof. Fix $\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r}$ and $\mathbf{r}'' \in \mathbf{r}' + \mathcal{D}$. Then $\pi(\mathbf{r}') \in \pi(\Lambda) \setminus \{\mathbf{0}\}$ and $\pi(\mathbf{r}'') \in \mathbf{r}' + \mathcal{D}_1$. In particular, we have

$$d(\mathbf{r}', \mathbb{R}\mathbf{r}) = \|\pi(\mathbf{r}')\| \geq \lambda_1(\pi(\Lambda))$$

[10] For convenience, we recall that this function is given by the formula $\mu(p_1 \cdots p_k) = (-1)^k$ if $p_1, \ldots, p_k$ are distinct primes, and $\mu(n) = 0$ if n is not square-free.

and thus

$$d(\mathbf{r}'', \mathbb{R}\mathbf{r}) \le d(\mathbf{r}', \mathbb{R}\mathbf{r}) + \|\pi(\Lambda)\| \le \left(1 + \frac{\|\pi(\Lambda)\|}{\lambda_1(\pi(\Lambda))}\right) d(\mathbf{r}', \mathbb{R}\mathbf{r}).$$

But since $\pi(\Lambda)^* = \Lambda^* \cap \mathbf{r}^\perp$, we have $\|\pi(\Lambda)\| \approx \lambda_1^{-1}(\Lambda^* \cap \mathbf{r}^\perp)$ and $\lambda_1(\pi(\Lambda)) \approx \|\Lambda^* \cap \mathbf{r}^\perp\|^{-1} \approx \lambda_{d-1}^{-1}(\Lambda^* \cap \mathbf{r}^\perp)$. Thus by Minkowski's Second Theorem, the ratio $\|\pi(\Lambda)\|/\lambda_1(\pi(\Lambda))$ is bounded from above by a constant depending only on $\mathrm{Irr}(\mathbf{r})$. This completes the proof. ◁

Denote the implied constant by k_1 (so that k_1 depends on $\mathrm{Irr}(\mathbf{r})$). We now split the proof into two cases according to whether or not $n = 1$:

Completion of the proof if $n = 1$. Without loss of generality suppose that $q > 0$ (where $\mathbf{r} = (\mathbf{p}, q)$) and $\|\cdot\|_\nu = |\cdot|$, so that $\|q\| = q = Q_1$. Now fix $\mathbf{r}' = (\mathbf{p}', q') \in \Lambda \setminus \mathbb{Z}\mathbf{r}$ such that $\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}') \neq \varnothing$ and $Q_1 \le q' \le Q_2$. Since

$$\Delta_\psi(\mathbf{r}) = B(\mathbf{p}/q, \Psi(q)), \quad \Delta_\psi(\mathbf{r}') = B(\mathbf{p}'/q', \Psi(q')), \tag{5.27}$$

we get

$$\left\| \frac{\mathbf{p}'}{q'} - \frac{\mathbf{p}}{q} \right\| \le \Psi(q) + \Psi(q') \le 2\Psi(q)$$

and thus

$$d(\mathbf{r}', \mathbb{R}\mathbf{r}) \le d\big((\mathbf{p}', q'), (q'\mathbf{p}/q, q')\big) \le 2q'\Psi(q).$$

Thus by (5.22),

$$\begin{aligned}
&\sum_{\substack{\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r} \\ Q_1 \le q' \le Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}')\right) \lessapprox \sum_{\substack{\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r} \\ Q_1 \le q' \le Q_2 \\ d(\mathbf{r}', \mathbb{R}\mathbf{r}) \le 2q'\Psi(q)}} \Psi^m(q') \\
&\le \int_{-(Q_2+\|\mathcal{D}\|)}^{Q_2+\|\mathcal{D}\|} \int_{\mathbb{R}^m} \widetilde{\Psi}^m(|q'| - \|\mathcal{D}\|) \Big[d(\mathbf{r}', \mathbb{R}\mathbf{r}) \le 2k_1|q'|\Psi(q) \Big] \, \mathrm{d}\mathbf{p}' \, \mathrm{d}q' \\
&\qquad\qquad \text{(by Observation 5.5 and Claim 5.14)} \\
&\le \int_{-(Q_2+\|\mathcal{D}\|)}^{Q_2+\|\mathcal{D}\|} \widetilde{\Psi}^m(|q'| - \|\mathcal{D}\|) \, \lambda_{\mathbb{R}^m}\left(B(q'\mathbf{p}/q, 2k_1k_2|q'|\Psi(q))\right) \mathrm{d}q' \\
&\qquad\qquad \text{(since } d(\mathbf{r}', \mathbb{R}\mathbf{r}) \le k_2\|\mathbf{p}' - q'\mathbf{p}/q\| \text{ for some constant } k_2) \\
&\approx_{\mathrm{Irr}(\mathbf{r})} \int_0^{Q_2+\|\mathcal{D}\|} \widetilde{\Psi}^m(q' - \|\mathcal{D}\|)\big(q'\Psi(q)\big)^m \, \mathrm{d}q' \\
&= \Psi^m(q) \int_{-\|\mathcal{D}\|}^{Q_2} \widetilde{\Psi}^m(q')(q' + \|\mathcal{D}\|)^m \, \mathrm{d}q' \\
&\lessapprox \Psi^m(q) \int_{-2Q_1}^{Q_2} \widetilde{\Psi}^m(q')(Q_1 \vee q')^m \, \mathrm{d}q' && \text{(by (5.26))} \\
&= \Psi^m(q)[F_\psi(Q_1, Q_2) + 3Q_1\psi^m(Q_1)] && \text{(since } n = 1) \\
&= \Psi^m(\|\mathbf{r}\|)[F_\psi(Q_1, Q_2) + 3\phi(Q_1)]. && \text{(by Lemma 5.10)}
\end{aligned}$$

This completes the proof of Lemma 5.13 in the case $n = 1$. □

Completion of the proof if $n \ge 2$.

Sublemma 5.15. *For all $\mathbf{r}' = (\mathbf{p}', \mathbf{q}') \in \Lambda \setminus \mathbb{Z}\mathbf{r}$ with $\|\mathbf{q}'\| \ge \|\mathbf{q}\|$, we have*

$$\lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}')\right) \lessapprox \left(\frac{\psi(\|\mathbf{r}\|)\psi(\|\mathbf{r}'\|)}{\|\mathbf{r} \wedge \mathbf{r}'\|}\right)^m \approx \left(\frac{\psi(\|\mathbf{r}\|)}{\|\mathbf{r}\|} \frac{\psi(\|\mathbf{r}'\|)}{d(\mathbf{r}', \mathbb{R}\mathbf{r})}\right)^m, \tag{5.28}$$

where $\wedge$ denotes the wedge product.

Proof. Let $\mathcal{G}(n,d)$ denote the Grassmanian space consisting of all subspaces of $\mathbb{R}^d$ of dimension n. Define the map $\iota : \mathcal{K} \to \mathcal{G}(n,d)$ via the formula

$$\iota(A) = \{(A\mathbf{q}, \mathbf{q}) : \mathbf{q} \in \mathbb{R}^n\},$$

and note that for all $\mathbf{r} = (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^d$ and $A \in \mathcal{K}$, we have $\|A\mathbf{q} - \mathbf{p}\| \approx d(\mathbf{r}, \iota(A))$. Also note that $\iota_*[\lambda_\mathcal{K}] \lessapprox \lambda_\mathcal{G}$, where $\lambda_\mathcal{G}$ denotes the natural measure on $\mathcal{G}(n,d)$ (i.e. the unique measure on $\mathcal{G}(n,d)$ invarant under the action of $\mathrm{SO}(d)$).[11] So

$$\lambda_\mathcal{K}\left(\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}')\right) \lessapprox \lambda_\mathcal{G}\left(\left\{V \in \mathcal{G}(n,d) : \begin{array}{c} d(\mathbf{r}, V) \le k_3\psi(\|\mathbf{r}\|) \\ d(\mathbf{r}', V) \le k_3\psi(\|\mathbf{r}'\|) \end{array}\right\}\right)$$

for some constant $k_3 > 0$. Here we have used Lemma 5.10 to replace $\psi(\|\mathbf{q}\|)$ and $\psi(\|\mathbf{q}'\|)$ by $\psi(\|\mathbf{r}\|)$ and $\psi(\|\mathbf{r}'\|)$, respectively, in the case where the left-hand side is nonzero.

