



# Wave fronts and cascades of soliton interactions in the periodic two dimensional Volterra system



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## HIGHLIGHTS

- The explicit formulas for soliton solutions of arbitrary rank are derived.
- A new class of exact solutions corresponding to wave fronts is presented.
- A full classification of rank 1 solutions is given.
- Soliton solutions similar to breathers resemble soliton webs in the KP theory.
- The full classification is associated with the Schubert decomposition of the Grassmannians.

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## ABSTRACT

In the paper we develop the dressing method for the solution of the two-dimensional periodic Volterra system with a period  $N$ . We derive soliton solutions of arbitrary rank  $k$  and give a full classification of rank 1 solutions. We have found a new class of exact solutions corresponding to wave fronts which represent smooth interfaces between two nonlinear periodic waves or a periodic wave and a trivial (zero) solution. The wave fronts are non-stationary and they propagate with a constant average velocity. The system also has soliton solutions similar to breathers, which resembles soliton webs in the KP theory. We associate the classification of soliton solutions with the Schubert decomposition of the Grassmannians  $Gr_{\mathbb{R}}(k, N)$  and  $Gr_{\mathbb{C}}(k, N)$ .

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## 1. Introduction

Constructing explicit exact solutions for integrable systems is an important and well developed area of research. Various methods have been designed to tackle this problem and many studies have considered soliton solutions of rank one. In the present study, we propose a dressing method for application to a periodic two-dimensional Volterra system and we derive explicit soliton solutions of arbitrary rank. We find solutions that resemble breathers (in the theory of the sine-Gordon equation), nonlinear periodic waves, and new types of exact solutions for integrable systems, which comprise smooth interfaces between two nonlinear periodic waves, or a periodic wave and a trivial (zero) solution.

The dressing method for Lax integrable systems was originally formulated and developed by [1,2]. Its predecessor was proposed

by Bargmann (1949) [3], who performed dressings of the Schrödinger operator and discovered potentials, which we now associate with the profiles of one and two soliton solutions for the Korteweg de-Vries (KdV) equation. The connection between the potentials of the Schrödinger operators and solutions of the KdV equation was established much later by Gardner, Greene, Kruskal, and Miura, who discovered the inverse spectral transform [4]. A year later, an elegant interpretation of their results was given by Lax in [5], where the concept of a Lax pair was first proposed.

In this study, we propose a dressing method and consider exact solutions for the two-dimensional generalization of the periodic Volterra lattice [6,7]

$$\begin{cases} \phi_t^{(i)} = \theta_x^{(i)} + \theta^{(i)} \phi_x^{(i)} - e^{2\phi^{(i-1)}} + e^{2\phi^{(i+1)}}, \\ \phi^{(i+N)} = \phi^{(i)}, \quad \theta^{(i+N)} = \theta^{(i)}, \\ \theta^{(i+1)} - \theta^{(i)} + \phi_x^{(i+1)} + \phi_x^{(i)} = 0, \\ \sum_{i=1}^N \phi^{(i)} = \sum_{i=1}^N \theta^{(i)} = 0. \end{cases} \quad (1)$$

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System (1) can be regarded as an integrable discretization of the Kadomtsev–Petviashvili (KP) equation (see Section 3.3). The KP equation was originally derived for ion-acoustic waves of small amplitude in plasma [8] and it is a 2 + 1-dimensional integrable generalization of the KdV equation. Its mathematical theory had a major impact on the theory of integrable equations and led to useful notions such as the  $\tau$  function and Sato Grassmannian [9]. The KP equation has a rich set of exact solutions, the classification of which require advanced techniques from cluster algebra, tropical geometry, and combinatorics, as developed by [10–12].

Eq. (1) was first derived in 1979 based on the reduction group theory for Lax representation [6]. For a fixed period  $N$ , the variables  $\theta^{(i)}$  can be eliminated, and thus (1) can be rewritten as a system of  $(N - 1)$ -component second order evolutionary equations. In the simplest nontrivial case where  $N = 3$ , system (1) becomes

$$\begin{cases} 3\phi_t^{(1)} = \phi_{xx}^{(1)} + 2\phi_{xx}^{(2)} + 2\phi_x^{(1)}\phi_x^{(2)} + \phi_x^{(1)2} \\ \quad + 3e^{2\phi^{(2)}} - 3e^{-2\phi^{(1)}-2\phi^{(2)}} \\ 3\phi_t^{(2)} = -2\phi_{xx}^{(1)} - \phi_{xx}^{(2)} - 2\phi_x^{(1)}\phi_x^{(2)} - \phi_x^{(2)2} \\ \quad - 3e^{2\phi^{(1)}} + 3e^{-2\phi^{(1)}-2\phi^{(2)}} \end{cases} \quad (2)$$

and after a point transformation it takes the form of a nonlinear Schrödinger type equation (system “u4” in [13])

$$iu_t = u_{xx} + (u_x^*)^2 + e^{-2u-2u^*} + \omega^* e^{-2\omega u-2\omega^* u^*} \\ + \omega e^{-2\omega^* u-2\omega u^*}, \quad \omega = e^{\frac{2\pi i}{3}},$$

where  $\star$  denotes complex conjugation. In this case, the system is bi-Hamiltonian. A recursion operator and bi-Hamiltonian structure for system (2) were explicitly constructed from its Lax representation by [14]. A certain class of Darboux transformations for arbitrary fixed period  $N$  was constructed recently by [15].

For infinite  $N$ , Eq. (1) is an integrable differential-difference equation in 2 + 1 dimensions, which was described by [16], who classified a family of equations with non-locality of intermediate long wave type. This equation’s infinitely many symmetries and conserved densities were constructed using its master symmetry by [17]. This case was studied using the inverse scattering transform by [18] who obtained some special solutions.

Bargmann’s potentials correspond to a rational (in the wave number) factor relative to the Jost function [3]. In our dressing method, we also start with a rational in the spectral parameter  $\lambda$  matrix factor  $\Phi(\lambda)$ , which modifies the fundamental solution of the “undressed” Lax pair. In the case of system (1), the Lax operators contain  $N \times N$  matrices and are invariant with respect to a reduction group isomorphic to  $\mathbb{Z}_2 \times \mathbb{D}_N$ . We construct the reduction group invariant dressing factors  $\Phi(\lambda)$ , which have  $N$  or  $2N$  simple poles that belong to the orbits generated by the transformations  $\lambda \mapsto \omega\lambda$ ,  $\lambda \mapsto \lambda^*$ , where  $\omega = \exp(\frac{2\pi i}{N})$ . The case of  $N$  simple poles leads to a new class of solutions, which we call kink solutions, and solutions corresponding to the orbits with  $2N$  poles are referred to as breathers. This terminology is borrowed from the sine-Gordon theory, where a kink solution corresponds to a dressing factor with one pole and two poles factor leads to a breather solution [19,20]. We could also construct  $(n, m)$  multisoliton solutions with  $n$  kinks and  $m$  breathers, but this generalization is rather straightforward, and thus we focus on solutions corresponding to a single orbit (i.e., one kink and one breather solutions) in the present study.

A kink solution can be parameterized by a real number  $v \notin \{\pm 1, 0\}$  and a point on a real Grassmannian  $\text{Gr}_{\mathbb{R}}(k, N)$ , whereas a breather solution can be uniquely parameterized by a complex number  $\mu \in \mathbb{C}$  such that  $|\mu| \notin \{1, 0\}$ ,  $\text{Im} \mu^N \neq 0$  and a point on a complex Grassmannian  $\text{Gr}_{\mathbb{C}}(k, N)$ . The number  $k$  in  $\text{Gr}(k, N)$  is the rank of the soliton solution. There is a difference between the cases with even and odd  $N$ . When  $N$  is even, there are two different orbits with  $N$  points, i.e.,  $\{v\omega^k\}_{k=1}^N$  and  $\{v\omega^{k+\frac{1}{2}}\}_{k=1}^N$ . This results in

two different kink solutions. A fine classification of wave interfaces (in the kink case) and soliton interactions (in the breather case) can be obtained naturally in terms of the invariant Schubert cell decomposition of the Grassmannian. In particular, elementary line breathers and periodic kink solutions correspond to the one-dimensional invariant Schubert cells of the Grassmannians.

Kink solutions represent regions filled by nonlinear periodic waves with moving interfaces between the regions (Fig. 1). Thus, we also call them wave fronts. Breathers correspond to a cascade of soliton decays and fusions. The density plots of a breather solution in the  $(x, t)$ -plane resemble soliton webs of the KP equations in the  $(x, y)$ -plane for a fixed moment in time (Fig. 2). In this study, we give explicit and detailed derivations for these two types of soliton solutions of arbitrary rank for the two-dimensional Volterra system (1) as well as giving a complete classification of the rank 1 kink and breather solutions.

The remainder of this paper is organized as follows. In Section 2, we recall the Lax representation for Eq. (1) and its reduction group. In Section 3, we discuss the dressing method in the presence of the reduction group. We explicitly derive both kink and breather solutions for Eq. (1) in a trivial background using the dressing method. The solutions obtained are all regular, which is proved in the Appendix. All exact solutions emerging from the dressing method can be written in the form

$$\phi^{(i)} = \frac{1}{2} \ln \frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}. \quad (3)$$

Under a certain continuous limit  $N \rightarrow \infty$ , Eq. (1) converges to the well known KP equation. In this limit,  $\tau_i$  in (3) can be related to the Hirota  $\tau$ -function for the bilinear form of the KP equation [21]. In Section 4, we classify both kink and breather solutions of rank 1 according to the eigenspaces of the constant matrix  $\Delta$  in the Lax operators of Eq. (1). For rank 1 kink solutions, we start with a description of all possible rank 1 kink solutions in the cases where  $N = 3, 4$  and we also prove the general results for arbitrary dimensions. For rank 1 breather solutions, we present some typical configurations and the general result based on the number of possible distinct configurations. Our definition soliton graphs based on tropicalization were motivated by [10], although we do not have the structure associated with the Wronskian of solutions. In the Conclusion, we summarize our results and discuss the feasibility of a full classification of higher rank solutions.

## 2. Lax representation and the dihedral reduction group

Let us consider general matrix operators of the form

$$\hat{L}(\lambda) = D_x + \mathbf{X}(x, t, \lambda), \quad \hat{M}(\lambda) = D_t + \mathbf{T}(x, t, \lambda), \quad (4)$$

where  $D_x$  and  $D_t$  are the total derivatives in  $x$  and  $t$  respectively,  $\lambda \in \mathbb{C}$  is a spectral parameter,  $\mathbf{X}$  and  $\mathbf{T}$  are real  $N \times N$  traceless matrices

$$\mathbf{X}(x, t, \lambda) = \mathbf{X}_0 + \lambda^{-1}\mathbf{U} + \lambda\bar{\mathbf{U}} \\ \mathbf{T}(x, t, \lambda) = \mathbf{T}_0 + \lambda^{-1}\mathbf{A} + \lambda\bar{\mathbf{A}} + \lambda^{-2}\mathbf{B} + \lambda^2\bar{\mathbf{B}},$$

and the matrices  $\mathbf{X}_0$ ,  $\mathbf{U}$ ,  $\bar{\mathbf{U}}$ ,  $\mathbf{T}_0$ ,  $\mathbf{A}$ ,  $\bar{\mathbf{A}}$ ,  $\mathbf{B}$  and  $\bar{\mathbf{B}}$  are functions of  $x$  and  $t$ . The compatibility condition  $[\hat{L}, \hat{M}] = 0$ , i.e.,

$$\mathbf{T}_x - \mathbf{X}_t + [\mathbf{X}, \mathbf{T}] = 0, \quad (5)$$

gives  $7(N^2 - 1)$  partial differential equations (coefficients of  $\lambda^{-3}, \dots, \lambda^3$ ) for  $7N^2$  matrix entries.

We define a group of automorphisms generated by the following two transformations for an operator  $\mathbf{d}(\lambda)$ , where the first is

$$\iota : \mathbf{d}(\lambda) \mapsto -\mathbf{d}^\dagger(\lambda^{-1}), \quad (6)$$

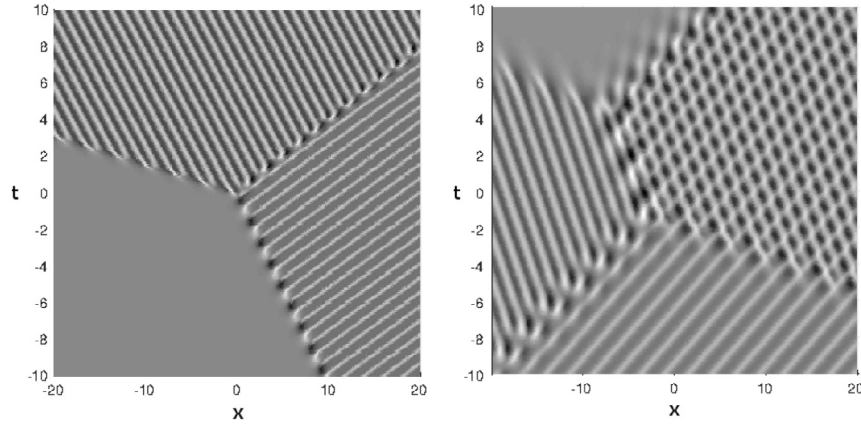


Fig. 1. Density plots of  $\phi^{(1)}(x, t)$  for  $N = 5$  of rank 1 and rank 2 kink solutions.

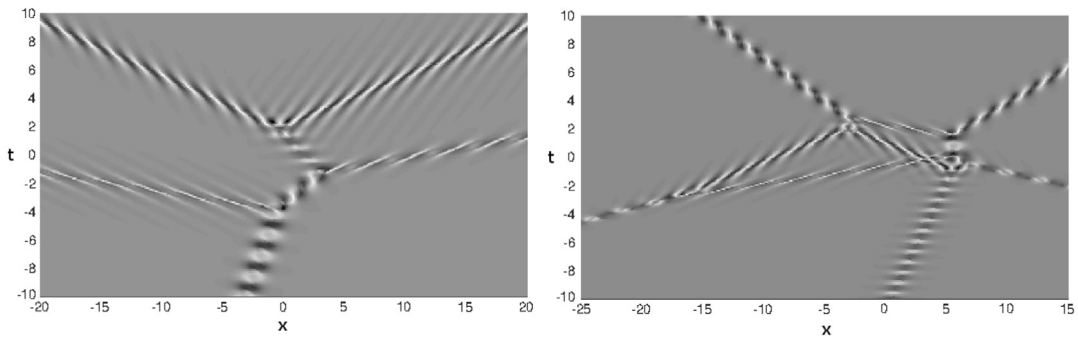


Fig. 2. Density plots of  $\phi^{(1)}(x, t)$  for  $N = 5$  of rank 1 and rank 2 breather solutions.

where  $\mathbf{d}^\dagger$  is the adjoint operator of the operator  $\mathbf{d}$ ; and the second is

$$s : \mathbf{d}(\lambda) \mapsto Q\mathbf{d}(\omega^{-1}\lambda)Q^{-1}, \quad (7)$$

$$Q = \text{diag}(\omega^i), \quad \omega = \exp \frac{2\pi i}{N}.$$

These two non-commuting transformations satisfy

$$\iota^2 = s^N = \text{id}, \quad \iota s \iota = s^{-1},$$

and thus they generate the dihedral group denoted by  $\mathbb{D}_N$ . We refer to the group generated by transformations (6) and (7) as the reduction group [6,7,22–25]. Note that transformation  $\iota$  (6) is an outer automorphism of the Lie algebra  $sl(n)$  over the Laurent polynomial ring  $\mathbb{C}[\lambda, \lambda^{-1}]$ .

**Proposition 1.** *If the linear operators (4) are invariant with respect to the reduction group  $\mathbb{D}_N$ , then they can be written in the following form*

$$\hat{L} = D_x + \lambda^{-1}\mathbf{u}\Delta - \lambda\Delta^{-1}\mathbf{u} \quad (8)$$

$$\hat{M} = D_t + \lambda^{-1}\mathbf{a}\Delta - \lambda\Delta^{-1}\mathbf{a} + \lambda^{-2}\mathbf{b}\Delta^2 - \lambda^2\Delta^{-2}\mathbf{b}, \quad (9)$$

where  $\mathbf{u}, \mathbf{a}, \mathbf{b}$  are diagonal matrices and  $\Delta$  is the shift operator given by

$$\Delta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (10)$$

**Proof.** From the invariance under the transformation  $\iota$  (6), i.e.,

$$(\hat{L}(\lambda), \hat{M}(\lambda)) = (-\hat{L}^\dagger(\lambda^{-1}), -\hat{M}^\dagger(\lambda^{-1})),$$

it follows that

$$\begin{aligned} \bar{\mathbf{U}} &= -\mathbf{U}^{\text{tr}}, & \bar{\mathbf{A}} &= -\mathbf{A}^{\text{tr}}, & \bar{\mathbf{B}} &= -\mathbf{B}^{\text{tr}}, \\ \mathbf{X}_0 &= -\mathbf{X}_0^{\text{tr}}, & \mathbf{T}_0 &= -\mathbf{T}_0^{\text{tr}}, \end{aligned} \quad (11)$$

where  $\text{tr}$  denotes matrix transposition. The invariance under the transformation  $s$  implies that  $\mathbf{X}_0, \mathbf{T}_0$  are diagonal and that the matrices  $\mathbf{U}, \mathbf{A}$  and  $\mathbf{B}$  have the form of

$$\mathbf{U} = \mathbf{u}\Delta, \quad \mathbf{A} = \mathbf{a}\Delta, \quad \mathbf{B} = \mathbf{b}\Delta^2,$$

where  $\mathbf{u}, \mathbf{a}$  and  $\mathbf{b}$  are diagonal matrices and  $\Delta$  is given by (10). By combining with (11), we obtain  $\mathbf{X}_0 = \mathbf{T}_0 = 0$  as well as expressions (8) and (9) in the statement.  $\square$

Let  $\mathbf{u} = \text{diag}(u^{(i)})$ ,  $\mathbf{a} = \text{diag}(a^{(i)})$  and  $\mathbf{b} = \text{diag}(b^{(i)})$ . Then, the compatibility condition of Lax operators (8) and (9) leads to  $3N$  equations

$$u^{(i)}b^{(i+1)} - b^{(i)}u^{(i+2)} = 0, \quad (12)$$

$$D_x(b^{(i)}) + u^{(i)}a^{(i+1)} - a^{(i)}u^{(i+1)} = 0, \quad (13)$$

$$D_t(u^{(i)}) = D_x(a^{(i)}) - u^{(i-1)}b^{(i-1)} + b^{(i)}u^{(i+2)} \quad (14)$$

in  $3N$  variables  $u^{(i)}, a^{(i)}$  and  $b^{(i)}$ , where  $i = 1, \dots, N$ . In this study, we assume that all the upper indices, which take values from 1 to  $N$ , are counted modulo  $N$ , unless stated otherwise. Take

$$\begin{aligned} \mathbf{u} &= \text{diag}(\exp(\phi^{(1)}), \dots, \exp(\phi^{(N)})), \\ \mathbf{a} &= \text{diag}(\theta^{(1)}\exp(\phi^{(1)}), \dots, \theta^{(N)}\exp(\phi^{(N)})). \end{aligned} \quad (15)$$

From (12)–(14), it follows that we can set  $b^{(i)} = \exp(\phi^{(i)} + \phi^{(i+1)})$  and  $\sum_{i=1}^N \phi^{(i)} = \sum_{i=1}^N \theta^{(i)} = 0$  without any loss of generality. In the variables  $\phi^{(i)}$  and  $\theta^{(i)}$ , the system of equations (12)–(14) leads to the two-dimensional generalization of the Volterra system (1).

The corresponding operators (8) and (9) can be expressed as the invariant operators under the reduction group  $\mathbb{D}_N$ , i.e.,

$$\begin{aligned}\mathcal{L} &= D_x + U, & U &= \lambda^{-1}\mathbf{u}\Delta - \lambda\Delta^{-1}\mathbf{u}, \\ \mathcal{M} &= D_t + V, \\ V &= \lambda^{-2}\mathbf{u}\Delta\mathbf{u}\Delta + \lambda^{-1}\mathbf{a}\Delta - \lambda\Delta^{-1}\mathbf{a} - \lambda^2\Delta^{-1}\mathbf{u}\Delta^{-1}\mathbf{u},\end{aligned}\quad (16)$$

where  $\mathbf{u}$  and  $\mathbf{a}$  are defined by (15), and the matrix  $\Delta$  is given by (10). The condition of the commutativity of these operators

$$[\mathcal{L}, \mathcal{M}] = D_x(V) - D_t(U) + [U, V] = 0 \quad (17)$$

leads to the two-dimensional generalization of the Volterra lattice (1) [6,7]. This is often called a zero curvature representation or Lax representation of Eq. (1). These two operators,  $\mathcal{L}$  and  $\mathcal{M}$ , are usually called the Lax pair.

