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Simmons, David orcid.org/0000-0002-9136-6635 and Solomon, Yaar (2016) A Danzer set for Axis Parallel Boxes. Proceedings of the American Mathematical Society. pp. 2725-2729. ISSN 0002-9939

https://doi.org/10.1090/proc/12911

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arXiv:1409.0926v3 [cs.CG] 18 Aug 2015

A DANZER SET FOR AXIS PARALLEL BOXES

DAVID SIMMONS AND YAAR SOLOMON

Abstract. We present concrete constructions of discrete sets in $\mathbb{R}^d$ ($d \geq 2$) that intersect every aligned box of volume 1 in $\mathbb{R}^d$, and which have optimal growth rate $O(T^d)$.

1. Introduction

A set $D \subseteq \mathbb{R}^d$ is called a *Danzer set* if there exists an $s > 0$ such that D intersects every convex set of volume s. The question whether a discrete Danzer set in $\mathbb{R}^d$ of growth rate $O(T^d)$ exists is due to Danzer, see [CFG, Go, GL], and has been open since the sixties.

There are several variants of this question. One is to weaken the Danzer property in the following sense. We say that $Y \subseteq \mathbb{R}^d$ is a *dense forest* if there is a function $\varepsilon = \varepsilon(T) \xrightarrow{T\to\infty} 0$ so that for every $x \in \mathbb{R}^d$ and for every direction $v \in \mathcal{S}^{d-1}$, the distance between Y and the line segment of length T which starts at x and proceeds in direction v is less than $\varepsilon(T)$. Intuitively, as it was presented in [Bi], T is the maximal distance that a man can see when standing in a forest with a trunk of radius ε located at each element of Y. Note that every Danzer set is a dense forest with $\varepsilon(T) = O(T^{-1/(d-1)})$, and a dense forest with $\varepsilon(T) = O(T^{-(d-1)})$ is a Danzer set.[1] A construction of a dense forest of growth rate $O(T^d)$ is given in [SW], and another construction in the plane follows from the proof of [Bi, Lemma 2.4].

One other interesting direction is to look for Danzer sets with faster growth rates. A Danzer set of growth rate $O(T^d(\log T)^{d-1})$ is given in [BW]; this bound was improved recently in [SW] by a probabilistic construction that gives growth rate $O(T^d \log T)$.

[1]The second statement is proven as follows: let D be a dense forest with $\varepsilon(T) = O(T^{-(d-1)})$, and let $R \subseteq \mathbb{R}^d$ be a box (i.e. a parallelotope with adjacent faces orthogonal) with volume s and shortest edge length 2ε. Since the volume of a box is the product of the length of its sides, R has an edge of length at least $T := \left(\frac{s}{2\varepsilon}\right)^{1/(d-1)}$. Let L be the line segment parallel to this edge, passing through the center of R, and of length $T - 2\varepsilon$. If R does not contain any points of D, then the distance from L to D is at least ε, which implies that $\varepsilon \leq O(T^{-(d-1)}) = O(\varepsilon/s)$. For s sufficiently large, this is a contradiction, so every box of sufficiently large volume intersects D. Since every convex set contains a box of volume at least a constant times the volume of the convex set, this shows that D is a Danzer set.

Another approach in trying to weaken the Danzer problem is by hitting a smaller family of sets, instead of all the convex sets. John's theorem [Jo] implies that replacing convex sets by boxes[2] gives an equivalent question. In this note we consider a question that arises naturally from the Danzer problem. We say that $D \subseteq \mathbb{R}^d$ is an *align-Danzer set* if there is an $s > 0$ such that D intersects every aligned box of volume s. In our main results, Theorem 1.1 and Theorem 1.3 below, we present simple constructions for align-Danzer sets in $\mathbb{R}^d$ of growth rate $O(T^d)$. Neither of these constructions is new, but the viewpoint of seeing them as connected with Danzer's problem is new.

We denote by $\{0,1\}^{\mathbb{Z}}_{Fin}$ the subset of $\{0,1\}^{\mathbb{Z}}$ consisting of those bi-infinite sequences that contain only finitely many 1s.

Theorem 1.1. *The set*

$$D \stackrel{\text{def}}{=} \left\{ \left(\pm \sum_{n\in\mathbb{Z}} a_n 2^n, \pm \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}$$

is an align-Danzer set in $\mathbb{R}^2$ of growth rate $O(T^2)$.

The set in Theorem 1.1 is a variant of the binary version of the well-known van der Corput sequence (see e.g. [vdC]).

Although the set D in Theorem 1.1 is given very explicitly, and the proof is by elementary means, it only solves the problem in dimension 2, and no simple higher-dimensional extension comes to mind. To solve the problem in higher dimensions we use a dynamical approach.

For a fixed $d \geq 2$ let $A \subseteq \mathrm{SL}_d(\mathbb{R})$ be the subgroup of diagonal matrices with positive entries, and let Ω be the space of all lattices in $\mathbb{R}^d$.

Definition 1.2 ([Sk, p.6])**.** A lattice $\Lambda \in \Omega$ is *admissible* if its orbit under A is precompact in Ω.