Now, one way of selecting a point $V \in \mathcal{G}(n,d)$ randomly with respect to $\lambda_\mathcal{G}$ is to independently select m vectors $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathbb{S}^{d-1}$ randomly with respect to $\lambda_\mathbb{S}$ (normalized Lebesgue measure on $\mathbb{S}$) and then to let $V = \bigcap_{i=1}^m \mathbf{s}_i^\perp$. (This can be verified by checking that the measure resulting from this method is invariant under the action of $\mathrm{SO}(d)$.) In this case, $d(\mathbf{r}, V) \ge |\mathbf{r} \cdot \mathbf{s}_i|$ for all i. It follows that the probability that $d(\mathbf{r}, V) \le k_3\psi(\|\mathbf{r}\|)$ and $d(\mathbf{r}', V) \le k_3\psi(\mathbf{r}')$ is less than the probability that for all i, we have $\|\mathbf{r} \cdot \mathbf{s}_i\| \le k_3\psi(\|\mathbf{r}\|)$ and $\|\mathbf{r}' \cdot \mathbf{s}_i\| \le k_3\psi(\|\mathbf{r}'\|)$. Translating this into symbols, we have

$$\lambda_\mathcal{G}\left(\left\{V \in \mathcal{G}(n,d) : \begin{array}{c} d(\mathbf{r}, V) \le k_3\psi(\|\mathbf{r}\|) \\ d(\mathbf{r}', V) \le k_3\psi(\|\mathbf{r}'\|) \end{array}\right\}\right) \le \left(\lambda_\mathbb{S}(R)\right)^m,$$

where

$$R = \left\{\mathbf{s} \in \mathbb{S}^{d-1} : |\mathbf{r} \cdot \mathbf{s}| \le k_3\psi(\|\mathbf{r}\|),\ \ |\mathbf{r}' \cdot \mathbf{s}| \le k_3\psi(\|\mathbf{r}'\|)\right\}.$$

The region R is approximately[12] the product of a $(d-3)$-dimensional unit sphere with a parallelogram whose distances between pairs of opposing sides are $k_3\psi(\|\mathbf{r}\|)/\|\mathbf{r}\|$ and $k_3\psi(\|\mathbf{r}'\|)/\|\mathbf{r}'\|$, and whose angle between sides is approximately $\|\mathbf{r} \wedge \mathbf{r}'\|/(\|\mathbf{r}\| \cdot \|\mathbf{r}'\|)$. Computing the volume of this shape completes the proof. ◁

Thus

$$\begin{aligned}
&\frac{1}{\Psi^m(\|\mathbf{r}\|)} \sum_{\substack{\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r} \\ Q_1 \le \|\mathbf{q}'\| \le Q_2}} \lambda_\mathcal{K}(\Delta_\psi(\mathbf{r}) \cap \Delta_\psi(\mathbf{r}')) \\
&\lessapprox \sum_{\substack{\mathbf{r}' \in \Lambda \setminus \mathbb{Z}\mathbf{r} \\ \|\mathbf{r}'\| \le Q_2}} \left(\frac{\widetilde{\psi}(\|\mathbf{r}'\|)}{d(\mathbf{r}', \mathbb{R}\mathbf{r})}\right)^m && \text{(by (5.28) and Lemma 5.10)} \\
&\le \int_{B(\mathbf{0}, Q_2 + \|\mathcal{D}\|)} \left(\frac{\widetilde{\psi}(\|\mathbf{r}'\| - \|\mathcal{D}\|)}{d(\mathbf{r}', \mathbb{R}\mathbf{r})/k_1}\right)^m \mathrm{d}\mathbf{r}' && \text{(by Observation 5.5 and Claim 5.14)} \\
&\lessapprox_{\mathrm{Irr}(\mathbf{r})} \int_0^{Q_2+\|\mathcal{D}\|} \int_0^{Q_2+\|\mathcal{D}\|} \left(\frac{\widetilde{\psi}(x \vee z - \|\mathcal{D}\|)}{x}\right)^m x^{d-2}\, \mathrm{d}x\, \mathrm{d}z && \left(\begin{array}{c}\text{cylindrical coordinates} \\ x = d(\mathbf{r}', \mathbb{R}\mathbf{r}),\ z = d(\mathbf{r}', \mathbf{r}^\perp)\end{array}\right) \\
&= \int_0^{Q_2+\|\mathcal{D}\|} \widetilde{\psi}^m(R - \|\mathcal{D}\|) \int_0^R (R^{n-2} + r^{n-2})\, \mathrm{d}r\, \mathrm{d}R && (R = x \vee z,\ r = x \wedge z) \\
&\approx \int_0^{Q_2+\|\mathcal{D}\|} \widetilde{\psi}^m(R - \|\mathcal{D}\|) R^{n-1}\, \mathrm{d}R && \text{(since } n \ge 2)
\end{aligned}$$

[11] Further details: The Radon–Nikodym derivative $\frac{\mathrm{d}\,\iota_*[\lambda_\mathcal{K}]}{\mathrm{d}\,\lambda_\mathcal{G}}$ can be computed as the quotient of the two volume forms $\iota_*[\omega_\mathcal{K}]$ and $\omega_\mathcal{G}$ corresponding to $\iota_*[\lambda_\mathcal{K}]$ and $\lambda_\mathcal{G}$. Both of these are positive continuous volume forms, so their quotient is a positive continuous function on the compact set $\iota(\mathcal{K})$. If C is an upper bound for this function, then we have $\iota_*[\lambda_\mathcal{K}] \le C\,\lambda_\mathcal{G}$.

[12] Here two Riemannian manifolds are "approximately the same" if each can be embedded into the other in a way that does not distort size too much.

$$= \int_{-\|\mathcal{D}\|}^{Q_2} \widetilde{\psi}^m(R)(R+\|\mathcal{D}\|)^{n-1}\,\mathrm{d}R$$
$$\lessapprox \int_{-2Q_1}^{Q_2} \widetilde{\psi}^m(R)(Q_1 \vee R)^{n-1}\,\mathrm{d}R = F_\psi(Q_1,Q_2) + 3\phi(Q_1). \qquad \text{(by (5.26))}$$

This completes the proof of Lemma 5.13 in the case $n \geq 2$. □

We are now ready to complete the proof of Theorem 5.1:

Proof of Theorem 5.1. The $\lesssim$ direction follows immediately from (5.14), so we prove the $\gtrsim$ direction. Fix $K \geq 1$ satisfying $Q_1 \geq K\operatorname{Irr}(\Lambda)$, and consider the function $f : \mathcal{K} \to \mathbb{R}$ defined by the formula

$$f(A) = \sum_{\substack{\mathbf{r}\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r})\leq K\\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \big[A \in \Delta_\psi(\mathbf{r})\big].$$

Then

$$\begin{aligned}
\|f\|_2^2 - \|f\|_1 &= \int (f^2 - f)\,\mathrm{d}\,\lambda_{\mathcal{K}}\\
&= \sum_{\substack{\mathbf{r}\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r})\leq K\\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \sum_{\substack{\mathbf{r}'\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r}')\leq K\\ Q_1\leq\|\mathbf{q}'\|\leq Q_2\\ \mathbf{r}'\neq\mathbf{r}}} \int \big[A \in \Delta_\psi(\mathbf{r})\big]\cdot\big[A \in \Delta_\psi(\mathbf{r}')\big]\,\mathrm{d}\,\lambda_{\mathcal{K}}(A)\\
&\leq 2 \sum_{\substack{\mathbf{r}\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r})\leq K\\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \sum_{\substack{\mathbf{r}'\in\Lambda\setminus\mathbb{Z}\mathbf{r}\\ \|\mathbf{q}\|\leq\|\mathbf{q}'\|\leq Q_2}} \lambda_{\mathcal{K}}\left(\Delta_\psi(\mathbf{r})\cap\Delta_\psi(\mathbf{r}')\right)\\
&\lessapprox_K \sum_{\substack{\mathbf{r}\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r})\leq K\\ Q_1\leq\|\mathbf{r}\|\leq Q_2}} \Psi^m(\|\mathbf{r}\|)\overline{F}_\psi(Q_1,Q_2) \qquad \text{(by Lemma 5.13)}\\
&\leq \overline{F}_\psi(Q_1,Q_2) \sum_{\substack{\mathbf{r}\in\Lambda\\ Q_1\leq\|\mathbf{r}\|\leq Q_2}} \Psi^m(\|\mathbf{r}\|)\\
&\lessapprox \overline{F}_\psi(Q_1,Q_2)\left[Q_1^d\Psi^m(Q_1) + \int_{Q_1}^{Q_2} r^{d-1}\Psi^m(r)\,\mathrm{d}r\right] \qquad \text{(by Corollary 5.6)}\\
&= \overline{F}_\psi(Q_1,Q_2)^2.
\end{aligned}$$