If we assume that the functions  $\phi^{(k)}, \theta^{(k)}$  in (15) are real, then the Lax operators  $\mathcal{L}, \mathcal{M}$  are also invariant with respect to transformation

$$r : \mathcal{L}(\lambda) \mapsto \mathcal{L}^*(\lambda^*), \quad \mathcal{M}(\lambda) \mapsto \mathcal{M}^*(\lambda^*), \quad (18)$$

where  $*$  denotes its complex conjugate. This transformation extends the dihedral group. The group generated by  $s, \iota, r$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{D}_N$ .

### 3. Rational dressing method for the generalized Volterra lattice

In this section, we use the rational dressing method [1,2,7] to construct new exact solutions of (1) starting from a known exact solution. The regularity of all these solutions is proved in the Appendix.

Let us denote  $U_0, V_0$  as the matrices  $U, V$  where  $\phi^{(i)}$  are replaced by the known exact solution  $\phi_0^{(i)}, i = 1, \dots, N$  of (1), i.e.,

$$\begin{aligned}U_0 &= \lambda^{-1}\mathbf{u}_0\Delta - \lambda\Delta^{-1}\mathbf{u}_0, \\ V_0 &= \lambda^{-2}\mathbf{u}_0\Delta\mathbf{u}_0\Delta + \lambda^{-1}\mathbf{a}_0\Delta - \lambda\Delta^{-1}\mathbf{a}_0 - \lambda^2\Delta^{-1}\mathbf{u}_0\Delta^{-1}\mathbf{u}_0.\end{aligned}$$

The corresponding overdetermined linear system

$$\mathcal{L}_0\psi_0 = (D_x + U_0)\psi_0 = 0, \quad \mathcal{M}_0\psi_0 = (D_t + V_0)\psi_0 = 0 \quad (19)$$

has a common fundamental solution  $\psi_0(\lambda, x, t)$ . Following [1,2], we assume that the fundamental solution  $\psi(\lambda, x, t)$  for the new (“dressed”) linear problems

$$\mathcal{L}\psi = (D_x + U)\psi = 0, \quad \mathcal{M}\psi = (D_t + V)\psi = 0 \quad (20)$$

is of the form

$$\psi = \Phi(\lambda)\psi_0, \quad \det \Phi \neq 0, \quad (21)$$

where the dressing matrix  $\Phi(\lambda)$  is assumed to be rational in the spectral parameter  $\lambda$  and invariant with respect to the symmetries

$$\Phi^{-1}(\lambda^{-1}) = \Phi^{\text{tr}}(\lambda); \quad (22)$$

$$Q\Phi(\omega^{-1}\lambda)Q^{-1} = \Phi(\lambda). \quad (23)$$

Conditions (22) and (23) guarantee that the corresponding Lax operators  $\mathcal{L}$  and  $\mathcal{M}$  are invariant under transformations (6) and (7).

We now derive real solutions for the real equation. Thus, we also require

$$\Phi^*(\lambda^*) = \Phi(\lambda). \quad (24)$$

From (19), (20), and (21), it follows that

$$\Phi(D_x + U_0)\Phi^{-1} = U; \quad (25)$$

$$\Phi(D_t + V_0)\Phi^{-1} = V. \quad (26)$$

These equations allow us to specify the form of the dressing matrix  $\Phi$  and construct the corresponding “dressed” solution  $\phi^{(i)}$  of Eq. (1).

Let us consider the most trivial case where the dressing matrix  $\Phi$  does not depend on the spectral parameter  $\lambda$ . In this case, the dressing matrix does not result in any new solutions.

**Proposition 2.** Assume that  $\Phi$  is a  $\lambda$  independent dressing matrix for the two-dimensional generalization of the Volterra lattice (1) and  $\phi_0^{(i)}$  is a real solution. If it satisfies (22)–(24), then the matrix is  $\Phi = \pm I_N$ , where  $I_N$  is the  $N \times N$  identity matrix and the real solutions on the background  $\phi_0^{(i)}$  are  $\phi^{(i)} = \phi_0^{(i)}$ .

**Proof.** Under the assumption that the dressing matrix  $\Phi$  is independent of the spectral parameter  $\lambda$ , from (25) and (26), it follows that

$$\begin{aligned}D_x\Phi &= 0; \quad D_t\Phi = 0; \quad \Phi\mathbf{u}_0\Delta = \mathbf{u}\Delta\Phi; \quad \Phi\Delta^{-1}\mathbf{u}_0 = \Delta^{-1}\mathbf{u}\Phi; \\ \Phi\mathbf{a}_0\Delta &= \mathbf{a}\Delta\Phi; \quad \Phi\Delta^{-1}\mathbf{a}_0 = \Delta^{-1}\mathbf{a}\Phi.\end{aligned}\quad (27)$$

It is obvious that the matrix  $\Phi$  is independent of  $x$  and  $t$ .  $\Phi$  satisfies (22)–(24), so we deduce that matrix  $\Phi$  is real,  $\Phi\Phi^{\text{tr}} = I_N$ , and  $\Phi$  is diagonal. Thus, the constant matrix  $\Phi$  has  $\pm 1$  on the diagonal. By substituting this  $\Phi$  into (27), we obtain  $\Phi = \pm I_N$  and  $\phi^{(i)} = \phi_0^{(i)}$  because both  $\phi^{(i)}$  and  $\phi_0^{(i)}$  are real.  $\square$

A  $\lambda$ -dependent dressing matrix  $\Phi(\lambda)$ , which is invariant with respect to the symmetries (22)–(24), has poles at the orbits of the reduction group. The simplest “one soliton” dressing corresponds to the cases where the matrix  $\Phi(\lambda)$  has only simple poles belonging to a single orbit.

Note that if  $\Phi(\lambda)$  is invariant under the reduction group, then so is  $\Phi^{-1}(\lambda)$ . Instead of specifying the poles for  $\Phi(\lambda)$ , we first specify the poles for  $\Phi^{-1}$ , and then determine  $\Phi$  from the relation (22). If  $\Phi^{-1}(\lambda)$  has a pole at the point  $\mu$ , then by the second relation (23) (for  $\Phi^{-1}$ ), it must also have poles at the points  $\omega^{-1}\mu, \omega^{-2}\mu, \dots, \omega^{-(N-1)}\mu$ . Due to (24), there are two non-trivial cases.

- (1) The matrix  $\Phi^{-1}(\lambda)$  has  $N$  poles:
  - (i) for arbitrary  $N$  poles at  $\omega^{-k}\mu, k = 0, \dots, N-1, \mu \neq 0, \mu \neq \pm 1, \mu = \mu^*$ ;
  - (ii) when  $N$  is even, i.e.,  $N = 2m$ , poles at  $\omega^{-k}\mu, k = 0, \dots, N-1, \mu = v \exp(\frac{\pi i}{N}), v \in \mathbb{R}, v \notin \{\pm 1, 0\}$ .
- (2) The matrix  $\Phi^{-1}(\lambda)$  has  $2N$  complex poles at  $\omega^{-k}\mu$  and  $\omega^{-k}\mu^*$ , where  $k = 0, \dots, N-1$  and
 
$$\mu \in \mathbb{C}, |\mu| \neq 1, \mu \neq \omega^k \mu^*, k = 1, \dots, N.$$

Note that when  $N = 2m + 1$  in case (ii), then  $\omega^m \mu = -v \in \mathbb{R}$ . Hence, this case is included in case (i) for odd  $N$ . The extra conditions on  $\mu$  aim to ensure that the poles are distinct for  $\Phi$  and  $\Phi^{-1}$ . These cases correspond to the “kink” and “breather” solutions, respectively.

The explicit forms of the matrix  $\Phi^{-1}(\lambda)$  corresponding to the two cases above as well as being invariant with respect to the symmetries (23) and (24) are

- (1) (i)  $\Phi^{-1}(\lambda) = C + \sum_{k=0}^{N-1} \frac{Q^{-k}AQ^k}{\lambda\omega^k - \mu}, \quad A = A^*,$   

$$\mu = \mu^*, \mu \neq 0, \mu \neq \pm 1;$$
- (ii)  $\Phi^{-1}(\lambda) = C + \sum_{k=0}^{2m-1} \frac{Q^{-k}AQ^k}{\lambda\omega^k - \mu}, \quad N = 2m,$   

$$A^* = \omega^{-1}Q^{-1}AQ, \mu = v \exp\left(\frac{\pi i}{2m}\right), v \in \mathbb{R}, v \notin \{\pm 1, 0\};$$
- (2)  $\Phi^{-1}(\lambda) = C + \sum_{k=0}^{N-1} \left( \frac{Q^{-k}AQ^k}{\lambda\omega^k - \mu} + \frac{Q^{-k}A^*Q^k}{\lambda\omega^k - \mu^*} \right),$   

$$|\mu| \neq 1, \mu \neq \omega^k \mu^*, k = 1, \dots, N,$$

where  $C$  and  $A$  are  $\lambda$ -independent matrices of size  $N \times N$ . Moreover, to satisfy (23) and (24), we have  $C = QCQ^{-1}$ , which implies that  $C$  is diagonal and  $C = C^*$ . Hence, we assume that

$$C = \text{diag}(c_1, \dots, c_N), \quad (28)$$

where  $c_i, i = 1, \dots, N$  are real functions of  $x$  and  $t$ .

We now derive the conditions on the matrices  $A$  and  $C$  such that  $\Phi^{-1}(\lambda)$  satisfies (22). In this case, we have  $\Phi(\lambda) = (\Phi^{-1}(\lambda^{-1}))^{\text{tr}}$ . Thus, it follows that

$$(1) \quad \Phi(\lambda) = C + \sum_{k=0}^{N-1} \frac{Q^k A^{\text{tr}} Q^{-k}}{\lambda^{-1} \omega^k - \mu}, \quad A = A^*, \quad \mu = \mu^*, \text{ or}$$

$$N = 2m, \quad \mu = \nu \exp\left(\frac{\pi i}{N}\right), \quad A^* = \omega^{-1} Q^{-1} A Q; \quad (29)$$

$$(2) \quad \Phi(\lambda) = C + \sum_{k=0}^{N-1} \left( \frac{Q^k A^{\text{tr}} Q^{-k}}{\lambda^{-1} \omega^k - \mu} + \frac{Q^k A^*{}^{\text{tr}} Q^{-k}}{\lambda^{-1} \omega^k - \mu^*} \right). \quad (30)$$

**Proposition 3.** Let  $I_N$  denote the  $N \times N$  identity matrix. The dressing matrix satisfies (22) if and only if matrix  $A$  and the real diagonal matrix  $C$  satisfy the relations:

$$\lim_{\lambda \rightarrow \infty} \Phi(\lambda) = C^{-1}; \quad \Phi(\mu)A = 0. \quad (31)$$

**Proof.** We verify that  $\Phi(\lambda)$  above is indeed the inverse matrix of  $\Phi^{-1}(\lambda)$  by checking that  $\Phi(\lambda)\Phi^{-1}(\lambda) = I_N$ . The product is a rational matrix function of  $\lambda$ . By taking the limit at  $\lambda = \infty$  we obtain  $\lim_{\lambda \rightarrow \infty} \Phi(\lambda)C = I_N$ , which implies the first equation in (31). Under the assumptions on  $\mu$ , the poles of both  $\Phi$  and  $\Phi^{-1}$  are simple and distinct. Therefore,  $\Phi(\lambda)\Phi^{-1}(\lambda)$  has simple poles. By requiring the vanishing of the residue at  $\lambda = \mu$ , we obtain the second equation in (31). The residues at all other points of the reduction group orbit will vanish due to the manifest invariants of the expression with respect to the reduction group.  $\square$

We now investigate the conditions (25) and (26) for  $\Phi(\lambda)$ , which follow from the fact that it is a dressing matrix. Note that  $\Phi$ ,  $U_0$  and  $\Phi^{-1}$  have distinct simple poles. Thus, the left-hand side of (25) only has simple poles. First, we compare the residues at the pole  $\mu$ . It follows that

$$\lim_{\lambda \rightarrow \mu} (\lambda - \mu)\Phi(D_x + U_0(\lambda))\Phi^{-1} = \Phi(\mu)(D_x + U_0(\mu))A = 0. \quad (32)$$

Thus,  $(D_x + U_0(\mu))A \in \ker \Phi(\mu)$ . In a similar manner, from condition (26), it follows that

$$\Phi(\mu)(D_t + V_0(\mu))A = 0,$$

that is,  $(D_t + V_0(\mu))A \in \ker \Phi(\mu)$ . (33)

We compute the residue at  $\lambda = \infty$  on both sides of (25) and we have

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \Phi(\lambda)(D_x + U_0(\lambda))\Phi^{-1}(\lambda)$$

$$= - \lim_{\lambda \rightarrow \infty} \Phi(\lambda)\Delta^{-1}\mathbf{u}_0 C = -\Delta^{-1}\mathbf{u}.$$

Using (31), we have

$$C^{-1}\Delta^{-1}\mathbf{u}_0 C = \Delta^{-1}\mathbf{u}. \quad (34)$$

This formula provides the relation between  $\mathbf{u}_0$  and  $\mathbf{u}$ . However, it is necessary to determine the diagonal matrix  $C$  in the dressing matrix  $\Phi$ , which depends on the choice of the form of  $\Phi(\lambda)$ . We determine this when we compute the kink and breather solutions.

In the following two sections, we construct the exact solutions starting with the trivial solution  $\phi_0^{(i)} = 0, i = 1, \dots, N$  for Eq. (1). In this case,  $\mathbf{u}_0 = I_N$  and  $\mathbf{a}_0 = 0$ . Thus, we have

$$\mathcal{L}_0 = D_x + \lambda^{-1}\Delta - \lambda\Delta^{-1}; \quad \mathcal{M}_0 = D_t + \lambda^{-2}\Delta^2 - \lambda^2\Delta^{-2}.$$

In this case, it is easy to see that the fundamental solution for (19) is

$$\Psi_0(x, t, \lambda) = \exp((-\lambda^{-1}\Delta + \lambda\Delta^{-1})x$$

$$- (\lambda^{-2}\Delta^2 - \lambda^2\Delta^{-2})t). \quad (35)$$

Obviously, the matrix  $\Psi_0(x, t, \lambda)$  satisfies the reduction group symmetry conditions (22)–(24).

On the trivial background, (34) becomes  $C^{-1}\Delta^{-1}C = \Delta^{-1}\mathbf{u}$ . It follows that

$$\exp(\phi^{(j)}) = \frac{c_j}{c_{j+1}}. \quad (36)$$

We use this later to construct solutions for (1) on the trivial background.

### 3.1. Kink solutions

In this section, we derive the explicit formulae for kink solutions of arbitrary ranks. As discussed earlier, a kink solution for Eq. (1) corresponds to the invariant dressing matrix with  $N$  simple poles, which has the form of (29). There is a difference when the dimension of  $N$  is even or odd. If  $N$  is odd, there is only one case where  $A = A^*, \mu \in \mathbb{R}$  and  $\mu \notin \{\pm 1, 0\}$ . If  $N$  is even, there is an extra case where  $\mu = \nu \exp(\frac{\pi i}{N})$  with  $\nu \in \mathbb{R}$  and  $A^* = \omega^{-1} Q^{-1} A Q$ . This difference is caused by the real requirement (24). Hence, we first derive the expressions for  $\mu$  and  $A$ , and then add the conditions for them.

For all cases, the matrix  $C$  defined by (28) is diagonal with real functions  $c_i, i = 1, \dots, N$  on the diagonal. Moreover, from Proposition 3, it follows that

$$\lim_{\lambda \rightarrow \infty} \Phi(\lambda) = C - \frac{1}{\mu} \sum_{k=0}^{N-1} Q^k A^{\text{tr}} Q^{-k}$$

$$= C - \mu^{-1} N A_{\text{diag}} = C^{-1}; \quad (37)$$

$$\Phi(\mu)A = \left( C + \sum_{k=0}^{N-1} \frac{Q^k A^{\text{tr}} Q^{-k}}{\mu^{-1} \omega^k - \mu} \right) A = 0. \quad (38)$$

In (37), we use the identity  $\sum_{k=0}^{N-1} Q^k A^{\text{tr}} Q^{-k} = N A_{\text{diag}}$ , which can be proved by directly computing its entries (see Lemma 1 in Appendix of [15]).

**Proposition 4.** If matrix  $A$  is nondegenerate in the dressing matrix (29), then the real solutions for (1) are  $\phi^{(i)} = 0$  on the trivial background  $\phi_0^{(i)} = 0$ , where  $i = 1, \dots, N$ .

**Proof.** If  $\det A \neq 0$ , then from (38), we obtain

$$C + \sum_{k=0}^{N-1} \frac{Q^k A^{\text{tr}} Q^{-k}}{\mu^{-1} \omega^k - \mu} = 0,$$

which implies that matrix  $A$  is diagonal since  $C$  is diagonal, and thus

$$C = - \sum_{k=0}^{N-1} \frac{Q^k A^{\text{tr}} Q^{-k}}{\mu^{-1} \omega^k - \mu} = -\mu \sum_{k=0}^{N-1} \frac{1}{\omega^k - \mu^2} A = \frac{N\mu^{2N-1}}{\mu^{2N} - 1} A.$$

We use Lemma 2, which was proved in the appendix of [15], and we state that for  $x^N \neq 1$  and  $\omega = \exp(\frac{2\pi i}{N})$ ,

$$\sum_{j=0}^{N-1} \frac{\omega^{lj}}{x - \omega^j} = \frac{N x^{(l-1) \bmod N}}{x^N - 1}. \quad (39)$$

By substituting this into (37), we obtain

$$C^2 - \frac{N}{\mu} A C = \frac{1}{\mu^{2N}} C^2 = I_N, \quad \text{i.e.,} \quad C^2 = \mu^{2N} I_N.$$

Thus, we have  $c_i = \pm \mu^N$  when  $\mu \in \mathbb{R}$ , or  $c_i = \pm \nu^N$  when  $\mu = \nu \exp(\frac{\pi i}{N})$  with  $\nu \in \mathbb{R}$ . From (36), it follows that

$$\exp(\phi^{(i)}) = \frac{c_i}{c_{i+1}} = 1$$

since  $\phi^{(i)} \in \mathbb{R}$ . Thus, we obtain the trivial solutions for  $\phi^{(i)}$ , as given in the statement.  $\square$

In order to construct solutions that depend on  $x$  and  $t$ , we consider the case where matrix  $A$  is of rank  $r \leq N - 1$ , and thus it can be represented in the form of

$$A = \mathbf{nm}^{\text{tr}},$$

where  $\mathbf{n}$  and  $\mathbf{m}$  are both  $N \times r$  matrices of rank  $r$ . We define the rank of the kink solutions as the rank of matrix  $A$  in the dressing matrix. In this case, Eq. (38) becomes

$$\left( C + \sum_{k=0}^{N-1} \frac{Q^k \mathbf{nm}^{\text{tr}} Q^{-k}}{\mu^{-1} \omega^k - \mu} \right) \mathbf{n} = 0 \quad (40)$$

since  $\mathbf{m}$  is the  $N \times r$  matrix of rank  $r$ . We can use this to solve for  $\mathbf{m}$  in terms of  $\mathbf{n}$ . Furthermore, we can determine matrix  $C$  in terms of  $A$  by using (37).

**Remark 1.** The dressing matrix  $\Phi(\lambda)$  (29) is parameterized by a matrix  $\mathbf{n}$  lying on a Grassmannian. Indeed, it is assumed that we obtain  $\mathbf{m} = F(\mathbf{n})$  from (40). If we make a change  $\hat{\mathbf{n}} = \mathbf{n}W$ , where  $W$  is an invertible  $r \times r$  matrix, then the corresponding solution is  $\hat{\mathbf{m}} = F(\mathbf{n})(W^{-1})^{\text{tr}} = \mathbf{m}(W^{-1})^{\text{tr}}$ . Therefore, the matrix is

$$\hat{A} = \hat{\mathbf{n}}\hat{\mathbf{m}}^{\text{tr}} = \mathbf{n}W W^{-1} \mathbf{m}^{\text{tr}} = \mathbf{nm}^{\text{tr}} = A.$$

From (40), it follows that  $\mathbf{n} \in \ker \Phi(\mu)$ . From (32), we obtain

$$0 = \Phi(\mu)(D_x + U_0(\mu))\mathbf{nm}^{\text{tr}} \\ = \Phi(\mu)(D_x + U_0(\mu))(\mathbf{n})\mathbf{m}^{\text{tr}} + \Phi(\mu)(\mathbf{nm}_x^{\text{tr}}),$$

which implies that  $(D_x + U_0(\mu))(\mathbf{n}) \in \ker \Phi(\mu)$ . Thus, a scalar function  $\gamma(x, t)$  exists such that

$$(D_x + U_0(\mu))(\mathbf{n}) = \gamma(x, t)\mathbf{n}. \quad (41)$$

Similarly, we can show that a scalar function  $\delta(x, t)$  exists such that

$$(D_t + V_0(\mu))(\mathbf{n}) = \delta(x, t)\mathbf{n}. \quad (42)$$

The compatibility of the operators  $D_x + U_0$  and  $D_t + V_0$  implies that  $\gamma_t = \delta_x$ . Thus, we let  $\gamma = \eta_x$  and  $\delta = \eta_t$ , where  $\eta$  is a potential function, so we can deduce that

$$\mathbf{n} = \exp(\eta)\Psi_0(x, t, \mu)\mathbf{n}_0,$$

where  $\mathbf{n}_0$  is a constant  $N \times r$  matrix and  $\Psi_0(x, t, \mu)$  is the fundamental solution of the linear differential equations defined by  $\mathcal{L}_0(\mu)\Psi_0 = 0$  and  $\mathcal{M}_0(\mu)\Psi_0 = 0$ . According to Remark 1, the dressing matrix  $\Phi$  is invariant under a rescaling of the matrix  $\mathbf{n}$ , so we can simply take

$$\mathbf{n} = \Psi_0(x, t, \mu)\mathbf{n}_0. \quad (43)$$

In the following, we explicitly construct kink solutions of arbitrary ranks.