Theorem 1.3 (Corollary of [Sk, Theorem 1.2])**.** *For every $d \geq 2$ there exists an admissible lattice in $\mathbb{R}^d$, and every admissible lattice is an align-Danzer set.*

Although Theorem 1.3 is a direct consequence of [Sk, Theorem 1.2], we provide the proof since it is elementary. We also refer to the discussions in [GL, p. 24-31] for additional reading.

As a direct consequence we reprove a result in computational geometry, that follows from a result of Halton on low discrepancy sequences, see [Ha]. We remark that Corollary 1.4 is not stated in [Ha], but it is well known in the computational geometry and combinatorics communities that Halton's construction satisfies it.

[2]A *box* in $\mathbb{R}^d$ is the image of an aligned box $[a_1, b_1] \times \cdots \times [a_d, b_d]$ under an orthogonal matrix.

Corollary 1.4. *For every $\varepsilon > 0$ there are ε-nets of optimal sizes $O(1/\varepsilon)$ for the range space $(X, \mathcal{R})$, where $X = [0,1]^d$ and $\mathcal{R} = \{$aligned boxes$\}$.*

This Corollary follows directly from the above Theorems by restricting to a bounded cube and rescaling to $[0,1]^d$. We refer to [AS, Ma] for a more comprehensive reading about the notions in Corollary 1.4.

Remark 1.5. Align Danzer sets in $\mathbb{R}^d$ of growth rate $O(T^d)$ can also be constructed by modifying the proof of [SW, Theorem 1.4] to work for aligned boxes and then combining with the result of [Ha] or [vdC] in the unit cube. Nonetheless, our constructions here are simple and the proofs are straightforward.

1.1. **Acknowledgements.** We thank Sathish Govindarajan, Shakhar Smorodinsky, and Barak Weiss for useful discussions that helped us understand the status of the problem. We also thank the referee for helpful comments.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. We first show that D intersects every aligned box of volume 64. It suffices to show that

$$D_+ \stackrel{\text{def}}{=} \left\{ \left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}$$

intersects every aligned box of volume 16 that sits in $\mathbb{R}^2_+ \stackrel{\text{def}}{=} [0,\infty)^2$.

Let $R \subseteq \mathbb{R}^2_+$ be an aligned box of volume 16, and denote its lower left vertex by (x,y). Let $t > 0$ be such that the lower right and the upper left vertices of R are $(x+t, y)$ and $(x, y+\frac{16}{t})$ respectively. We define a sequence $(a_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}_{Fin}$ so that $\left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n}\right) \in R$.

For each integer k, we denote by $\{0,1\}^{\geq k}_{Fin}$ and $\{0,1\}^{<k}_{Fin}$ the subsets of $\{0,1\}^{\geq k}$ and $\{0,1\}^{<k}$, respectively, consisting of those sequences that contain only finitely many 1s. Here $\{0,1\}^{\geq k}$ is the set of all sequences in $\{0,1\}$ of the form $(a_k, a_{k+1}, \ldots)$, and $\{0,1\}^{<k}$ is the set of all sequences in $\{0,1\}$ of the form $(\ldots, a_{k-2}, a_{k-1})$.

Let $k \in \mathbb{Z}$ be such that $2^k \leq \frac{t}{2} < 2^{k+1}$. Observe that $\sum_{n<k} a_n 2^n < 2^k \leq \frac{t}{2}$ for any sequence (a_n) in $\{0,1\}^{<k}_{Fin}$, and that the interval $(x, x+\frac{t}{2})$ intersects the set

$$2^k\mathbb{N} = \left\{ \sum_{n\geq k} a_n 2^n : (a_n) \in \{0,1\}^{\geq k}_{Fin} \right\}.$$

Then we may choose the a_ns for $n \geq k$ so that $\sum_{n\geq k} a_n 2^n \in (x, x+\frac{t}{2})$, and thus for any choice of the a_ns for $n < k$ (and in particular for the choice described below) we have $\sum_{n\in\mathbb{Z}} a_n 2^n \in (x, x+t)$.

The analysis of the y coordinate is similar. Here $2^{-k-1} < \frac{2}{t} \leq 2^{-k}$, and therefore $2^{-k+1} < \frac{8}{t} \leq 2^{-k+2}$. We have $\sum_{n\geq k} a_n 2^{-n} < 2^{-k+1} < \frac{8}{t}$ for any sequence (a_n) in $\{0,1\}^{\geq k}_{Fin}$, and the interval $(y, y+\frac{8}{t})$ intersects the set

$$2^{-k+1}\mathbb{N} = \left\{ \sum_{n<k} a_n 2^{-n} : (a_n) \in \{0,1\}^{<k}_{Fin} \right\}.$$

Then we may choose the a_ns for $n < k$ so that $\sum_{n<k} a_n 2^{-n} \in (y, y+\frac{8}{t})$, and thus for any choice of the a_ns for $n \geq k$ (and in particular for the choice described above) we have $\sum_{n\in\mathbb{Z}} a_n 2^{-n} \in (y, y+\frac{16}{t})$.