Denote the implied constant by $k_4 = h(K)$, so that

$$\|f\|_2^2 - \|f\|_1 \leq k_4\overline{F}_\psi(Q_1,Q_2)^2.$$

Now consider the function $g : \mathcal{K} \to \{0,1\}$ which is the composition of f with the map $k \mapsto \big[k \geq 1\big]$. Equivalently, g is the characteristic function of the set

$$\bigcup_{\substack{\mathbf{r}\in\Lambda^+_{\mathrm{pr}}\\ \operatorname{Irr}(\mathbf{r})\leq K\\ Q_1\leq\|\mathbf{q}\|\leq Q_2}} \Delta_\psi(\mathbf{r}).$$

Since $f = fg$, Hölder's inequality gives

$$\|f\|_1 = \|fg\|_1 \leq \|f\|_2\cdot\|g\|_2$$

and thus

(5.29)
$$\lambda_{\mathcal{K}}\left(W_{\psi,\Lambda}(Q_1,Q_2)\right) \geq \|g\|_1 = \|g\|_2^2 \geq \frac{\|f\|_1^2}{\|f\|_2^2} \geq \frac{\|f\|_1^2}{\|f\|_1 + k_4\overline{F}_\psi(Q_1,Q_2)^2} = \frac{\|f\|_1}{1 + k_4\overline{F}_\psi(Q_1,Q_2)^2/\|f\|_1}.$$

It remains to work through some quantifier logic. Fix $1 < \gamma \leq 2$. By (5.16), there exists a constant $R_\gamma^{(1)} \geq 1$ depending only on γ which is large enough so that

$$\begin{cases} K \geq R_\gamma^{(1)} \\ Q_1 \geq K \operatorname{Irr}(\Lambda) \\ \phi(Q_1) \leq F_\psi(Q_1,Q_2)/R_\gamma^{(1)} \end{cases} \quad \Rightarrow \quad \|f\|_1 \geq \gamma^{-1}\eta_{\mu,\nu}F_\psi(Q_1,Q_2). \tag{5.30}$$

Letting $K = R_\gamma^{(1)}$ in (5.30) gives

$$\begin{cases} Q_1 \geq R_\gamma^{(1)} \operatorname{Irr}(\Lambda) \\ \phi(Q_1) \leq F_\psi(Q_1,Q_2)/R_\gamma^{(1)} \end{cases} \quad \Rightarrow \quad \|f\|_1 \geq \eta_{\mu,\nu}\gamma^{-1}F_\psi(Q_1,Q_2),$$

and since $R_\gamma^{(1)} \geq 1$ we have

$$\phi(Q_1) \leq F_\psi(Q_1,Q_2)/R_\gamma^{(1)} \quad \Rightarrow \quad \overline{F}_\psi(Q_1,Q_2) \leq 2F_\psi(Q_1,Q_2).$$

Combining with (5.29) gives

$$\begin{cases} Q_1 \geq R_\gamma^{(1)} \operatorname{Irr}(\Lambda) \\ \phi(Q_1) \leq F_\psi(Q_1,Q_2)/R_\gamma^{(1)} \end{cases} \quad \Rightarrow \quad \|g\|_1 \geq \eta_{\mu,\nu}\gamma^{-1}F_\psi(Q_1,Q_2)\frac{1}{1 + h(R_\gamma^{(1)})(8/\eta_{\mu,\nu})F_\psi(Q_1,Q_2)};$$

letting $R_\gamma^{(2)} = \frac{8h(R_\gamma^{(1)})}{\eta_{\mu,\nu}(\gamma-1)}$, we have

$$\begin{cases} Q_1 \geq R_\gamma^{(1)} \operatorname{Irr}(\Lambda) \\ \phi(Q_1) \leq F_\psi(Q_1,Q_2)/R_\gamma^{(1)} \\ F_\psi(Q_1,Q_2) \leq 1/R_\gamma^{(2)} \end{cases} \quad \Rightarrow \quad \lambda_{\mathcal{K}}\left(W_{\psi,\Lambda}(Q_1,Q_2)\right) \geq \|g\|_1 \geq \eta_{\mu,\nu}\gamma^{-2}F_\psi(Q_1,Q_2).$$

Since γ can be chosen arbitrarily close to 1, this completes the proof. □

6. Lower bound

In this section, we prove the lower bound in Theorem 1.8, namely we prove that there exists a function $C_{\mu,\nu}$ such that if $L_{f,\psi} > C_{\mu,\nu}(M_\psi)$, then $\mathcal{H}^f(B(\psi)) = \infty$. As in that theorem, we let f be a dimension function, and we let ψ be a nice approximation function such that $L_{f,\psi} > C_{\mu,\nu}(M_\psi)$. Let the notation be as in Section 4.

6.1. **Sketch of the proof.** Imagine a spaceship hurtling through the space Ω_d, flying in the direction of the (g_t) flow. It periodically makes corrections to its course by applying an operator from the (u_A) flow, where $A \in \mathcal{K}$. After an infinite amount of time, the path of the spaceship will be within a bounded distance of a path of the form $(g_t u_{A_\infty}\Lambda_*)_{t\geq 0}$ for some $A_\infty \in \mathcal{M}$.

Now fix $Q_0 > 0$, and suppose you want the spaceship to steer in a way such that $A_\infty \in B_{\psi,Q_0}$, but you have only limited control of its movements. Specifically, at certain times $t_1, t_2, \ldots$ you can make the spaceship apply a u_A correction where you cannot choose A, but you can enforce some restrictions on the possibilities for A as long as you allow e.g. at least 90% of the possible values for A. With these limitations, what strategy should you use to guarantee $A_\infty \in B_{\psi,Q_0}$?

The answer is to use the mixing property of the (g_t) flow to guarantee that at the times $t_1, t_2, \ldots$, the spaceship is in a bounded region whose size is independent of ψ. During the "in between times" where $t \neq t_k$ for any k, the spaceship can leave this bounded region but still stays in the time-dependent region $(K(\psi,t))_{t\geq 0}$ corresponding to ψ according to (4.16).

More rigorously, choose $R \geq 1$ large enough so that the set $K_R = \{\Lambda \in \Omega_d : \Delta(\Lambda) \leq R\}$ has measure close to 1, say $1 - \varepsilon$. Due to the mixing property of the (g_t) flow, if the gap sizes $t_{k+1} - t_k$ are large

enough then the set of potential directions A such that applying a u_A correction at time t_k will cause the spaceship to be in K_R at time t_{k+1} has size at least $1-2\varepsilon$. Thus by applying appropriate restrictions to the movement corrections, we can guarantee that the spaceship is in K_R at the times $t_1, t_2, \ldots$. Since the set of directions we must discard is at most size 2ε, this requirement does not significantly hamper our ability to control the spaceship. But then each time we control the ship, we start out in K_R, and since K_R is compact "all points look the same" – any calculations we do will apply with the same quantitative bounds to all points in K_R. But this means that the behavior of the ship at the intervals $[t_k, t_{k+1}]$ $(k \in \mathbb{N})$ can be treated as independent, so then the heuristic argument of Section 3 finishes the proof.

6.2. **The proof.** Fix $R \geq 1$ and $0 < \beta \leq 1$, and let $(N_k)_1^\infty$ be the sequence given by Lemma 4.2. Fix $k \in \mathbb{N}$. Recall that $N^k = \prod_{j=1}^k N_j$, and let

$$t_k = \delta \log(N^k), \qquad g^k = g_{t_k}.$$

This will be our sequence of times we can steer the spaceship. Note that by (4.13), $g^k u_A = u_{N^k A} g^k$ for all $A \in \mathcal{M}$.