### 3.1.1. Rank 1 kink solutions

We consider the matrix  $A = \mathbf{nm}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are vectors. As discussed earlier, we first solve for  $\mathbf{m}$  using Eq. (40), i.e.,

$$\left( C + \sum_{k=0}^{N-1} \frac{Q^k \mathbf{nm}^{\text{tr}} Q^{-k}}{\mu^{-1} \omega^k - \mu} \right) \mathbf{n} = 0. \quad (44)$$

We then determine the diagonal matrix  $C$  and write the rank 1 solutions as follows.

**Lemma 1.** Let matrix  $A$  be a bi-vector and  $A = \mathbf{nm}^{\text{tr}}$ . If the dressing matrix given by (29) satisfies (22), then the entries of the diagonal matrix  $C$  are given by

$$c_i^2 = \mu^2 \frac{\tau_{i-1}}{\tau_i}, \quad \tau_i = \frac{1}{\mu^{2N} - 1} \sum_{l=1}^N n_l^2 \mu^{2\{(i-l) \bmod N\}}, \quad (45)$$

where  $n_i$  are the components of the vector  $\mathbf{n}$ .

**Proof.** Under the assumption, we find that  $\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}$  is a scalar function. Thus, the matrix

$$W = \sum_{k=0}^{N-1} \frac{Q^k \mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\mu^{-1} \omega^k - \mu} = \sum_{k=0}^{N-1} \frac{\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n} Q^k}{\mu^{-1} \omega^k - \mu}$$

is diagonal and the entries on the diagonal are

$$W_{ii} = \sum_{k=0}^{N-1} \sum_{l=1}^N n_l^2 \omega^{-lk} \frac{\omega^{ik}}{\mu^{-1} \omega^k - \mu} = \mu \sum_{l=1}^N n_l^2 \sum_{k=0}^{N-1} \frac{\omega^{(i-l)k}}{\omega^k - \mu^2} \\ = -\mu \sum_{l=1}^N n_l^2 \frac{N \mu^{2\{(i-l-1) \bmod N\}}}{\mu^{2N} - 1},$$

where we use (39). Thus,  $W$  is invertible because  $\mu \neq 0$  and  $|\mu| \neq 1$ . By substituting this into (44), we obtain the vector  $\mathbf{m}$  with components

$$m_i = \frac{\mu^{2N} - 1}{\mu^N} \frac{c_i n_i}{\sum_{l=1}^N n_l^2 \mu^{2\{(i-l-1) \bmod N\}}}. \quad (46)$$

The matrix  $C$  can be determined using Eq. (37), which is equivalent to

$$c_i - c_i^{-1} = \mu^{-1} N m_i n_i = \frac{\mu^{2N} - 1}{\mu^2} \frac{c_i n_i^2}{\sum_{l=1}^N n_l^2 \mu^{2\{(i-l-1) \bmod N\}}}.$$

This leads to

$$c_i^2 = \frac{\mu^2 \sum_{l=1}^N n_l^2 \mu^{2\{(i-l-1) \bmod N\}}}{\mu^2 \sum_{l=1}^N n_l^2 \mu^{2\{(i-l-1) \bmod N\}} - (\mu^{2N} - 1) n_i^2} = \mu^2 \frac{\tau_{i-1}}{\tau_i},$$

where  $\tau_i$  is defined by (45).  $\square$

We now use (36) to derive the real solution for  $\phi^{(i)}$ . For the case where  $\mu = \mu^* = \nu$  and  $A = A^*$ , we only need to choose  $\mathbf{n}_0$  as a real valued constant vector. Using (43) and (35), we can determine the real vector  $\mathbf{n}$ , which leads to  $c_i^2 > 0$ . According to (36), the solutions are

$$\phi^{(i)} = \ln \left( \frac{c_i}{c_{i+1}} \right) = \frac{1}{2} \ln \left( \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \right).$$

Thus, we have the following result.

**Proposition 5.** Let  $\mathbf{n}_0$  be a constant real vector and  $\nu \in \mathbb{R}$ ,  $\nu \neq 0$ ,  $\nu \neq \pm 1$ . A rank 1 kink solution of the system (1) on a trivial background  $\phi^{(i)} = 0$ ,  $i = 1, \dots, N$ , is given by

$$\phi^{(i)} = \frac{1}{2} \ln \left( \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \right), \quad \tau_i = \frac{1}{\nu^{2N} - 1} \sum_{l=1}^N n_l^2 \nu^{2\{(i-l) \bmod N\}}, \quad (47)$$

where  $n_i$  are the components of the vector

$$\mathbf{n} = \exp((\nu \Delta^{-1} - \nu^{-1} \Delta)x - (\nu^{-2} \Delta^2 - \nu^2 \Delta^{-2})t) \mathbf{n}_0.$$

For the case where  $\mu = \nu \exp(\frac{\pi i}{N})$  with  $\nu \in \mathbb{R}$  and  $A^* = \omega^{-1} Q^{-1} A Q$ , we use the following statement to obtain the real solutions.

**Proposition 6.** Let  $\mathbf{n}_0$  be a constant vector that satisfies  $\mathbf{n}_0 = Q \mathbf{n}_0^*$ . For  $\mu = \nu \exp(\frac{\pi i}{N})$ , where  $\nu \in \mathbb{R}$  and  $\nu \neq 0$ ,  $\nu \neq \pm 1$ , a rank 1

kink solution of the system (1) on a trivial background  $\phi^{(i)} = 0, i = 1, \dots, N$ , is given by

$$\phi^{(i)} = \frac{1}{2} \ln \left( \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \right), \tag{48}$$

$$\tau_i = \frac{1}{\mu^{2N} - 1} \sum_{l=1}^N n_l^2 \mu^{2\{(i-l) \bmod N\}},$$

where  $n_i$  are the components of the vector

$$\mathbf{n} = \exp((\mu\Delta^{-1} - \mu^{-1}\Delta)x - (\mu^{-2}\Delta^2 - \mu^2\Delta^{-2})t)\mathbf{n}_0.$$

**Proof.** To prove the statement, we only need to show that  $c_i$  given by (45) are real. First, we note that

$$\mu = \omega\mu^*, \quad \Delta Q = \omega Q \Delta, \quad \Delta^{-1}Q = \omega^{-1}Q \Delta^{-1}.$$

Using these identities, we can show that

$$\begin{aligned} & ((\mu\Delta^{-1} - \mu^{-1}\Delta)x - (\mu^{-2}\Delta^2 - \mu^2\Delta^{-2})t) Q \\ &= Q \left( (v\omega^{-\frac{1}{2}}\Delta^{-1} - v^{-1}\omega^{\frac{1}{2}}\Delta)x - (v^{-2}\omega\Delta^2 - v^2\omega^{-1}\Delta^{-2})t \right) \\ &= Q \left( (\mu^*\Delta^{-1} - \mu^{*-1}\Delta)x - (\mu^{*-2}\Delta^2 - \mu^{*2}\Delta^{-2})t \right), \end{aligned}$$

which leads to  $\Psi_0(x, t, \mu)Q = Q\Psi_0(x, t, \mu^*)$ . Using (43), we have

$$\mathbf{n} = \Psi_0(x, t, \mu)\mathbf{n}_0 = \Psi_0(x, t, \mu)Q\mathbf{n}_0^* = Q\Psi_0(x, t, \mu^*)\mathbf{n}_0^* = Q\mathbf{n}^*,$$

i.e.,  $n_l = \omega^l n_l^*$ . By substituting these into (46), we obtain

$$m_i^* = \frac{\nu^{2N} - 1}{\mu\omega^{-1N}} \frac{c_i \omega^{-i} n_i}{\omega^{-2i+2} \sum_{l=1}^N n_l^2 \mu^{2\{(i-l-1) \bmod N\}}} = \omega^{i-1} m_i, \tag{49}$$

which implies that  $\mathbf{m} = \omega Q^{-1} \mathbf{m}^*$ . Thus,  $A = \mathbf{nm}^{\text{tr}} = \omega Q A^* Q^{-1}$ . Finally, we show that  $c_i$  are real. Indeed,

$$\tau_i^* = \frac{1}{\mu^{*2N} - 1} \sum_{l=1}^N n_l^{*2} \mu^{*2\{(i-l) \bmod N\}} = \omega^{-2i} \tau_i.$$

Therefore, we have  $c_i^{*2} = \mu^{*2} \frac{\tau_{i-1}^*}{\tau_i^*} = \mu^2 \frac{\tau_{i-1}}{\tau_i} = c_i^2$ . Using (36), we derive the real solution for  $\phi^{(i)}$ , as given in the statement.  $\square$

Note that this Proposition is valid for arbitrary dimension  $N$ . However, it only leads to new solutions that differ from those obtained in Proposition 5 when  $N$  is even.

### 3.1.2. Rank $r > 1$ kink solutions

In this case, the rank  $r$  matrix  $A = \mathbf{nm}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are  $N \times r$  matrices of rank  $r$ . As discussed earlier, we first solve  $\mathbf{m}$  in terms of  $\mathbf{n}$  using (40). From (37), it follows that

$$C = C^{-1} + \mu^{-1} \sum_{k=0}^{N-1} Q^k \mathbf{m} \mathbf{n}^{\text{tr}} Q^{-k}. \tag{50}$$

By substituting this into (40), we obtain

$$C^{-1} \mathbf{n} + \frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \mathbf{m} \mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\omega^k \mu^{-1} - \mu} = 0. \tag{51}$$

Let  $\tilde{\mathbf{m}} = C\mathbf{m}$ . Then, (50) and (51) become

$$\begin{aligned} C^2 &= I_N + \mu^{-1} N(\tilde{\mathbf{m}} \mathbf{n}^{\text{tr}})_{\text{diag}}; \\ \frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \tilde{\mathbf{m}} \mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{-1}} &= \mathbf{n}. \end{aligned} \tag{52}$$

We denote the  $j$ th rows of  $\mathbf{n}$  and  $\tilde{\mathbf{m}}$  as  $\mathbf{n}_j$  and  $\tilde{\mathbf{m}}_j$ , respectively. From the second identity in (52), it follows that

$$\begin{aligned} \mathbf{n}_j &= \tilde{\mathbf{m}}_j \frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} \mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{-1}} = \tilde{\mathbf{m}}_j \frac{1}{\mu^2} \mathbf{n}^{\text{tr}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} Q^{-k}}{\mu - \omega^k \mu^{-1}} \mathbf{n} \\ &= \frac{N}{\mu(\mu^{2N} - 1)} \tilde{\mathbf{m}}_j \mathbf{n}^{\text{tr}} S(j) \mathbf{n}, \end{aligned} \tag{53}$$

where  $S(j)$  is an  $N \times N$  diagonal matrix and the  $i$ th diagonal entry is equal to  $\mu^{2\{(j-i) \bmod N\}}$ . We can determine  $\mathbf{m}$  as well as matrix  $C$  in the dressing matrix as follows.

**Lemma 2.** Let rank  $r$  matrix  $A = \mathbf{nm}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are  $N \times r$  matrices of rank  $r$ . If the dressing matrix given by (29) satisfies (22), then the entries for diagonal matrix  $C$  are given by

$$c_j^2 = \mu^{2r} \frac{\tau_{j-1}}{\tau_j}, \quad \tau_j = \det R(j), \quad R(j) = \frac{1}{\mu^{2N} - 1} \mathbf{n}^{\text{tr}} S(j) \mathbf{n}, \tag{54}$$

where  $S(j)$  is an  $N \times N$  diagonal matrix and the  $i$ th diagonal entry is equal to  $\mu^{2\{(j-i) \bmod N\}}$ .

**Proof.** Using the notations given in the statement, from (53), it follows that

$$\tilde{\mathbf{m}}_j = \frac{\mu}{N} \mathbf{n}_j R(j)^{-1}.$$

We substitute this into the first equation in (52) and determine that the  $j$ th diagonal entry of the diagonal matrix  $C$  satisfies

$$c_j^2 = 1 + \sum_{\alpha, \beta=1}^r \mathbf{n}_{j\alpha} \{R(j)^{-1}\}_{\alpha, \beta} \mathbf{n}_{j\beta}. \tag{55}$$

The explicit formula for the entries at  $(\alpha, \beta)$  in  $R(j)$  is equal to

$$R(j)_{\alpha, \beta} = \frac{1}{\mu^{2N} - 1} \sum_{k=1}^N n_{k\alpha} n_{k\beta} \mu^{2\{(j-k) \bmod N\}}.$$

It is easy to show the identity between the entries in matrices  $R(j)$  and  $R(j-1)$ :

$$R(j)_{\alpha, \beta} = \mu^2 R(j-1)_{\alpha, \beta} - n_{j\alpha} n_{j\beta},$$

which implies that

$$R(j) = \mu^2 R(j-1) - \mathbf{n}_j^{\text{tr}} \mathbf{n}_j.$$

Using Sylvester's determinant theorem, this leads to

$$\mu^{2r} \det R(j-1) = \det R(j) (1 + \mathbf{n}_j R(j)^{-1} \mathbf{n}_j^{\text{tr}}).$$

By comparing this with (55), we obtain

$$c_j^2 = \frac{\mu^{2r} \det R(j-1)}{\det R(j)} = \frac{\mu^{2r} \tau_{j-1}}{\tau_j},$$

which is (54) in the statement.  $\square$

Using (36), we obtain the solutions in the statement, where  $\mathbf{n}$  is determined by (43) and (35). In the case where  $\mu = \mu^* = \nu$  and  $A = A^*$ , we only need to choose  $\mathbf{n}_0$  as a real valued constant matrix of size  $N \times r$  in order to guarantee that the solutions are real. We state the result as follows.

**Proposition 7.** Let  $\mathbf{n}_0$  be a rank  $r$  constant real matrix of size  $N \times r$  and  $\mu = \mu^* = \nu \in \mathbb{R}, \nu \neq 0, \nu \neq \pm 1$ . A rank  $r$  kink solution of the system (1) on a trivial background  $\phi^{(i)} = 0, i = 1, \dots, N$ , is given by

$$\begin{aligned} \phi^{(i)} &= \frac{1}{2} \ln \left( \frac{\tau_{j-1} \tau_{j+1}}{\tau_j^2} \right), \quad \tau_j = \det R(j), \\ R(j) &= \frac{1}{\nu^{2N} - 1} \mathbf{n}^{\text{tr}} S(j) \mathbf{n}, \end{aligned} \tag{56}$$

where  $S(j)$  is an  $N \times N$  diagonal matrix and the  $i$ th diagonal entry is equal to  $v^{2(j-i) \bmod N}$ , and

$$\mathbf{n} = \exp((v\Delta^{-1} - v^{-1}\Delta)x - (v^{-2}\Delta^2 - v^2\Delta^{-2})t)\mathbf{n}_0.$$

Note that by taking  $r = 1$  in Proposition 7, we obtain the results in Proposition 5.

In the case where  $\mu = v \exp(\frac{\pi i}{N})$  with  $v > 0$  and  $A^* = \omega^{-1}Q^{-1}AQ$ , we use a similar result to Proposition 6 to obtain kink solutions of rank  $r$ .

**Proposition 8.** Let  $\mathbf{n}_0$  be a rank  $r$  constant matrix of size  $N \times r$  that satisfies  $\mathbf{n}_0 = Q\mathbf{n}_0^*$ . For  $\mu = v \exp(\frac{\pi i}{N})$ , where  $v \neq 0$  and  $v \neq 1$ , a rank  $r$  kink solution of system (1) on a trivial background  $\phi^{(i)} = 0$ ,  $i = 1, \dots, N$ , is given by

$$\phi^{(i)} = \frac{1}{2} \ln \left( \frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2} \right), \quad \tau_j = \det R(j), \quad (57)$$

$$R(j) = \frac{1}{\mu^{2N} - 1} \mathbf{n}^{\text{tr}} S(j) \mathbf{n},$$

where  $S(j)$  is an  $N \times N$  diagonal matrix and the  $i$ th diagonal entry is equal to  $\mu^{2(j-i) \bmod N}$ , and

$$\mathbf{n} = \exp((\mu\Delta^{-1} - \mu^{-1}\Delta)x - (\mu^{-2}\Delta^2 - \mu^2\Delta^{-2})t)\mathbf{n}_0.$$

**Proof.** Similar to the proof of Proposition 6, under the assumption, we have  $\mathbf{n} = Q\mathbf{n}^*$ , which leads to

$$\tau_j^* = \det R(j)^* = \det \left( \frac{1}{\mu^{*2N} - 1} \mathbf{n}^{*\text{tr}} S(j)^* \mathbf{n}^* \right)$$

$$= \det \left( \frac{\omega^{-2j}}{\mu^{2N} - 1} \mathbf{n}^{\text{tr}} S(j) \mathbf{n} \right) = \omega^{-2j} \tau_j.$$

Therefore, we have  $c_j^{*2} = \mu^{*2r} \frac{\tau_{j-1}^*}{\tau_j^*} = \mu^{2r} \frac{\tau_{j-1}}{\tau_j} = c_j^2$ . Using (36), we can derive the real solution for  $\phi^{(i)}$  as given in the statement.  $\square$

Similar to Proposition 6, although this Proposition is valid for arbitrary dimension  $N$ , it only leads to new solutions that differ from those obtained in Proposition 7 when  $N$  is even.

### 3.2. Breather solutions

A breather solution corresponds to the simple poles at the points of a generic orbit of the reduction group. The corresponding dressing matrix has the form where  $A$  is a  $\lambda$ -independent matrix of size  $N \times N$  and the matrix  $C$  defined by (28) is diagonal with real functions  $c_i$ ,  $i = 1, \dots, N$  on the diagonal. Moreover, from Proposition 3, it follows that

$$\lim_{\lambda \rightarrow \infty} \Phi(\lambda) = C - \frac{1}{\mu} \sum_{k=0}^{N-1} Q^k A^{\text{tr}} Q^{-k} - \frac{1}{\mu^*} \sum_{k=0}^{N-1} Q^k A^{*\text{tr}} Q^{-k}$$

$$= C - \frac{1}{\mu} N A_{\text{diag}} - \frac{1}{\mu^*} N A_{\text{diag}}^* = C^{-1}; \quad (58)$$

$$\Phi(\mu)A = \left( C + \sum_{k=0}^{N-1} \frac{Q^k A^{\text{tr}} Q^{-k}}{\mu^{-1}\omega^k - \mu} + \sum_{k=0}^{N-1} \frac{Q^k A^{*\text{tr}} Q^{-k}}{\mu^{-1}\omega^k - \mu^*} \right) A$$

$$= 0. \quad (59)$$

If matrix  $A$  is invertible, then in a similar manner to Proposition 4 for the kink case, we can find the real solutions for  $\phi^{(i)} = 0$  on the trivial background. Hence, we assume that the rank of  $A$  is  $r \leq N - 1$ .

In the same manner as the case for kinks in Section 3.1, we present  $A = \mathbf{nm}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are both  $N \times r$  matrices of

rank  $r$ . It is obvious that the dressing matrix  $\Phi(\lambda)$  (30) is again parameterized by a matrix  $\mathbf{n}$  lying on a Grassmannian, and we also have

$$\mathbf{n} = \Psi_0(x, t, \mu)\mathbf{n}_0, \quad (60)$$

where  $\mathbf{n}_0$  is an  $N \times r$  matrix of rank  $r$ .

#### 3.2.1. Rank 1 breather solutions

In a similar manner to the case of the rank 1 kink, we consider the matrix  $A = \mathbf{nm}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are vectors. Then, (59) becomes

$$\left( C + \sum_{k=0}^{N-1} \frac{Q^k \mathbf{m} \mathbf{n}^{\text{tr}} Q^{-k}}{\mu^{-1}\omega^k - \mu} + \sum_{k=0}^{N-1} \frac{Q^k \mathbf{m}^* \mathbf{n}^{*\text{tr}} Q^{-k}}{\mu^{-1}\omega^k - \mu^*} \right) \mathbf{n} = 0. \quad (61)$$

First, we use this to determine  $\mathbf{m}$  in terms of  $\mathbf{n}$ . Then, we use (36) to compute the solutions  $\phi^{(i)}$  for Eq. (1).