It is left to show that D (or D_+) is of growth rate $O(T^2)$. To see that, consider the set

$$B \stackrel{\text{def}}{=} \left\{ \left(\sum_{n\geq 0} a_n 2^n, \sum_{n<0} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\}.$$

Observe that the mapping $g : D_+ \to B$ which is defined in the obvious way by

$$\left(\sum_{n\in\mathbb{Z}} a_n 2^n, \sum_{n\in\mathbb{Z}} a_n 2^{-n} \right) \stackrel{g}{\mapsto} \left(\sum_{n\geq 0} a_n 2^n, \sum_{n<0} a_n 2^{-n} \right)$$

is a bijection, and for any $(x,y) \in D_+$ we have $\|(x,y) - g(x,y)\|_2 \leq \sqrt{5}$ (where $\|\cdot\|_2$ denotes the Euclidean norm). But since $B = \mathbb{N} \times 2\mathbb{N}$, the assertion follows. $\square$

Remark 2.1. We want to stress that D is not a Danzer set in $\mathbb{R}^2$ and not even a dense forest. To see it, observe that symmetric sequences (a_n) correspond to points on the line $y = x$. On the other hand, non-symmetric sequences correspond to points (x,y) with $|x-y| > 1$, and in particular D misses a neighborhood of the line $y = x + \frac{1}{4}$.

3. Proof of Theorem 1.3

Fix $d \geq 2$. Let $V = \{\mathbf{t} \in \mathbb{R}^d : \sum_{i=1}^d t_i = 0\}$, and for each $\mathbf{t} \in V$ let $g_{\mathbf{t}} \in \mathrm{SL}_d(\mathbb{R})$ be the diagonal matrix whose entries are e^{t_i}. Then $\mathbf{t} \mapsto g_{\mathbf{t}}$ is a homomorphism.

Proof of Theorem 1.3. Let K be a totally real number field of degree d, and let $\mathcal{O}_K$ be its ring of integers. Let $\phi_1, \ldots, \phi_d : K \to \mathbb{R}$ be the Galois embeddings of K into $\mathbb{R}$, and let $\Phi : K \to \mathbb{R}^d$ be their direct sum. Then $\Lambda \stackrel{\text{def}}{=} \Phi(\mathcal{O}_K)$ is a lattice in $\mathbb{R}^d$. To see that Λ is admissible, fix $\mathbf{x} = \Phi(\alpha) \in \Lambda$, and observe that if $\mathbf{x} \neq 0$,

$$\prod_{i=1}^{d} |x_i| = \prod_{i=1}^{d} |\phi_i(\alpha)| = |N(\alpha)| \in \mathbb{Z} \smallsetminus \{0\}.$$

Here N denotes the norm in the field K. In particular, $\prod_{i=1}^{d} |x_i| \geq 1$ and thus $\prod_{i=1}^{d} |e^{t_i} x_i| \geq 1$ for all $\mathbf{t} \in V$. It follows that $|e^{t_i} x_i| \geq 1$ for some $i = 1, \ldots, d$ and thus $\|g_{\mathbf{t}}\mathbf{x}\| \geq 1$. Since $\mathbf{t}$, $\mathbf{x}$ were arbitrary, Mahler's compactness criterion shows that Λ is admissible.

For the second part of the proof, let Λ be an admissible lattice in $\mathbb{R}^d$. Let R be an aligned box disjoint from Λ. Then there exists $\mathbf{t} \in V$ such that $g_{\mathbf{t}}R$ is a cube. By assumption $g_{\mathbf{t}}\Lambda$ is in a compact subset $K \subseteq \Omega$, hence the codiameter[3] of $g_{\mathbf{t}}\Lambda$ is bounded above by a constant independent of $\mathbf{t}$. But since $g_{\mathbf{t}}R$ is disjoint from $g_{\mathbf{t}}\Lambda$, the distance from the center of $g_{\mathbf{t}}R$ to the complement of $g_{\mathbf{t}}R$, i.e. half the edge length of the cube $g_{\mathbf{t}}R$, is bounded above by the distance from the center of $g_{\mathbf{t}}R$ to $g_{\mathbf{t}}\Lambda$, which is in turn bounded above by the codiameter of $g_{\mathbf{t}}\Lambda$. Thus both the diameter and the volume of $g_{\mathbf{t}}R$ are bounded above by a constant independent of $\mathbf{t}$. Since $\mathrm{Vol}(R) = \mathrm{Vol}(g_{\mathbf{t}}R)$, the proof is complete. □

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[3]The *codiameter* of a lattice $\Gamma \subseteq \mathbb{R}^d$ is the diameter of the quotient space $\mathbb{R}^d/\Gamma$ (with respect to the quotient metric $d([\mathbf{x}], [\mathbf{y}]) = \min\{\|\mathbf{y}-\mathbf{x}\| : \mathbf{x}, \mathbf{y} \text{ representatives of } [\mathbf{x}], [\mathbf{y}]\}$), or equivalently the maximum of the function $\mathbb{R}^d \ni \mathbf{x} \mapsto d(\mathbf{x}, \Gamma)$. The codiameter is continuous as a function of the lattice.

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