Now when we steer the spaceship at time t_k, we will maneuver so as to avoid the rationals represented by points in the window

$$\mathcal{Q}_k = \{(\mathbf{p}, \mathbf{q}) \in \Lambda_* : e^{-2R} Q^k \leq \|\mathbf{q}\| < e^{-2R} Q^{k+1}\}.$$

We will do this under the assumption that the lattice represented by the spaceship at time t_k is contained in the compact set

$$K_R = \{\Lambda \in \Omega_d : \Delta(\Lambda) \leq R\}.$$

Specifically, if

$$\mathcal{B}_k = \{A \in \mathcal{K} : g^k u_A \Lambda_* \in K_R\}$$
$$\mathcal{B}'_k = \mathcal{B}_k \setminus \bigcup_{\mathbf{r} \in \mathcal{Q}_k} \Delta_\psi(\mathbf{r}) = \mathcal{B}_k \setminus W_\psi(e^{-2R} Q^k, e^{-2R} Q^{k+1})$$

then, assuming that the current trajectory of the spaceship is represented by $(g_t u_A \Lambda_*)_{t \geq 0}$ for some $A \in \mathcal{B}_k$, we will steer so that the new trajectory is represented by a point of $\mathcal{B}'_{k+1}$. Fix $k_0 \in \mathbb{N}$, and for each $k \geq k_0$ let

$$T^k = \{\omega \in E^k : \mathcal{K}_{\omega \restriction j} \subseteq \mathcal{B}'_j \text{ for all } j = k_0, \ldots, k\}.$$

(For $0 \leq k < k_0$, let $T^k = E^k$.) Note that:

- The sets $(\mathcal{Q}_k)_{k_0}^\infty$ are disjoint and their union is $\{\mathbf{r} \in \Lambda_* : \|\mathbf{q}\| \geq e^{-2R} Q^{k_0}\}$.
- If T^∞ is the set of infinite branches through the tree $T^* = \bigcup_k T^k \subseteq E^*$, then $B(\psi) \supseteq \pi(T^\infty)$. To see why this is the case, fix $\omega \in T^\infty$ and $\mathbf{r} \in \Lambda_*$ such that $\|\mathbf{q}\| \geq e^{-2R} Q^{k_0}$; there exists k such that $\mathbf{r} \in \mathcal{Q}_k$, and
$$\pi(\omega) \in \mathcal{K}_{\omega \restriction k} \subseteq \mathcal{B}'_k \subseteq \mathcal{K} \setminus \Delta_\psi(\mathbf{r}).$$
Since $\mathbf{r}$ was arbitrary, we have $\pi(\omega) \in B_{\psi, Q_0} \subseteq B(\psi)$, where $Q_0 = e^{-2R} Q^{k_0}$.

6.3. **Bounding the evaporation rate of T^*.** Fix $\omega \in T^*$, and let $k = |\omega|$. We partition the set $E_{k+1} \setminus T_\omega$ into two subsets:

$$T'_\omega = \{a \in E_{k+1} : \mathcal{K}_{\omega a} \not\subseteq \mathcal{B}_{k+1}\},$$
$$T''_\omega = E_{k+1} \setminus (T_\omega \cup T'_\omega) = \bigcup_{\mathbf{r} \in \mathcal{Q}_k} T_{\omega, \mathbf{r}},$$

where

$$T_{\omega, \mathbf{r}} = \{a \in E_{k+1} \setminus T'_\omega : \mathcal{K}_{\omega a} \cap \Delta_\psi(\mathbf{r}) \neq \varnothing\}.$$

Claim 6.1. *We have*

$$\frac{\#(T'_\omega)}{(N_{k+1})^D} \underset{\substack{R\to\infty \\ M_\psi/\beta\to 0\ (R)}}{\ll} 1 \tag{6.1}$$

and

$$\frac{\#(T''_\omega)}{(N_{k+1})^D} \underset{\substack{\beta\to 0 \\ R\to\infty \\ M_\psi/\beta\to 0\ (R)}}{\lesssim} \eta_{\mu,\nu}\beta. \tag{6.2}$$

In particular, the upper evaporation rate $(P_k^+)_1^\infty$ *of* T^* *satisfies*

$$\sup_{k\geq 1} P_k^+ \underset{\substack{\beta\to 0 \\ R\to\infty\ (\beta) \\ M_\psi/\beta\to 0\ (R,\beta)}}{\lesssim} \eta_{\mu,\nu}\beta.$$

Proof of (6.1). Fix $\varepsilon > 0$. Let λ_Ω denote the unique probability Haar measure on the space of lattices Ω_d. Since $\bigcup_{R\geq 1} K_R = \Omega_d$, there exists $R_\varepsilon \geq 1$ such that

$$\lambda_\Omega(K_{R_\varepsilon}) > 1 - \varepsilon.$$

We observe that the group $G = \mathrm{SL}_d(\mathbb{R})$ can be written as the product of the two subgroups

$$U \stackrel{\mathrm{def}}{=} \{u_A : A \in \mathcal{M}\}$$

$$H \stackrel{\mathrm{def}}{=} \{M \in G : M_{i,j} = 0 \ \ \forall i = 1,\ldots,m \ \ \forall j = m+1,\ldots,m+n\}.$$

Let ν_1 be an absolutely continuous and compactly supported probability measure on H, and let ν_2 be the image of $\lambda_\mathcal{K}$ under the map $A \mapsto u_A$. Let ν denote the image of $\nu_1 \times \nu_2$ under the map $(h,u) \mapsto hu$. Then ν is an absolutely continuous and compactly supported probability measure on G. It follows that the map $f : \Omega_d \to \mathcal{M}(\Omega_d)$ defined by the formula

$$f(\Lambda) \stackrel{\mathrm{def}}{=} \int \delta_{g\Lambda}\ \mathrm{d}\nu(g) \in \mathcal{M}(\Omega_d)$$

is continuous with respect to the norm topology on $\mathcal{M}(\Omega_d)$, the space of measures on Ω_d. (Here δ_x denotes the Dirac delta mass at x.) Equivalently, $f(\Lambda)$ is the measure on Ω_d such that for all $S \subseteq \Omega_d$, we have $f(\Lambda)[S] = \nu(\{g \in G : g\Lambda \in S\})$. Thus the family of functions

$$f_t(\Lambda) \stackrel{\mathrm{def}}{=} f(\Lambda)[g_{-t}K_{R_\varepsilon}] = \nu(\{g : g_t g\Lambda \in K_{R_\varepsilon}\}) \ \ (t \in \mathbb{R})$$

is equicontinuous. On the other hand, by the Howe–Moore theorem [18, Theorem 5.2], the matrix coefficients of the natural representation of G on $L^2_0(\Omega_d, \lambda_\Omega)$ vanish at infinity, meaning that the action of G on Ω_d is mixing with respect to the natural measure λ_Ω. It follows that the (g_t) flow on Ω_d is also mixing, so for any fixed Λ we have

$$f_t(\Lambda) = \int \big[g_t x \in K_{R_\varepsilon}\big] \frac{\mathrm{d}[g \mapsto g\Lambda]_*\nu}{\mathrm{d}\lambda_\Omega}(x)\ \mathrm{d}\lambda_\Omega(x) \xrightarrow[t\to\infty\ (\Lambda)]{} \lambda_\Omega(K_{R_\varepsilon}) > 1 - \varepsilon.$$

Equicontinuity implies that this convergence is uniform when Λ is restricted to any compact set.

Since the map $h \mapsto g_t h g_{-t}$ is nonexpanding on H and since Δ is uniformly continuous, there exists a constant $k_6 > 0$ such that

$$\Delta(g_t h u_A \Lambda) \geq \Delta(g_t u_A \Lambda) - k_6 \ \ \forall A \in \mathcal{K} \ \ \forall h \in \mathrm{Supp}(\nu_1) \ \ \forall \Lambda \in \Omega_d \ \ \forall t \geq 0. \tag{6.3}$$

Also, there exists a constant $k_7 > 0$ such that

$$|\Delta(u_{A-B}\Lambda) - \Delta(\Lambda)| \leq k_7 \ \ \forall A, B \in \mathcal{K} \ \ \forall \Lambda \in \Omega_d. \tag{6.4}$$

Fix $R \geq R_\varepsilon + k_6 + k_7$, and we will show that

$$\#(T'_\omega) \underset{M_\psi/\beta\to 0\ (R)}{\lesssim} \varepsilon(N_{k+1})^D, \tag{6.5}$$

which suffices to prove (6.1). Indeed, let t_R be large enough so that $f_t(\Lambda) \geq 1-\varepsilon$ for all $\Lambda \in K_R$ and $t \geq t_R$. Equivalently,

$$(\nu_1 \times \lambda_{\mathcal{K}})\left(\{(h,A) : \Delta(g_t h u_A \Lambda) > R_\varepsilon\}\right) \leq \varepsilon \ \ \forall \Lambda \in K_R \ \ \forall t \geq t_R.$$

Applying (6.3) gives

$$\lambda_{\mathcal{K}}\left(\{A : \Delta(g_t u_A \Lambda) > R_\varepsilon + k_6\}\right) \leq \varepsilon \ \ \forall \Lambda \in K_R \ \ \forall t \geq t_R. \tag{6.6}$$