**Proposition 9.** Let  $\mathbf{n}_0$  be a constant vector and  $\mu \in \mathbb{C}$ ,  $|\mu| \neq 1$ ,  $\mu \neq \omega^k \mu^*$ ,  $k = 1, \dots, N$ . A rank 1 breather solution of the system (1) on a trivial background  $\phi^{(i)} = 0$ ,  $i = 1, \dots, N$ , is given by

$$\phi^{(i)} = \frac{1}{2} \ln \left( \frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2} \right), \quad \tau_i = \rho(i)^2 - |\sigma(i)|^2, \quad (62)$$

where

$$\sigma(i) = \frac{1}{\mu^{2N} - 1} \sum_{l=1}^N n_l^2 \mu^{2\{(i-l) \bmod N\}}; \quad (63)$$

$$\rho(i) = \frac{1}{|\mu|^{2N} - 1} \sum_{l=1}^N |n_l|^2 |\mu|^{2\{(i-l) \bmod N\}}$$

and  $n_i$  are the components of the vector

$$\mathbf{n} = \exp((\mu\Delta^{-1} - \mu^{-1}\Delta)x - (\mu^{-2}\Delta^2 - \mu^2\Delta^{-2})t)\mathbf{n}_0.$$

**Proof.** Under the assumption, we find that  $\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}$  and  $\mathbf{n}^{*\text{tr}} Q^{-k} \mathbf{n}^*$  are scalar functions. Thus, we define

$$D = \sum_{k=0}^{N-1} \frac{\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n} Q^k}{\mu^{-1}\omega^k - \mu}; \quad E = \sum_{k=0}^{N-1} \frac{\mathbf{n}^{*\text{tr}} Q^{-k} \mathbf{n}^* Q^k}{\mu^{-1}\omega^k - \mu^*},$$

which are diagonal and the entries on the diagonal are

$$D_{ii} = \sum_{k=0}^{N-1} \sum_{l=1}^N n_l^2 \omega^{-lk} \frac{\omega^{ik}}{\mu^{-1}\omega^k - \mu} = \mu \sum_{l=1}^N n_l^2 \sum_{k=0}^{N-1} \frac{\omega^{(i-l)k}}{\omega^k - \mu^2}$$

$$= -\mu \sum_{l=1}^N n_l^2 \frac{N \mu^{2\{(i-l-1) \bmod N\}}}{\mu^{2N} - 1};$$

$$E_{ii} = \sum_{k=0}^{N-1} \sum_{l=1}^N |n_l|^2 \omega^{-lk} \frac{\omega^{ik}}{\mu^{-1}\omega^k - \mu^*} = \mu \sum_{l=1}^N |n_l|^2 \sum_{k=0}^{N-1} \frac{\omega^{(i-l)k}}{\omega^k - |\mu|^2}$$

$$= -\mu \sum_{l=1}^N |n_l|^2 \frac{N |\mu|^{2\{(i-l-1) \bmod N\}}}{|\mu|^{2N} - 1},$$

where we use (39). Using the notations defined by (63), we rewrite them as

$$D_{ii} = -\mu N \sigma(i-1); \quad E_{ii} = -\mu N \rho(i-1). \quad (64)$$

By writing the entries of (61), we have

$$c_i n_i + D_{ii} m_i + E_{ii} m_i^* = 0.$$

We solve this for  $m_i$  and it follows that

$$m_i = c_i \frac{D_{ii}^* n_i - E_{ii} n_i^*}{|E_{ii}|^2 - |D_{ii}|^2}.$$



The matrix  $C$  can be determined using Eq. (58), which is equivalent to

$$\begin{aligned} c_i - c_i^{-1} &= N \left( \frac{1}{\mu} m_i n_i + \frac{1}{\mu^*} m_i^* n_i^* \right) \\ &= \frac{N c_i}{|E_{ii}|^2 - |D_{ii}|^2} \left( \frac{1}{\mu} D_{ii}^* n_i^2 + \frac{1}{\mu^*} D_{ii} n_i^{*2} \right. \\ &\quad \left. - \left( \frac{1}{\mu} E_{ii} + \frac{1}{\mu^*} E_{ii}^* \right) |n_i|^2 \right). \end{aligned}$$

This leads to

$$c_i^2 = \frac{|E_{ii}|^2 - |D_{ii}|^2}{|E_{ii}|^2 - |D_{ii}|^2 - \frac{N}{\mu} D_{ii}^* n_i^2 - \frac{N}{\mu^*} D_{ii} n_i^{*2} + N \left( \frac{1}{\mu} E_{ii} + \frac{1}{\mu^*} E_{ii}^* \right) |n_i|^2}.$$

By substituting (64) into this, we obtain

$$c_i^2 = |\mu|^4 \frac{\rho(i-1)^2 - |\sigma(i-1)|^2}{\rho(i)^2 - |\sigma(i)|^2},$$

where  $\sigma(i)$  and  $\rho(i)$  are defined in (63). We use the identities between  $\sigma(i-1)$  and  $\sigma(i)$ , and  $\rho(i-1)$  and  $\rho(i)$  as follows:

$$\mu^2 \sigma(i-1) - n_i^2 = \sigma(i); \quad |\mu|^2 \rho(i-1) - |n_i|^2 = \rho(i).$$

From (36), it follows that

$$\phi^{(i)} = \ln \left( \frac{c_i}{c_{i+1}} \right) = \frac{1}{2} \ln \left( \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \right),$$

where  $\tau_i$  is defined by (62). The vector  $\mathbf{n}$  is determined using (60) and (35).  $\square$

### 3.2.2. Rank $r > 1$ breather solutions

In this case, the rank  $r$  matrix  $A = \mathbf{n}\mathbf{m}^{\text{tr}}$ , where  $\mathbf{n}$  and  $\mathbf{m}$  are  $N \times r$  matrices. From (58), it follows that

$$C = C^{-1} + \frac{1}{\mu} \sum_{k=0}^{N-1} Q^k \mathbf{m}\mathbf{n}^{\text{tr}} Q^{-k} + \frac{1}{\mu^*} \sum_{k=0}^{N-1} Q^k \mathbf{m}^* \mathbf{n}^{*\text{tr}} Q^{-k}. \quad (65)$$

By substituting this into (59), we obtain

$$\begin{aligned} C^{-1} \mathbf{n} + \frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \mathbf{m}\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\omega^k \mu^{-1} - \mu} \\ + \frac{1}{\mu^{*2}} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \mathbf{m}^* \mathbf{n}^{*\text{tr}} Q^{-k} \mathbf{n}}{\omega^k \mu^{*-1} - \mu} = 0. \end{aligned} \quad (66)$$

Let  $\tilde{\mathbf{m}} = C\mathbf{m}$ . Then, (65) and (66) become

$$C^2 = I_N + \frac{1}{\mu} N(\tilde{\mathbf{m}}\mathbf{n}^{\text{tr}})_{\text{diag}} + \frac{1}{\mu^*} N(\tilde{\mathbf{m}}^* \mathbf{n}^{*\text{tr}})_{\text{diag}}; \quad (67)$$

$$\frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \tilde{\mathbf{m}}\mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{-1}} + \frac{1}{\mu^{*2}} \sum_{k=0}^{N-1} \frac{\omega^k Q^k \tilde{\mathbf{m}}^* \mathbf{n}^{*\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{*-1}} = \mathbf{n}. \quad (68)$$

We denote the  $j$ th rows of  $\mathbf{n}$  and  $\tilde{\mathbf{m}}$  by  $\mathbf{n}_j$  and  $\tilde{\mathbf{m}}_j$ , respectively. From (68), it follows that

$$\begin{aligned} \mathbf{n}_j &= \tilde{\mathbf{m}}_j \frac{1}{\mu^2} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} \mathbf{n}^{\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{-1}} + \tilde{\mathbf{m}}_j^* \frac{1}{\mu^{*2}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} \mathbf{n}^{*\text{tr}} Q^{-k} \mathbf{n}}{\mu - \omega^k \mu^{*-1}} \\ &= \tilde{\mathbf{m}}_j \frac{1}{\mu^2} \mathbf{n}^{\text{tr}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} Q^{-k}}{\mu - \omega^k \mu^{-1}} \mathbf{n} \\ &\quad + \tilde{\mathbf{m}}_j^* \frac{1}{\mu^{*2}} \mathbf{n}^{*\text{tr}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} Q^{-k}}{\mu - \omega^k \mu^{*-1}} \mathbf{n}. \end{aligned} \quad (69)$$

Let us introduce the following notations for the  $r \times r$  matrices where the entry at  $(\alpha, \beta)$  is

$$\begin{aligned} R(j)_{\alpha,\beta} &= \frac{\mu}{N} \left( \frac{1}{\mu^2} \mathbf{n}^{\text{tr}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} Q^{-k}}{\mu - \omega^k \mu^{-1}} \mathbf{n} \right)_{\alpha,\beta} \\ &= \frac{1}{\mu^{2N} - 1} \sum_{l=1}^N n_{l\alpha} n_{l\beta} \mu^{2\{(j-l) \bmod N\}}; \end{aligned} \quad (70)$$

$$\begin{aligned} P(j)_{\alpha,\beta} &= \frac{\mu^*}{N} \left( \frac{1}{\mu^{*2}} \mathbf{n}^{*\text{tr}} \sum_{k=0}^{N-1} \frac{\omega^{(j+1)k} Q^{-k}}{\mu - \omega^k \mu^{*-1}} \mathbf{n} \right)_{\alpha,\beta} \\ &= \frac{1}{|\mu|^{2N} - 1} \sum_{l=1}^N n_{l\alpha}^* n_{l\beta} |\mu|^{2\{(j-l) \bmod N\}}. \end{aligned} \quad (71)$$

Note that  $R(j) = R(j)^{\text{tr}}$  and  $P(j)^\dagger = P(j)^{* \text{tr}} = P(j)$ , where the  $\dagger$  notation denotes the Hermitian transpose of a matrix. It is easy to show the identity between the entries in  $R(j)$  and  $R(j-1)$ , and in  $P(j)$  and  $P(j-1)$ :

$$\begin{aligned} R(j)_{\alpha,\beta} &= \mu^2 R(j-1)_{\alpha,\beta} - n_{j\alpha} n_{j\beta}; \\ P(j)_{\alpha,\beta} &= |\mu|^2 P(j-1)_{\alpha,\beta} - n_{j\alpha}^* n_{j\beta}^*, \end{aligned}$$

which imply that

$$R(j) = \mu^2 R(j-1) - \mathbf{n}_j^{\text{tr}} \mathbf{n}_j; \quad P(j) = |\mu|^2 P(j-1) - \mathbf{n}_j^\dagger \mathbf{n}_j. \quad (72)$$

From (69), it follows that

$$\mathbf{n}_j = \frac{N}{\mu} \tilde{\mathbf{m}}_j R(j) + \frac{N}{\mu^*} \tilde{\mathbf{m}}_j^* P(j) = \hat{\mathbf{m}}_j R(j) + \hat{\mathbf{m}}_j^* P(j),$$

where  $\hat{\mathbf{m}} = \frac{N}{\mu} \tilde{\mathbf{m}}$ , and thus we obtain the solution for  $\hat{\mathbf{m}}$ , which leads to

$$\hat{\mathbf{m}}_j = (\mathbf{n}_j P(j)^{-1} - \mathbf{n}_j^* R(j)^{* -1}) (R(j) P(j)^{-1} - P(j)^* R(j)^{* -1})^{-1}.$$

We substitute this into (67) and determine that the  $j$ th diagonal entry in the diagonal matrix  $C$  satisfies

$$\begin{aligned} c_j^2 &= 1 + \sum_{\alpha=1}^r (\hat{m}_{j\alpha} n_{j\alpha} + \hat{m}_{j\alpha}^* n_{j\alpha}^*) = 1 + \hat{\mathbf{m}}_j \mathbf{n}_j^{\text{tr}} + \hat{\mathbf{m}}_j^* \mathbf{n}_j^\dagger \\ &= 1 + \mathbf{n}_j (R(j) - P(j)^* R(j)^{* -1} P(j))^{-1} \mathbf{n}_j^{\text{tr}} \\ &\quad + \mathbf{n}_j^* (P(j)^* - R(j) P(j)^{-1} R(j)^*)^{-1} \mathbf{n}_j^{\text{tr}} \\ &\quad + \mathbf{n}_j^* (R(j)^* - P(j) R(j)^{-1} P(j)^*)^{-1} \mathbf{n}_j^\dagger \\ &\quad + \mathbf{n}_j (P(j) - R(j)^* P(j)^{* -1} R(j))^{-1} \mathbf{n}_j^\dagger. \end{aligned} \quad (73)$$

**Lemma 3.** The matrices  $R(j)$  and  $P(j)$  defined by (70) and (71), respectively, and the scalar  $c_j$  given by (73) satisfy the identity

$$\begin{aligned} \tau_{j-1} &= |\mu|^{-4r} \tau_j c_j^2, \\ \tau_j &= \det(P(j) P(j)^* - P(j) R(j) P(j)^{-1} R(j)^*). \end{aligned} \quad (74)$$

**Proof.** First, we apply Sylvester's determinant theorem to (72), which leads to

$$|\mu|^{2r} \det P(j-1) = \det P(j) (1 + \mathbf{n}_j P(j)^{-1} \mathbf{n}_j^\dagger). \quad (75)$$

Using the Sherman–Morrison formula, we find that

$$\begin{aligned} P(j-1)^{-1} &= |\mu|^2 P(j)^{-1} \left( 1 - \frac{1}{1 + \alpha} \mathbf{n}_j^\dagger \mathbf{n}_j P(j)^{-1} \right), \\ \alpha &= \mathbf{n}_j P(j)^{-1} \mathbf{n}_j^\dagger. \end{aligned}$$

Using  $\alpha$ , we rewrite (75) as

$$\det P(j-1) = \frac{1}{|\mu|^{2r}} \det P(j)(1+\alpha). \quad (76)$$

Using (72), we now compute

$$\begin{aligned} & P(j-1)^* - R(j-1)P(j-1)^{-1}R(j-1)^* \\ &= \frac{1}{|\mu|^2} \left( P(j)^* + \mathbf{n}_j^{\text{tr}} \mathbf{n}_j^* - (R(j) + \mathbf{n}_j^{\text{tr}} \mathbf{n}_j) P(j)^{-1} \right. \\ &\quad \left. \times \left( 1 - \frac{1}{1+\alpha} \mathbf{n}_j^{\dagger} \mathbf{n}_j P(j)^{-1} \right) (R(j)^* + \mathbf{n}_j^{\dagger} \mathbf{n}_j^*) \right) \\ &= \frac{1}{|\mu|^2} \left( P(j)^* - R(j)P(j)^{-1}R(j)^* \right. \\ &\quad \left. + \frac{1}{1+\alpha} (\mathbf{n}_j^{\text{tr}} - R(j)P(j)^{-1} \mathbf{n}_j^{\dagger}) (\mathbf{n}_j^* - \mathbf{n}_j P(j)^{-1} R(j)^*) \right). \end{aligned}$$

Let  $W(j) = P(j)^* - R(j)P(j)^{-1}R(j)^*$ . Using Sylvester's determinant theorem, we obtain

$$\begin{aligned} \det W(j-1) &= \frac{1}{|\mu|^{2r}} \det W(j) \\ &\quad \times \left( 1 + \frac{1}{1+\alpha} (\mathbf{n}_j^* - \mathbf{n}_j P(j)^{-1} R(j)^*) W(j)^{-1} (\mathbf{n}_j^{\text{tr}} - R(j)P(j)^{-1} \mathbf{n}_j^{\dagger}) \right). \end{aligned}$$

By combining this with (76) and using the notation in (74), we have

$$\begin{aligned} \tau_{j-1} &= \det P(j-1) \det W(j-1) \\ &= \frac{1}{|\mu|^{4r}} \det P(j) \det W(j) \left( 1 + \alpha + (\mathbf{n}_j^* - \mathbf{n}_j P(j)^{-1} R(j)^*) \right. \\ &\quad \left. \times W(j)^{-1} (\mathbf{n}_j^{\text{tr}} - R(j)P(j)^{-1} \mathbf{n}_j^{\dagger}) \right). \end{aligned}$$

We compare the scalar expression inside the bracket with  $c_j^2$  given by (73), and use the identity

$$\begin{aligned} & P(j)^{-1} + P(j)^{-1}R(j)^*W(j)^{-1}R(j)P(j)^{-1} \\ &= (P(j) - R(j)^*P(j)^{-1}R(j))^{-1} \end{aligned}$$

to obtain the formula (74) in the statement.  $\square$

Note that

$$\begin{aligned} \tau_j &= \det \left( \begin{pmatrix} P(j) & 0 \\ R(j) & I_r \end{pmatrix} \begin{pmatrix} I_r & P(j)^{-1}R(j)^* \\ 0 & P(j)^* - R(j)P(j)^{-1}R(j)^* \end{pmatrix} \right) \\ &= \det H(j), \end{aligned} \quad (77)$$

$$H(j) = \begin{pmatrix} P(j) & R(j)^* \\ R(j) & P(j)^* \end{pmatrix},$$

where the  $r \times r$  matrices  $R(j)$  and  $P(j)$  are defined by (70) and (71), respectively. We can now write the rank  $r$  breather solutions as follows.

**Proposition 10.** Let  $\mathbf{n}_0$  be a rank  $r$  constant matrix of size  $N \times r$  and  $\mu \in \mathbb{C}$ ,  $|\mu| \neq 1$ ,  $\mu \neq \omega^k \mu^*$ ,  $k = 1, \dots, N$ . A rank  $r$  breather solution of the system (1) on a trivial background ( $\phi^{(i)} = 0$ ,  $i = 1, \dots, N$ ) is given by

$$\phi^{(j)} = \frac{1}{2} \ln \left( \frac{\tau_{j-1} \tau_{j+1}}{\tau_j^2} \right), \quad \tau_j = \det H(j), \quad (78)$$

where the  $2r \times 2r$  matrices  $H(j)$  are given by (77) and

$$\mathbf{n} = \exp((\mu \Delta^{-1} - \mu^{-1} \Delta)x - (\mu^{-2} \Delta^2 - \mu^2 \Delta^{-2})t) \mathbf{n}_0.$$

**Proof.** From Lemma 3, it follows that

$$c_j^2 = |\mu|^{4r} \frac{\tau_{j-1}}{\tau_j}.$$

By using (36), we obtain the solutions in the statement, where  $\mathbf{n}$  is determined by (60) and (35).  $\square$

Note that by taking  $r = 1$  in Proposition 10, we obtain the results in Proposition 9.

### 3.3. The $\tau$ -function and continuous limits

In the previous two sections, we showed that both kink and breather solutions can be expressed in the form

$$\phi^{(i)} = \ln \frac{c_i}{c_{i+1}} = \frac{1}{2} \ln \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \quad (79)$$

according to Propositions 5–10. In this section, we derive the equations for  $c_i$  and  $\tau_i$ . To simplify the notations, we drop the index  $i$  and introduce the shift operator  $\mathcal{S}$  for mapping the index  $i$  to  $i+1$ , i.e.,  $c_i = c$ ,  $c_{i+1} = \mathcal{S}c$ ,  $c_{i-1} = \mathcal{S}^{-1}c$ , and the same for  $\tau_i$ . The shift operator satisfies  $\mathcal{S}^N \tau = \tau$  and  $\mathcal{S}^N c = c$ .