Now fix $\omega \in T^*$, and let $k = |\omega|$. Since $\omega \in T^k$, we have $\pi(\omega) \in \mathcal{B}'_k$ and thus $\Lambda_\omega \stackrel{\text{def}}{=} g^k u_{\pi(\omega)} \Lambda_* \in K_R$. Moreover, if we assume $M_\psi/\beta \leq \delta/(\alpha t_R)$ (a valid assumption given the convergence assumption $M_\psi/\beta \to 0$ (R)), then (4.10) and (4.11) together imply that $t_{k+1} - t_k = \delta \log(N_{k+1}) \geq t_R$. Thus, we can plug $\Lambda = \Lambda_\omega$ and $t = t_{k+1} - t_k = \delta \log(N_{k+1})$ into (6.6). Now by (4.13), we have

$$g_t u_A \Lambda = g_t u_A g^k u_{\pi(\omega)} \Lambda_* = g^{k+1} u_{\pi(\omega)+(N^k)^{-1}A} \Lambda_* = g^{k+1} u_{\Phi_\omega(A)} \Lambda_*.$$

Thus, applying a change of variables to (6.6) yields

$$(N^k)^D \, \lambda_{\mathcal{K}}\left(\left\{A \in \mathcal{K}_\omega : \Delta(g^{k+1} u_A \Lambda_*) > R_\varepsilon + k_6\right\}\right) \leq \varepsilon.$$

Fix $a \in E_{k+1}$; then for all $A, B \in \mathcal{K}_{\omega a}$ we have

$$|\Delta(g^{k+1} u_B \Lambda_*) - \Delta(g^{k+1} u_A \Lambda_*)| = |\Delta(u_{N^{k+1}(B-A)} g^{k+1} u_A \Lambda_*) - \Delta(g^{k+1} u_A \Lambda_*)| \leq k_7.$$

It follows that

$$\begin{aligned}\left\{A \in \mathcal{K}_\omega : \Delta(g^{k+1} u_A \Lambda_*) > R_\varepsilon + k_6\right\} &\supseteq \bigcup \left\{\mathcal{K}_{\omega a} : \exists A \in \mathcal{K}_{\omega a} \ \ \Delta(g^{k+1} u_A \Lambda_*) > R_\varepsilon + k_6 + k_7\right\} \\ &\supseteq \bigcup_{a \in T'_\omega} \mathcal{K}_{\omega a} \qquad\qquad (\text{since } R \geq R_\varepsilon + k_6 + k_7)\end{aligned}$$

and thus

$$(N^k)^D \, \lambda_{\mathcal{K}}\left(\bigcup_{a \in T'_\omega} \mathcal{K}_{\omega a}\right) \leq \varepsilon.$$

Direct computation and rearrangement now yields (6.5). □

Proof of (6.2). Fix $\mathbf{r} = (\mathbf{p}, \mathbf{q}) \in \mathcal{Q}_k$ and $a \in T_{\omega,\mathbf{r}}$. By definition, there exists $A \in \mathcal{K}_{\omega a} \cap \Delta_\psi(\mathbf{r})$. Since $a \notin T'_\omega$, we have $A \in \mathcal{B}_{k+1}$ and thus

$$g^{k+1} u_A \mathbf{r} \notin B(\mathbf{0}, e^{-R}).$$

Equivalently,

$$\frac{1}{Q^{k+1}} \|\mathbf{q}\| > e^{-R} \text{ or } (N^{k+1})^{\delta/m} \|A\mathbf{q} - \mathbf{p}\| > e^{-R}. \tag{6.7}$$

On the other hand, since $\mathbf{r} \in \mathcal{Q}_k$ we have

$$\|\mathbf{q}\| \leq e^{-2R} Q^{k+1}.$$

In particular, this contradicts the first half of the dichotomy (6.7), so it must be the last half which holds. Thus

$$\|A\mathbf{q} - \mathbf{p}\| \geq e^{-R} (N^{k+1})^{-\delta/m} = e^{-R} \frac{Q^{k+1}}{N^{k+1}} \geq e^R \frac{\|\mathbf{q}\|}{N^{k+1}}.$$

Since $\operatorname{diam}(\mathcal{K}_{\omega a}) = \|\mathcal{K}\|/N^{k+1}$, we have for all $B \in \mathcal{K}_{\omega a}$

$$\begin{aligned}\|B\mathbf{q} - \mathbf{p}\| &\leq \|A\mathbf{q} - \mathbf{p}\| + \|B - A\| \cdot \|\mathbf{q}\| \\ &\leq \|A\mathbf{q} - \mathbf{p}\| + \frac{\|\mathcal{K}\| \cdot \|\mathbf{q}\|}{N^{k+1}} \\ &\leq (1 + \|\mathcal{K}\| e^{-R}) \|A\mathbf{q} - \mathbf{p}\| \leq (1 + \|\mathcal{K}\| e^{-R}) \psi(\|\mathbf{q}\|).\end{aligned}$$

Letting $\varepsilon = \varepsilon_R = \|\mathcal{K}\| e^{-R}$, we have

$$\mathcal{K}_{\omega a} \subseteq \Delta_{(1+\varepsilon)\psi}(\mathbf{r}) \subseteq W_{(1+\varepsilon)\psi}(e^{-2R} Q^k, e^{-2R} Q^{k+1}) \ \ \forall \mathbf{r} \in \mathcal{Q}_k \ \ \forall a \in T_{\omega,\mathbf{r}}$$

and thus

$$\frac{\#(T''_\omega)}{(N^{k+1})^D} = \sum_{a\in T''_\omega} \lambda_{\mathcal{K}}(\mathcal{K}_{\omega a}) \le \lambda_{\mathcal{K}}\left(\mathcal{K}_\omega \cap W_{(1+\varepsilon)\psi}(e^{-2R}Q^k, e^{-2R}Q^{k+1})\right).$$

Now let $g_\omega = g^k u_{\pi(\omega)}$ and $\Lambda_\omega = g_\omega \Lambda_*$ be as in Corollary 5.2, and let $\psi'(q) = (Q^k)^{n/m}\psi(Q^k q)$ be as in the proof of Corollary 5.2. Fix $K \ge 1$. We have

$$\frac{\#(T''_\omega)}{(N_{k+1})^D} \le \underbrace{\lambda_{\mathcal{K}}\left(W_{(1+\varepsilon)\psi',\Lambda_\omega}(e^{-2R}, K)\right)}_{\text{Term 1}} + \underbrace{(N^k)^D\, \lambda_{\mathcal{K}}\left(\mathcal{K}_\omega \cap W_{(1+\varepsilon)\psi}(KQ^k, e^{-2R}Q^{k+1})\right)}_{\text{Term 2}}.$$

Now since $\omega \in T^k$, we have $\pi(\omega) \in \mathcal{B}_k$ and thus $\Lambda_\omega \in K_R$. So $\#(\Lambda_\omega \cap B_K(\mathbf{0}))$ is bounded depending only on R and K. Thus by (5.22),

$$\text{Term 1} \lessapprox ((1+\varepsilon)\Psi')^m(e^{-2R})\#(\Lambda_\omega \cap B_K(\mathbf{0})) \lessapprox_{R,K} \phi'(e^{-2R}) = \phi(e^{-2R}Q^k) \le \phi(1) = M_\psi.$$

On the other hand, we have

$$\begin{aligned} K/\operatorname{Irr}(\Lambda_\omega) &\xrightarrow[K\to\infty\ (R)]{} \infty && \text{(since } \Lambda_\omega \in K_R\text{)} \\ F_{(1+\varepsilon)\psi}(Q^k, KQ^k) &\le (1+\varepsilon)^m \log(K) M_\psi \\ F_{(1+\varepsilon)\psi}(e^{-2R}Q^{k+1}, Q^{k+1}) &\le (1+\varepsilon)^m 2R M_\psi \end{aligned}$$

and thus by Corollary 5.2,

$$\text{Term 2} \underset{\substack{K\to\infty\ (R)\\ \beta\to 0\\ M_\psi/\beta\to 0\ (R,K)}}{\sim} \eta_{\mu,\nu}(1+\varepsilon_R)\beta \underset{R\to\infty}{\sim} \eta_{\mu,\nu}\beta.$$

Standard quantifier logic completes the proof. □

6.4. **Completion of the proof.** For each $\gamma > 1$, we have

$$\exp(-\gamma\eta_{\mu,\nu}\beta) \le 1 - \gamma^{1/2}\eta_{\mu,\nu}\beta$$

for all sufficiently small β, and thus by Claim 6.1 there exist $h_1(\gamma), h_2(\gamma), h_3(\gamma) > 0$ (depending only on γ and m, n, μ, ν) such that

$$\begin{cases} \beta = h_1(\gamma) \\ R = h_2(\gamma) \\ M_\psi/\beta = h_3(\gamma) \end{cases} \quad \Rightarrow \quad \left[1 - \sup_{k\ge 1} P_k^+ \ge \exp(-\gamma\eta_{\mu,\nu}\beta)\right]. \tag{6.8}$$