Let  $c = e^u$  and  $\tau = e^{2v}$ . From (79), it follows that

$$\phi = (1 - \mathcal{S})u = \mathcal{S}^{-1}(\mathcal{S} - 1)^2 v, \quad (80)$$

which leads to

$$\theta = -(\mathcal{S} - 1)^{-1}(\mathcal{S} + 1)\phi_x = (\mathcal{S} + 1)u_x. \quad (81)$$

By substituting (80) and (81) into (1), we obtain

$$\begin{aligned} (1 - \mathcal{S})u_t &= (\mathcal{S} + 1)u_{xx} + ((\mathcal{S} + 1)u_x) ((1 - \mathcal{S})u_x) \\ &\quad + (\mathcal{S}^2 - 1)e^{2(\mathcal{S}^{-1}-1)u}. \end{aligned}$$

Thus, we have

$$u_t = (\mathcal{S} + 1)(1 - \mathcal{S})^{-1}u_{xx} + u_x^2 - (\mathcal{S} + 1) \left( e^{2(\mathcal{S}^{-1}-1)u} - 1 \right), \quad (82)$$

where we select the integration constant as 1 such that  $u = 0$  is a solution of (82).  $u = \ln c$ , so we obtain the following equation for function  $c$ :

$$\begin{aligned} \frac{c_t}{c} &= (\mathcal{S} + 1)(1 - \mathcal{S})^{-1} \left( \frac{c_{xx}}{c} - \frac{c_x^2}{c^2} \right) + \frac{c_x^2}{c^2} \\ &\quad - (\mathcal{S} + 1) \left( e^{2(\mathcal{S}^{-1}-1) \ln c} - 1 \right). \end{aligned} \quad (83)$$

We now derive the equation for  $v$ . From (80), we have  $u = (\mathcal{S}^{-1} - 1)v$ . Note that

$$u_x^2 = (1 + \mathcal{S}^{-1})(v_x^2 - v_x \mathcal{S} v_x) + (1 - \mathcal{S}^{-1})v_x \mathcal{S} v_x.$$

By substituting these into (82), we obtain

$$\begin{aligned} v_t &= (\mathcal{S} + 1)(1 - \mathcal{S})^{-1} \left( v_{xx} + v_x^2 - v_x \mathcal{S} v_x - e^{2(\mathcal{S}^{-1}-1)v} + 1 \right) \\ &\quad - v_x \mathcal{S} v_x. \end{aligned} \quad (84)$$

Now, the  $\tau$ -function is related to  $v$  by  $v = \frac{1}{2} \ln \tau$ . From (84), it follows that

$$\begin{aligned} \frac{\tau_t}{\tau} &= (\mathcal{S} + 1)(1 - \mathcal{S})^{-1} \left( \frac{\tau_{xx}}{\tau} - \frac{1}{2} \left( \frac{\tau_x^2}{\tau^2} + \frac{\tau_x \mathcal{S} \tau_x}{\tau \mathcal{S} \tau} \right) \right. \\ &\quad \left. - 2 \frac{(\mathcal{S} \tau)(\mathcal{S}^{-1} \tau)}{\tau^2} + 2 \right) - \frac{1}{2} \frac{\tau_x \mathcal{S} \tau_x}{\tau \mathcal{S} \tau}. \end{aligned} \quad (85)$$

Thus, we have proved the following statement.

**Proposition 11.** *If the function  $c$  satisfies (83), then  $\phi = \ln \frac{c}{\delta c}$  satisfies the two-dimensional Volterra equation (1). If the function  $\tau$  satisfies (85), then  $\phi = \ln \frac{(\delta\tau)(\delta^{-1}\tau)}{\tau^2}$  satisfies the two-dimensional Volterra equation (1).*

We know that the continuous limit of system (1) goes to the KP equation. Indeed, for the continuous limit as  $N \rightarrow \infty$  and  $h = N^{-1}$ , we set

$$T = h^3 t, \quad X = ih + 4ht, \quad Y = h^2 x, \tag{86}$$

which implies that

$$\frac{\partial}{\partial t} = h^3 \frac{\partial}{\partial T} + 4h \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial x} = h^2 \frac{\partial}{\partial Y}. \tag{87}$$

Let  $\phi^{(i)}(x, t) = h^2 w(X, Y, T)$ . In the new variables, system (1) takes the form of

$$w_T = \frac{2}{3} w_{XXX} + 8w w_X - 2D_X^{-1} w_{YY} + O(h^2),$$

and it goes to the KP equation in the limit  $h \rightarrow 0$ . We can compute the continuous limits of Eqs. (82) and (84) by setting

$$u(x, t) = h\hat{u}(X, Y, T), \quad v(x, t) = \hat{v}(X, Y, T) \tag{88}$$

respectively. Note that

$$\mathcal{S}u = \hat{u}(X + h, Y, T) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \frac{\partial^n \hat{u}}{\partial X^n} = e^{h \frac{\partial}{\partial X}} \hat{u}.$$

Hence, we replace the shift operator  $\mathcal{S}$  by  $e^{h \frac{\partial}{\partial X}}$ , which leads to

$$h \frac{\partial}{\partial X} (1 + \mathcal{S})(1 - \mathcal{S})^{-1} = -2 - \frac{h^2}{6} \frac{\partial^2}{\partial X^2} + O(h^4). \tag{89}$$

By substituting (87), (88), and (89) into (82), we obtain

$$h^5 \hat{u}_{XT} + 4h^3 \hat{u}_{XX} = \left( -2 - \frac{h^2}{6} \frac{\partial^2}{\partial X^2} \right) h^5 \hat{u}_{YY} + 4h^3 \hat{u}_{XX} + \frac{2}{3} h^5 \hat{u}_{XXX} - 8h^5 \hat{u}_X \hat{u}_{XX} + O(h^6),$$

which implies that

$$\hat{u}_{XT} + 2\hat{u}_{YY} - \frac{2}{3} \hat{u}_{XXX} + 8\hat{u}_X \hat{u}_{XX} = O(h).$$

In the same manner, we substitute (87), (88), and (89) into (84), and obtain its continuous limit

$$\hat{v}_{XT} + 2\hat{v}_{YY} - \frac{2}{3} \hat{v}_{XXX} - 4\hat{v}_{XX}^2 = O(h).$$

In the variable  $\tilde{\tau} = e^{\hat{v}}$  and in the limit  $h \rightarrow 0$ , this becomes

$$3(\tilde{\tau} \tilde{\tau}_{XT} - \tilde{\tau}_X \tilde{\tau}_T) + 6(\tilde{\tau} \tilde{\tau}_{YY} - \tilde{\tau}_Y^2) - 2\tilde{\tau} \tilde{\tau}_{4X} + 8\tilde{\tau}_X \tilde{\tau}_{3X} - 6\tilde{\tau}_{XX}^2 = 0,$$

which is the standard bilinear form for the KP equation, thereby providing a link between the Hirota  $\tau$ -function for the bilinear form and the functions  $\tau_i$  defined in Propositions 5–10 in the continuous limit  $\tau_i = \tilde{\tau}^2$ .

#### 4. Classification of rank 1 solutions

In this section, we describe and analyze the kink and breather solutions given in Propositions 5 and 6, and 9. The solutions are characterized completely by the choice of the poles for the dressing matrix  $\Phi(\lambda)$  and a constant matrix  $\mathbf{n}_0$ . In the case of kink solutions, the invariant dressing matrix has  $N$  poles, whereas in the case of breather solutions, the dressing matrix has  $2N$  poles.

It is convenient to use the basis

$$\mathbf{e}_k = (\omega^k, \omega^{2k}, \dots, \omega^{(N-1)k}, 1)^{\text{tr}}, \quad k = 1, \dots, N \tag{90}$$

of the eigenvectors  $\Delta \mathbf{e}_k = \omega^k \mathbf{e}_k$  for the matrix  $\Delta$  to represent the vector

$$\mathbf{n}_0 = \sum_{k=1}^N \alpha_k \mathbf{e}_k. \tag{91}$$

In this basis, we have

$$\Psi_0(x, t, \mu) \mathbf{e}_k = \exp((\mu \omega^{-k} - \mu^{-1} \omega^k)x + (\mu^2 \omega^{-2k} - \mu^{-2} \omega^{2k})t) \mathbf{e}_k$$

and thus

$$\mathbf{n} = \Psi_0(x, t, \mu) \mathbf{n}_0 = \sum_{k=1}^N \alpha_k \exp((\mu \omega^{-k} - \mu^{-1} \omega^k)x + (\mu^2 \omega^{-2k} - \mu^{-2} \omega^{2k})t) \mathbf{e}_k.$$

Obviously,  $\alpha_k = N^{-1} \mathbf{e}_{-k}^{\text{tr}} \mathbf{n}_0$  in (91). The vector  $\mathbf{n}_0$  in this basis is given by a matrix  $\alpha = (\alpha_1, \dots, \alpha_N)$ .

#### 4.1. Classification of rank 1 kink solutions

In this section, we classify the kink solutions of rank 1 given by Propositions 5 and 6. First, we describe the possible kink solutions in the cases where  $N = 3, 4$  and then provide an overview of the general case. We note that the properties of the solutions for even and odd values of  $N$  are slightly different. In particular, in the case of an even  $N$ , there is an obvious solution

$$\phi^{(j)} = (-1)^j f(x), \quad \theta_j = 0, \quad j = 1, \dots, 2N \tag{92}$$

of the system (1), where  $f(x)$  is an arbitrary differentiable function of  $x$ . Moreover, Proposition 6 gives new solutions only when  $N$  is even.

In the case of the kink solutions obtained in Proposition 5 when  $\mu = \mu^* = \nu \in \mathbb{R}$  and  $\nu \notin \{\pm 1, 0\}$ , the vector  $\mathbf{n}_0$  is real, and thus we require that

$$\alpha_N = \alpha_N^*, \quad \alpha_{N-k} = \alpha_k^*, \quad k = 1, \dots, N-1.$$

In the case of the kink solutions obtained in Proposition 6 when  $\mu = \nu \exp(\frac{\pi i}{N})$ ,  $\nu \in \mathbb{R}$  and  $\nu \notin \{\pm 1, 0\}$ , the vector  $\mathbf{n}_0 = Q \mathbf{n}_0^*$ . Note that  $Q \mathbf{n}_0^* = \alpha_1^* \mathbf{e}_N + \alpha_2^* \mathbf{e}_{N-1} + \dots + \alpha_N^* \mathbf{e}_1$  and thus we require that

$$\alpha_k = \alpha_{N-k+1}^*, \quad k = 1, \dots, N.$$

In particular, when  $N = 2m$ , this reduces to

$$\alpha_k = \alpha_{2m-k+1}^*, \quad k = 1, \dots, m. \tag{93}$$

#### 4.1.1. Classification of rank 1 kink solutions in the case where $N = 3$

In this section, we set  $N = 3$  for Eq. (1), i.e., Eq. (2). In this case, it is sufficient to study the rank 1 solutions because  $\text{Gr}_{\mathbb{R}}(1, 3) \simeq \text{Gr}_{\mathbb{R}}(2, 3)$ .

We classify the possible solutions in terms of the constant matrix  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ , which represents the real vector  $\mathbf{n}_0$ . In the variables

$$\xi = (\nu - \nu^{-1})x - (\nu^{-2} - \nu^2)t;$$

$$\eta = \frac{\sqrt{3}}{2}((\nu + \nu^{-1})x - (\nu^{-2} + \nu^2)t),$$

we have

$$\Psi_0(x, t, \nu) \mathbf{e}_1 = e^{-\frac{\xi}{2} - i\eta} \mathbf{e}_1; \quad \Psi_0(x, t, \nu) \mathbf{e}_2 = e^{-\frac{\xi}{2} + i\eta} \mathbf{e}_2; \quad \Psi_0(x, t, \nu) \mathbf{e}_3 = e^{\xi} \mathbf{e}_3.$$

To obtain real solutions, it is required that  $\mathbf{n}$  and  $\nu$  are real. Hence, there are three cases:

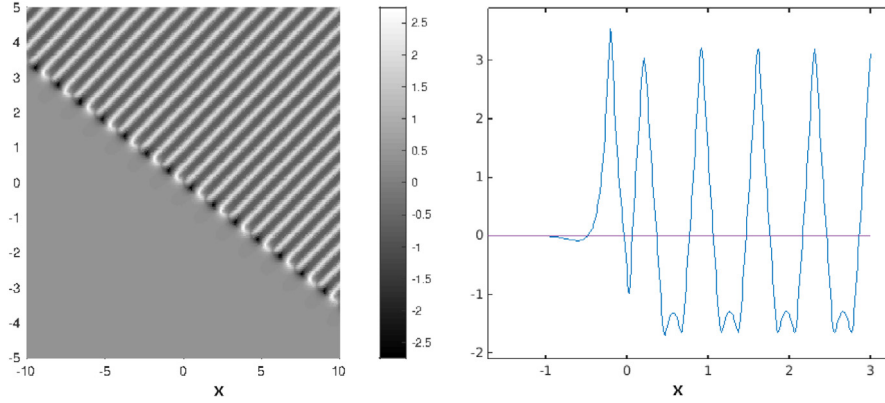


Fig. 3. Density plot of  $\phi^{(1)}(x, t)$  and a snapshot of  $\phi^{(1)}$  at  $t = 0$  ( $\alpha = (1, 1, 1)$ ,  $\nu = 0.4$ ).

1.  $\alpha_3 \neq 0 \in \mathbb{R}$ ,  $\alpha_1 = \alpha_2 = 0$ . Thus, we have

$$\mathbf{n} = \alpha_3 e^{\xi} \mathbf{e}_3 = \alpha_3 e^{\xi} (1, 1, 1)^{\text{tr}},$$

which leads to  $\tau_i = \alpha_3^2 e^{2\xi} (1 + \nu^2 + \nu^4) / (\nu^6 - 1)$  in Proposition 5.

From (47), it follows that the solution is trivial, i.e.,  $\phi^{(i)} = 0$ . When we consider the classification of solutions later, we do not consider this case further.

2.  $\alpha_3 = 0$ ,  $\alpha_2 = \alpha_1^* \neq 0$ . Without any loss of generality, we take  $\alpha_1 = e^{i\beta}$ , where  $\beta \in \mathbb{R}$  is constant. Then, we take  $\alpha_2 = e^{-i\beta}$  such that  $\mathbf{n}$  is real. Indeed,

$$\mathbf{n} = 2e^{-\frac{\nu\xi}{2}} \left( \cos\left(\eta - \beta - \frac{2\pi}{3}\right), \cos\left(\eta - \beta + \frac{2\pi}{3}\right), \cos(\eta - \beta) \right)^{\text{tr}}.$$

Using (47), we obtain the solution

$$\phi^{(j)} = \frac{1}{2} \ln \left( \frac{\tau_{j-1} \tau_{j+1}}{\tau_j^2} \right),$$

$$\tau_j = \cos^2 \left( \eta - \beta - \frac{2\pi}{3} \right) \nu^{2(j-1) \bmod 3} + \cos^2 \left( \eta - \beta + \frac{2\pi}{3} \right) \nu^{2(j-2) \bmod 3} + \cos^2(\eta - \beta) \nu^{2j \bmod 3}.$$

In this case, the solutions are periodic functions of the variable  $\eta$ .

3.  $\alpha_1 \alpha_2 \alpha_3 \neq 0$ . Let  $\alpha_1 = e^{i\beta+\gamma}$ , where  $\beta, \gamma$  are constants. We take  $\alpha_3 = e^{\delta} \in \mathbb{R}$  and  $\alpha_2 = e^{-i\beta+\gamma}$  in order  $\mathbf{n}$  as real.

$$\mathbf{n} = e^{-\frac{1}{2}\xi+\gamma} \begin{pmatrix} 2 \cos\left(\eta - \beta - \frac{2\pi}{3}\right) + e^{\frac{3\xi}{2}+\delta-\beta} \\ 2 \cos\left(\eta - \beta + \frac{2\pi}{3}\right) + e^{\frac{3\xi}{2}+\delta-\beta} \\ 2 \cos(\eta - \beta) + e^{\frac{3\xi}{2}+\delta-\beta} \end{pmatrix}.$$

Using (47), we can write the solutions for  $\phi^{(i)}$ . We omit the tedious formula and only show the density plot. Note that when  $\xi \rightarrow +\infty$ , the solutions  $\phi^{(i)} \rightarrow 0$ ; and when  $\xi \rightarrow -\infty$ , the contributions of  $\alpha_1$  and  $\alpha_2$  are dominant, which leads to periodic solutions. A line on the  $(x, t)$ -plane given by  $e^{\frac{3\xi}{2}+\delta-\beta} = 2$  corresponds to the wave front propagation, which has a slope equal to  $-\nu/(1 + \nu^2)$ .

We now choose  $\nu = 0.4$  and  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ . On the left in Fig. 3, we show a density plot of  $\phi^{(1)}$  in the  $(x, t)$ -plane and a snapshot of the solution  $\phi^{(1)}$  at  $t = 0$  is shown on the right. Note that the solution is a periodic oscillating wave, which oscillates

in half of the space (the  $x$ -axis) only and it moves to the left as time progresses. Furthermore, the frontier of the wave does not have a stationary profile and it oscillates in a rather complicated manner.

Therefore, in the case where  $N = 3$ , we only have two types of kink solutions. To the best of our knowledge, the wave front solutions (Fig. 3) represent a new class of exact solutions for integrable models.

#### 4.1.2. Classification of rank 1 kink solutions in the case where $N = 4$

For  $N = 4$ , Eq. (1) can be rewritten in the form

$$2\phi_t^{(i)} = \phi_{xx}^{(i+1)} - \phi_{xx}^{(i+3)} + \phi_x^{(i)} (\phi_x^{(i+1)} - \phi_x^{(i+3)}) + 2e^{2\phi^{(i+1)}} - 2e^{2\phi^{(i-1)}}, \quad (94)$$

where  $\phi^{(i+4)} = \phi^{(i)}$  and  $\sum_{i=1}^4 \phi^{(i)} = 0$ . We now classify its all possible rank 1 kink solutions.

First, we consider the case when  $\mu = \mu^* = \nu$  and the constant real vector

$$\mathbf{n}_0 = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 + \alpha_4 \mathbf{e}_4,$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ ,  $\alpha_3 = \alpha_1^* \in \mathbb{C}$ ,  $\alpha_2, \alpha_4 \in \mathbb{R}$ . In the variables,  $\xi = (\nu - \nu^{-1})x$ ,  $\zeta = (\nu + \nu^{-1})x$ , and  $\eta = (\nu^{-2} - \nu^2)t$ , we have

$$\Psi_0(x, t, \nu) \mathbf{e}_1 = e^{-\zeta+i\eta} \mathbf{e}_1; \quad \Psi_0(x, t, \nu) \mathbf{e}_2 = e^{-\xi-\eta} \mathbf{e}_2;$$

$$\Psi_0(x, t, \nu) \mathbf{e}_3 = e^{\xi+i\eta} \mathbf{e}_3; \quad \Psi_0(x, t, \nu) \mathbf{e}_4 = e^{\xi-\eta} \mathbf{e}_4.$$

There are four cases (excluding the trivial solutions):

1.  $\alpha_2$  and  $\alpha_4$  are both non-zero real numbers, and  $\alpha_1 = \alpha_3 = 0$ . We can take  $\alpha_4 = 1$ , and thus

$$\mathbf{n} = (e^{\xi-\eta} - \alpha_2 e^{-\xi-\eta}, e^{\xi-\eta} + \alpha_2 e^{-\xi-\eta}, e^{\xi-\eta} - \alpha_2 e^{-\xi-\eta}, e^{\xi-\eta} + \alpha_2 e^{-\xi-\eta})^{\text{tr}} = e^{\xi-\eta} (1 - \alpha_2 e^{-2\xi}, 1 + \alpha_2 e^{-2\xi}, 1 - \alpha_2 e^{-2\xi}, 1 + \alpha_2 e^{-2\xi})^{\text{tr}},$$

which leads to

$$\tau_j = e^{2\xi-2\eta} \left( (1 - \alpha_2 e^{-2\xi})^2 (\nu^{2(j-1) \bmod 4} + \nu^{2(j-3) \bmod 4}) + (1 + \alpha_2 e^{-2\xi})^2 (\nu^{2(j-2) \bmod 4} + \nu^{2j \bmod 4}) \right) / (\nu^8 - 1) = e^{2\xi-2\eta} \left( \frac{1 + \alpha_2^2 e^{-4\xi}}{\nu^2 - 1} - (-1)^j \frac{2\alpha_2 e^{-2\xi}}{\nu^2 + 1} \right).$$

Using (47), we obtain the solution

$$\phi^{(j)} = \frac{1}{2} \ln \left( \frac{\tau_{j-1} \tau_{j+1}}{\tau_j^2} \right) = \ln \left| \frac{(1 + \alpha_2^2 e^{-4\xi})(\nu^2 + 1) + 2\alpha_2 e^{-2\xi} (-1)^j (\nu^2 - 1)}{(1 + \alpha_2^2 e^{-4\xi})(\nu^2 + 1) - 2\alpha_2 e^{-2\xi} (-1)^j (\nu^2 - 1)} \right|,$$

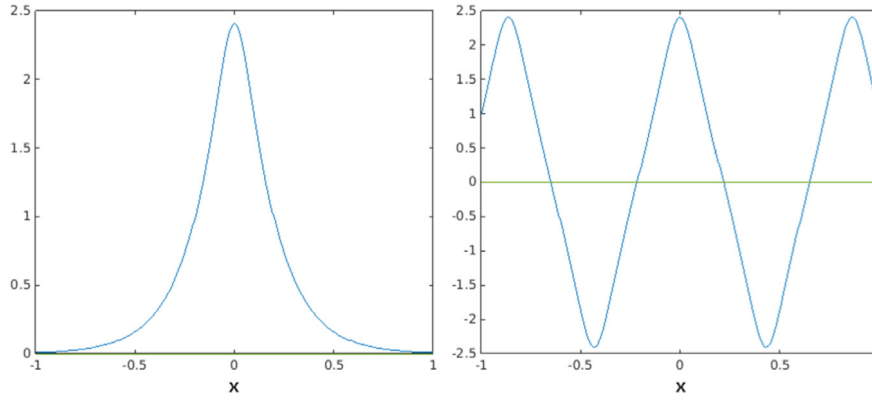


Fig. 4. Graph of  $\phi^{(1)}(x, t)$  for kink solutions with  $\nu = 0.3$ :  $\alpha = (0, 1, 0, 1)$  on the left and  $\alpha = (1, 0, 1, 0)$  on the right.

which is independent of time  $t$  (see the plot on the left in Fig. 4). This solution is of type (92).

- $\alpha_2 = \alpha_4 = 0, \alpha_1\alpha_3 \neq 0$ . Without any loss of generality, we take  $\alpha_1 = e^{i\beta} = \alpha_3^*$ , where  $\beta \in \mathbb{R}$  is a real constant. Then,

$$\mathbf{n} = 2e^\eta(\sin(\zeta - \beta), -\cos(\zeta - \beta), -\sin(\zeta - \beta), \cos(\zeta - \beta))^{\text{tr}}.$$

Using (47), we obtain the solution

$$\phi^{(j)} = \frac{1}{2} \ln \left( \frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2} \right),$$

$$\tau_j = \sin^2(\zeta - \beta)(\nu^{2(j-1) \bmod 4} - \nu^{2(j-3) \bmod 4}) + \cos^2(\zeta - \beta)(\nu^{2j \bmod 4} - \nu^{2(j-2) \bmod 4}),$$

where we obtain periodic solutions (see the plot on the right in Fig. 4). This is also a solution of type (92).

- Only one of  $\alpha_2$  and  $\alpha_4$  is nonzero, and  $\alpha_1 = e^{i\beta}, \alpha_3 = e^{-i\beta}$ , where  $\beta \in \mathbb{R}$  is constant. Using Proposition 5, we can write the solutions for  $\phi^{(j)}$ . We ignore the tedious formula and only show their plots (see the first two density plots in Fig. 5).
- $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$ . Let  $\alpha_1 = e^{i\beta}, \alpha_3 = e^{-i\beta}$ , where  $\beta \in \mathbb{R}$  is constant. We take  $\alpha_2, \alpha_4 \in \mathbb{R}$ . The plot on the right in Fig. 5 is its density plot.

For an even dimension, we also need to consider the case stated in Proposition 6. Let  $\mu = \nu \exp(\frac{\pi i}{4})$  and the constant vector

$$\mathbf{n}_0 = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 + \alpha_4 \mathbf{e}_4,$$

where  $\alpha_i \in \mathbb{C}$ . Note that

$$\mathbf{n}_0^* = \alpha_1^* \mathbf{e}_3 + \alpha_2^* \mathbf{e}_2 + \alpha_3^* \mathbf{e}_1 + \alpha_4^* \mathbf{e}_4, \quad Q\mathbf{e}_i = \mathbf{e}_{i+1}.$$

The requirement that  $\mathbf{n}_0 = Q\mathbf{n}_0^* = \alpha_1^* \mathbf{e}_4 + \alpha_2^* \mathbf{e}_3 + \alpha_3^* \mathbf{e}_2 + \alpha_4^* \mathbf{e}_1$  implies that  $\alpha_1 = \alpha_4^*$  and  $\alpha_2 = \alpha_3^*$ .

In the variables,  $\xi = \frac{\sqrt{2}}{2}(\nu - \nu^{-1})x, \zeta = \frac{\sqrt{2}}{2}(\nu + \nu^{-1})x$  and  $\eta = (\nu^{-2} + \nu^2)t$ , we have

$$\begin{aligned} \Psi_0(x, t, \mu)\mathbf{e}_1 &= e^{\xi - (\zeta + \eta)i} \mathbf{e}_1; & \Psi_0(x, t, \mu)\mathbf{e}_2 &= e^{-\xi - (\zeta - \eta)i} \mathbf{e}_2; \\ \Psi_0(x, t, \mu)\mathbf{e}_3 &= e^{-\xi + (\zeta - \eta)i} \mathbf{e}_3; & \Psi_0(x, t, \mu)\mathbf{e}_4 &= e^{\xi + (\zeta + \eta)i} \mathbf{e}_4. \end{aligned}$$

There are three cases:

- $\alpha_1 = \alpha_4 = 0$ . Without any loss of generality, we take  $\alpha_2 = e^{i\beta} = \alpha_3^*$ , where  $\beta \in \mathbb{R}$ . Then,

$$\mathbf{n} = e^{-\xi} \begin{pmatrix} (1+i)(\sin(\zeta - \eta - \beta) - \cos(\zeta - \eta - \beta)) \\ -2i \sin(\zeta - \eta - \beta) \\ (-1+i)(\sin(\zeta - \eta - \beta) + \cos(\zeta - \eta - \beta)) \\ 2 \cos(\zeta - \eta - \beta) \end{pmatrix}.$$

Let  $\theta = 2(\zeta - \eta - \beta)$ . Using (48), we obtain

$$\begin{aligned} \tau_1 &= 2ie^{-2\xi} ((1 - \cos \theta)\nu^6 + (1 + \sin \theta)\nu^4 + (1 + \cos \theta)\nu^2 + 1 - \sin \theta); \\ \tau_2 &= -2e^{-2\xi} ((1 + \sin \theta)\nu^6 + (1 + \cos \theta)\nu^4 + (1 - \sin \theta)\nu^2 + 1 - \cos \theta); \\ \tau_3 &= -2ie^{-2\xi} ((1 - \cos \theta)\nu^2 + 1 + \sin \theta + (1 + \cos \theta)\nu^6 + (1 - \sin \theta)\nu^4); \\ \tau_4 &= 2e^{-2\xi} ((1 + \sin \theta)\nu^2 + 1 + \cos \theta + (1 - \sin \theta)\nu^6 + (1 - \cos \theta)\nu^4). \end{aligned}$$

Hence, we obtain periodic solutions. The plot on the left in Fig. 6 is its density plot.

- $\alpha_2 = \alpha_3 = 0$ . Without any loss of generality, we take  $\alpha_1 = e^{i\beta} = \alpha_4^*$ , where  $\beta \in \mathbb{R}$ . In this case, we also obtain a periodic solution in a similar manner to the case above. The middle plot in Fig. 6 is its density plot.
- $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$ . Let  $\alpha_1 = e^{i\beta} = \alpha_4^*$  and  $\alpha_2 = \rho e^{i\gamma} = \alpha_3^*, \rho \neq 0$ , where  $\beta, \gamma, \rho \in \mathbb{R}$ . We ignore the tedious formula and only show the density plots (see the plot on the right in Fig. 6).

Therefore, in the case where  $N = 4$ , we have eight different rank 1 kink solutions.

#### 4.1.3. Classification of rank 1 kink solutions for arbitrary dimensions

In this section, we classify all the possible rank 1 kink solutions for arbitrary dimension  $N$ . We have already explored the solutions for lower dimensions where  $N = 3, 4$ . There is a difference between the dimensions when  $N$  is even or odd.

For arbitrary  $N$ , according to Proposition 5, our kink solution depends only on  $\mathbf{n}_0 \in \mathbb{R}^N$  and  $\nu \in \mathbb{R}, \nu \notin \{\pm 1, 0\}$ . We can decompose  $\mathbb{R}^N$  as a direct sum of the invariant subspaces of  $\Delta$  as follows:

$$N = 2m - 1, \quad \mathbb{R}^N = E_0^1 \bigoplus_{p=1}^{m-1} E_p^2;$$

$$N = 2m, \quad \mathbb{R}^N = E_0^1 \bigoplus_{p=1}^{m-1} E_p^1 \bigoplus_{p=1}^{m-1} E_p^2,$$

where

$$E_0^1 = \text{span}_{\mathbb{R}}(\mathbf{e}_N), \quad E_m^1 = \text{span}_{\mathbb{R}}(\mathbf{e}_m), \quad E_p^2 = \text{span}_{\mathbb{R}}(\text{Re}(\mathbf{e}_p), \text{Im}(\mathbf{e}_p)).$$

We define ‘‘elementary waves’’ as solutions corresponding to the case where  $\mathbf{n}_0$  is simply a combination of two eigenvectors. There are  $m$  elementary waves when  $N = 2m$ : one pair of real eigenvalues and  $m - 1$  pairs of complex conjugate eigenvalues. There are  $m - 1$  elementary wave solutions when  $N = 2m - 1$

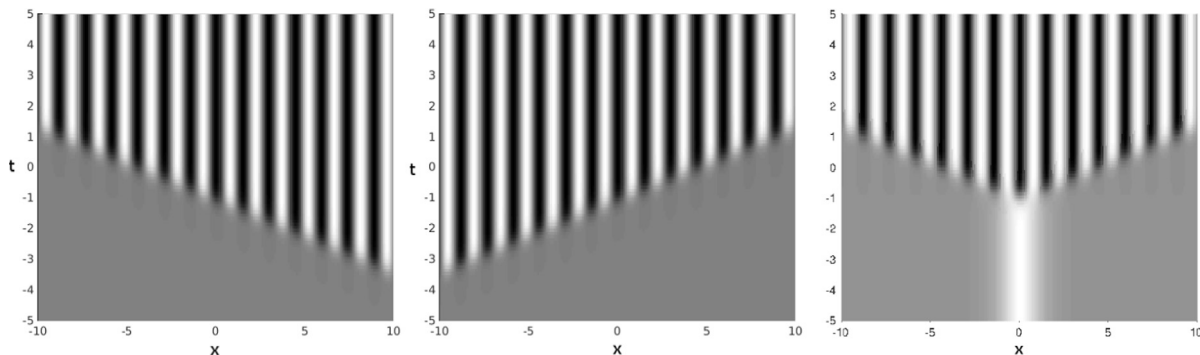


Fig. 5. Density plot of  $\phi^{(1)}(x, t)$  for kink solutions where  $v = 0.7$  and  $\alpha$  equals  $(1, 1, 1, 0)$  in the left plot,  $(1, 0, 1, 1)$  in the middle plot, and  $(1, 1, 1, 1)$  in the right plot.

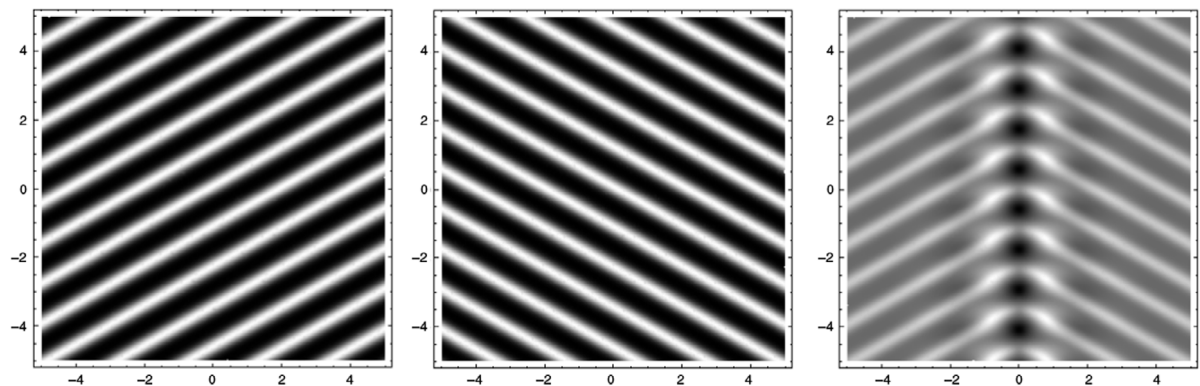


Fig. 6. Density plot of  $\phi^{(1)}(x, t)$  for kink solutions where  $\mu = \frac{3}{2}e^{\frac{\pi i}{4}}$  and  $\alpha$  equals  $(0, 1, 1, 0)$  in the left plot,  $(1, 0, 0, 1)$  in the middle plot, and  $(1, 1, 1, 1)$  in the right plot.

because there is only one real eigenvalue, which leads to trivial solutions and thus it is excluded. The other solutions can be built from these elementary wave solutions together with the trivial solutions.

We can write the elementary wave solutions for arbitrary  $N$  where we use the following identity. For fixed  $p \in \mathbb{N}$  and  $\omega^N = 1$ , by direct computation, we obtain

$$\sum_{l=1}^N \omega^{pl} \mu^{2\{(j-l) \bmod N\}} = \frac{\mu^{2N} - 1}{\mu^2 \omega^{-p} - 1} \omega^{pj}, \quad \mu \in \mathbb{C}. \quad (95)$$

**Theorem 1.** For any given nonzero constants  $\beta, v \in \mathbb{R}, N \in \mathbb{N}, v^2 \neq 1, N > 2$  and  $p \in \{1, \dots, \lfloor \frac{N-1}{2} \rfloor\}$ , system (1) has an elementary periodic wave solution of rank 1 given by

$$\phi^{(j)} = \frac{1}{2} \ln \frac{\tau_{j-1} \tau_{j+1}}{\tau_j^2},$$

$$\tau_j = 2e^{2a} \left( \frac{v^2 \cos(2b - \frac{4p(j+1)\pi}{N}) - \cos(2b - \frac{4pj\pi}{N})}{|v^2 \omega^{2p} - 1|^2} + \frac{1}{v^2 - 1} \right),$$

where

$$a = \left( v - \frac{1}{v} \right) \cos \left( \frac{2p\pi}{N} \right) x + \left( v^2 - \frac{1}{v^2} \right) \cos \left( \frac{4p\pi}{N} \right) t;$$

$$b = \left( v + \frac{1}{v} \right) \sin \left( \frac{2p\pi}{N} \right) x + \left( v^2 + \frac{1}{v^2} \right) \sin \left( \frac{4p\pi}{N} \right) t - \beta. \quad (96)$$

For even  $N = 2m$ , there is also a time independent rank 1 elementary kink solution of the form

$$\phi^{(j)} = \ln \left| \frac{(\beta^2 + e^{-4\xi})(v^2 + 1) + 2\beta e^{-2\xi} (-1)^j (v^2 - 1)}{(\beta^2 + e^{-4\xi})(v^2 + 1) - 2\beta e^{-2\xi} (-1)^j (v^2 - 1)} \right|,$$

$$\xi = \left( v - \frac{1}{v} \right) x.$$

**Proof.** Let us take  $\mathbf{n}_0 = e^{i\beta} \mathbf{e}_p + e^{-i\beta} \mathbf{e}_{N-p}$ , then the  $k$ th component of the vector  $\mathbf{n} = \Psi_0(x, t, v) \mathbf{n}_0$  can be written as follows:

$$\mathbf{n}_k = e^{\left( \frac{v}{\omega^p} - \frac{\omega^p}{v} \right) x + \left( \frac{v^2}{\omega^{2p}} - \frac{\omega^{2p}}{v^2} \right) t + \beta i} (\mathbf{e}_p)_k$$

$$+ e^{\left( v\omega^p - \frac{1}{\omega^p v} \right) x + \left( v^2 \omega^{2p} - \frac{1}{\omega^{2p} v^2} \right) t - \beta i} (\mathbf{e}_{N-p})_k$$

$$= e^a \left( e^{-bi} \omega^{kp} + e^{bi} \omega^{-kp} \right),$$

where  $a$  and  $b$  are defined by (96). From (47), it follows that

$$\tau_j = \frac{1}{v^{2N} - 1} e^{2a} \sum_{k=1}^N \left( (e^{-bi} \omega^{kp} + e^{bi} \omega^{-kp})^2 \right) v^{2\{(j-k) \bmod N\}}$$

$$= e^{2a} \left( e^{-2bi} \frac{\omega^{2pj}}{v^2 \omega^{-2p} - 1} + e^{2bi} \frac{\omega^{-2pj}}{v^2 \omega^{2p} - 1} + \frac{2}{v^2 - 1} \right)$$

$$= 2e^{2a} \left( \frac{v^2 \cos(2b - \frac{4p(j+1)\pi}{N}) - \cos(2b - \frac{4pj\pi}{N})}{|v^2 \omega^{2p} - 1|^2} + \frac{1}{v^2 - 1} \right),$$

which leads to the periodic solutions for  $\phi^{(j)}$  given in the statement.

Similarly, in the case where  $N = 2m$ , we compute the solution corresponding to  $\mathbf{n}_0 = \mathbf{e}_m + \beta \mathbf{e}_{2m}$ . Now, we have

$$\mathbf{n}_k = \beta e^{\left( v - \frac{1}{v} \right) x + \left( v^2 - \frac{1}{v^2} \right) t} + (-1)^k e^{\left( -v + \frac{1}{v} \right) x + \left( v^2 - \frac{1}{v^2} \right) t}$$

$$= e^{\left( v - \frac{1}{v} \right) x + \left( v^2 - \frac{1}{v^2} \right) t} \left( \beta + (-1)^k e^{-2\left( v - \frac{1}{v} \right) x} \right),$$

where  $k = 1, \dots, N = 2m$ , which leads to

$$\tau_j = e^{2\left( v - \frac{1}{v} \right) x + 2\left( v^2 - \frac{1}{v^2} \right) t}$$

$$\times \left( \frac{\beta^2 + e^{-4\left( v - \frac{1}{v} \right) x}}{v^2 - 1} + (-1)^{j+1} \frac{2\beta e^{-2\left( v - \frac{1}{v} \right) x}}{v^2 + 1} \right),$$

and this yields the solutions  $\phi^{(j)}$  independent of time  $t$  given in the statement.  $\square$

When  $N = 2m$ , according to Proposition 6, to obtain the kink solutions, we take  $\mu = \nu e^{\frac{\pi i}{2m}}$  and  $\mathbf{n}_0 = Q\mathbf{n}_0^*$ . From (93), it follows there are also  $m$  elementary waves. Similar to Theorem 1, we explicitly derive the elementary solutions in this case.

**Theorem 2.** Let  $N = 2m$ , where the integer  $m \geq 2$ . For any given nonzero constants  $\beta, \nu \in \mathbb{R}$  and  $p \in \{1, \dots, m\}$ , system (1) has an elementary periodic wave solution of rank 1 given by

$$\phi^{(j)} = \frac{1}{2} \ln \frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2},$$

$$\tau_j = \frac{\nu^2 \cos\left(2b - \frac{(2p-1)(j+1)\pi}{m}\right) - \cos\left(2b - \frac{(2p-1)j\pi}{m}\right)}{|\nu^2\omega^{2p-1} - 1|^2} + \frac{1}{\nu^2 - 1},$$

where

$$b = \left(\nu + \frac{1}{\nu}\right) \sin\left(\frac{(2p-1)\pi}{2m}\right)x + \left(\nu^2 + \frac{1}{\nu^2}\right) \sin\left(\frac{(2p-1)\pi}{m}\right)t - \beta. \tag{97}$$

**Proof.** For  $\mu = \nu \exp\left(\frac{\pi i}{2m}\right)$ , we take  $\mathbf{n}_0 = e^{i\beta}\mathbf{e}_p + e^{-i\beta}\mathbf{e}_{2m-p+1}$  following (93), and thus the  $k$ th component of the vector  $\mathbf{n} = \Psi_0(x, t, \nu)\mathbf{n}_0$  can be written as follows:

$$\mathbf{n}_k = e^{\left(\frac{\nu}{\omega^{p-\frac{1}{2}} - \frac{\omega^{p-\frac{1}{2}}}{\nu}}\right)x + \left(\frac{\nu^2}{\omega^{2p-1} - \frac{\omega^{2p-1}}{\nu^2}}\right)t + \beta i} (\mathbf{e}_p)_k + e^{\left(\frac{\nu}{\omega^{p-\frac{1}{2}} - \frac{\omega^{\frac{1}{2}-p}}{\nu}}\right)x + \left(\nu^2\omega^{2p-1} - \frac{1}{\omega^{2p-1}\nu^2}\right)t - \beta i} (\mathbf{e}_{2m-p+1})_k = e^a (e^{-bi}\omega^{kp} + e^{bi}\omega^{-k(p-1)}),$$

where

$$a = \left(\nu - \frac{1}{\nu}\right) \cos\left(\frac{(2p-1)\pi}{2m}\right)x + \left(\nu^2 - \frac{1}{\nu^2}\right) \cos\left(\frac{(2p-1)\pi}{m}\right)t$$

and  $b$  is defined by (97). From (48), it follows that

$$\tau_j = \frac{1}{\nu^{2N} - 1} e^{2a} \sum_{k=1}^N \left( (e^{-bi}\omega^{kp} + e^{bi}\omega^{-k(p-1)})^2 \right) \mu^{2\{(j-k) \bmod N\}}$$

$$= e^{2a} \left( e^{-2bi} \frac{\omega^{2pj}}{\mu^2\omega^{-2p} - 1} + e^{2bi} \frac{\omega^{(2-2p)j}}{\mu^2\omega^{2p-2} - 1} + \frac{2\omega^j}{\mu^2\omega^{-1} - 1} \right)$$

$$= 2e^{2a}\omega^j \left( \frac{\nu^2 \cos\left(2b - \frac{(2p-1)(j+1)\pi}{m}\right) - \cos\left(2b - \frac{(2p-1)j\pi}{m}\right)}{|\nu^2\omega^{2p-1} - 1|^2} + \frac{1}{\nu^2 - 1} \right),$$

which leads to the periodic solutions for  $\phi^{(j)}$  given in the statement.  $\square$

The elementary rank 1 kink solutions correspond to the two-dimensional  $\Delta$ -invariant subspaces of  $\mathbb{R}^N$ . The other rank 1 solutions correspond to invariant subspaces of dimension 3, 4, ...,  $N$ . The number of all possible  $\Delta$ -invariant subspaces gives us the number of all rank 1 solutions.