Without loss of generality, we may assume that h_1, h_2, h_3 are homeomorphisms from $[1,\infty]$ to $[0,\beta_0]$, $[R_0,\infty]$, and $[0,\delta_0]$ for some $\beta_0, R_0, \delta_0 > 0$ (with h_1, h_3 increasing and h_2 decreasing). In particular, $h_1(1) = h_3(1) = 0$ while $h_2(1) = \infty$. Then the function $C_{\mu,\nu}$ in the statement of Theorem 1.8 can be defined by the formula

$$C_{\mu,\nu}(M) = \begin{cases} ((h_1 h_3)^{-1}(M))\eta_{\mu,\nu} & \text{if } M \le \beta_0\delta_0 \\ \infty & \text{if } M \ge \beta_0\delta_0 \end{cases}.$$

Suppose that $L_{f,\psi} > C_{\mu,\nu}(M_\psi)$ (in particular this implies $C_{\mu,\nu}(M_\psi) < \infty$ and thus $M_\psi < \beta_0\delta_0$), and let $\gamma \stackrel{\text{def}}{=} (h_1h_3)^{-1}(M_\psi)$, $\beta \stackrel{\text{def}}{=} h_1(\gamma)$, and $R \stackrel{\text{def}}{=} h_2(\gamma)$, so that $C_{\mu,\nu}(M_\psi) = \gamma\eta_{\mu,\nu}$. By (6.8), we have $1 - \sup_{k\ge1} P_k^+ \ge \exp(-\gamma\eta_{\mu,\nu}\beta)$. Thus the function f_+ from §4.2 satisfies

$$f_+(\rho) \le \left(\prod_{j=1}^{k(\rho)} \exp(\gamma\eta_{\mu,\nu}\beta)\right)\rho^D = \exp(\gamma\eta_{\mu,\nu}\beta k(\rho))\rho^D.$$

We can relate $\beta k(\rho)$ to ρ via the definition of the sequence $(N_k)_1^\infty$:

$$\begin{aligned} F_\psi(\rho^{-\alpha}) &\geq F_\psi\big((N^{k(\rho)})^\alpha\big) = F_\psi(Q^{k(\rho)}) \\ &= \sum_{k=0}^{k(\rho)-1} F_\psi(Q^k, Q^{k+1}) \\ &\geq \sum_{k=0}^{k(\rho)-1} \beta \qquad \text{(by (4.11))} \\ &= \beta k(\rho). \end{aligned}$$

Combining gives

$$f_+(\rho) \leq \exp(\gamma\eta_{\mu,\nu} F_\psi(\rho^{-\alpha}))\rho^D$$

and thus by Proposition 4.1(i),

$$\mathcal{H}^f(B(\psi)) \gtrapprox \liminf_{\rho\to 0} \frac{\rho^{-D} f(\rho)}{\exp(\gamma\eta_{\mu,\nu} F_\psi(\rho^{-\alpha}))} = \liminf_{\rho\to 0} \frac{\exp\log\left(\frac{f(\rho)}{f_*(\rho)}\right)}{\exp(\gamma\eta_{\mu,\nu} F_\psi(\rho^{-\alpha}))}.$$

Since $L_{f,\psi} > C_{\mu,\nu}(M_\psi) = \gamma\eta_{\mu,\nu}$, the right-hand side is equal to ∞, which completes the proof.

7. Upper bound

In this section, we prove the upper bound in Theorem 1.8, namely we prove that there exists a function $c_{\mu,\nu}$ such that if $L_{f,\psi} < c_{\mu,\nu}(M_\psi)$, then $\mathcal{H}^f(B(\psi)) = 0$. As in that theorem, we let f be a dimension function, and we let ψ be a nice approximation function such that $L_{f,\psi} < c_{\mu,\nu}(M_\psi)$. Let the notation be as in Section 4.

7.1. **Sketch of the proof.** As before, imagine a spaceship flying through Ω_d in the direction of the (g_t) flow, periodically making (u_A) corrections, whose path eventually corresponds to a path of the form $(g_t u_{A_\infty}\Lambda_*)_{t\geq 0}$ for some $A_\infty \in \mathcal{M}$. But this time, suppose that you want to steer in a way such that $A_\infty \in W_{\psi,Q_0}$. Instead of being able to direct the movement of the spaceship at all of the times $t_1, t_2, \ldots$ when it makes (u_A) corrections, you can only restrict its movement at one of those times – better make it count! In this scenario, what is the right strategy?

The answer again depends on the spaceship being in a bounded region, but this time, we don't have to work to move it into that region – if the spaceship ever leaves the time-dependent region $(K(\psi,t))_{t\geq 0}$ defined by (4.16), then by definition we will have $A_\infty \in W(\psi)$ regardless of any choices we make. So we can assume that at time t_k, the spaceship is not in $K(\psi,t_k)$. From this vantage point, we can attempt to move into W_{ψ,Q_0} by moving into $\Delta_\psi(\mathbf{r})$ for some $\mathbf{r} \in \Lambda_*$ with $\|\mathbf{q}\| \geq Q_0$. According to Corollary 5.2, this strategy will work as long as the disadvantage that comes from being in $K(\psi,t_k)$ rather than a fixed compact region is small in comparison to the total size of W_{ψ,Q_0} relative to Λ, i.e. as long as

$$F_\psi(Q^k, \mathrm{Irr}(\Lambda_\omega)Q^k)$$

is small. But by (4.9),

$$F_\psi(Q^k, \mathrm{Irr}(\Lambda_\omega)Q^k) \leq \phi(Q^k)\log(\mathrm{Irr}(\Lambda_\omega)) \lessapprox \phi(Q^k) r_\psi(t_k) \sim \phi(Q^k)|\log\phi(Q^k)| \xrightarrow[M_\psi\to 0]{} 0,$$

where the last convergence holds because $\phi(Q^k) \leq M_\psi$.

7.2. **The proof.** Fix $0 < \beta \leq 1$, let the sequence $(N_k)_1^\infty$ (depending on ψ and β) be given by Lemma 4.2, and let the notation be as in Section 4. Fix $Q_0 > 0$, and for each k let

$$T^k = \{\omega \in E^k : \mathcal{K}_\omega \cap B_{\psi,Q_0} \neq \varnothing\}$$

(cf. (3.1)). Then $B_{\psi,Q_0} = \pi(T^\infty)$.

Lemma 7.1. *The lower evaporation rate $(P_k^-)_1^\infty$ of T^* satisfies*

$$\liminf_{k\to\infty} P_k^- \underset{\substack{M_\psi^{1/2}/\beta\to 0\\ \beta\to 0}}{\gtrsim} \eta_{\mu,\nu}\beta.$$

Proof. The idea is to choose numbers $Q_{1,k}, Q_{2,k} \in [Q^k, Q^{k+1}]$ such that $F_\psi(Q^k, Q_{1,k})$ and $F_\psi(Q_{2,k}, Q^{k+1})$ are both small in comparison to $F_\psi(Q^k, Q^{k+1}) \geq \beta$. Then for $\varepsilon > 0$ small and $\omega \in T^k$, the measure of $\mathcal{K}_\omega \cap W_{(1-\varepsilon)\psi}(Q_{1,k}, Q_{2,k})$ can be bounded from below by Corollary 5.2.

Choice of $Q_{1,k}$. For each $k \in \mathbb{N}$ let $Q_{1,k} = \exp(\phi^{-1/2}(Q^k))Q^k$. This choice is made so as to maximize $Q_{1,k}$ while leaving $F_\psi(Q^k, Q_{1,k})$ relatively small:

$$\begin{aligned} F_\psi(Q^k, Q_{1,k}) &\leq \phi(Q^k)\log(Q_{1,k}/Q^k) && \text{(by (4.9))}\\ &= \phi(Q^k)\phi^{-1/2}(Q^k) = \phi^{1/2}(Q^k) \leq M_\psi^{1/2}. \end{aligned}$$

Choice of $Q_{2,k}$. Fix $0 < \varepsilon \leq \|\mathcal{K}\|$ small. Since Ψ is continuous, for all k sufficiently large there exists $Q_{2,k} \in [1, Q^{k+1}]$ such that

$$\varepsilon\Psi(Q_{2,k}) = \frac{\|\mathcal{K}\|}{N^{k+1}} = \|\mathcal{K}\|(Q^{k+1})^{-1/\alpha}. \tag{7.1}$$

This choice is motivated by the implication

$$\mathcal{K}_\omega \cap W_{(1-\varepsilon)\psi}(Q_{1,k}, Q_{2,k}) \neq \varnothing \quad\Rightarrow\quad \mathcal{K}_\omega \subseteq W_\psi(Q_{1,k}, Q_{2,k}), \tag{7.2}$$

which holds for all $\omega \in E^{k+1}$ (since for such ω, we have $\operatorname{diam}(\mathcal{K}_\omega) = \|\mathcal{K}\|/N^{k+1}$). The following calculation shows that $F_\psi(Q_{2,k}, Q^{k+1})$ is small in comparison to β:

$$\begin{aligned} F_\psi(Q_{2,k}, Q^{k+1}) &\leq \phi(Q_{2,k})\log(Q^{k+1}/Q_{2,k}) && \text{(by (4.9))}\\ &= \phi(Q_{2,k})\log\left(\frac{(\|\mathcal{K}\|/(\varepsilon\Psi(Q_{2,k})))^\alpha}{Q_{2,k}}\right) && \text{(by (7.1))}\\ &= \phi(Q_{2,k})[\alpha\log(\|\mathcal{K}\|/\varepsilon) + (1/d)\log(1/\phi(Q_{2,k}))]\\ &\lessapprox_\varepsilon \phi^{1/2}(Q_{2,k}) && \text{(since } \phi(Q_{2,k}) \leq M_\psi \leq 1)\\ &\leq M_\psi^{1/2}. \end{aligned}$$

Application of Corollary 5.2. The above calculations show that

$$M_\psi/\beta, F_\psi(Q^k, Q_{1,k})/\beta, F_\psi(Q_{2,k}, Q^{k+1})/\beta \xrightarrow[M_\psi^{1/2}/\beta\to 0\ (\varepsilon)]{} 0.$$

In particular, if $M_\psi^{1/2}/\beta$ is sufficiently small (depending on ε), then we have $Q^k \leq Q_{1,k} \leq Q_{2,k} \leq Q^{k+1}$ and so we can apply Corollary 5.2 to get

$$(N^k)^D\,\lambda_{\mathcal{K}}\left(\mathcal{K}_\omega \cap W_{(1-\varepsilon)\psi}(Q_{1,k}, Q_{2,k})\right) \underset{\substack{\exp(\phi^{-1/2}(Q^k))/\operatorname{Irr}(\Lambda_\omega)\to\infty\\ \beta\to 0\\ M_\psi^{1/2}/\beta\to 0\ (\varepsilon)}}{\sim} \eta_{\mu,\nu}(1-\varepsilon)\beta, \tag{7.3}$$

where $\omega \in T^k$, $g_\omega = g^k u_{\pi(\omega)}$, and $\Lambda_\omega = g_\omega\Lambda_*$ are as in Corollary 5.2.

Claim 7.2. *The top convergence assumption can be replaced by the convergence assumption $k \to \infty$ (ψ, Q_0), since*

$$\exp(\phi^{-1/2}(Q^k))/\operatorname{Irr}(\Lambda_\omega) \xrightarrow[\substack{M_\psi\to 0\\ k\to\infty\ (\psi,Q_0)}]{} \infty. \tag{7.4}$$

Proof. Fix $\mathbf{r} \in \Lambda_*$ and let $\widetilde{\mathbf{r}} = g_\omega\mathbf{r}$. We bound $\|\widetilde{\mathbf{r}}\|$ from below in four cases:

1. Suppose $Q_0 \le \|\mathbf{q}\| \le Q^k$. By assumption $\mathcal{K}_\omega \not\subseteq W_{\psi,Q_0}$, so we may choose $A \in \mathcal{K}_\omega \cap B_{\psi,Q_0}$. Then
$$\begin{aligned}\|\widetilde{\mathbf{r}}\| \approx \|g^k u_A \mathbf{r}\| &\ge (Q^k)^{n/m}\|A\mathbf{q}-\mathbf{p}\| > (Q^k)^{n/m}\psi(\|\mathbf{q}\|)\\ &\ge (Q^k)^{n/m}\psi(Q^k) = \phi^{1/m}(Q^k).\end{aligned}$$
2. If $\|\mathbf{q}\| \ge Q^k$, then $\|\widetilde{\mathbf{r}}\| \ge \|\widetilde{\mathbf{q}}\| = (Q^k)^{-1}\|\mathbf{q}\| \ge 1$.
3. Suppose $1 \le \|\mathbf{q}\| \le Q_0$. Applying the argument of Case 1 to $Q_0\widetilde{\mathbf{r}}$ shows that
$$Q_0\|\widetilde{\mathbf{r}}\| \gtrapprox (Q^k)^{n/m}\psi(Q_0\|\mathbf{q}\|) \ge (Q^k)^{n/m}\psi(Q_0^2)$$
and thus
$$\|\widetilde{\mathbf{r}}\| \gtrapprox (Q^k)^{n/m}Q_0^{-1}\psi(Q_0^2) \xrightarrow[k\to\infty\ (\psi,Q_0)]{} \infty.$$
4. If $\mathbf{q} = \mathbf{0}$ but $\mathbf{p} \ne \mathbf{0}$, then $\|\widetilde{\mathbf{r}}\| = (Q^k)^{n/m}\|\mathbf{r}\| \gtrapprox 1$.

So if k is large enough (depending on ψ and Q_0), then
$$\lambda_1(\Lambda_\omega) \gtrapprox \phi^{1/m}(Q^k)$$
and thus
$$\mathrm{Irr}(\Lambda_\omega) \lessapprox \phi^{-(2d-1)/m}(Q^k) \underset{M_\psi\to 0}{\ll} \exp(\phi^{-1/2}(Q^k)). \qquad \lhd$$

Completion of the proof of Lemma 7.1. By (7.2), we have
$$W_{(1-\varepsilon)\psi,\Lambda_*}(Q_{1,k},Q_{2,k}) \subseteq \bigcup_{\substack{a\in E_{k+1}\\ \mathcal{K}_{\omega a}\subseteq W_\psi(Q_{1,k},Q_{2,k})}} \mathcal{K}_{\omega a} \subseteq \bigcup_{\substack{a\in E_{k+1}\\ \omega a\notin T^{k+1}}} \mathcal{K}_{\omega a}$$
where the second inclusion holds for all k sufficiently large (depending on ψ and Q_0). Thus
$$\begin{aligned}&\frac{1}{(N_{k+1})^D}\#\{a\in E_{k+1} : \omega a \notin T^{k+1}\}\\ &\underset{k\to\infty\ (\psi,Q_0)}{\gtrsim} (N^k)^D\,\lambda_{\mathcal{K}}\left(\mathcal{K}_\omega \cap W_{(1-\varepsilon)\psi}(Q_{1,k},Q_{2,k})\right)\\ &\underset{\substack{\beta\to 0\\ M_\psi^{1/2}/\beta\to 0\ (\varepsilon)\\ k\to\infty\ (\psi,Q_0)}}{\sim} \eta_{\mu,\nu}(1-\varepsilon)\beta \qquad\qquad \text{(by (7.3))}\\ &\underset{\varepsilon\to 0}{\sim} \eta_{\mu,\nu}\beta.\end{aligned}$$
Standard quantifier arguments yield
$$\frac{1}{(N_{k+1})^D}\#\{a\in E_{k+1} : \mathcal{K}_{\omega a}\subseteq W_{\psi,Q_0}\} \underset{\substack{M_\psi^{1/2}/\beta\to 0\\ \beta\to 0\\ k\to\infty\ (\psi,Q_0)}}{\gtrsim} \eta_{\mu,\nu}\beta,$$
and taking the infimum over all $\omega \in T^k$ and then the liminf as $k\to\infty$ completes the proof of Lemma 7.1. $\square$

For each $0<\gamma<1$, by Lemma 7.1 there exists $\xi>0$ (depending only on γ and m,n,μ,ν) such that
$$\begin{cases} M_\psi^{1/2}/\beta \le \xi \\ \beta \le \xi \end{cases} \quad\Rightarrow\quad \begin{cases} \liminf_{k\to\infty} P_k^- > \gamma\eta_{\mu,\nu}\beta \\ M_\psi \alpha \log(2) \le (\gamma^{-1}-1)\beta \end{cases}. \tag{7.5}$$
It follows that there exist $\xi_0>0$ and a decreasing homeomorphism $h:[0,1]\to[0,\xi_0]$ (in particular with $h(1)=0$) such that for all $0<\gamma<1$, (7.5) holds with $\xi = h(\gamma)$. Then the function $c_{\mu,\nu}$ in the statement of Theorem 1.8 can be defined by the formula
$$c_{\mu,\nu}(M) = \begin{cases} (h^{-1}(M^{1/4}))^2\eta_{\mu,\nu} & \text{if } M \le \xi_0^4 \\ 0 & \text{if } M \ge \xi_0^4 \end{cases}.$$