**Theorem 3.** Eq. (1) with odd  $N = 2m - 1$  has  $2^m - 2$  different rank 1 kink solutions. In the case of even  $N = 2m$ , it has  $3 \cdot 2^m - 4$  different rank 1 kink solutions.

**Proof.** When  $N = 2m - 1$ , there are  $m - 1$  elementary solutions from  $\mathbf{n}_0 \in E_p^2$  for each  $p = 1, 2, \dots, m - 1$ , and one constant solution from  $\mathbf{n}_0 \in E_0^1$ . We can build other solutions by taking any combination of them. For example, there are  $C_m^2 = \frac{m(m-1)}{2}$  different solutions if we take any two combinations. Thus, the total number of different rank 1 kink solutions is

$$m - 1 + \sum_{k=2}^m C_m^k = 2^m - 2.$$

When  $N = 2m$  and  $\mu = \nu \in \mathbb{R}$ , there are  $m - 1$  elementary solutions from  $\mathbf{n}_0 \in E_p^2$  for each  $p = 1, 2, \dots, m - 1$ , one elementary solution from  $\mathbf{n}_0 \in E_0^1 \oplus E_m^1$ , and two constant solutions from  $\mathbf{n}_0 \in E_0^1$  or  $\mathbf{n}_0 \in E_m^1$ . We can build other solutions by taking any combinations of the  $m - 1$  elementary solutions alone, or with either one real or both real eigenvectors. Thus, the total number of different rank 1 kink solutions in this case is

$$1 + 4 \sum_{k=1}^m C_{m-1}^k = 4(2^{m-1} - 1) + 1 = 2^{m+1} - 3.$$

When  $N = 2m$  and  $\mu = \nu e^{\frac{\pi i}{2m}}$ ,  $\nu \in \mathbb{R}$ , there are also  $m$  elementary solutions. In this case, the total number of different rank 1 kink solutions is

$$\sum_{k=1}^m C_m^k = 2^m - 1.$$

Hence, when  $N = 2m$ , the total number of different rank 1 kink solutions is  $3 \cdot 2^m - 4$ . Thus, we have completed the proof.  $\square$

Note that the statement is consistent with the actual results for  $N = 3$  and  $N = 4$ . We plot some density plots of  $\phi^{(1)}$  and show snapshots for  $N = 5$  when  $\nu = 0.4$  and  $\alpha$  in Figs. 7 and 8.

#### 4.1.4. Tropicalization and wave front trajectories

In the general case of rank 1 ‘‘kink’’ solutions, the trajectories of the wave fronts can be understood geometrically. According to Proposition 5, the  $(x, t)$  dependence of the solution is determined by the vector  $\mathbf{n}$ , which can be presented in the form

$$\mathbf{n} = \sum_{k=1}^N e^{\Theta_k(x,t)} \mathbf{e}_k,$$

where

$$\Theta_k(x, t) = (\nu\omega^{-k} - \nu^{-1}\omega^k)x + (\nu^2\omega^{-2k} - \nu^{-2}\omega^{2k})t + \log \alpha_k.$$

The imaginary part  $\Theta_k^{Im}(x, t)$  is responsible for the oscillations of the solution, whereas the real part

$$\Theta_k^{Re}(x, t) = (\nu - \nu^{-1}) \cos\left(\frac{2\pi k}{N}\right)x + (\nu^2 - \nu^{-2}) \cos\left(\frac{4\pi k}{N}\right)t + \log |\alpha_k|$$

indicates the term in the sum that is dominant at a given point  $(x, t)$ . In a region where only one term in the sum is dominant, we can ignore other terms and the solution is close to the trivial (zero) solution. In regions where two terms have the same real exponent ( $\Theta_k^{Re}(x, t) = \Theta_{-k}^{Re}(x, t)$ ), we observe elementary waves. The boundaries of these regions correspond to the wave fronts. Thus, the wave fronts can be described as follows. We consider a set of linear functions  $\Theta_k^{Re}(x, t)$ ,  $k = 1, \dots, N$  and define a continuous piecewise linear function

$$\Theta(x, t) = \max(\Theta_1^{Re}(x, t), \dots, \Theta_N^{Re}(x, t)). \tag{98}$$

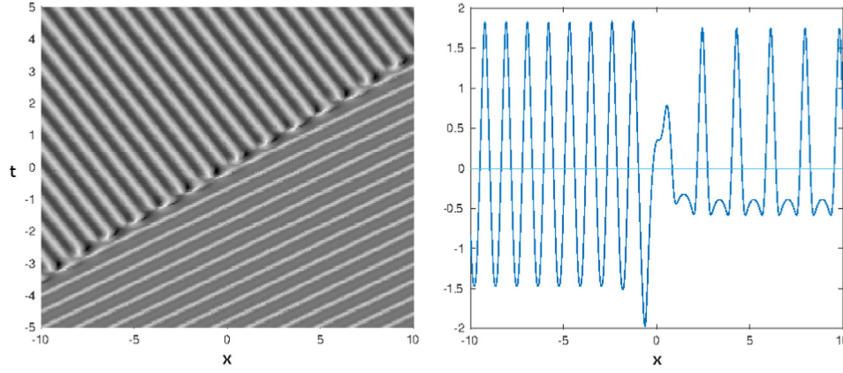


Fig. 7. Density plot of  $\phi^{(1)}(x, t)$  and a snapshot of  $\phi^{(1)}$  at  $t = 0$  ( $\alpha = (1, 1, 1, 1, 0)$ ,  $\nu = 0.4$ ).

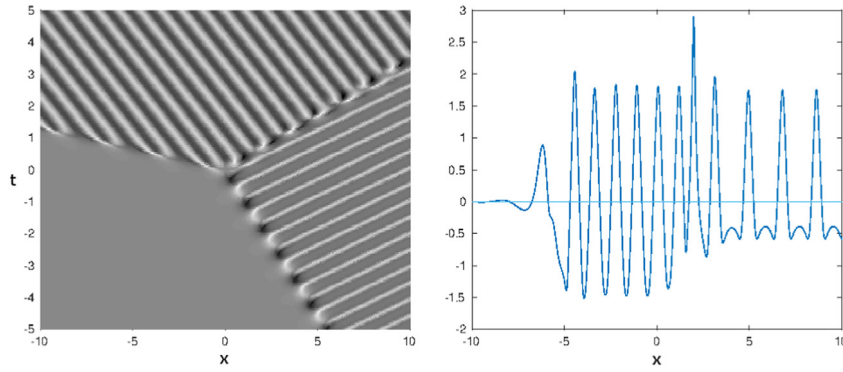


Fig. 8. Density plot of  $\phi^{(1)}(x, t)$  and a snapshot of  $\phi^{(1)}$  at  $t = 0.7$  ( $\alpha = (1, 1, 1, 1, 1)$ ,  $\nu = 0.4$ ).

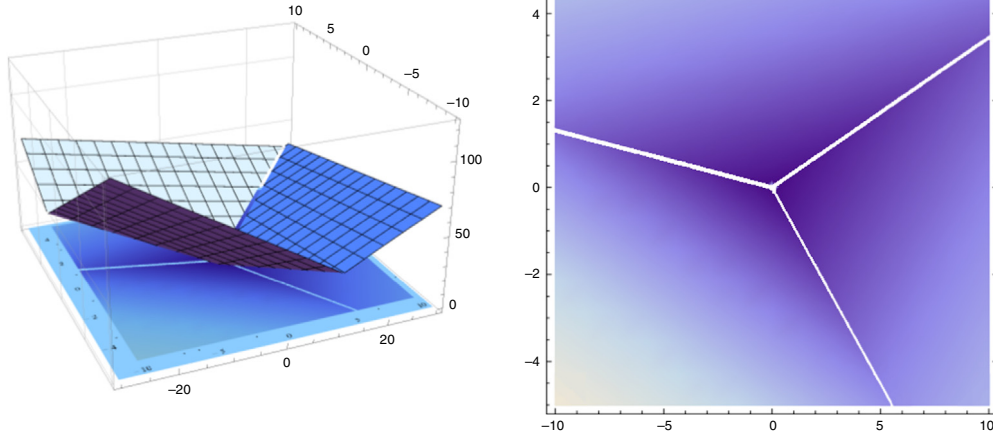


Fig. 9. Density and three-dimensional plots for  $\Theta(x, t)$ ,  $N = 5$ ,  $\alpha = (1, 1, 1, 1, 1)$ ,  $\nu = 0.4$  (compare with Fig. 8).

The locus where the function  $\Theta(x, t)$  is not smooth corresponds to the wave fronts. To compare the numerical result for wave fronts with the locus described above, we can compare Figs. 8 and 9. This construction is similar to tropicalization and the soliton graphs proposed by Kodama and Williams in the case of KP solitons [12], although there is a slight difference because we do not use rescaling in our definition and we retain the logarithmic term  $\log |\alpha_k|$ , which disappears in the scaling limit.

#### 4.2. Classification of rank 1 breather solutions for arbitrary dimensions

In this section, we classify all possible rank 1 breather solutions for arbitrary dimension  $N$ . According to Proposition 9, our soliton

solution depends only on  $\mathbf{n}_0 \in \mathbb{C}^N$ ,  $\mu \in \mathbb{C}$ , and  $|\mu| \notin \{0, 1\}$ . In a similar manner to the kinks, a natural way of classifying the possible solutions in terms of  $\mathbf{n}_0$  is to first consider the eigenvectors and eigenvalues of the constant matrix  $\Delta$ . We decompose  $\mathbb{C}^N$  as a direct sum of the invariant subspaces of  $\Delta$ , as follows:

$$\mathbb{C}^N = \bigoplus_{p=1}^N E_p^1, \quad E_p^1 = \text{span}_{\mathbb{C}}(\mathbf{e}_p).$$

The vector  $\mathbf{n}_0$  in this basis

$$\mathbf{n}_0 = \sum_{p=1}^N \alpha_p \mathbf{e}_p, \quad \alpha_p \in \mathbb{C} \tag{99}$$

is given by a matrix  $\alpha = (\alpha_1, \dots, \alpha_N)$ . We immediately obtain the following result.



$$\begin{aligned}
 |\gamma| &= \left| \frac{\alpha_q}{\alpha_p} \right| \left| e^{\left(\frac{\mu}{\omega^q} - \frac{\omega^q}{\mu} - \frac{\mu}{\omega^p} + \frac{\omega^p}{\mu}\right)x + \left(\frac{\mu^2}{\omega^{2q}} - \frac{\omega^{2q}}{\mu^2} - \frac{\mu^2}{\omega^{2p}} + \frac{\omega^{2p}}{\mu^2}\right)t} \right| \\
 &= e^{\left(|\mu| - \frac{1}{|\mu|}\right)\left(\cos\left(\delta - \frac{2\pi q}{N}\right) - \cos\left(\delta - \frac{2\pi p}{N}\right)\right)x + \left(|\mu|^2 - \frac{1}{|\mu|^2}\right)\left(\cos\left(2\delta - \frac{4\pi q}{N}\right) - \cos\left(2\delta - \frac{4\pi p}{N}\right)\right)t + \ln \left| \frac{\alpha_q}{\alpha_p} \right|} \\
 &= e^{2\left(|\mu| - \frac{1}{|\mu|}\right) \sin\left(\delta - \frac{(p+q)\pi}{N}\right) \sin \frac{(q-p)\pi}{N} (x - v_{pq}t - x_{pq}^0)}.
 \end{aligned}$$

**Box I.**

**Proposition 12.** For a constant complex vector  $\mathbf{n}_0$  in the form of (99), if only one  $\alpha_p$  is nonzero, then the solutions for (1) are trivial, i.e.,  $\phi^{(i)} = 0$ .

**Proof.** From (95), it follows that

$$\begin{aligned}
 \frac{1}{\mu^{2N} - 1} \sum_{l=1}^N \omega^{2pl} \mu^{2\{(i-l) \bmod N\}} &= \frac{\omega^{2p(i+1)}}{\mu^2 - \omega^{2p}}; \\
 \frac{1}{|\mu|^{2N} - 1} \sum_{l=1}^N |\mu|^{2\{(i-l) \bmod N\}} &= \frac{1}{|\mu|^2 - 1}.
 \end{aligned}$$

Thus, we can compute  $\tau_i$  in (62) as follows:

$$\begin{aligned}
 \tau_i &= |\alpha_p|^4 \left| e^{\left(\frac{\mu}{\omega^p} - \frac{\omega^p}{\mu}\right)x + \left(\frac{\mu^2}{\omega^{2p}} - \frac{\omega^{2p}}{\mu^2}\right)t} \right|^4 \\
 &\quad \times \left( \frac{1}{(|\mu|^2 - 1)^2} - \frac{1}{|\mu^2 - \omega^{2p}|^2} \right),
 \end{aligned}$$

which is independent of  $i$ . According to Proposition 9, we obtain the solutions  $\phi^{(i)} = 0$ , as stated above.  $\square$

Now, we consider the case where there are only two nonzero components, e.g.,  $\alpha_p$  and  $\alpha_q$ , among all  $\alpha_i$ ,  $i = 1, \dots, N$ , i.e.,

$$\mathbf{n}_0 = \alpha_p \mathbf{e}_p + \alpha_q \mathbf{e}_q, \quad \alpha_p \alpha_q \neq 0, \quad q > p. \tag{100}$$

It follows that the  $k$ -component of vector  $\mathbf{n}$  is

$$\begin{aligned}
 n_k &= \alpha_p e^{\left(\frac{\mu}{\omega^p} - \frac{\omega^p}{\mu}\right)x + \left(\frac{\mu^2}{\omega^{2p}} - \frac{\omega^{2p}}{\mu^2}\right)t} \omega^{kp} + \alpha_q e^{\left(\frac{\mu}{\omega^q} - \frac{\omega^q}{\mu}\right)x + \left(\frac{\mu^2}{\omega^{2q}} - \frac{\omega^{2q}}{\mu^2}\right)t} \omega^{kq} \\
 &= A(\omega^{kp} + \gamma \omega^{kq}),
 \end{aligned}$$

where we introduce notations for  $A, \gamma \in \mathbb{C}$  to shorten the expressions for  $\sigma(j)$  and  $\rho(j)$  defined by (63). We have

$$\begin{aligned}
 \sigma(j) &= A^2 \left( \frac{\omega^{2p(j+1)}}{\mu^2 - \omega^{2p}} + 2\gamma \frac{\omega^{(p+q)(j+1)}}{\mu^2 - \omega^{p+q}} + \gamma^2 \frac{\omega^{2q(j+1)}}{\mu^2 - \omega^{2q}} \right); \\
 \rho(j) &= |A|^2 \left( (1 + |\gamma|^2) \frac{1}{|\mu|^2 - 1} + \gamma^* \frac{\omega^{(p-q)(j+1)}}{|\mu|^2 - \omega^{p-q}} \right. \\
 &\quad \left. + \gamma \frac{\omega^{(q-p)(j+1)}}{|\mu|^2 - \omega^{q-p}} \right).
 \end{aligned}$$

According to Proposition 9, the breather solutions depend on  $\gamma$  and  $\mu$  because  $A$  is cancelled when we compute the solutions. Let  $\mu = |\mu|e^{i\delta}$ . The breather trajectory is determined by the condition that  $|\gamma| = 1$ , where (see equation in Box I).

This reflects the balance between the exponents. Thus, the speed of the breather is given by

$$v_{pq} = -4 \left( \frac{1}{|\mu|} + |\mu| \right) \cos \left( \delta - \frac{\pi(p+q)}{N} \right) \cos \left( \frac{\pi(q-p)}{N} \right)$$

and it is shifted to the right along the  $x$ -axis by

$$x_{pq}^0 = \frac{\ln \left| \frac{\alpha_p}{\alpha_q} \right|}{2\left(|\mu| - \frac{1}{|\mu|}\right) \sin\left(\delta - \frac{(p+q)\pi}{N}\right) \sin \frac{(q-p)\pi}{N}}.$$

It is localized in  $x$  and has a size of  $L_{pq}$

$$L_{pq}^{-1} = 2\left(|\mu| - |\mu|^{-1}\right) \sin \left( \delta - \frac{(p+q)\pi}{N} \right) \sin \frac{(q-p)\pi}{N}.$$

The rank 1 breather solutions can be obtained in the following manner.

- There are  $C_N^2$  possible choices of two-dimensional  $\Delta$ -invariant subspaces in  $C^N$ , and thus there are  $C_N^2$  elementary breathers.
- Solutions corresponding to three-dimensional invariant subspaces, i.e.,

$$\mathbf{n}_0 = \alpha_p \mathbf{e}_p + \alpha_q \mathbf{e}_q + \alpha_r \mathbf{e}_r, \quad \alpha_p \alpha_q \alpha_r \neq 0$$

represent the decays or fusions of breathers (“Y” shape), and there are  $C_N^3$  of these solutions.

- Solutions corresponding to four-dimensional invariant subspaces, i.e.,

$$\mathbf{n}_0 = \alpha_p \mathbf{e}_p + \alpha_q \mathbf{e}_q + \alpha_r \mathbf{e}_r + \alpha_s \mathbf{e}_s, \quad \alpha_p \alpha_q \alpha_r \alpha_s \neq 0$$

represent solutions combining two “Y” shapes (“2Y” shape solutions). There are  $C_N^4$  of these solutions, etc.

Examples of “Y”, “2Y”, and “3Y” configurations in the case where  $N = 5$  are presented in Fig. 10.

The total number of possible distinct configurations for a breather solution given by Proposition 9 is

$$\sum_{k=2}^N C_N^k = 2^N - N - 1.$$

The type of breather solution depends on the choice of the matrix  $\alpha = (\alpha_1, \dots, \alpha_N)$ . The explicit expression for the solution is given in Proposition 9 and it is quite complicated, but the tropicalization method, which we used in Section 4.1.4, allows us to give a simple description of the soliton graph. We consider the observation that the vector

$$\mathbf{n} = \sum_{k=1}^N e^{\Theta_p(x,t)} \mathbf{e}_p,$$

where

$$\begin{aligned}
 \Theta_p(x, t) &= \left( \frac{\mu}{\omega^p} - \frac{\omega^p}{\mu} \right) x + \left( \frac{\mu^2}{\omega^{2p}} - \frac{\omega^{2p}}{\mu^2} \right) t \\
 &\quad + \log \alpha_p, \quad \mu = |\mu|e^{i\delta}
 \end{aligned}$$

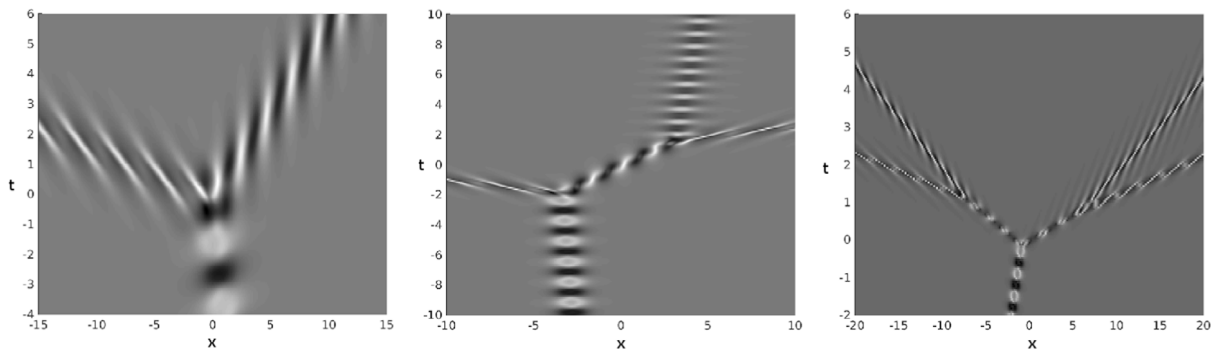
completely determines the  $(x, t)$  dependence of the solution. In the regions where only one term is dominant, the solution is exponentially small. We can define the tropical graph of the breather as a locus where two or more terms are in balance. In particular, let us consider the real part of  $\Theta_p(x, t)$ , i.e.,

$$\begin{aligned}
 \Theta_p^{\text{Re}}(x, t) &= (|\mu| - |\mu|^{-1}) \cos \left( \delta - \frac{2\pi p}{N} \right) x \\
 &\quad + (|\mu|^2 - |\mu|^{-2}) \cos \left( 2\delta - \frac{4\pi p}{N} \right) t + \log |\alpha_p|
 \end{aligned}$$

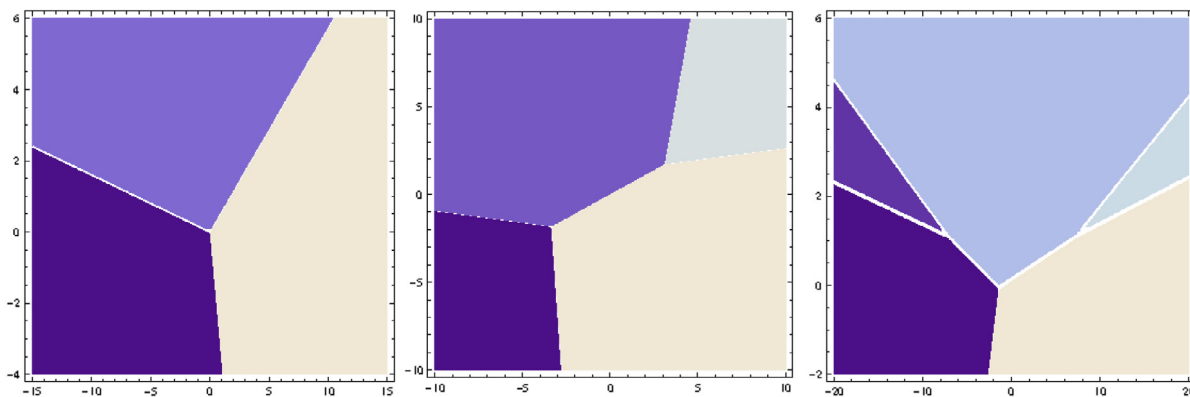
and a piecewise linear continuous function of variables  $(x, t)$ :

$$\Theta(x, t) = \max_p \Theta_p^{\text{Re}}(x, t).$$

**Definition 1.** For rank one breather solutions, the tropical soliton graph is defined as a locus of points where the function  $\Theta(x, t)$  is not smooth.