Suppose that $L_{f,\psi} < c_{\mu,\nu}(M_\psi)$ (in particular this implies $c_{\mu,\nu}(M_\psi) > 0$ and thus $M_\psi < \xi_0^4$), and let $\beta \stackrel{\text{def}}{=} \xi \stackrel{\text{def}}{=} M_\psi^{1/4}$ and $\gamma \stackrel{\text{def}}{=} h^{-1}(\xi)$, so that $c_{\mu,\nu}(M_\psi) = \gamma^2\eta_{\mu,\nu}$. Since $M_\psi^{1/2}/\beta = \beta = \xi$, by (7.5) we have $\liminf_{k\to\infty} P_k^- > \gamma\eta_{\mu,\nu}\beta$, say $P_k^- \geq \gamma\eta_{\mu,\nu}\beta$ for all $k \geq k_0 = k_0(\psi, \beta, Q_0, \gamma)$. Then the function f_- from §4.2 satisfies

$$f_-(\rho) \geq \left(\prod_{j=k_0}^{k(\rho)} \frac{1}{1-\gamma\eta_{\mu,\nu}\beta}\right) \rho^D \gtrapprox_{\psi,\beta,Q_0,\gamma} \exp(\gamma\eta_{\mu,\nu}\beta k(\rho))\rho^D.$$

We can relate $\beta k(\rho)$ to ρ via the definition of the sequence $(N_k)_1^\infty$:

$$\begin{aligned}
F_\psi(\rho^{-\alpha}) &\leq F_\psi\big((N^{k(\rho)+1})^\alpha\big) = F_\psi(Q^{k(\rho)+1}) \\
&= \sum_{k=0}^{k(\rho)} F_\psi(Q^k, Q^{k+1}) \\
&\leq \sum_{k=0}^{k(\rho)} [\beta + M_\psi\alpha\log(2)] && \text{(by (4.11))} \\
&\leq (k(\rho)+1)\gamma^{-1}\beta && \text{(by (7.5))} \\
&\approx_{+,\gamma,\beta} \gamma^{-1}\beta k(\rho).
\end{aligned}$$

Combining gives

$$f_-(\rho) \gtrapprox_{\psi,\beta,Q_0,\gamma} \exp(\gamma^2\eta_{\mu,\nu}F_\psi(\rho^{-\alpha}))\rho^D$$

and thus by Proposition 4.1(ii),

$$\mathcal{H}^f(B_{\psi,Q_0}) \lessapprox_{\psi,\beta,Q_0,\gamma} \liminf_{\rho\to 0} \frac{\rho^{-D}f(\rho)}{\exp(\gamma^2\eta_{\mu,\nu}F_\psi(\rho^{-\alpha}))} = \liminf_{\rho\to 0} \frac{\exp\log\left(\frac{f(\rho)}{f_*(\rho)}\right)}{\exp(\gamma^2\eta_{\mu,\nu}F_\psi(\rho^{-\alpha}))}.$$

Since $L_{f,\psi} < c_{\mu,\nu}(M_\psi) = \gamma^2\eta_{\mu,\nu}$, the right-hand side is equal to zero. Since Q_0 was arbitrary, we have $\mathcal{H}^f(B(\psi)) = 0$, which completes the proof.

Appendix A. Index of Notations

A.1. **Norms.** Throughout this paper, $m, n \in \mathbb{N}$ are fixed, as are norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Whenever $\|\cdot\|$ is used without a subscript, it may indicate one of the norms $\|\cdot\|_\mu$ or $\|\cdot\|_\nu$, the norm

$$\|\mathbf{r}\| = \|\mathbf{p}\|_\mu \vee \|\mathbf{q}\|_\nu \quad (\mathbf{r} = (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{m+n}),$$

or the norm

$$\|A\| = \max_{\|\mathbf{q}\|_\nu = 1} \|A\mathbf{q}\|_\mu \quad (A \in \mathcal{M}),$$

depending on context. If S is a set, then $\|S\| = \max_{\mathbf{x}\in S}\|\mathbf{x}\|$, unless S is a lattice in which case $\|S\| = \max_{\mathbf{x}\in\mathbb{R}^d} d(\mathbf{x}, S)$.

A.2. **Shorthand.** The following mathematical objects depend only on m, n, μ, ν:

$$\begin{aligned}
V_\mu &= \text{volume of } B_\mu(\mathbf{0}, 1) = \{\mathbf{p} \in \mathbb{R}^m : \|\mathbf{p}\|_\mu \leq 1\} \\
d &= m + n & D &= mn \\
\delta &= D/d & \alpha &= m/d \\
\eta = \eta_{\mu,\nu} &= n\frac{V_\mu V_\nu}{2\zeta(d)} & \theta = \theta_{\mu,\nu} &= \frac{mn}{m+n} \cdot \frac{V_\mu V_\nu}{2\zeta(d)} \\
\psi_*(q) &= q^{-n/m} & f_*(\rho) &= \rho^{mn} \\
\Lambda_* &= \mathbb{Z}^{m+n}.
\end{aligned}$$

The following objects also depend on ψ and/or the sequence $(N_k)_1^\infty$ and/or the dimension function f:

$$\phi(q) = (\psi(q)/\psi_*(q))^m = q^n\psi^m(q), \qquad \Psi(q) = \psi(q)/q$$

$$F_\psi(Q_1,Q_2) = \int_{Q_1}^{Q_2} q^{n-1}\psi^m(q)\,\mathrm{d}q = \int_{Q_1}^{Q_2}\phi(q)\frac{\mathrm{d}q}{q} \qquad \widetilde{\psi}(q) = \psi(Q_1 \vee q)$$

$$\overline{F}_\psi(Q_1,Q_2) = F_\psi(Q_1,Q_2) + \phi(Q_1) \qquad M_\psi = \psi^m(1) = \phi(1)$$

$$\Delta_\psi(\mathbf{r}) = \{A \in \mathcal{M} : \|A\mathbf{q} - \mathbf{p}\| \le \psi(\|\mathbf{q}\|)\}$$

$$W_{\psi,\Lambda}(Q_1,Q_2) = \bigcup_{\substack{\mathbf{r}\in\Lambda \\ Q_1\le\|\mathbf{q}\|\le Q_2}} \Delta_\psi(\mathbf{r}) \qquad W_\psi(Q_1,Q_2) = W_{\psi,\Lambda_*}(Q_1,Q_2)$$

$$W_{\psi,Q_0} = \bigcup_{Q\ge Q_0} W_\psi(Q_0,Q) \qquad W(\psi) = \bigcap_{Q_0\in\mathbb{N}} W_{\psi,Q_0}$$

$$N^k = \prod_{j=1}^{k} N_j \qquad Q^k = (N^k)^\alpha$$

$$t_k = \delta\log(N^k) = n\log(Q^k) \qquad g^k = g_{t_k}$$

$$L_{f,\psi} = \liminf_{\rho\to 0} \frac{\log\left(\frac{f(\rho)}{f_*(\rho)}\right)}{F_\psi(\rho^{-\alpha})}$$

Remark. In the special case $\psi(q) = \kappa^{1/m}\psi_*(q)$, we get:

$$\phi(q) = \kappa \qquad \Psi(q) = \kappa^{1/m}q^{-1/\alpha}$$

$$N_k = N_* \stackrel{\text{def}}{=} 2^{\lceil\beta/(\kappa\alpha\log(2))\rceil} \qquad N^k = N_*^k \qquad Q^k = N_*^{k\alpha}$$

$$M_\psi = \phi(1) = \kappa \qquad L_{f,\psi} = \frac{mn-s}{\alpha\kappa} \qquad F_\psi(Q_1,Q_2) = \kappa\log(Q_2/Q_1)$$

A.3. **Miscellaneous notation organized by section of introduction.**

- §1: $\mathcal{M}$, $c_{\mu,\nu}$, $C_{\mu,\nu}$
- §2: $\mathcal{H}^f$, $\underline{\mathcal{B}}^f$, $\mathrm{N}_\rho(S)$, $B_\rho(x)$
- §3: $\mathcal{N}(S,\rho)$
- §4: $\approx$, $\lessapprox$, $\sim$, $\lesssim$, $\ll$
- §4.1: $\mathcal{K}$, π, $\mathcal{K}_\omega$, Φ_ω, E_k, E^k
- §4.2: T_ω
- §4.4: Ω_d, g_t, u_A
- §5: $\mathrm{Irr}(\Lambda)$
- §5.2: $\mathrm{Irr}(\mathbf{r})$

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