**Fig. 10.** Density plots of  $\phi^{(1)}(x, t)$  for breather solutions where  $N = 5$  with  $\mu = 0.7 + 0.15i$ ,  $\alpha = (1, 0, 1, 0, 1)$ , and  $\mu = 0.5 + 0.15i$ ,  $\alpha = (1, 0.01, 1, 0, 0.0001)$ , and  $\mu = 0.3 + 0.15i$ ,  $\alpha = (0.01, 1, 10, 10i, 0.1)$ .



**Fig. 11.** Tropical plots of  $\phi^{(1)}(x, t)$  for breather solutions where  $N = 5$  with parameters  $\mu$  and  $\alpha$  corresponding to the plots in Fig. 10.

In order to visualize the tropical plot, we show the density plot for the piecewise constant function  $\partial_x \Theta(x, t)$  in Fig. 11, where the plots correspond to the solutions plotted in Fig. 10.

This definition does not reflect the fact that we are dealing with a system of equations, and thus the graphs corresponding to the variables  $\phi^{(i)}(x, t)$ ,  $i = 1, \dots, N$  are slightly different (they may depend on the index  $i$ ). Thus, this can be considered as a first approximation that captures the trajectories of the solitons well (breathers).

The general approach for visualizing the rank  $r$  solutions is similar to the case of rank one. We use the fact that a rank  $r$  solution is a function of the point

$$\mathbf{n}(x, t) = \exp((\mu \Delta^{-1} - \mu^{-1} \Delta)x - (\mu^{-2} \Delta^2 - \mu^2 \Delta^{-2})t) \mathbf{n}_0$$

on the Grassmannian  $\text{Gr}(r, N)$ , where  $\mathbf{n}(x, t)$  is a  $N \times r$  full rank matrix (because this follows from Proposition 10). In the basis  $\mathbf{e}_k$  (90), the matrix  $\mathbf{n}(x, t)$  can be represented as

$$\mathbf{n}(x, t) = (\mathbf{e}_1, \dots, \mathbf{e}_N) \alpha^{\text{tr}}(x, t),$$

where  $\alpha(x, t)$  is an  $r \times N$  matrix of full rank and

$$(\alpha(x, t))_{pq} = \alpha_{pq}^{(0)} \exp((\mu \omega^{-q} - \mu^{-1} \omega^q)x - (\mu^{-2} \omega^{2q} - \mu^2 \omega^{-2q})t), \quad 1 \leq p \leq r, 1 \leq q \leq N.$$

Let

$$I = \{i_1 < i_2 < \dots < i_r\} \subset [1, \dots, N]$$

and let  $\Delta_I(x, t)$  denote the minor of  $\alpha(x, t)$  with columns  $i_1, \dots, i_r$  (a Plücker coordinate on the Grassmannian  $\text{Gr}(r, N)$ ). Let us define  $\Theta_I(x, t) = \log |\Delta_I(x, t)|$  if  $\Delta_I(x, t) \neq 0$ , and for  $I$  such that  $\Delta_I(x, t) = 0$ , we set  $\Theta_I(x, t) = -\infty$ . The function  $\Theta_I(x, t)$  is a linear function of the coordinates  $(x, t)$ . If there is only one nonzero minor  $\Delta_I(x, t)$ , then it is easy to show that the corresponding solution is  $\phi^{(i)}(x, t) = 0$ ,  $i = 1, \dots, N$  (similar to Proposition 12).

The solution is concentrated near the points where two or more Plücker coordinates are in balance and we can give the following definition of the tropical soliton graph in the case of rank  $r$  breather solutions.

**Definition 2.** For rank  $r$  breather solutions, the tropical soliton graph is defined as a locus of points where the function  $\Theta(x, t) = \max_I \Theta_I(x, t)$  is not smooth.

Using the definition, we plot the tropical soliton graph for

$$N = 5, \quad \mu = 0.57 + 0.2i, \\ \alpha = \begin{pmatrix} 1 & 10 & 10^4 & 10^3 & 1 \\ 10^6 & 1 & 10^6 & 10^4 & 99.9 \end{pmatrix}$$

and compare it with the actual density plot for  $\phi^{(1)}(x, t)$  in Fig. 12. The definition of a tropical soliton graph given above is not perfect (it does not reflect the dependence of the graph on the index  $i$  for different components  $\phi^{(i)}(x, t)$ ), but it reflects the breather interactions well. It also allows the classification of possible configurations in the multi-soliton solutions of arbitrary rank.

## 5. Conclusion

In this study, we proposed a dressing method for the two-dimensional Volterra system (1). We constructed two types of exact solutions for the system. The first type is rather unusual because it represents the propagation of wave fronts. To the best of our knowledge, this is a new class of solutions in integrable models. The second type resembles breathers in the sine-Gordon equation. Nonlinear wave (“kink”) solutions are parameterized by a real parameter  $\nu$  and a point on a real Grassmannian  $\text{Gr}_{\mathbb{R}}(r, N)$ . In the case of breathers, the parameters  $\mu$  and Grassmannian  $\text{Gr}_{\mathbb{C}}(r, N)$  are complex. The integer  $r$  is the rank of the solution. We show that

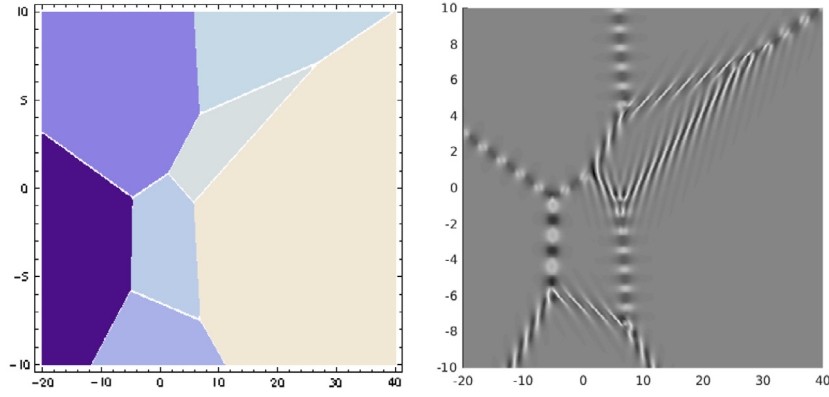


Fig. 12. Tropical and density plots of  $\phi^{(1)}(x, t)$  for rank 2 breather solutions  $N = 5$ .

all of these solutions are regular in the Appendix. We also studied the detailed properties and configurations of rank 1 solutions, where the Grassmannians are real and complex projective spaces, respectively. The classification of rank  $r$  solutions was linked with the classification of  $\Delta$ -invariant Schubert decompositions of the Grassmannians, where  $\Delta$  is the cyclic shift matrix from the Lax representation of the two-dimensional Volterra system (1).

In this study, we did not provide a classification of higher rank solutions, but we claim that their properties are quite different from those of the solutions of rank one. For example, the nonlinear wave (“kink”) solutions of rank 2 may represent nonlinear interference of waves (see Fig. 1, right), which is impossible in the case of rank one solutions. In the case where  $N = 4$ , we presented all the possible rank 1 kink solutions in Section 4.1.2. We show density plots for two kink solutions of rank 2 when  $N = 4$  in Fig. 13 and some snapshots in Fig. 14, which do not resemble any rank 1 solutions. Breather solutions of rank 1 do not have closed loops, but instead rank 2 loops exist (see Fig. 2).

To study the structure and classifications of higher rank wave front and breather solutions as well as multi-soliton solutions (with a finite number of orbits of the poles in the dressing matrix  $\Phi(\lambda)$ ), we need to develop methods similar to those proposed by Kodama et al. for the KP equation [10–12]. There is also an interesting and as yet unsolved problem regarding finding the solutions of (1), which approximate the solutions of the KP equation for large  $N$ .

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### Appendix. The regularity of solutions

In Section 3, we derived rank  $r$  ( $r \geq 1$ ) solutions for both kinks and breathers, which are all expressed as

$$\phi^{(i)} = \frac{1}{2} \ln \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \quad (101)$$

according to Propositions 5–10. In this section, we show that  $\frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} > 0$  for all of the solutions obtained, which guarantees that there are no singularities for the solutions, i.e., the solutions are regular.

We only prove this for  $\tau_i$  in Propositions 7, 8 and 10 associated with the rank  $r$  solutions. The rank 1 solutions are special cases. For completeness, we first write the Cauchy–Binet formula from [26] (on page 9). We also introduce some notations.

For an  $m \times n$  matrix  $A$  and an  $n \times m$  matrix  $B$ , the Cauchy–Binet formula is

$$\det(AB) = \sum_{J \in \mathcal{K}} \det(A_{[m],J}) \det(B_{J,[m]}), \quad (102)$$

where  $\mathcal{K} = \binom{[n]}{m}$  denotes the set of  $m$ -combinations of the set  $\{1, \dots, n\}$  denoted by  $[n]$ . For  $J \in \mathcal{K}$ ,  $A_{[m],J}$  is the  $m \times m$  matrix where the columns are the columns of  $A$  at indices from  $J$ , and  $B_{J,[m]}$  is the  $m \times m$  matrix where the rows are the rows of  $B$  at indices from  $J$ . We use this to prove the regularity of solutions (101).

**Proposition 13.** Let  $\mathbf{n}$  be a rank  $r$  real matrix of size  $N \times r$  and  $v \in \mathbb{R}$ ,  $v \neq 0$ ,  $v \neq \pm 1$ . Then,

$$\det(\mathbf{n}^{\text{tr}} S(j) \mathbf{n}) > 0,$$

where  $S(j)$  is an  $N \times N$  diagonal matrix and the  $i$ th diagonal entry is equal to  $v^{2\{(j-i) \bmod N\}}$ .

**Proof.** We compute the determinant using the Cauchy–Binet formula (102):

$$\begin{aligned} \det(\mathbf{n}^{\text{tr}} S(j) \mathbf{n}) &= \sum_{J \in \mathcal{K}} \det(\mathbf{n}_{[r],J}^{\text{tr}}) \det(S(j)_{J,J}) \det(\mathbf{n}_{J,[r]}) \\ &= \sum_{J \in \mathcal{K}} \det(S(j)_{J,J}) \det(\mathbf{n}_{J,[r]})^2, \end{aligned}$$

where  $S(j)_{J,J}$  is a diagonal matrix with diagonal entries at indices from  $J$ . Note that all the diagonal entries of matrix  $S(j)$  are positive, which implies that  $\det(S(j)_{J,J}) > 0$ . Hence, we have proved the statement.  $\square$

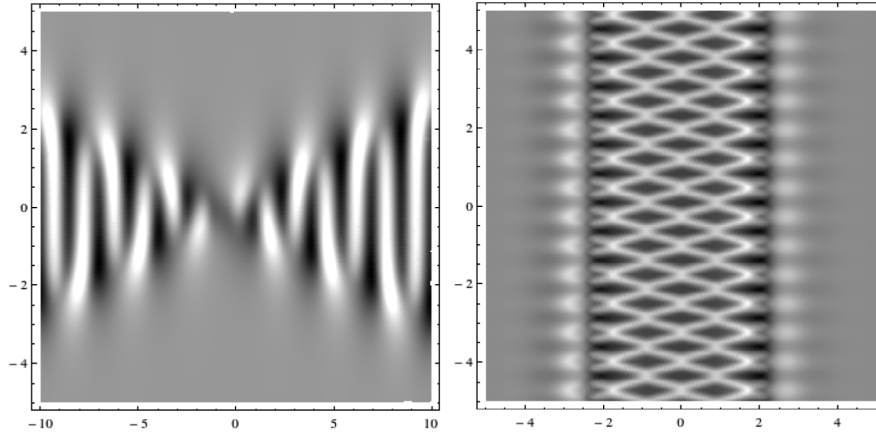
For the solutions obtained in Proposition 8, we know that

$$\mu = v \exp\left(\frac{\pi i}{N}\right), \quad \mu^{2\{(j-i) \bmod N\}} = \omega^{\{(j-i) \bmod N\}} v^{2\{(j-i) \bmod N\}},$$

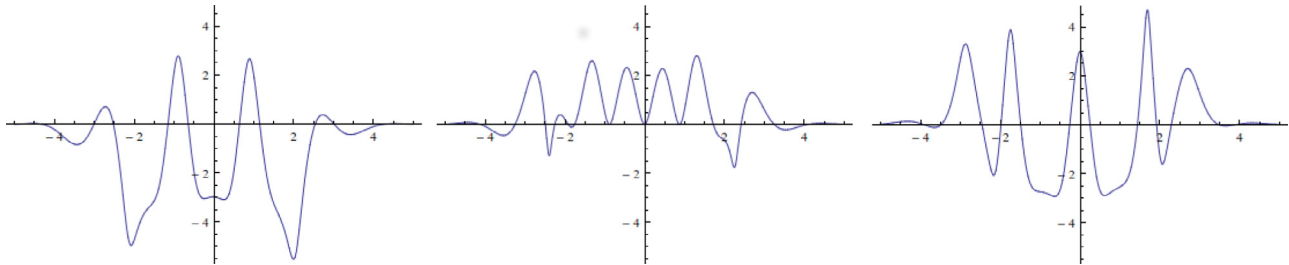
$$\mathbf{n} = \text{diag}(\omega^{\frac{j}{2}}) \tilde{\mathbf{n}},$$

where  $\omega = \exp\left(\frac{2\pi i}{N}\right)$  and  $\tilde{\mathbf{n}}$  is a rank  $r$  real matrix of size  $N \times r$ . Thus, we have

$$\begin{aligned} \tau_j &= \det\left(\frac{1}{\mu^{2N} - 1} \mathbf{n}^{\text{tr}} S(j) \mathbf{n}\right) \\ &= \frac{\omega^j}{(\mu^{2N} - 1)^r} \det(\tilde{\mathbf{n}}^{\text{tr}} \text{diag}(v^{2\{(j-i) \bmod N\}}) \tilde{\mathbf{n}}) = \frac{\omega^j}{(\mu^{2N} - 1)^r} \tilde{\tau}_j, \end{aligned}$$



**Fig. 13.** Density plot of  $\phi^{(1)}(x, t)$  for a kink solution of rank 2 where  $N = 4$ :  $\nu = 0.8$ ,  $\mathbf{n}_0 = (i\mathbf{e}_1 - 2\mathbf{e}_2 - i\mathbf{e}_3 + \mathbf{e}_4, 2\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3 + \mathbf{e}_4)$  in the left plot, and  $\mu = 2e^{\frac{\pi i}{4}}$ ,  $\mathbf{n}_0 = (\mathbf{e}_1 + (1 + 100i)\mathbf{e}_2 + \mathbf{e}_3 + (1 - 100i)\mathbf{e}_4, 10i\mathbf{e}_1 + 5i\mathbf{e}_2 - 10i\mathbf{e}_3 - 5i\mathbf{e}_4)$  in the right plot.



**Fig. 14.** Snapshots of  $\phi^{(1)}(x, t)$  for the rank 2 kink solution of the plot on the right in Fig. 13:  $t = 0.0945$  for the left plot,  $t = -0.09075$  for the middle plot, and  $t = -0.276$  for the right plot.

where  $\tilde{\tau}_j > 0$  according to Proposition 13. Hence, we obtain

$$\frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2} = \frac{\tilde{\tau}_{j-1}\tilde{\tau}_{j+1}}{\tilde{\tau}_j^2} > 0$$

to complete the proof for the regularity of the solutions in Proposition 8.

Finally, we show the regularity for the breather solutions obtained in Proposition 10.

**Proposition 14.** Let  $\mathbf{n}$  be a rank  $r$  constant matrix of size  $N \times r$  and  $\mu \in \mathbb{C}$ ,  $|\mu| \neq 1$ ,  $\mu \neq \omega^k \mu^*$ ,  $k = 1, \dots, N$ . Then,  $\tau_j > 0$  defined by (77) for all  $j = 1, \dots, N$ .

**Proof.** Recall that  $\tau_j = \det H(j)$ , and the  $r \times r$  matrices  $R(j)$  and  $P(j)$  are defined as

$$P(j) = \mathbf{n}^\dagger \text{diag} \left( \frac{|\mu|^{2(j-i) \bmod N}}{||\mu|^{2N} - 1|} \right) \mathbf{n} = \mathbf{n}^\dagger \alpha(j) \mathbf{n};$$

$$R(j) = \mathbf{n}^{\text{tr}} \text{diag} \left( \frac{\mu^{2(j-i) \bmod N}}{\mu^{2N} - 1} \right) \mathbf{n} = \mathbf{n}^{\text{tr}} \beta(j) \mathbf{n},$$

where  $\alpha(j)$  and  $\beta(j)$  are the corresponding diagonal matrices. We substitute  $P(j)$  and  $R(j)$  into the matrix  $H(j)$ , which then becomes

$$H(j) = \begin{pmatrix} P(j) & R(j)^* \\ R(j) & P(j)^* \end{pmatrix} = \begin{pmatrix} \mathbf{n}^\dagger & 0 \\ 0 & \mathbf{n}^{\text{tr}} \end{pmatrix} \begin{pmatrix} \alpha(j) & \beta(j)^* \\ \beta(j) & \alpha(j) \end{pmatrix} \begin{pmatrix} \mathbf{n} & 0 \\ 0 & \mathbf{n}^* \end{pmatrix}.$$

We introduce the following notations.

$$A = \begin{pmatrix} \mathbf{n} & 0 \\ 0 & \mathbf{n}^* \end{pmatrix}; \quad B = \begin{pmatrix} \alpha(j) & \beta(j)^* \\ \beta(j) & \alpha(j) \end{pmatrix}.$$

Note that the matrix  $B$  is Hermitian. Thus, a unitary matrix  $U$  exists such that  $B = U^\dagger \gamma U$ , where  $\gamma$  is a diagonal matrix and its real

eigenvalues are

$$\begin{aligned} & \frac{|\mu|^{2(j-i) \bmod N}}{||\mu|^{2N} - 1|} \pm \frac{|\mu|^{2(j-i) \bmod N}}{|\mu^{2N} - 1|} \\ & = |\mu|^{2(j-i) \bmod N} \left( \frac{1}{||\mu|^{2N} - 1|} \pm \frac{1}{|\mu^{2N} - 1|} \right) > 0. \end{aligned}$$

We now use the Cauchy–Binet formula to compute

$$\begin{aligned} \tau_j & = \det(A^\dagger U^\dagger \gamma U A) = \sum_{J \in \mathcal{K}} \det((UA)_{[2r], J}^\dagger) \det(\gamma_{J, J}) \det((UA)_{J, [2r]}) \\ & = \sum_{J \in \mathcal{K}} \det(\gamma_{J, J}) |\det((UA)_{J, [2r]})|^2, \end{aligned}$$

where  $\mathcal{K} = \binom{[2N]}{2r}$  for the set of  $2r$ -combinations of  $[2N]$ , and  $\gamma_{J, J}$  is a diagonal matrix with diagonal entries at indices from  $J$ . Note that the rank of  $U$  is  $N$ , the rank of  $A$  is  $2r$ , and all the diagonal entries of matrix  $\gamma$  are positive, which implies that  $\det(\gamma_{J, J}) > 0$ . Hence,  $\tau_j > 0$  and thus we have proved the statement.  $\square$

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